Lower bounds on the Q-factor for small oversampled superdirective arrays over a ground plane

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Abstract

Base station antennas for next generation mobile communication networks will be required to have a wide bandwidth for compatibility and support multi-beam applications. Antenna arrays are one possible candidate for a next generation base station antenna since they can obtain high directivity, support multi-beam and wide angle scanning. However, a wide-band antenna array faces several problems at the low frequency limit. The electrical size of the antenna array becomes smaller at the low frequency and it becomes harder to obtain high directivity. Another issue is that the bandwidth becomes narrower for electrically small antenna arrays under superdirectivity constraints. The $Q$-factor is a measure of losses and is proportional to the ratio of the stored energy to the dissipated power. It is also inversely proportional to the fractional bandwidth of an antenna array when $Q \gg 1$.

In this thesis, we investigate the relation between the $Q$-factor and the directivity and extend our analysis to a superdirective antenna array. The model we are using is based on the antenna designed for base station applications with the frequency range from 700MHz to 4.2GHz and we assume that it is placed over an infinite ground plane. We calculate the $Q$-factor at 745MHz, which is the center frequency of GSM 700MHz bands using convex optimization. Here, we use the CVX as the convex optimization tool, which can be easily integrated with MATLAB. The expressions of the stored energy and the radiated power are formulated in a matrix form based on the Method of Moment (MoM) using Rao-Wilton-Gilson (RWG) basis functions for the convex optimization. We show the trade-off between the directivity and the $Q$-factor, or the bandwidth at the low frequency limit. The results are investigated forth cases; one is a vertical model, where each array elements is placed vertically above an infinite ground plane, and the other is a horizontal model, where each array elements is placed horizontally. We show that the $Q$-factor is slightly lower for the vertical case than the horizontal case.
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Chapter 1

Introduction

Today, several frequency bands are used for the cellular communication, including Global System for Mobiles (GSM) and the latest network, LTE-Advanced. GSM was introduced back in 1982 first but it is still widely used all over the world due to its reliability and easy implementation[1]. Therefore, for compatibility, a base station antenna for next generation is expected to operate at a wide range of frequency bands covering from 700MHz to several GHz. Demands for high data rate communications are also increasing due to the rapid growth of mobile traffic. One possible solution to this problem is the multi-beam communication. With multi-beam communication, an antenna can send a signal to multiple users simultaneously and provide high data rate communications. In multi-beam communications, highly directive beams are required to avoid the interference between users and to reduce the wasted energy, thus a new generation base station antenna is also required to support multi-beam and wide angle scanning. Antenna arrays are one possible candidate for a next generation base station antenna since they can obtain high directivity and support multi-beam and wide angle scanning. However, a wide-band antenna array faces several problems at the low frequency limit. The electrical size of the antenna array becomes smaller at the low frequency and it becomes harder to obtain high directivity. Another issue is that the bandwidth becomes narrower for electrically small antenna array in particular if it is under high directivity constraints. The $Q$-factor is a measure of bandwidth and it is proportional to the ratio of the stored energy to the dissipated power[2]. In antenna application, the dissipated power is considered as the radiated power for a lossless antenna[2], thus a low $Q$-factor is desired[3]. The $Q$-factor is inversely proportional to the fractional bandwidth of an antenna array when the $Q$-factor is high[2], thus the $Q$-factor is an important parameter to estimate the bandwidth. In this thesis, we investigate the relation between the $Q$-factor and the directivity and extend our analysis to a superdirective antenna array. We assume that the superdirective antenna array is placed above an infinite ground plane. In Hansen book[4], the superdirectivity is defined as “directivity higher than that obtained with the same array length and elements uniformly excited”. However, a superdirective antenna array has very narrow bandwidth because of highly oscillating currents and the efficiency decreases due to the low radiation resistance[4]. Moreover, the excitation is sensitive to disturbances. These facts make it
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difficult to design superdirective array antennas. Therefore, it is important to know a-priori information about the maximum bandwidth, or the \( Q \)-factor for the superdirective antenna array. Here, we also need an interpretation of what the \( Q \)-factor represents for an array antenna which usually have many ports for individual elements. The \( Q \)-factor is a measure on the frequency stability of the high-directiving current solution. A way to view this stability measure in a traditional impedance bandwidth is to imagine a feeding network from one single port to all currents port on the array. This network is such that when we feed the single port we excite a current with the right phase and amplitude at all part of the aperture, yielding the superdirective radiation pattern. In this single port we have a narrow band response since the optimal currents depend strongly on the frequency.

There are several expressions for stored energy and they are not unique[2] [5]-[7]. Here, we use the expressions by Vandenbosch [6]. These expressions are expressed in terms of the electric current density of an antenna array and can be applied to arbitrary structures. However, it is known that these expressions can give negative stored energy[8]. This fact can be interpreted as the subtraction of the far-field energy inside the sphere with radius \( a \), where \( a \) is the smallest radius of a sphere which circumscribes all current distributions[7]. The different definitions mean that there is an uncertainty of the \( Q \)-factor of the order \( ka[7] \), where \( k \) is the wavenumber. The proposed \( Q_p \) satisfies \( \max(Q,0) \leq Q_p \leq Q + ka \), thus the here presented \( Q \) is a lower bound for \( Q_p \). For electrically small antennas, \( ka \ll 1 \) and \( Q \gg 1 \), thus the uncertainty would be negligible. In this thesis, we investigate superdirective antenna array with \( ka \approx 2 \). Although \( ka \) is not small, we argue that the method mentioned above can be applied since the superdirective antenna array has high \( Q \)-factor and the approximation are still valid. The \( Q \)-factor is calculated using convex optimization [9]. Here, we use the CVX[10] as the convex optimization tool, which can be easily integrated with MATLAB. The expressions of the stored energy and the radiated power are formulated in a matrix form based on the Method of Moment (MoM)[11] using Rao-Wilton-Gilson (RWG) basis functions[12] to carry out the convex optimization. These formulations are extended to an infinite ground plane case using the image theory[13]. In the convex optimization, the stored energy is minimized for all current distribution on the antenna array and the optimal current distribution which minimizes the \( Q \)-factor is determined. Note that the obtained optimal current distribution is not unique[3] and there might exist other optimal current distributions, which produces the same \( Q \)-factor. The model we are using is based on the antenna designed for base station applications with the frequency range from 700MHz to 4.2GHz [14]. In this thesis, we calculate at 745MHz, which is the center frequency of the 700MHz bands for GSM[15]. The calculated results include the directivity v.s. the \( Q \)-factor for several cases; array antennas vertically and horizontally placed above an infinite ground plane in the broad-side direction and scanning cases. These results illustrate the trade-off between the bandwidth and the directivity.

