Abstract. A. Postnikov and B. Shapiro introduced a class of commutative algebras which enumerate forests and trees of graphs. Our main result is that the algebra counting forests depends only on graphical matroid and converse.

Also we generalize algebras for a hypergraph. For this, we define spanning forests and trees of a hypergraph and the corresponding "hypergraphical" matroid.

1. Introduction

The famous matrix-tree theorem of Kirchhoff (see [12] and p. 138 in [18]) claims that the number of spanning trees of a given graph $G$ equals to the determinant of the Laplacian matrix of $G$. It is also well known that the number of maximal spanning forests of $G$ (or equivalently trees for connected $G$) equals to $T_G(1, 1)$ while the number of all spanning forests of $G$ equals to $T_G(2, 1)$, where $T_G$ is the Tutte polynomial of $G$ (see p. 237 in [18]).

There exist many generalizations of the classical matrix-tree theorem, e.g. for directed graphs, matrix-forest theorems, etc (see e.g. [7], see [5] and references therein). In particular, in [15] A. Postnikov and B. Shapiro constructed several algebras associated to $G$ whose total dimensions are equal to the number of either spanning trees or forests of $G$.

We use the standard notation: $E(G)$ and $e(G)$ are the set and the number of edges of graph $G$; $V(G)$ and $v(G)$ are the set and the number of vertices of $G$; $T_G(x, y)$ is the Tutte polynomial of $G$.

Notation 1. Take an undirected graph $G$ with $n$ vertices and some field $\mathbb{K}$ of characteristic 0.

(1) Let $\Phi^F_G$ be the commutative algebra over $\mathbb{K}$ generated by \{\phi_e : e \in E(G)\} satisfying the relations $\phi^2_e = 0$, for any $e \in E(G)$.

Fix any linear order of vertices of $G$. For $i = 1, \ldots, n$, set

$$X_i = \sum_{e \in G} c_{i,e} \phi_e,$$

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where \( c_{i,e} = \pm 1 \) for vertices incident to \( e \) (for the smaller vertex \( v_i \), \( c_{i,e} = 1 \); for the larger vertex \( v_j \), \( c_{j,e} = -1 \)) and 0 otherwise. Denote by \( C_G^F \) the subalgebra of \( \Phi_G^F \) generated by \( X_1, \ldots, X_n \).

(II) Consider the ideal \( J_G^F \) in the ring \( \mathbb{K}[x_1, \ldots, x_n] \) generated by

\[
p_I^F = \left( \sum_{i \in I} x_i \right)^{D_I+1} ,
\]

where \( I \) ranges over all nonempty subsets of vertices, and \( D_I \) is the total number of edges between vertices in \( I \) and vertices outside \( I \). Define the algebra \( B_G^F \) as the quotient \( \mathbb{K}[x_1, \ldots, x_n]/J_G^F \).

**Remark 1.** In the above definition of \( C_G^F \), we can separately choose the "smaller" vertex for each edge. In other words, if we change the signs of all \( c_{i,e} \) for some edges, then we obtain an isomorphic algebra.

For an orientation \( \overline{G} \) of graph \( G \), we define the subalgebra \( C_G^F \) of \( \{ \phi_e : e \in E(G) \} \), where for each edge arrow goes to the "smaller" vertex.

The field \( \mathbb{K} \) of characteristic 0 is fixed throughout this paper. By a graph we always mean an undirected graph without loops (multiple edges are allowed).

**Notation 2.** Fix some linear order on the set \( E(G) \) of edges of \( G \). Let \( F \) be any spanning forest in \( G \). By \( \text{act}_G(F) \) denote the number of all externally active edges of \( F \), i.e. the number of edges \( e \in E(G) \setminus F \) such that (i) subgraph \( F + e \) has a cycle; (ii) \( e \) is the minimal edge in this cycle in the above linear order.

Denote by \( F^+ \) the set of edges of the forest \( F \) together with all externally active edges, and denote by \( F^- = E(G) \setminus F^+ \) the set of externally nonactive edges.

The following result was proved in [15].

**Theorem 1** (cf. [15]). For any graph \( G \), algebras \( B_G^F \) and \( C_G^F \) are isomorphic, their total dimension over \( \mathbb{K} \) is equal to the number of spanning forests in \( G \).

Moreover, the dimension of the \( k \)-th graded component of these algebras equals the number of spanning forests \( F \) of \( G \) with external activity \( e(G) - e(F) - k \).

In fact, the second part of Theorem 1 claims that the above Hilbert polynomial is a specialization of the Tutte polynomial of \( G \).

**Corollary 1.** The dimension of the \( k \)-th graded component of \( B_G^F \) is equal to the coefficient of the monomial \( y^{e(G) - e(F) - c(G) - k} \) in the polynomial \( T_G \left( 1 + \frac{1}{y}, y \right) \), (where \( c(G) \) is the number of connected components of \( G \)).
There are a few generalizations of $\mathcal{B}_F^G$ and $\mathcal{B}_T^G$ in the literature (the latter algebra counts trees in $G$ instead of forests, see notation in section 5), see two popular papers [2] and [9], in both papers the main object of the generalizations is the algebra of second definitions (quotient algebra). In [10] the definitions of $\mathcal{B}_\Delta^G$ and $\mathcal{C}_\Delta^G$ were given in terms of some simplex $\Delta$ on the set of vertices of $G$. With this definition algebras $\mathcal{B}_F^G$ and $\mathcal{B}_T^G$ become particular cases corresponding to different simplices.

In § 2 we prove that the original Postnikov-Shapiro algebra depends only on the graphical matroid of $G$. And conversely, we can reconstruct the matroid from such an algebra, see Theorem 2. It means that these algebras store almost all information about the graphs. As a consequence, the same is true for $t$-algebras.

In § 3 we generalize Theorem 1 for $t > 1$. In fact, these new algebras are isomorphic to the original algebras for the graph $\hat{G}$, where each edge of $G$ corresponds to $t$ edges from $\hat{G}$. We think this direction is important, because on one hand it is not a new object, but on the other hand we can reconstruct the Tutte polynomial from the Hilbert series of algebra corresponding to large $t$, see Theorem 7.

In § 4 we construct a family of algebras corresponding to a given hypergraph, and we also present a natural definition of a matroid of a hypergraph. For usual graphs, our definition gives graphical matroids. For a given hypergraph, we prove that almost all algebras in our family have the same Hilbert series, and this Hilbert series is a specialization of the Tutte polynomial corresponding to the hypergraphical matroid. This family of algebras is motivated by Postnikov-Shapiro-Shapiro algebras for vector configurations (see [16]).

In § 5 we discuss on similar problems for algebras counting spanning trees.

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2. Algebras and matroids

Obviously, the original Postnikov-Shapiro algebra corresponding to a disconnected graph $G$ are the Cartesian products of the algebras corresponding to the connected components of $G$. In particular, it is also true for 2-connected components (maximal connected subgraphs such that they remain connected, after removal of any vertex). The same fact is also true for matroids. In this section we prove the following result.