This thesis is organised as follows. In chapter 2, the basic theory used this thesis are introduced. The formulation of the stored energy and the radiated power in a matrix form and the implementation of MoM and CVX codes are discussed in chapter 3. The numerical results are shown in chapter 4. Conclusion and future work are mentioned in chapter 5.
Chapter 2

Background theory

In this chapter, the background knowledge used in this project are discussed. Firstly, the basic theory of antennas such as the directivity, the Q-factor, array antennas and superdirectivity are mentioned. Secondly, the basic concepts of the convex optimization are introduced. An efficient convex optimization solver called CVX[9] is used in this project. Finally, the formulations of the method of moment with RWG edge elements are discussed.

2.1 Basic theory of antennas

2.1.1 Antenna directivity

The antenna directivity is defined as “the ratio of the radiation intensity in a given direction from the antenna to the radiation intensity averaged over all directions”[13]. In other words, the antenna directivity is the ratio of the antenna radiation intensity in a given direction to that of an isotropic antenna. The antenna directivity $D$ can be calculated from the following formula

$$D = \frac{U}{U_0} = 4\pi \frac{U}{P_{rad}},$$

(2.1)

where $U$ and $P_{rad}$ are the radiation intensity and the total radiation power respectively. Figure 2.1 shows an example of the antenna directivity. Let the red line show the radiation intensity of an antenna which we want to measure and the blue line show a radiation intensity of an isotropic antenna, the antenna directivity in a given direction is given by the ratio of the amplitude of the red line to that of the blue line.
If we know the far-fields pattern $F(\hat{r})$ of an antenna in all direction, the antenna directivity can also be calculated as

$$D(\hat{r}) = 4\pi \int_0^{2\pi} \int_0^\pi F(\hat{r}) \sin \theta \, d\theta \, d\phi.$$  \hfill (2.2)

The antenna gain is another parameter, which is similar to the antenna directivity. The difference between the antenna directivity and the antenna gain is that the antenna gain takes the effects of conductor and dielectric losses and impedance mismatch into account but the antenna directivity does not.

### 2.1.2 \textit{Q}-factor

An antenna cannot radiate the whole energy which is input to it. Some is reflected due to the matching, but some of it is stored to the structure and one part vanish as heat due to losses. The space around an antenna is divided into three part[13]: Far-field region ($R > 2d^2/\lambda$), Radiating near-field region ($2d^2/\lambda > R > 0.62\sqrt{d^3/\lambda}$) and Reactive near-field region ($0.62\sqrt{d^3/\lambda} > R$), where $R$ is the distance from the antenna and $d$ is the largest dimension of the antenna[13]. The stored energy is usually stored in the reactive near-field region. The antenna \textit{Q}-factor is defined as the ratio of the stored energy to the radiated energy and can be written as [16]
CHAPTER 2. BACKGROUND THEORY

\[ Q = \max \left( Q^{(E)}, Q^{(M)} \right), \]

\[ Q^{(E)} = \frac{2\omega W_e}{P_{rad}}, \quad Q^{(M)} = \frac{2\omega W_m}{P_{rad}}, \]  

(2.3)

where \( \omega \) is the angular frequency, \( W_e \) is the stored electric energy and \( W_m \) is the stored magnetic energy. The high \( Q \)-factor means that the stored energy is considerable compared to the radiated power. For the antenna applications we want an antenna to radiate as much as possible; therefore, the low \( Q \)-factor is desired.

The \( Q \)-factor is inversely proportional to the fractional bandwidth \( B = (f_2 - f_1)/f_0 \) [2], where \( f_0 = (f_2 + f_1)/2 \), which means that a high \( Q \)-factor gives a narrow bandwidth. The stored energy of the antenna can be modelled locally as a series RLC single resonance circuits when the \( Q \)-factor is large and the relation between the fractional bandwidth and the \( Q \)-factor is given by [3]

\[ B \approx \frac{2}{Q} \frac{\Gamma_0}{\sqrt{1 - \Gamma_0^2}}, \]  

(2.4)

where \( \Gamma_0 \) is the threshold of the reflection coefficient. Note that the half power bandwidth (\( \Gamma_0 = 1/\sqrt{2} \)) is \( B \approx 2/Q \).
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2.1.3 $D/Q$ quotient

The antenna directivity and the Q-factor have been mentioned above. In the antenna applications, the antenna directivity and the bandwidth ($Q$-factor) are the crucial parameters. The directivity and $Q$-factor quotient give a balance between the antenna directivity and bandwidth. From the equation (2.1) and (2.3), the directivity and $Q$-factor quotient can be expressed as

$$\frac{D(\hat{r}, \hat{e})}{Q} = \frac{2\pi U(\hat{r}, \hat{e})}{c_0k \max(W_e, W_m)},$$

where $c_0$ is the speed of light, $k$ is the wave number, $\hat{e}$ is the polarization vector.

2.1.4 Array antenna

An array antenna can be used to achieve multiple beams and high directive patterns. A single element gives a wider radiation pattern and a low directivity. If a higher directivity or a specific radiation pattern are required, usually an array antennas is used in order to achieve the required specifications. A simple array case of two infinitesimal dipoles shown in Fig.2.3 is explained in [13].

![Figure 2.3: Two dipoles array][13]

The two dipoles are placed on the z-axis directing +y direction. If the two dipoles are excited with the same amplitude and the phase difference $\beta$, the total field can be
expressed as

$$E_t \approx E_1 + E_2 = \hat{\theta} j \eta_0 \frac{k I_0 l}{4\pi} \left\{ \frac{e^{-j(kr_1 - \beta/2)}}{r_1} + \frac{e^{-j(kr_1 + \beta/2)}}{r_2} \right\},$$  \hspace{1cm} (2.6)

where $I_o$ is the excitation coefficient, $k$ is the wavenumber, $l$ is the length of the dipole, $\eta_0$ is the wave impedance in the free space. The above expression is just an approximation since the effects of the mutual coupling are not considered. If the observation point is very far from the origin compared to the distance between the dipole, the far-fields approximation can be applied as follows.

$$\theta_1 \simeq \theta_2 \sim \theta, \hspace{1cm} (2.7)$$

$$r_1 = r - \frac{d}{2} \cos \theta, \hspace{1cm} (2.8)$$

$$r_2 = r + \frac{d}{2} \cos \theta. \hspace{1cm} (2.9)$$

Moreover, for the amplitude variations, we can approximate $r_1$ and $r_2$ as

$$r_1 \simeq r_2 = r. \hspace{1cm} (2.10)$$

By substituting the equation (2.7) - (2.10) into (2.6), the total field can be rewritten as

$$E_t \approx \hat{\theta} j \eta_0 \frac{k I_0 l e^{(-j kr)}}{4\pi r} \cos \theta \left\{ 2 \cos \theta \left[ \frac{kd \cos \theta + \beta}{2} \right] \right\}. \hspace{1cm} (2.11)$$

This result implies that the total field of an array antenna is given by the field of a single element multiplied by a factor (in the above case, $2 \cos \theta \left[ \frac{kd \cos \theta + \beta}{2} \right]$). This factor is called the array factor. The array factor depends on the array geometry and the phase excitation difference (in the above case, $d$ and $\beta$ respectively). Thus, the radiation pattern of an array antenna excited with the same amplitude can be controlled by the geometry of the antenna and the excitation of each element.

An array antenna which has the main lobe in the direction perpendicular to the axis of the array is called broadside array (Fig.2.4), whereas one which has the main lobe in the direction parallel to the axis of the array is called end-fire array[13] (Fig.2.5). If the space between the elements is larger than a certain value, multiple lobes which have the same amplitude as the main lobe are created[13]. These lobes are called grating lobes and degrade the antenna directivity. In order to avoid the grating lobes, the space between the elements must be less than one wavelength for a broadside array and a half wavelength for an end-fire array.
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Figure 2.4: Broadside array

Figure 2.5: End-fire array
2.1.5 Superdirectivity

There is an upper limit of the directivity for the uniformly excited aperture antennas and can be expressed as [13]

\[ D_0 = \frac{4\pi U_{\text{max}}}{P_{\text{rad}}} = \frac{4\pi}{\lambda^2} A, \]  

(2.12)

where \( A \) is the area of an aperture antenna. However, the larger directivity than the limit mentioned above can be achieved by allowing higher exciting modes. These type of antennas are called superdirective antennas. In Hansen’s book, the definition of the superdirectivity is given by “directivity higher than that obtained with the same array length and elements uniformly excited”[4]. These higher mode excitations make a trade-off between the bandwidth and the directivity. Allowing the higher order mode excitation means that the amplitude of the current might be very large and the phase of the current changes rapidly. This fact leads the very high Q-factor (narrow bandwidth). In Hansen’s book, it is also mentioned that theoretically any desired directivity value can be achieved for a fixed aperture size if we accept the very high Q-factor[4].

2.2 Convex optimization

In this project, convex optimizations are used to find optimal currents. There are two main advantages of using convex optimizations. Firstly, if a function \( f \) is a convex function and we find a minimum, then that minimum is always a global minimum [3]. If a function \( f \) is not convex function, then there is no guarantee that the minimum we found is a global minimum, and it might be just a local minimum. For convex optimizations, there is no need for checking whether the minimum we found is a global minimum or not. Secondly, there exist an efficient solver called CVX [10]. It can be easily implemented in MATLAB and is widely used.

A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) which satisfies the following relation is called a convex function [9].

\[ f(\theta x + (1 - \theta) y) \leq \theta f(x) + (1 - \theta) f(y). \]  

(2.13)

If \(-f\) is a convex function, then \( f \) is called a concave function.[9] The illustration of a convex function is shown in Fig.2.6.

![Convex function](image)
In this project, linear form \( f(x) = bx \), quadratic form \( f(x) = x^T A x \) are mainly used. For the quadratic form, a matrix \( A \) must be positive semi-definite and symmetric to solve convex optimization problems[3].

2.3 Method of Moment

2.3.1 Integral Equation: Method of Moment

There are several ways to solve electromagnetic problems. One widely used method is the Method of Moment [11]. Consider the following equation,

\[
LJ(r) = F, \tag{2.14}
\]

where \( L \) is a linear operator. The vector \( F \) is known and \( J \) is unknown which we want to solve. In the Method of Moment, the unknown functions are expanded as [11]

\[
J(r) = \sum_{n=1}^{N} a_n \psi_n(r), \tag{2.15}
\]

where \( a_n \) is an unknown expansion coefficient and \( \psi_n \) is a known function called a basis function. Once we find the expansion coefficients for all \( n \), we can say that the equation (2.14) are solved. By substituting the equation (2.15) into the equation (2.14),

\[
\sum_{n=1}^{N} a_n L \psi_n(r) \approx F. \tag{2.16}
\]

Since \( L \) is a linear operator, we can move it inside the summation. Taking the inner product between the equation (2.16) and other basis functions \( w_m(r) \) called test function leads to the following relation[11].

\[
\sum_{n=1}^{N} a_n \langle w_m(r), L \psi_n(r) \rangle \approx \langle w_m(r), F \rangle. \tag{2.17}
\]

Here the inner product \( \langle w_m(r), \psi_n(r) \rangle \) is defined as [13]

\[
\langle w_m(r), \psi_n(r) \rangle = \int_S w_m^*(r) \cdot \psi_n(r) dS, \tag{2.18}
\]

where \( w_m^*(r) \) is the complex conjugate of \( w_m(r) \). The equation (2.17) can be written in a matrix form as

\[
ZI = V, \tag{2.19}
\]

where

\[
Z = \begin{bmatrix}
\langle w_1(r), L \psi_1(r) \rangle & \langle w_1(r), L \psi_2(r) \rangle & \cdots \\
\langle w_2(r), L \psi_1(r) \rangle & \langle w_2(r), L \psi_2(r) \rangle & \cdots \\
\vdots & \vdots & \ddots
\end{bmatrix},
\]
\[ I = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}, \quad V = \begin{bmatrix} \langle w_1(r), F \rangle \\ \langle w_2(r), F \rangle \\ \vdots \\ \langle w_N(r), F \rangle \end{bmatrix}. \] 

The expansion coefficients, or \( I \) vector can be solved as \([13]\)

\[ I = Z^{-1} V. \] 

If the basis functions \( \psi_n(r) \) used for the expansion in the equation (2.15) and weighting functions \( w_m(r) \) are the same, \( Z \) matrix is symmetric and consequently the computation time can be reduced. This technique is called Galerkin’s Method\([13]\).