**Theorem 2.** Algebras $\mathcal{B}_{G_1}^F$ and $\mathcal{B}_{G_2}^F$ for graphs $G_1$ and $G_2$ are isomorphic if and only if the graphical matroids of these graphs coincide.
In the proof of Theorem 2 we use the following theorem of H. Whitney.

**Theorem 3** (Whitney’s 2-isomorphism theorem, see [19] or [14]). Let $G_1$ and $G_2$ be two graphs. Then their graphical matroids are isomorphic if and only if $G_1$ can be transformed to a graph, which is isomorphic to $G_2$ by a sequence of operations of vertex identification, cleaving and twisting.

These three operation are defined below.

1a) **Identification**: Let $v$ and $v'$ be vertices from different connected components of the graph. We modify the graph by identifying $v$ and $v'$ as a new vertex $v''$.

1b) **Cleaving** (the inverse of identification): A graph can only be cleft at a cut-vertex.

2) **Twisting**: Suppose that the graph $G$ is obtained from two disjoint graphs $G_1$ and $G_2$ by identifying vertices $u_1$ of $G_1$ and $u_2$ of $G_2$ as the vertex $u$ of $G_1$ and additionally identifying vertices $v_1$ of $G_1$ and $v_2$ of $G_2$ as the vertex $v$ of $G$. In a twisting of $G$ about $\{u,v\}$, we identify $u_1$ with $v_2$ and $u_2$ with $v_1$ to get a new graph $G'$.

We split our proof of Theorem 2 in two parts presented in §3.1 and in §3.2.

2.1. **Algebras are isomorphic if their matroids are isomorphic.** Because algebras $B_{G_1}^F$ and $C_{G}^F$ are isomorphic for any graph by Theorem 1, it suffices to prove Theorem 2 for algebras $C_{G_1}^F$ and $C_{G_2}^F$.

**Lemma 1.** If graphs $G$ and $G'$ differ by a sequence of Whitney’s deformations, then the algebras $C_{G_1}^F$ and $C_{G_2}^F$ are isomorphic.

**Proof.** It is sufficient to check the claim for each deformation separately.

1° **Identification and Cleaving.** We need to prove our fact only for cleaving, because identification is the inverse of cleaving. In this case algebras doesn’t change, because the linear subspace defined by $X_i$ for vertices doesn’t change. This holds, because if we split a vertex $k$ into $k'$ and $k''$, then in the new graph, $X_{k'}$ equals to the minus sum of $X_i$ corresponding to the vertices from its component except $k'$ (sum of $X$ from one connected component is zero), i.e. $X_{k'}$ belongs to the linear space $<X_1,\ldots,X_{k-1},X_{k+1},\ldots,X_n>$. Similarly $X_{k''}$ belongs to the linear space $<X_1,\ldots,X_{k-1},X_{k+1},\ldots,X_n>$. Hence, $<X_1,\ldots,X_{k'},X_{k''},\ldots,X_n>$ is a subspace of the linear space $<X_1,\ldots,X_n>$. The equation, $X_k = X_{k'} + X_{k''}$ implies that these linear spaces coincide.

2° **Whitney’s deformation of the second kind.** Define the digraph $\overrightarrow{G}$ as the orientation of the graph $G$, where each arrow goes to the "smaller" vertex (see Remark 1).
Let us make a twist of the vertices $u$ and $v$. Let $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ be the orientations of $G_1$ and $G_2$ corresponding to $\overrightarrow{G}$. Let $\overrightarrow{G'}$ be the orientation of $G'$ corresponding to the gluing $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ with reversing each arrow from $\overrightarrow{G_2}$. Vertex $u'$ in $G'$ is obtained by gluing of $u_1$ and $v_2$; $v'$ is obtained by gluing of $v_1$ and $u_2$.

Let $X_k$, $X_{1,k}$, $X_{2,k}$ and $X'_k$ be the sums of edges with signs of incident to vertex $k$ in graphs $\overrightarrow{G}$, $\overrightarrow{G_1}$, $\overrightarrow{G_2}$ and $\overrightarrow{G'}$.

For a vertex $k$ of $G_1$ except $u_1$ and $v_1$, we get $X_k = X_{1,k} = X'_k$.

For a vertex $k$ of $G_2$ except $u_2$ and $v_2$, we get $X_k = X_{2,k} = -X'_k$, because we reverse the orientation in the second part of twisting.

For other vertices we have:

- $X_u = X_{1,u_1} + X_{2,u_2}$;
- $X_v = X_{1,v_1} + X_{2,v_2}$;
- $X'_u = X_{1,u_1} - X_{2,v_2} = X_u - (X_{2,u_2} + X_{2,v_2})$;
- $X'_v = X_{1,v_1} - X_{2,u_2} = X_v - (X_{2,u_2} + X_{2,v_2})$.

We know that the sum of variables corresponding to the vertices of any graph is zero, because each edge goes with a plus to one vertex and with a minus to another. We have

$$\sum_{k \in G_2} X_{2,k} = 0,$$

$$\sum_{k \in G_2 \setminus \{u_2, v_2\}} X_{2,k} + X_{2,u_2} + X_{2,v_2} = 0,$$

$$X_{2,u_2} + X_{2,v_2} = - \sum_{k \in G_2 \setminus \{u_2, v_2\}} X_{2,k} = - \sum_{k \in G_2 \setminus \{u_2, v_2\}} X_k.$$

Hence, $X'_u$ and $X'_v$ belong to the linear space generated by $X_k$, where $k \in G$. In other words, the linear space for $\overrightarrow{G'}$ is a linear subspace of the space for $\overrightarrow{G}$. Similarly we can prove the converse. Then, the linear spaces coincide, and since we have the same relations ($\phi^2_e = 0$ for any edge), the algebras in these bases coincide. \hfill $\square$

We have proved the first part of Theorem 2, because if the corresponding graphical matroids are isomorphic, then there exists such a sequence of Whitney’s operations.

**Corollary 2.** The algebra corresponding to a graph $G$ is the Cartesian product of the algebras corresponding to the 2-connected components of $G$.

### 2.2. Reconstructing matroid from the algebra.

**Lemma 2.** It is possible to reconstruct the matroid of a graph $G$ from the algebra $\mathcal{C}_F^G$. 
Remark 2. We assume that we only know \( C_F^G \) as an algebra. I.e. we assume that we do not know the basis corresponding to the vertices of \( G \), and that we have no information about the graded components of \( C_F^G \).

Proof. For an element \( Y \in C_F^G \), we define its length \( \ell(Y) \) as the minimal number such that \( Y^{\ell+1} \) is zero (the length can be infinite).

We call an element \( Y \in C_F^G \) irreducible if there is no representation \( Y = \sum_i Z_{2i-1}Z_{2i} \) such that \( \ell(Z_j) \) is finite for any \( j \).

Consider a basis \( \{ Y_1, ..., Y_m \} \) of the algebra \( C_F^G \) with the following conditions:

- Each \( Y_j \) is irreducible;
- For any \( k \leq m \), different nonzero numbers \( r_1, ..., r_k \in \mathbb{K} \), any \( r'_1, ..., r'_k \in \mathbb{K} \) and different \( i_1, ..., i_k \in [1, m] \), we have \( \ell(r_1Y_{i_1} + \cdots + r_kY_{i_k}) \geq \ell(r'_1Y_{i_1} + \cdots + r'_kY_{i_k}) \);
- For any linear combination \( Y \) of \( \{ Y_1, ..., Y_m \} \) and a reducible \( Z \), \( \ell(Y) \leq \ell(Y + Z) \);
- \( \sum_i \ell(Y_i) \) is minimal.