### 2.3.2 RWG edge elements

In order to calculate the impedance matrix, we need to choose a set of basis functions and expand the electric currents. In this project, Rao-Wilton-Gilson (RWG) edge elements\([12]\) are used. The structure of RWG edge elements is shown in Fig. 2.7.

![Figure 2.7: RWG edge element](image-url)
Each element has one interior edge and two triangles $T^+_n$ and $T^-_n$. The vector basis functions are defined as\[12\]

\[
\psi_n(r) = \begin{cases} 
\frac{l_n}{2A^+_n} \rho^+_n & (r \text{ in } T^+_n), \\
\frac{l_n}{2A^-_n} \rho^-_n & (r \text{ in } T^-_n), \\
0 & \text{(otherwise)}, 
\end{cases}
\] (2.22)

where $l_n$ is the length of the interior edge and $A^+_n, A^-_n$ are the area of the triangles $T^+_n, T^-_n$ respectively. The vector $\rho^+_n$ is defined as the vector from the free vertex of the triangle $T^+_n$ to the observation point $r$ and $\rho^-_n$ is defined as the vector from the observation point $r$ to the free vertex of the triangle $T^-_n$. From the definition of the vector basis functions (2.22), it can be seen that there is no current perpendicular to the surface of the triangles $T^+_n, T^-_n$ and the current normal to the interior edge is constant and continuous. These facts mean that there is no line charge on all edges of two triangles\[12\]. Moreover, the divergence of the vector basis functions can be easily calculated as\[12\]

\[
\nabla_s \cdot \psi_n(r) = \begin{cases} 
\frac{l_n}{A^+_n} & (r \text{ in } T^+_n), \\
\frac{l_n}{A^-_n} & (r \text{ in } T^-_n), \\
0 & \text{(otherwise)}, 
\end{cases}
\] (2.23)

therefore, the charge density is constant over the surface of the two triangles.

In order to drive the $Z$ matrix expressed by RWG edge elements, first one need to drive the electric field integral equation, or EFIE. The scattered fields $E_s$ by an antenna can be expressed as

\[
E_s(r) = -j\omega A(r) - \nabla \Phi(r),
\] (2.24)

where $A$ and $\Phi$ are the vector potential and the scalar potential respectively expressed as

\[
A(r) = \frac{\mu}{4\pi} \int_S J(r') e^{-jk|r-r'|} dS',
\] (2.25)

\[
\Phi(r) = -\frac{1}{4\pi\varepsilon} \int_S \sigma(r') e^{-jk|r-r'|} dS' - \frac{1}{4\pi j\omega\varepsilon} \int_S \nabla_s \cdot J(r') e^{-jk|r-r'|} dS',
\] (2.26)

where $J(r)$ is the induced current on the surface $S$. The boundary condition on the surface of an antenna $\hat{n} \times E = 0$ and the equation (2.24) give the following relation\[12\],

\[
-\hat{n} \times E_s(r) = \hat{n} \times (-j\omega A(r) - \nabla \Phi(r)) \quad (r \text{ on } S).
\] (2.27)
CHAPTER 2. BACKGROUND THEORY

Taking an inner product between (2.27) and RWG vector basis functions $\psi_m(r)$ yields the following expression[12].

$$
j \omega l_m \left[ A(r_m^+) \cdot \frac{\rho_m^+}{2} + A(r_m^-) \cdot \frac{\rho_m^-}{2} \right] + l_m \left[ \Phi(r_m^-) - \Phi(r_m^+) \right]
= l_m \left[ E^i(r_m^+) \cdot \frac{\rho_m^+}{2} + E^i(r_m^-) \cdot \frac{\rho_m^-}{2} \right],
$$

(2.28)

where $r_m^+$ and $r_m^-$ are the vector from the origin to the centroid of triangles $T_m^+$, $T_m^-$ respectively and the following approximation were used.

$$\frac{1}{A_m} \int_{T_m^+} f(r) dS \approx f(r_m^+),
$$

(2.29)

$$\frac{1}{A_m} \int_{T_m^-} f(r) dS \approx f(r_m^-).
$$

(2.30)

By substituting (2.15) into (2.28), finally we can get the matrix equation (2.19) where

$$V_m = l_m \left( E_m^i(r_m^+) \cdot \frac{\rho_m^+}{2} + E_m^i(r_m^-) \cdot \frac{\rho_m^-}{2} \right),
$$

(2.31)

$$Z_{mn} = l_m \left[ j \omega \left( A_{mn}^+ \cdot \frac{\rho_m^+}{2} + A_{mn}^- \cdot \frac{\rho_m^-}{2} \right) + \phi_m^- - \phi_m^+ \right],
$$

(2.32)

$$A_{mn}^\pm = \frac{\mu}{4\pi} \int_S \psi_n(r') e^{-jkR_{mn}^\pm} \frac{dS'}{R_{mn}^\pm},
$$

(2.33)

$$\phi_{mn}^\pm = -\frac{1}{4\pi j \omega} \int_S \nabla \cdot \psi_n(r') e^{-jkR_{mn}^\pm} \frac{dS'}{R_{mn}^\pm},
$$

(2.34)

$$R_{mn}^\pm = |r_m^\pm - r'|.
$$

(2.35)
Chapter 3

Implementation of MoM and CVX codes

3.1 An overview of the implemented codes

In Makarov’s book[17], he developed the MoM codes using RWG basis functions in MATLAB and demonstrated a wide range of cases from simple dipoles to array antennas. Codes used in this project are based on the codes from his book. The flowchart of the implemented codes is shown in Fig.3.1.

![Flowchart of the implemented codes](image)

Figure 3.1: The flowchart of the implemented codes
CHAPTER 3. IMPLEMENTATION OF MOM AND CVX CODES

The main purpose of the implemented codes is to find the current distributions which minimize $Q$-factor or maximize the $D/Q$ quotient. The short explanations of the implemented codes are mentioned below.

First, parameters (such as the geometry of the antenna, frequency, polarization, direction of the radiation and discretization parameters) are input to the codes. According to the antenna geometry and the discretization parameters, the antenna structures are discretized and the RWG basis functions are created. These parts are identical to that of the codes in the Makarov’s book. Then, the stored energy and the radiated power are calculated in an analogous way to that of the impedance matrix (mentioned in Section 3.2). The far field is also calculated here. The directivity is formulated by the far field (radiation intensity) and the radiated power, and the $Q$-factor is formulated by the stored energy and the radiated power. Finally, applying the convex optimization by the CVX codes gives the optimal current distributions. Once we find the current distributions, basic antenna parameters such as the far field patterns, directivity and $Q$-factor can be calculated easily from the current distributions.

3.2 Implementation of the MoM codes

3.2.1 Calculation of the impedance matrix

In this section, the calculation procedures of the impedance matrix are discussed. For the optimization problems, it is not necessary to calculate the impedance matrix itself, however the stored energy and the radiated power can be calculated in a similar way to that of the impedance matrix.

The impedance matrix has been derived in Section 2.3.2. To calculate it on a computer, one need to derive an expression which is suitable for a numerical calculation.

Substituting (2.33) and (2.34) into (2.32) yields the following expression.

$$Z_{mn} = \frac{j\omega \mu_m}{2} \left[ \int_S \psi_n(r') \cdot \rho_m e^{\frac{-jkR_n}{4\pi R_m}} dS' + \int_S \psi_n(r') \cdot \rho_m e^{\frac{-jkR_m}{4\pi R_m}} dS' \right] + \frac{1}{j\omega\epsilon_0} \left[ \int_S \nabla' \cdot \psi_n(r') \frac{e^{-jkR_n}}{4\pi R_m^+} dS' - \int_S \nabla' \cdot \psi_n(r') \frac{e^{-jkR_m}}{4\pi R_m^-} dS' \right].$$  \hspace{1cm} (3.1)

Since $\psi_n(r')$ is nonzero only on $T_n^+$ and $T_n^-$, the surface integral in (3.1) can be written as

$$\int_S f(r') dS' = \int_{T_n^+} f(r') dS' + \int_{T_n^-} f(r') dS'.$$  \hspace{1cm} (3.2)

In the algorithm mentioned in the Makarov’s book[17], the surface integral over a triangle $T_p$ (Fig.3.2) is calculated first and added to the corresponding components of the impedance matrix.
In the example of Fig.3.2, three RWG edge elements share the same triangle $T_p$ and $T_p$ is considered as $T_i^+$, $T_j^+$ and $T_k^-$. Therefore, the integral over the triangle $T_p$ is added to $Z_{m,n=[i,j,k]}$ as the following manner.

$$Z_{m,n=[i,j]} = Z_{m,n=[i,j]}^+ + \frac{j \omega \mu_m}{4} \frac{l_n}{A_n^+} \int_{T_n^+} \left[ \rho_m^+(r') \cdot \rho_m^e \frac{e^{-jkR_m^+}}{4\pi R_m^+} + \rho_n^+(r') \cdot \rho_n^e \frac{e^{-jkR_m^+}}{4\pi R_m^+} \right] dS' + \frac{1}{j \omega \epsilon_0} \frac{l_n}{A_n^+} \int_{T_n^+} \left[ \frac{e^{-jkR_m^+}}{4\pi R_m^+} - \frac{e^{-jkR_m^-}}{4\pi R_m^-} \right] dS', \quad (3.3)$$

$$Z_{m,n=k} = Z_{m,n=k}^+ - \frac{j \omega \mu_m}{4} \frac{l_n}{A_n^-} \int_{T_n^-} \left[ \rho_n^-(r') \cdot \rho_m^e \frac{e^{-jkR_m^-}}{4\pi R_m^-} + \rho_n^-(r') \cdot \rho_n^e \frac{e^{-jkR_m^-}}{4\pi R_m^-} \right] dS' - \frac{1}{j \omega \epsilon_0} \frac{l_n}{A_n^-} \int_{T_n^-} \left[ \frac{e^{-jkR_m^+}}{4\pi R_m^+} - \frac{e^{-jkR_m^-}}{4\pi R_m^-} \right] dS'. \quad (3.4)$$

The processes mentioned above are done over all triangles $T_p=[1,2,\ldots]$. To improve the accuracy of the calculation, a technique called barycentric subdivision of an arbitrary triangle [18] is used. In this technique, a triangle is divided into 9 small triangles (Fig.3.3) and the surface integral over these triangles is given by [17]

$$\int_{T_p} f(r)dS = \frac{A_p}{9} \sum_{k=1}^{9} f(r_k^c), \quad (3.5)$$

where $r_k^c$ is the vector from the origin to the centroid of each sub-triangles.
3.2.2 Calculation of the stored energy and radiated power

In Section 2.1.2, the brief explanation of the $Q$-factor was mentioned. In order to calculate the $Q$-factor, first it is necessary to calculate the stored energy. There are several way to express the stored energy. In this project, the expressions introduced by Vandenbosch [6] are used and given by

$$W_e = \frac{\eta_0}{4\omega} \int_{\Omega} \int_{\Omega} \nabla_1 \cdot \mathbf{J}(r_1) \nabla_2 \cdot \mathbf{J}^*(r_2) \frac{\cos(kR_{12})}{4\pi kR_{12}}$$
$$- (k^2 \mathbf{J}(r_1) \cdot \mathbf{J}^*(r_2) - \nabla_1 \cdot \mathbf{J}(r_1) \nabla_2 \cdot \mathbf{J}^*(r_2)) \frac{\sin(kR_{12})}{8\pi} dV_1 dV_2, \quad (3.6)$$

$$W_m = \frac{\eta_0}{4\omega} \int_{\Omega} \int_{\Omega} k^2 \mathbf{J}(r_1) \cdot \mathbf{J}^*(r_2) \frac{\cos(kR_{12})}{4\pi kR_{12}}$$
$$- (k^2 \mathbf{J}(r_1) \cdot \mathbf{J}^*(r_2) - \nabla_1 \cdot \mathbf{J}(r_1) \nabla_2 \cdot \mathbf{J}^*(r_2)) \frac{\sin(kR_{12})}{8\pi} dV_1 dV_2, \quad (3.7)$$

where $W_e$ is the stored electric energy, $W_m$ is the stored magnetic energy and $R_{12} = |r_1 - r_2|$. In his paper, the expression of the radiated power $P_r$ is also mentioned and given by

$$P_r = \frac{\eta_0}{2k} \int_{\Omega} \int_{\Omega} (k^2 \mathbf{J}(r_1) \cdot \mathbf{J}^*(r_2) - \nabla_1 \cdot \mathbf{J}(r_1) \nabla_2 \cdot \mathbf{J}^*(r_2)) \frac{\sin(kR_{12})}{4\pi k} dV_1 dV_2. \quad (3.8)$$
These expressions are expressed by the current distributions on the antenna structures; therefore they are suitable for the optimization problems. In the above expressions, by expanding the electric current with the vector basis functions $\psi_n(r)$ we can get the following matrix expressions [3].