Such a basis of \( C_F^G \) always exists. For example, the basis \( X_1, ..., X_n \) (corresponding to the vertices) satisfies the first three conditions. However, the sum of lengths of \( X_i \) is not always minimal.

For an element \( Y \in C_F^G \), we define its linear part as \( \hat{Y} \) and its remainder as \( \hat{Y} \), where \( Y = \hat{Y} + \hat{Y} \), \( \hat{Y} \) belongs to the 1-graded component of \( C_F^G \) and \( \hat{Y} \) belongs to the linear span of the other graded components. Observe that we do not know this decomposition explicitly, because we do not know the 1-graded component. We say that an edge \( e \) belongs to \( Y \) if \( Y \) includes the variable \( \phi_e \) corresponding to edge \( e \) with a nonzero coefficient.

Proposition 4. A basis \( \{ Y_1, ..., Y_m \} \) as above satisfies additionally the following conditions:

1. The set \( \{ \bar{Y}_1, ..., \bar{Y}_m \} \) is a basis;
2. For any linear combination \( Y \) of \( \{ Y_1, ..., Y_m \} \), \( \ell(Y) = \ell(\bar{Y}) \);
3. Each edge belongs to one or two \( \overline{Y}_i \) and \( \overline{Y}_j \). If it belongs to two, then with the opposite coefficients;
4. Each \( \overline{Y}_j \) has edges only from one 2-connected component.

Proof. (1). We know that \( \{ Y_1, ..., Y_m \} \) is a basis. Hence, the 1-graded component of \( C_F^G \) coincides with a linear span of \( \langle \bar{Y}_1, ..., \bar{Y}_m \rangle \). Then, \( \langle \overline{Y}_1, ..., \overline{Y}_m \rangle \) is also a basis.

(2). For any \( i \in [1, m] \), the part \( \hat{Y}_i \) is reducible, otherwise \( \ell(Y) \) is infinite, and the sum in last condition is infinite. However for the basis corresponding to the vertices this sum is finite. Then for any linear combination \( Y \) of \( \{ Y_1, ..., Y_m \} \), we have \( \ell(Y) \leq \ell(\bar{Y}) < \infty \). However,
it is clear that $\ell(Y) \geq \ell(\overline{Y})$. Hence, $\ell(Y) = \ell(\overline{Y})$ for any $Y$ from the linear space $<Y_1, \ldots, Y_m>$.

(3). Obviously, each edge belongs to at least one $\overline{Y_i}$, because any edge belongs to some $X_i$ and this $X_i$ is a linear combination of $Y_1, \ldots, \overline{Y_m}$.

Assume that there is an edge $e$, which belongs to $\overline{Y_i}$, $\overline{Y_j}$ and $\overline{Y_k}$. Then there are different nonzero numbers $r_1, r_2$ and $r_3$ such that, for $Y = r_1Y_1 + r_2Y_2 + r_3Y_3$, $\overline{Y}$ without $e$. Then $\ell(\overline{Y})$ is at most the number of different edges in $\overline{Y_i}, \overline{Y_j}$ and $\overline{Y_k}$ minus one.

Consider general $r_1', r_2'$ and $r_3'$, then, for $Y' = r_1'Y_1 + r_2'Y_2 + r_3'Y_3$, $\ell(\overline{Y'})$ is the number of different edges in $\overline{Y_{i_1}}, \overline{Y_{i_2}}$ and $\overline{Y_{i_3}}$. We have

$$\ell(r_1Y_1 + r_2Y_2 + r_3Y_3) < \ell(r_1'Y_1 + r_2'Y_2 + r_3'Y_3).$$

Using (2) we also have

$$\ell(r_1Y_1 + r_2Y_2 + r_3Y_3) < \ell(r_1'Y_1 + r_2'Y_2 + r_3'Y_3),$$

which contradicts our choice of a basis.

We proved the first part of condition (3); the proof of the second part is the same, but only for two different $r_1$ and $r_2$.

(4). Assume the opposite, i.e. that there is $\overline{Y_i}$ which has edges belonging to two different 2-connected components.

We know that the algebra is the Cartesian product of the subalgebras corresponding to 2-connected components. Then there are $Z_1$ and $Z_2$ from the 1-graded component, such that $\overline{Y_i} = Z_1 + Z_2$ and $\ell(Z_1), \ell(Z_2) < \ell(\overline{Y_i}).$ (For example, $Z_1$ is a part corresponding to some 2-connected component and $Z_2$ is another part).

Because $Z_1$ and $Z_2$ belong to the linear space $<\overline{Y_1}, \ldots, \overline{Y_m}>$, we can change $\overline{Y_i}$ to $Z_1$ or $Z_2$ in the basis $\{\overline{Y_1}, \ldots, \overline{Y_m}\}$. (Indeed if we can not do it, then $Z_1$ and $Z_2$ belong to $<\overline{Y_1}, \ldots, \overline{Y_{i-1}}, \overline{Y_{i+1}}, \ldots, \overline{Y_m}>$. Therefore $\overline{Y_i} = Z_1 + Z_2$ also belongs to the latter space). Let us $\{\overline{Y_1}, \ldots, \overline{Y_{i-1}}, Z_1, \overline{Y_{i+1}}, \ldots, \overline{Y_m}\}$ is also a basis.

We have a new basis $\{\overline{Y_1}, \ldots, \overline{Y_{i-1}}, Z_1, \overline{Y_{i+1}}, \ldots, \overline{Y_m}\}$, whose sum of lengths is less than the sum of lengths of $\{\overline{Y_1}, \ldots, \overline{Y_m}\}$, which equals to the sum of lengths of $\{Y_1, \ldots, Y_m\}$. And for this basis, the first three conditions of a choice a basis are hold, and the sum of lengths is smaller. Then we should choose the basis $\{\overline{Y_1}, \ldots, \overline{Y_{i-1}}, Z_1, \overline{Y_{i+1}}, \ldots, \overline{Y_m}\}$ instead of $\{Y_1, \ldots, Y_m\}$. Contradiction.

Let us now construct the cut space of $G$. This will finish the proof, because we can define the graphical matroid in terms of the cut space of a graph. By a cut we mean a set $C$ of edges such that the subgraph $G \setminus C$ has more connected components than $G$. By an elementary cut we mean a minimal cut, i.e. a cut, whose arbitrary subset is not a cut.
The sum $\sum 2^i Y_i = \sum 2^i Y_i$ has each edge with a nonzero coefficient by (2) of Proposition 4. Hence,

$$e(G) = \ell \left( \sum_{i=1}^m 2^i Y_i \right) = \ell \left( \sum_{i=1}^m 2^i Y_i \right).$$

Therefore we know the number of edges in the graph.