$$ W_e \approx \frac{1}{4\omega} \mathbf{I}^H \mathbf{X}^e \mathbf{I} = \frac{1}{8} \mathbf{I}^H \left( \frac{\partial \mathbf{X}}{\partial \omega} - \frac{\mathbf{X}}{\omega} \right) \mathbf{I}, $$

$$ W_m \approx \frac{1}{4\omega} \mathbf{I}^H \mathbf{X}^m \mathbf{I} = \frac{1}{8} \mathbf{I}^H \left( \frac{\partial \mathbf{X}}{\partial \omega} + \frac{\mathbf{X}}{\omega} \right) \mathbf{I}, $$

$$ P_r \approx \frac{1}{2} \mathbf{I}^H \mathbf{R} \mathbf{I}, $$

where $\mathbf{I}^H$ denotes the Hermitian transpose and $\mathbf{R}$ and $\mathbf{X}$ are the real and imaginary part of the impedance matrix respectively. These expressions were first introduced by Harrington and Mautz [19] and the results are identical to the expressions by Vandenbosch. The electric reactance matrix $\mathbf{X}^e$ and magnetic reactance matrix $\mathbf{X}^m$ are given by

$$ \mathbf{X}^e = \eta_0 \int_S \int_S (\nabla_1 \cdot \psi_m(r_1) \nabla_2 \cdot \psi_n(r_2)) \cos(kR_{21}) \frac{dS_1 dS_2}{4\pi kR_{21}} $$

$$ - \eta_0 \int_S \int_S (k^2 \psi_m(r_1) \cdot \psi_n(r_2) - \nabla_1 \cdot \psi_m(r_1) \nabla_2 \cdot \psi_n(r_2)) \frac{\sin(kR_{21})}{8\pi} dS_1 dS_2, $$

(3.12)

$$ \mathbf{X}^m = \eta_0 \int_S \int_S (k^2 \psi_m(r_1) \cdot \psi_n(r_2)) \cos(kR_{21}) \frac{dS_1 dS_2}{4\pi kR_{21}} $$

$$ - \eta_0 \int_S \int_S (k^2 \psi_m(r_1) \cdot \psi_n(r_2) - \nabla_1 \cdot \psi_m(r_1) \nabla_2 \cdot \psi_n(r_2)) \frac{\sin(kR_{21})}{8\pi} dS_1 dS_2. $$

(3.13)

From the approximation (2.29) - (2.30) and the expressions of the vector basis functions (2.22) - (2.23), the components of the electric reactance matrix $\mathbf{X}^e$ and magnetic reactance matrix $\mathbf{X}^m$ can be written as

$$ X_{mn}^e = -\frac{\eta_0 k^2 l_m}{2} \left[ \int_S \psi_n(r') \cdot \mathbf{\rho}_m^e \frac{\sin(kR_{m}^+)}{8\pi} dS' + \int_S \psi_n(r') \cdot \mathbf{\rho}_m^e \frac{\sin(kR_{m}^-)}{8\pi} dS' \right] $$

$$ + \eta_0 l_m \left[ \int_S \nabla_1' \cdot \psi_n(r') \left( \cos\left(\frac{kR_{m}^+}{4\pi kR_{m}^+}\right) + \sin\left(\frac{kR_{m}^-}{8\pi}\right) \right) dS' \right. $$

$$ \left. - \int_S \nabla_2' \cdot \psi_n(r') \left( \cos\left(\frac{kR_{m}^-}{4\pi kR_{m}^-}\right) + \sin\left(\frac{kR_{m}^-}{8\pi}\right) \right) dS' \right] $$

(3.14)
functions. In the case of Fig. 3.2, the contribution of the surface integral over the triangle

\[
X^m_{mn} = \frac{\eta_0 k^2 l_m}{2} \left[ \int_S \psi_n(r') \cdot \rho^+_m \left( \frac{\cos(kR^+_{mn})}{4\pi k R^+_{mn}} - \frac{\sin(kR^+_{mn})}{8\pi} \right) dS' \right. \\
+ \left. \int_S \psi_n(r') \cdot \rho^-_m \left( \frac{\cos(kR^-_{mn})}{4\pi k R^-_{mn}} - \frac{\sin(kR^-_{mn})}{8\pi} \right) dS' \right] \\
+ \eta_0 l_m \left[ \int_S \nabla' \cdot \psi_n(r') \frac{\sin(kR^+_{mn})}{8\pi} dS' - \int_S \nabla' \cdot \psi_n(r') \frac{\sin(kR^-_{mn})}{8\pi} dS' \right].
\]

(3.15)

By comparing the impedance matrix (3.1) with the reactance matrices (3.14) and (3.15), it can be seen that these expressions are very similar to each other and the only difference is the Green’s functions; therefore we can calculate the reactance matrices in a similar way to that of the impedance matrix only with the modification of the Green’s functions. In the case of Fig. 3.2, the contribution of the surface integral over the triangle \( T_p \) to the reactance matrix is given by

\[
X^e_{m,n=[i,j]} = X^e_{m,n=[i,j]} + \\
- \frac{\eta_0 k^2 l_m}{4} l_n \int_{T_p^+} \left[ \rho^+_n(r') \cdot \rho^+_m \frac{\cos(kR^+_{mn})}{4\pi k R^+_{mn}} + \rho^-_n(r') \cdot \rho^+_m \frac{\cos(kR^-_{mn})}{4\pi k R^-_{mn}} \right] dS' \\
+ \eta_0 l_m \int_{T_p^+} \left[ \left( \frac{\cos(kR^+_{mn})}{4\pi k R^+_{mn}} + \frac{\sin(kR^+_{mn})}{8\pi} \right) - \left( \frac{\cos(kR^-_{mn})}{4\pi k R^-_{mn}} + \frac{\sin(kR^-_{mn})}{8\pi} \right) \right] dS',
\]

(3.16)

\[
X^e_{m,n=k} = X^e_{m,n=k} + \\
- \frac{\eta_0 k^2 l_m}{4} l_n \int_{T_p^-} \left[ \rho^+_n(r') \cdot \rho^+_m \frac{\cos(kR^+_{mn})}{4\pi k R^+_{mn}} + \rho^-_n(r') \cdot \rho^+_m \frac{\cos(kR^-_{mn})}{4\pi k R^-_{mn}} \right] dS' \\
- \frac{\eta_0 l_m}{A_n} \int_{T_p^-} \left[ \left( \frac{\cos(kR^+_{mn})}{4\pi k R^+_{mn}} + \frac{\sin(kR^+_{mn})}{8\pi} \right) - \left( \frac{\cos(kR^-_{mn})}{4\pi k R^-_{mn}} + \frac{\sin(kR^-_{mn})}{8\pi} \right) \right] dS',
\]

(3.17)

\[
X^m_{m,n=[i,j]} = X^m_{m,n=[i,j]} + \\
\frac{\eta_0 k^2 l_m}{4} \frac{l_n}{A_n} \int_{T_p^+} \left[ \rho^+_n(r') \cdot \rho^+_m \left( \frac{\cos(kR^+_{mn})}{4\pi k R^+_{mn}} - \frac{\sin(kR^+_{mn})}{8\pi} \right) \\
+ \rho^-_n(r') \cdot \rho^+_m \left( \frac{\cos(kR^-_{mn})}{4\pi k R^-_{mn}} - \frac{\sin(kR^-_{mn})}{8\pi} \right) \right] dS' \\
+ \eta_0 l_m \frac{l_n}{A_n} \int_{T_p^+} \left[ \frac{\sin(kR^+_{mn})}{8\pi} - \frac{\sin(kR^-_{mn})}{8\pi} \right] dS'.
\]

(3.18)
\[ X_{m,n=k}^m = X_{m,n=k}^m + \frac{\eta_0 k^2 l_m}{4} \frac{l_n}{A_n} \int_{T_n}^T \left[ \rho_n^-(r') \cdot \rho_m^+ \left( \frac{\cos (kR_m^+)}{4\pi kR_m^+} - \frac{\sin (kR_m^+)}{8\pi} \right) + \rho_n^-(r') \cdot \rho_m^- \left( \frac{\cos (kR_m^-)}{4\pi kR_m^-} - \frac{\sin (kR_m^-)}{8\pi} \right) \right] dS' \]

\[ - \eta_0 l_m \frac{l_n}{A_n} \int_{T_n}^T \left[ \frac{\sin (kR_m^+)}{8\pi} - \frac{\sin (kR_m^-)}{8\pi} \right] dS'. \]  

(3.19)

These expressions are not exactly symmetric because the different approximations for the surface integral are used, (2.29), (2.30) and (3.5). To make these matrices symmetric, they are modified by

\[ X_{\text{new}} = \frac{X + X^H}{2} \]  

(3.20)

The expressions of the stored energy by Vandenbosch can produce the negative stored energy\[3\] and the reactance matrices are not positive semidefinite especially for electrically large structures. In this thesis, the negative stored energy is considered as zero\[3\]. This is performed by replacing the negative eigenvalue with 0. To perform this procedure, first the reactance matrices are decomposed by an eigenvalue decomposition\[3\].

\[ X_{\text{new}} = U A U^H, \]  

(3.21)

where \( A \) is a diagonal matrix and its components are the eigenvalues of \( X \). If \( A \) contains negative eigenvalues, these eigenvalues are replaced by 0 \[3\].

3.2.3 Calculation of the radiation intensity

In Section 2.1, the antenna directivity and \( D/Q \) quotient have been introduced. To formulate these quantities for the optimization problems, one needs to express the radiation intensity \( U \) in terms of the current distributions. In this section, the formulation of the radiation intensity is discussed.

In the far field region, the partial radiation intensity in the direction \( \hat{r} \) and with the polarization \( \hat{e} \) is given by

\[ U(\hat{r}, \hat{e}) = \frac{\eta_0^2}{2} |\hat{e}^* \cdot E(r)|^2, \]

(3.22)

where \( r \) is the position of the observation point and \( r = |r| \), and the electric fields \( E(\hat{r}) \) is given by [21]

\[ E(r) = \frac{e^{-jkr}}{r} \left( \hat{r} \times F(\hat{r}) \right) \times \hat{r}, \]

(3.23)

\[ F(\hat{r}) = -\frac{j\eta_0}{4\pi} \int_V J(r')e^{jk\hat{r}' \cdot r'} dV'. \]

(3.24)
Since \( \hat{r} \) and \( \hat{e} \) are perpendicular to each other in the far field region, the inner product between \( \hat{e}^* \) and (3.23) can be expressed as

\[
\hat{e}^* \cdot E(r) = \frac{e^{-jkr}}{r}\hat{e}^* \cdot F(\hat{r}).
\]  
(3.25)

By substituting (3.25) into (3.22) and expanding the current distribution \( \mathbf{J}(r) \) with the RWG vector functions \( \psi_n(r) \), the partial radiation intensity can be expressed as the following matrix form[20]

\[
U(\hat{r}, \hat{e}) \approx \frac{1}{2\eta_0} |\mathbf{F}\mathbf{I}|^2, \quad (3.26)
\]

where \( \mathbf{I} \) is (2.20) and the components of \( \mathbf{F} \) is

\[
F_n = -\frac{jk\eta_0}{4\pi} \int_{S} \hat{e}^* \cdot \psi_n(r') e^{jk\hat{r} \cdot r'} dS'.
\]  
(3.27)

This surface integral can be calculated with the use of the barycentric subdivision of an arbitrary triangle technique mentioned in Section (3.2).

### 3.3 Implementation of the CVX codes

The formulations of the stored energy, the radiation power and the radiation intensity expressed in the current distributions have been derived. The main goal of the implemented CVX codes is to find the optimal currents which give the physical bounds. In this section, two types of convex optimization problems with simple examples are discussed. One is the maximization of the \( D/Q \) quotient and the other is the maximization of the \( D/Q \) quotient for superdirective antennas.