Consider the set $\tau = \{ \psi_1, \ldots, \psi_{e(G)} \}$ consisting of $e(G)$ elements and a family of subsets $K_1, \ldots, K_m$ constructed by the following rules.

- For each pair $i$ and $j$, we choose $\frac{\ell(Y_i) + \ell(Y_j) - \ell(Y_i + Y_j)}{2}$ own elements from $\tau$ and add them to both $Z_i$ and $Z_j$;
- For every $i$, we choose $\ell(Y_i) - \sum_{j \neq i} \frac{\ell(Y_i) + \ell(Y_j) - \ell(Y_i + Y_j)}{2}$ own elements from $\tau$ and add them to $Z_i$.

In fact, for any edge $e$ from $G$, we choose the corresponding element $\psi_{k(e)}$ and add it to $Z_i$ if and only if $e$ belongs to $Y_i$.

Consider the space $\Gamma$ of subsets in $\tau$ with the operation $\Delta$ (symmetric difference) generated by $Z_1, \ldots, Z_m$. We want to prove that $\Gamma$ is isomorphic to the cut space of $G$.

Let $C$ be an elementary cut of $G$. Let $X_C$ be the sum of $X_i$ corresponding to the vertices, which belong to some new component of $G \setminus C$ (hence, $X_C$ in 1-graded component). Then $X_C$ has an edge with a nonzero coefficient if and only if the edge belongs to $C$. Consider the minimal $t$ such that there is a linear combination $X_C = a_1 Y_{i_1} + \ldots + a_t Y_{i_t}$; consider the sum $X'_C = \sum_{i_1}^{\Delta} Y_{i_1}$. Obviously, $X'_C$ is nonzero and has edges only from the cut $C$, because an edge belongs to sum $X'_C$ if and only if it is exactly in one of $Y_{i_1}, \ldots, Y_{i_t}$.

Assume that $X'_C$ has not all edges from $X_C$. Let $C'$ be a subset of the edges of $C$ belonging to $X'_C$. The set of edges $C'$ is not a cut of $G$, because $C$ is an elementary cut. Hence, for any edge $e \in C'$, there is a path $e_1, \ldots, e_k$ in $G \setminus C'$, such that $e, e_1, \ldots, e_k$ is a simple cycle. Let the edge $e$ connect vertices $b_k$ and $b_0$, and the edge $e_i$ connect $b_{i-1}$ and $b_i$ for any $i \in [1, k]$. Because $X'_C$ belongs to the 1-graded component, then $X'_C = \sum_{i=1}^n \hat{a}_i X_i$, where $\hat{a}_i \in \mathbb{R}$ and $X_i$ are the elements corresponding to vertices of $G$. For $i \in [1, k]$, the edge $e_i$ does not belong to $X'_C$, however variable $\phi_{e_i}$ belongs only to $X_{b_{i-1}}$ and $X_{b_i}$ from $\{ X_1, \ldots, X_n \}$. Furthermore it belongs to one of $X_{b_{i-1}}$, $X_{b_i}$ with coefficient 1 and to another with $-1$, hence, $\hat{a}_{b_{i-1}} = \hat{a}_{b_i}$. Then, we also have $\hat{a}_{b_0} = \hat{a}_{b_k}$. Hence the variable $\phi_e$ belongs to $X'_C$ with a zero coefficient. Contradiction.

We concluded that a subset of $\tau$ corresponding to an elementary cut belongs to the space $\Gamma$. To finish the proof, we need to show, that if a subset of $\tau$ belongs to $\Gamma$, that it either corresponds to a cut or to the empty set.

Assume the contrary, i.e. assume that there is $Z_{i_1} \Delta Z_{i_2} \Delta \cdots \Delta Z_{i_s}$ not belonging to a cut. Let $C$ be a set of edges from $\sum Y_{i_1} + \sum Y_{i_2} + \cdots + \sum Y_{i_s}$,
then $C$ is not a cut in $G$. By Proposition 4 we can split the summation into the summations inside individual connected components (and even inside 2-connected component). Then, for any connected component it is not a cut.

Let $Y_{i_1} + Y_{i_2} + \cdots + Y_{i_r}$ corresponds to a connected component $G'$, and has the set of edges $C'$. Then

$$Y_{i_1} + Y_{i_2} + \cdots + Y_{i_r} = \sum_{v \in V(G')} a_v X_v.$$ 

We know that $G' \setminus C'$ is connected. Therefore there is a spanning tree $T$ in $G' \setminus C'$. For any edge $v_i v_j$ from $T$, we have $a_{v_i} = a_{v_j}$; otherwise the edge $v_i v_j$ belongs to $\sum_{v \in V(G')} a_v X_v$ with a nonzero coefficient. Since $T$ is a spanning tree of $G'$, then all coefficients $a_v$ are the same. Thus $\sum_{v \in V(G')} a_v X_v = a(\sum_{v \in V(G')} X_v) = 0$; the last sum is zero, because the sum of variables corresponding to vertices from a connected component is zero. Then we also have $Y_{i_1} + Y_{i_2} + \cdots + Y_{i_r} = 0$, hence, $Z_{i_1} \Delta Z_{i_2} \Delta \cdots \Delta Z_{i_r}$ is the empty set.

Therefore the space $\Gamma$ is isomorphic to the cut space of $G$, i.e. there is a unique graphical matroid corresponding to $\mathcal{C}_F^G$. $\square$

3. Algebras associated with $t$-labelled forests

In this section we substitute the square-free algebra $\Phi_G^F$ for the $(t+1)$-free algebra $\Phi_G^{F_i}$.

**Notation 3.** Take an undirected graph $G$ with $n$ vertices. Let $t > 0$ be positive integer number.

(I) Let $\Phi_G^{F_i}$ be the commutative algebra over $\mathbb{K}$ generated by $\{\phi_e : e \in E(G)\}$ satisfying the relations $\phi_e^{t+1} = 0$, for any $e \in E(G)$.

Fix any linear order of vertices of $G$. For $i = 1, \ldots, n$, set

$$X_i = \sum_{e \in G} c_{i,e} \phi_e,$$

where $c_{i,e} = \pm 1$ as in notation 1. Denote by $\mathcal{C}_G^{F_i}$ the subalgebra of $\Phi_G^{F_i}$ generated by $X_1, \ldots, X_n$.

(II) Consider the ideal $J_G^F$ in the ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by

$$p_I^{F_i} = \left( \sum_{i \in I} x_i \right)^{tD_I + 1},$$

where $I$ ranges over all nonempty subsets of vertices, and $D_I$ is the total number of edges between vertices in $I$ and vertices outside $I$. Define the algebra $\mathcal{B}_G^{F_i}$ as the quotient $\mathbb{K}[x_1, \ldots, x_n]/J_G^F$.

It enumerates the so-called $t$-labelled spanning forests.

Consider a finite labelling set $\{1, 2, \ldots, t\}$ containing $t$ different labels; each label being to a number from 1 to $t$. 
Definition 4. A spanning forest of a graph $G$ with a label from \( \{1, 2, \ldots, t\} \) on each edge is called a $t$-labelled forest. The weight of a $t$-labelled forest $F$, denoted by $\omega(F)$, is the sum of the labels of all its edges.

Theorem 5. For any graph $G$ and a positive integer $t$, algebras $B_{G}^{F_{t}}$ and $C_{G}^{F_{t}}$ are isomorphic. Their total dimension over $\mathbb{K}$ is equal to the number of $t$-labelled forests in $G$.

The dimension of the $k$-th graded component of the algebra $B_{G}^{F_{t}}$ is equal to the number of $t$-labelled forests $F$ of $G$ with the weight $t \cdot (e(G) - \text{act}_{G}(F)) - k$.

Proof. Denote by $\hat{G}$ the graph on $n$ vertices and with $t \cdot e(G)$ edges such that each edge of $G$ corresponds to its $t$ clones in the graph $\hat{G}$. In other words, each edge of $G$ is substituted by its $t$ copies with labels $1, 2, \ldots, t$. For each edge $e \in E(G)$, its clones $e_1, \ldots, e_t \in E(\hat{G})$ are ordered according to their numbers; clones of different edges have the same linear order as the original edges. Thus we obtain a linear order of the edges of $\hat{G}$.

Consider the following bijection between $t$-labelled forests in $G$ and usual forests in $\hat{G}$: each $t$-labelled forest $F \subset G$ corresponds to the forest $F' \subset \hat{G}$, such that for each edge $e \in E(F)$, the forest $F'$ has the clone of the edge $e$ whose number is identical to the label of edge $e$ in the forest $F$.

Obviously,

$$act_{\hat{G}}(F') = t \cdot act_{G}(F) + \omega(F) - e(F),$$

and $e(\hat{G}) = t \cdot e(G)$. Since $B_{G}^{F_{t}}$ and $B_{\hat{G}}^{F_{t}}$ are isomorphic, the Hilbert series of the algebra $B_{G}^{F_{t}}$ coincides with the Hilbert series of the algebra $B_{\hat{G}}^{F_{t}}$, which settles the second part of Theorem 5.

To prove the first part of this Theorem, observe that $B_{G}^{F_{t}}$ and $B_{\hat{G}}^{F_{t}}$ are isomorphic, and algebras $C_{G}^{F_{t}}$ and $C_{\hat{G}}^{F_{t}}$ are isomorphic. Thus we must show that algebras $C_{G}^{F_{t}}$ and $C_{\hat{G}}^{F_{t}}$ are isomorphic. This indeed true, because for every edge $e \in E(G)$, the elements $\phi_{e_1}, \ldots, \phi_e^t$ are linearly independent in the algebra $\Phi_{G}^{F_{t}}$ with coefficients containing no $\phi_e$. Also elements $(\phi_{e_1} + \cdots + \phi_{e_t}), \ldots, (\phi_{e_1} + \cdots + \phi_{e_t})^t$ are linearly independent in the algebra $\Phi_{\hat{G}}^{F_{t}}$ with coefficients containing no $\phi_{e_1}, \ldots, \phi_{e_t}$, and $(\phi_{e_1} + \cdots + \phi_{e_t})^{t+1} = 0$. Moreover elements $\phi_{e_i}$ only occur in the sum $(\phi_{e_1} + \cdots + \phi_{e_t})$ in the algebra $\Phi_{\hat{G}}^{F_{t}}$.

In fact Hilbert Series of $B_{G}^{F_{t}}$ was calculated in papers [2] and [13]. Furthermore, for graph $\hat{G}$ was presented there, where each edge is replaced by its own number of edges. The Hilbert series is a specialization of the multivariate Tutte polynomial (see definition in [17]). When "$t"
is the same for every edge, the multivariate Tutte polynomial is calculated from the usual Tutte polynomial. So in our case the Hilbert Series of $B_{F_t}^{G}$ is a specialization of the Tutte polynomial of $G$.

**Theorem 6.** The dimension of the $k$-th graded component of $B_{F_t}^{G}$ is equal to the coefficient of the monomial $y^t e(G) - v(G) + c(G) - k$ in the polynomial

$$
\left( \frac{y^t - 1}{y - 1} \right)^{v(G) - c(G)} \cdot T_G \left( \frac{y^{t+1} - 1}{y^{t+1} - y}, y^t \right).
$$

Consider the graph $\hat{G}$ constructed in the proof of Theorem 5. We need the following technical lemma which was proved in [4].

**Lemma 3** (Lemma 6.3.24 in [4]).

$$
T_{\hat{G}}(x, y) = \left( \frac{y^t - 1}{y - 1} \right)^{v(G) - c(G)} \cdot T_G \left( \frac{y^t - y + x(y - 1)}{y^t - 1}, y^t \right).
$$

After substitution $x \rightarrow 1 + \frac{1}{y}$, we get the following equality.

**Corollary 3.**

$$
T_{\hat{G}}(1 + \frac{1}{y}, y) = \left( \frac{y^t - 1}{y - 1} \right)^{v(G) - c(G)} \cdot T_G \left( \frac{y^{t+1} - 1}{y^{t+1} - y}, y^t \right).
$$

**Proof of Theorem 6.** Algebra $B_{F_t}^{G}$ is isomorphic to algebra $B_{\hat{F_t}}^{\hat{G}}$ (which was shown in the proof of Theorem 5), furthermore they are isomorphic as graded algebras. So it is enough to show that dimension of the $k$-th graded component of $B_{\hat{F_t}}^{\hat{G}}$ is equal to the coefficient of the monomial $y^t e(\hat{G}) - v(\hat{G}) + c(\hat{G}) - k$ in the polynomial $T_{\hat{G}}(1 + \frac{1}{y}, y)$. This fact is true by Corollary 1.

**Theorem 7.** For any positive integer $t \geq n$, it is possible to restore the Tutte polynomial of any connected graph $G$ on $n$ vertices knowing only the dimensions of each graded component of the algebra $B_{F_t}^{G}$.

**Proof.** By Theorem 5 we know that the degree of the maximal nonempty graded component of $B_{F_t}^{G}$ equals to the maximum of $t \cdot (e(G) - act_G(F)) - \omega(F)$ taken over $F$. It attains its maximal value for the empty forest (i.e., $F = \emptyset$). Then we know the value of $t \cdot e(G)$, and hence, we know the number of edges of the graph $G$.

Since we know $t \cdot e(G) - v(G) + c(G) = t \cdot e(G) - n + 1$ ($G$ is connected, i.e., $c(G) = 1$), by Theorem 6 we can calculate the polynomial

$$
\left( \frac{y^t - 1}{y - 1} \right)^{v(G) - c(G)} \cdot T_G \left( \frac{y^{t+1} - 1}{y^{t+1} - y}, y^t \right)
$$

from the Hilbert series. Then we also can calculate $T_G \left( \frac{y^{t+1} - 1}{y^{t+1} - y}, y^t \right)$. 


It is well known that for any graph $G$, its Tutte polynomial $T_G(x,y)$ is equal to $\sum_F (x-1)^{c(F)-c(G)} y^{\text{act}_G(F)}$, where the summation is taken over all spanning forests of $G$. Then we obtain

$$T_G\left(\frac{y^{t+1} - 1}{y^{t+1} - y}, y^t\right) = \sum_F \left(\frac{y^{t+1} - 1}{y^{t+1} - y} - 1\right)^{n-1-e(F)} y^{t-\text{act}_G(F)} =$$

$$= \sum_F \left(\frac{1}{y(y^{t-1} + \ldots + 1)}\right)^{n-1-e(F)} y^{t-\text{act}_G(F)}.$$

Hence, we can restore polynomial

$$\sum_F (y^{t-1} + \ldots + 1)^{e(F)} y^{t-\text{act}_G(F)+e(F)}.$$  \((\ast)\)

Since $e(F) < t$, we can compute the number of usual spanning forests with a fixed pair of parameters $e(F)$ and $\text{act}_G(F)$. Indeed, consider the monomial of the minimal degree in polynomial $(\ast)$, and present it in the form $s \cdot y^m$. Observe that $s$ is the number of spanning forests $F$ such that $F \equiv m \pmod{t}$ and $\text{act}_G(F) = \left\lceil \frac{m}{t} \right\rceil$. Remove from the polynomial $(\ast)$ all summands for these spanning forests, and repeat this operation until we get 0.

Note again that $T_G(x,y) = \sum_{a,b} \#\{F : e(F) = a, \text{act}(F) = b\} \cdot (x - 1)^{n-1-a} \cdot y^b$. Therefore we know the whole Tutte polynomial of $G$, since we know the number of usual spanning forests with any fixed number of edges and any fixed external activity. \(\square\)

4. Vector configurations and hypergraphs

4.1. Algebra corresponding to vector configuration. The following algebra was introduced by A. Postnikov, B. Shapiro and M. Shapiro in [16].

**Notation 5.** Given a finite set $A = \{a_1, \ldots, a_m\}$ of vectors in $\mathbb{K}^n$, let $\Phi^F_m$ be the commutative algebra over $\mathbb{K}$ generated by $\{\phi_i : i \in [1,m]\}$ with relations $\phi_i^2 = 0$, for each $i \in [1,m]$.

For $i = 1, \ldots, n$, set $X_i = \sum_{k \in [1,m]} a_{k,i} \phi_k$. Denote by $\mathcal{C}_A$ the subalgebra of $\Phi^F_m$ generated by $X_1, \ldots, X_n$.

The Hilbert series of $\mathcal{C}_A$ also corresponds to a specialization of the Tutte polynomial of the corresponding vector matroid, see Theorem 3 in [16].

**Theorem 8** (cf. [16]). *The dimension of algebra $\mathcal{C}_A$ is equal to the number of independent subsets in $V$. Moreover, the dimension of the $k$-th graded component is equal to the number of independent subsets $S \subset A$ such that $k = m - |S| - \text{act}(S)$.*
Corollary 4. Given a vector configuration $A$ in $\mathbb{K}^n$, the Hilbert series of algebra $C_A$ is

$$HS_{C_A}(t) = T_A \left(1 + t, \frac{1}{t}\right) \cdot t^{|A| - \text{rk}(A)},$$

where $T_A$ is the Tutte polynomial corresponding to $A$, $|A|$ is the number of vectors in the configuration and $\text{rk}(A)$ is the dimension of the linear span of these vectors.

The set of different vector configurations depends on continuous parameters. Additionally, there are uncountably many non-isomorphic algebras each corresponding to its vector configuration. At the same time the number of matroids is countable, and it is finite for a fixed number of vectors. It means that there are many different vector configurations with the same corresponding matroid. In other words, it is in principle impossible to reconstruct a vector configuration and its algebra from the corresponding matroid.

4.2. Family of algebras for a hypergraph. In this subsection we present a family of algebras corresponding to a hypergraph. Almost all algebras from this family (generic algebras) have the same Hilbert series and this generic Hilbert series counts forests of this hypergraph. There are many definitions of spanning trees of a hypergraph, for example: a spanning cacti in [1]; a hypertree in [6] (also known as an arboreal hypergraph in [3]). However all these definitions do not have one of the main properties of a spanning tree of a usual graph, such trees of a hypergraph can have different number of edges. We define spanning trees such that this property holds and also other natural properties hold. Also we define the hypergraphical matroid and the corresponding Tutte polynomial, whose points $T(2, 1)$ and $T(1, 1)$ calculate the numbers of spanning forests and of spanning trees, resp. There is another definition of the Tutte polynomial for a hypergraph in [11], however, it is not our case.

First we define the family of algebras.

Given a hypergraph $H$ on $n$ vertices, let us associate commuting variables $\phi_e, e \in H$ to all edges of $H$.

Set $\Phi_H$ to be the algebra generated by $\{\phi_e : e \in H\}$ with relations $\phi_e^2 = 0$, for any $e \in H$.

Define $C = \{c_{i,e} \in \mathbb{K} : i \in [1, n], e \in H\}$ as a set of parameters of $H$, for any edge $e \in H$, $c_{i,e} = 0$ for vertices non-incident to $e$, and $\sum_{i=1}^n c_{i,e} = 0$.

For $i = 1, \ldots, n$, set

$$X_i = \sum_{e \in H} c_{i,e} \phi_e,$$
Denote by $C_{H(C)}^F$ the subalgebra of $\Phi_H$ generated by $X_1, \ldots, X_n$, and denote by $\widehat{C_{H(C)}^F}$ the family of such subalgebras.

The following trivial properties hold for this family of algebras.

**Proposition 9.** (I) For a hypergraph $H$, the dimension of the space of parameters is $\sum_{e \in E}(|e| - 1)$.

(II) Given a set of parameters $C$ and non-zero numbers $a_e$, $e \in E$, let $C'$ be the set of parameters such that $c'_{i,e} = a_e c_{i,e}$ for any $i \in [1, n]$ and $e \in E$. Then the subalgebras for $C$ and for $C'$ are isomorphic.

**Corollary 5.** For a usual graph $G$, almost all algebras from $\widehat{C_G}$ are isomorphic to $C_G$.

We define a hypergraphical matroid using the definition of an independent set of edges of a hypergraph.

**Definition 6.** Let $H$ be a hypergraph on $n$ vertices. A set $F$ of edges is called independent if there is a set of parameters $C$ of $H$, such that vectors corresponding to edges from $F$ are linearly independent. In other words, $F$ is independent if, for a generic set of parameters of $H$, vectors are linearly independent. Define the hypergraphical matroid of $H$ as the matroid with ground set $E(H)$.

There is a combinatorial definition of an independent set of edges. First we need to define a cycle of $H$.

**Definition 7.** A subset of edges $C \subseteq E$ is called a cycle if

- $|C| = |\cup_{e \in C} e|$
- There is no subset $|C' \subset C|$, such that, for $C'$, the first property holds.

Definitions of dependence and of cycle are similar.

**Theorem 10.** A subset of edges $X \subseteq E$ is dependent if and only if there is a cycle $C \subset X$.

We present a proof of this theorem after Theorem 13.

**Definition 8.** A set of edges $F$ is called a spanning forest if $F$ has no cycles, in other words, $F$ is forest if and only if $F$ is an independent set (by Theorem 10). Furthermore, all maximal spanning forests of a hypergraph $H$ have the same size. A set of edges $T \subseteq H$ is called a spanning tree if it is a forest and $T$ has exactly $v(H) - 1$ edges.

Hypergraph $H$ is called strongly connected if it has at least one spanning tree.

**Proposition 11.** Maximal spanning forests of hypergraph have the same number of edges. In fact, if $H = (V, E)$ is strongly connected hypergraph, then for any spanning forest $F \subseteq E$ there is a spanning tree $T$ which contains $F$ (i.e. $F \subseteq T \subseteq E$).
Proof. By Theorem 10 we know that a spanning forest is the same that independent set of edges. Then we can add edges to a forest until the number of edges is less than the dimension of the linear space generated by generic vectors $b_1, b_2, \ldots, b_n$. Then all maximal spanning forests have the same size. Hence, if a hypergraph is strongly connected, then any spanning forest is contained in some spanning tree. □

The Hilbert series of algebras in $\hat{\mathcal{C}}^F_H$ are also counting forests of $H$.

**Theorem 12.** For a hypergraph $H$, generic algebras from $\hat{\mathcal{C}}^F_H$ have the same Hilbert series. The dimension of the $k$-th graded component of generic algebra equals the number of spanning forests $F$ in $H$ with external activity $|H| - |F| - k$.

**Proof.** By Theorem 10 we can change definition of spanning forests to independent sets. Consider generic set of parameters $C$. By Theorem 8 we know the Hilbert series of $\mathcal{C}^F_{H(C)}$ and it is same for all generic sets of parameters. □

We define the Tutte polynomial of $H$ as the Tutte polynomial of the corresponding hypergraphical matroid. By the theorems above we know

- $T_H(2, 1)$ is the number of spanning forests;
- $T_H(1, 1)$ is the number of maximal spanning forests. In fact, $T_H(1, 1)$ is the number of spanning trees if $H$ has at least one spanning tree.

By Theorem 12, we get that a generic Hilbert series is a specialization of the Tutte polynomial of $H$.

**Corollary 6.** Given hypergraph $H$ and its generic set of parameters $C$, the Hilbert series of algebra $\mathcal{C}_{H(C)}$ is given by

$$HS_{\mathcal{C}_{H(C)}}(t) = T_G \left(1 + t, \frac{1}{t}\right) \cdot t^{e(H) - rk_H},$$

where $rk_H$ is the size of a maximal spanning forest of $H$.

There is another definition of forests/trees of $H$, which again shows that it is a generalization of forests/trees of a usual graph.

**Theorem 13.** A subset of edges $X \subset E$ is a forest (tree) if and only if there is map from edges to pairs: $e_k \rightarrow (i, j)$, where $v_i, v_j \in e_k$, such that these pairs form a forest (tree) in the graph $K_{n+1}$.

**Proof.** Consider our forest $F$ and hypergraph $H$, add to them $n - 1 - |F|$ full edges, i.e. edges of type $V$. We get a new hypergraph $H'$ and a subset of edges $F'$. It is clear that $F'$ is a tree, because there is no cycle without new edges, and with at least one new edge we need to cover all vertices and their number is bigger than number of edges. So let $F' = T$ and we will prove our Theorem for a spanning tree.
Consider the bipartite graph $B$ with two sets of vertices: the first set are edges of $T$ and the second are vertices $V \setminus v_1$, where there is an edge between $e_i$ and $v_j$ if and only if $v_j \in e_i$.

There is a perfect matching in $B$, because we can use Hall’s marriage theorem (see [8] or any classic book). We know that $|T| = |V \setminus v_1|$ and, for any $X \subset T$, they cover at least $|X|$ vertices (otherwise these edges cover these vertices and may be $v_1$ and then $X$ has cycle). Consider a bijection $f$ form $T$ to $V \setminus v_1$ constructed by this perfect matching. Now we construct by recursion a map $g$ from $T$ to pairs of vertices:

1. $A := \{v_1\}$ and $B := T$
2. repeat until $A \neq V$:
   - chose the minimal edge $e_i$ from $B$ such that $e_i \cap A \neq \emptyset$
   - $g(e_i) = (u, f(e_i \cap A))$, where $u \in e_i \cap A$
   - $A := A \cup \{f(e_i)\}$ and $B := B \setminus \{e_i\}$

It works, otherwise we can not chose such an edge $e_i$, then either $B = \emptyset$ or $B \neq \emptyset$ at this moment. We know that $|A| + |B| = n$, then in the fist case we already have $A = V$; in the second case edges from $B$ have vertices only from $V \setminus A$, then there is a cycle on these edges, i.e. $T$ is not tree. Then this algorithm gives some usual tree.

**Proof of Theorem 10.** Assume the contrary, then there is a subset $X \subset E$, which is dependent and without cycles, i.e. $X$ is a spanning forest.

By Theorem 13 we know that there is map $f$ from $X$ to pairs of vertices, which gives an usual forest. Consider the vector set

$$\{a_e := z_{f_1(e)} - z_{f_2(e)}, \ e \in X\},$$

where $z_v = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the unit vertex corresponding to vertex $v$. Since $f$ gives the usual forest, these vectors are independent. Hence, generic vectors $\{b_e, \ e \in X\}$ are also linear independent. We get that edges are independent, contradiction.

By induced subgraph on vertices $V' \subset V$, we assume hypergraph $(V', E')$, where $E'$ is all edges of $E$, which has vertices only from $V'$ (i.e. $e \in E'$ if $e \subset V'$). This definition works well with colorings of hypergraphs, because if we want to color a hypergraph such away that there are no monochromatic edges, then it is the same as splitting vertices into sets with empty induced subgraphs. Also this definition works well with standard sense of connectivity.

**Proposition 14.** Let $V_1$ and $V_2$ be subsets of vertices such that the induced subgraphs of $H$ on $V_j$ are strongly connected and $V_1 \cap V_2 \neq \emptyset$. Then the induced subgraph of $H$ on $V_1 \cup V_2$ is also strongly connected.

**Proof.** For any vertex $v_i \in V$ we will consider the corresponding unit vector $z_{v_i} = (0, \ldots, 0, 1, 0, \ldots, 0)$. Fix vertex $u$, which lie in intersection $V_1 \cap V_2$. Let $H_1$ and $H_2$ be induced subgraphs on $V_1$ and $V_2$, resp.
Consider vectors \( b_i \) corresponding to \( e_i \in E_1 \). We know that there is a spanning tree of graph \( H_1 \), then dimension of linear space of such vectors is \( |V_1| - 1 \), furthermore any sum of coordinates of any vector is zero. Hence, for any \( v \in V_1 \), \( z_v - z_u \in \text{span}\{b_i : e_i \in E_1\} \). Similarly we get the same for \( H_2 \) and, hence, we have the same for \( H_1 \cup H_2 \).

We get that for any \( v \in V_1 \cup V_2 \), \( z_v - z_u \in \text{span}\{b_i : e_i \in E_1 \cup E_2\} \).

Hence, hypergraph \( H_1 \cup H_2 \) has \( |V_1 \cup V_2| - 1 \) independent edges, then has a spanning tree. We have that the induced subgraph of \( H \) on vertices \( V_1 \cup V_2 \) is strongly connected, since it has all edges from \( H_1 \cup H_2 \). \( \square \)

5. Algebras corresponding to spanning trees, problems

In this section we discuss analogous algebras counting spanning trees. Recall the definition of algebras \( \mathcal{B}^T_G \) and \( \mathcal{C}^T_G \) borrowed from [15].

**Notation 9.** Take an undirected graph \( G \) with \( n \) vertices.

(I) Let \( \Phi^T_G \) be the commutative algebra over \( \mathbb{K} \) generated by \( \{\phi_e : e \in G\} \) with relations \( \phi_e^2 = 0 \), for any \( e \in G \), and \( \prod_{e \in H} \phi_e = 0 \), for any \( H \subset E(G) \) such that \( G \setminus H \) is disconnected.

Fix a linear order of vertices of \( G \). For \( i = 1, \ldots, n \), set

\[
X_i = \sum_{e \in E(G)} c_{i,e} \phi_e,
\]

where \( c_{i,e} = \pm 1 \) for vertices incident to \( e \) (for the smaller vertex \( v_i \), \( c_{i,e} = 1 \), for the larger vertex \( v_j \), \( c_{j,e} = -1 \)) and 0 otherwise. Denote by \( \mathcal{C}^T_G \) the subalgebra of \( \Phi^T_G \) generated by \( X_1, \ldots, X_n \).

(II) Consider the ideal \( J^T_G \) in the ring \( \mathbb{K}[x_1, \ldots, x_n] \) generated by

\[
p^T_I = \left( \sum_{i \in I} x_i \right)^{D_I},
\]

where \( I \) ranges over all nonempty proper subsets of vertices, and \( D_I \) is the total number of edges between vertices in \( I \) and vertices outside \( I \). Define the algebra \( \mathcal{B}^T_G \) as the quotient \( \mathbb{K}[x_1, \ldots, x_n]/J^T_G \).

The case, when the graph \( G \) is disconnected, is not interesting, because both algebras are trivial. In paper [15] the following result was proved:

**Theorem 15** (cf. [15]). For any graph \( G \), algebras \( \mathcal{B}^T_G \) and \( \mathcal{C}^T_G \) are isomorphic; their total dimension over \( \mathbb{K} \) is equal to the number of spanning trees in \( G \).

Moreover, the dimension of the \( k \)-th graded component of these algebras equals the number of spanning trees of \( G \) with external activity \( e(G) - v(G) + 1 - k \).
5.1. **Algebras and matroids.** For graph $G$, we define its bridge-free matroid as the usual graphical matroid of graph $G'$ which is obtained from $G$ after removal all its bridges.

**Proposition 16.** For any pair of connected graphs $G_1$ and $G_2$ with isomorphic bridge-free matroids, their algebras $B^T_{G_1}$ and $B^T_{G_2}$ are isomorphic.

**Proof.** Notice that, if we add an edge $e$ and an vertex $v$ to $G$, such that $v$ is an endpoint of $e$ and another endpoint of $e$ is some vertex of $G$, then algebra $B^T$ does not change (this is obvious because $e$ is bridge, and hence, $\phi_e$ is one of generators of the ideal). This operation doesn’t change bridge-free matroid. Therefore it is enough to prove Proposition 16 only for graphs with the same number of edges.

Assume that $|E(G_1)| = |E(G_2)|$. In this case an isomorphism of bridge-free matroids is equivalent to an isomorphism of matroids of graphs $G_1$ and $G_2$.

In fact, in Lemma 1 we construct orientations $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ of graphs $G_1$ and $G_2$ on the same set of edges $E(G_1) = E(G_2)$ (it was constructed for graphs differ on one Whitney’s deformation, so we can to extend it to sequence of deformations), such that they give the same graphical matroid on edges and with these orientations the algebras $C^F_{G_1}$ and $C^F_{G_2}$ are coincide as subalgebras of $\Phi^F_{G_1}$ ($\Phi^F_{G_1}$ and $\Phi^F_{G_2}$ are the same, because graphs have common set of edges).

Let $I$ be the ideal generated by the products of edges from the cuts of $G_1$ in $\Phi^F_{G_1}$. Because the variables on edges in $G_1$ and $G_2$ are the same and $C$ is a cut in $G_1$ if and only if $C$ is a cut in $G_2$, then $I$ is also the ideal generated by the cuts of $G_2$.

Thus we have $\Phi^F_{G_1} = \Phi^F_{G_2}/I$, hence,

$$C^T_{G_1} = C^F_{G_1}/I,$$

similarly

$$C^T_{G_2} = C^F_{G_2}/I.$$  

It means that the algebras $C^T_{G_1}$ and $C^T_{G_2}$ are also the same in orientations $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$.  

**Remark 3.** Connectivity of graphs is important, because for disconnected graphs, algebras counting trees are trivial.

It is easy to check that if a connected graph $G$ has a bridge $e$ (i.e. $G - e$ is disconnected), then algebras $C^T_{G}$ and $C^T_{G/e}$ are isomorphic (graph $G/e$ is obtained from $G$ after contraction the edge $e$). So we formulate the following conjecture.

**Conjecture 1.** Algebras $B^T_{G_1}$ and $B^T_{G_2}$ for connected graphs $G_1$ and $G_2$ are isomorphic if and only if their bridge-free matroids are isomorphic.
5.2. $t$-labelled trees. It is possible to introduce similar algebras which enumerate $t$-labelled trees, but it is not very exciting. Let $B^T_G$ be an algebra in which we change the generators $(\sum_{i \in I} x_i)^{D_i}$ of the ideal by $(\sum_{i \in I} x_i)^{D_i}$. The definition of $C^T_G$ will change in a more complicated way.

However, there is no result about a reconstruction of the Tutte polynomial from the Hilbert series, because all trees have the same number of edges and then $HS_{B^T_G}(x) = (1 + x)^{n-1}HS_{B^T_G}(x^t)$, where $HS$ is the Hilbert series and $n$ is the number of vertices. Then, the Hilbert series of $B^T_G$ and of $B^T_G$ have the same information about the graph.

5.3. Algebras for hypergraphs. The main problem is to construct a family $\hat{C}^T_H$ of algebras, which count spanning trees of $H$.

By paper [16], for hypergraph $H$ and set of parameters $C$, we can present $C^F_{H(C)}$ as a quotient algebra, i.e, as $B^F_{H(C)}$. We can consider the algebra $B^T_{H(C)}$, which is obtained from $B^F_{H(C)}$ by changing the powers of the generators of the ideal (writing always one less). By paper [2] algebra $B^T_{H(C)}$ should count spanning trees of $H$. However at this moment, we can not present $\Phi^T_H$ such that its generic subalgebra $C^T_H(C)$ counts spanning trees of $H$.

Probably, we need to add to $\Phi^F_H$ relations corresponding to cuts, where a cut is a subset of edges such that without it $H$ has no a spanning tree. However, we need to prove it and if we want to do something similar the proof of Theorem 15, then we need to define $H$-parking functions for a hypergraph.

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