#### 3.3.1 Maximization of the \( D/Q \) quotient

By substituting (3.9), (3.10) and (3.26) into (2.5), we can express the \( D/Q \) quotient as the following matrix form [22].

\[
\frac{D(\hat{r}, \hat{e})}{Q} \approx \frac{4\pi |\mathbf{F}\mathbf{I}|^2}{\eta_0 \max(\mathbf{I}^H \mathbf{X}^c\mathbf{I}, \mathbf{I}^H \mathbf{X}^m\mathbf{I})} \quad (3.28)
\]

In the above expression, even if the current vector \( \mathbf{I} \) is scaled to \( \alpha \mathbf{I} \), the \( D/Q \) quotient will not change and we can choose \( |\mathbf{F}\mathbf{I}| \) arbitrary; therefore the maximization of (3.28) can be written as the following minimization problem [22].

\[
\begin{align*}
\text{minimize} & \quad \max(\mathbf{I}^H \mathbf{X}^c\mathbf{I}, \mathbf{I}^H \mathbf{X}^m\mathbf{I}), \\
\text{subject to} & \quad |\mathbf{F}\mathbf{I}|^2 = 1. \quad (3.29)
\end{align*}
\]
CHAPTER 3. IMPLEMENTATION OF MOM AND CVX CODES

Moreover, it is sufficient to consider the imaginary part of \( \mathbf{F} \mathbf{I} \) and (3.29) can be rewritten as [22]

\[
\begin{align*}
\text{minimize} & \quad \max(I^H \mathbf{X}^\text{e} \mathbf{I}, I^H \mathbf{X}^\text{m} \mathbf{I}), \\
\text{subject to} & \quad \text{Im} [\mathbf{F} \mathbf{I}] = 1.
\end{align*}
\] (3.30)

Although we can implement above minimization problem with CVX directly, the following reformulation is used for the improvement of the convergence [3].

\[
I^H \mathbf{X} \mathbf{I} = ||X^{1/2}\mathbf{I}||^2,
\] (3.31)

where \( ||\mathbf{A}|| \) is the 2-norm of the matrix \( \mathbf{A} \).

Numerical examples of the \( D/Q \) quotient and the \( Q \)-factor are discussed here. The calculation model is a simple strip dipole shown in Fig.3.4. The strip dipole has the length \( L \) and the width \( W \) and it is infinitely thin.

The results of the maximized \( D/Q \) quotient and the \( Q \)-factor for the direction \( \hat{r} = \hat{z} \) and polarization \( \hat{y} \) are shown in Fig.3.5. The structure is discretised with \( N_x = 1 \), \( N_y = 100 \) mesh. It can be seen that as the dipole becomes shorter, the \( Q \)-factor becomes larger and \( D/Q \) quotient becomes smaller. For the half wavelength dipole with \( L = 0.48\lambda \), the CVX code gives \( Q \approx 5 \), \( D/Q \approx 0.3 \) and \( D \approx 1.65 \). These values agree with the results in [3]. The current distributions on the dipole with \( L = \{0.25\lambda, 0.48\lambda\} \) are shown in Fig.3.6. The current distribution of the dipole with length \( L = 0.48\lambda \) is same as the current distribution of the center fed half wavelength dipole.

\[ W=0.02L \]

\[ L=\lambda/2 \]

\[ \text{Polarization} \hat{y} \]

Figure 3.4: Simple strip dipole
Figure 3.5: \( Q \)-factor and \( D/Q \) quotient of the strip dipole

Figure 3.6: Current distributions on the strip dipole
3.3.2 Maximization of the $D/Q$ quotient for superdirective antennas

In this section, the physical bounds of superdirective antennas are discussed. Extension to superdirective antennas can be done simply by adding the following new constraint to (3.30) [22].

$$D_{th} \leq D = \frac{4\pi U(\hat{r}, \hat{e})}{P_{rad}} \approx \frac{4\pi |\mathbf{FI}|^2}{\eta_0 \mathbf{I}^\mathsf{H} \mathbf{R} \mathbf{I}}. \quad (3.32)$$

Therefore, the maximization of $D/Q$ quotient for superdirective antennas can be written as [22]

$$\text{minimize} \quad \max(\mathbf{I}^\mathsf{H} \mathbf{X}^e \mathbf{I}, \mathbf{I}^\mathsf{H} \mathbf{X}^m \mathbf{I}),$$

$$\text{subject to} \quad \text{Im} [\mathbf{FI}] = 1, \quad (3.33)$$

$$\mathbf{I}^\mathsf{H} \mathbf{R} \mathbf{I} \leq \frac{4\pi}{\eta_0 D_{th}}.$$ 

Numerical examples of the $D/Q$ quotient and the $Q$-factor for a superdirective dipole are discussed here. The calculation model is the same as that of the previous section with the length $L = 0.48\lambda$, see Fig.3.4. The structure is discretised with $N_x = 1, N_y = 100$ mesh. The results of the $Q$-factor, the $D/Q$ quotient v.s. the directivity for the direction $\hat{r} = \hat{z}$ and polarization $\hat{y}$ are shown in Fig.3.7 and the current distributions on the dipole are given in Fig.3.8. The amplitude of the currents are normalized by the maximum value among the three current distributions.

We observe that the $Q$-factor increases as the directivity increases, and it begins to increase rapidly when the directivity exceeds around 2.85. This could be due to the next higher order mode excitation, see Fig.3.8. It can also be seen in Fig.3.8 that the same order modes are excited for $D = 2$ and $D = 2.5$. However, the amplitude of the current for $D = 2.5$ case is larger than that of $D = 2$ case. The $Q$-factor of superdirective antennas is considerably higher than that of the normal dipole mentioned in the previous section and these facts agree with the discussion in [4]; therefore, superdirective antennas have the very narrow bandwidth.
Figure 3.7: $Q$-factor and $D/Q$ quotient of the superdirective strip dipole

Figure 3.8: Current distribution on the superdirective strip dipole
3.4 Extension to a ground plane case

In this section, the implementations of the physical bounds calculation are extended to a ground plane case using the image theory[13]. Although it is possible to calculate the effects of the image currents by simply adding image structures to the mesh, the computation time will increase. Instead, it is calculated by modifying the Green’s function in this project. This method does not require the additional mesh and the computation time can be reduced.

3.4.1 Modification of the impedance matrix

The electric and magnetic fields from the electric currents and the electric charges above an infinite ground plane can be explained by introducing the image currents and the image charges, see Fig.3.9.

![Image](image.png)

**Figure 3.9: The image electric current and the image electric charge**

To apply the image theory, the source current must be decomposed into two components. One is horizontal component against the ground plane and another is vertical component against the ground plane. The horizontal components give the image currents directed in the opposite direction, whereas the vertical components give the image currents directed in the same direction. The electric charges give the same amount of the image charges with the opposite sign.

According to the image theory, the expressions of the vector potential (2.25) and scaler potential (2.26) in the case of Fig.3.9 are given by[13]
\[
A_{x,y}(\mathbf{r}) = \frac{\mu}{4\pi} \int_S J_{x,y}(\mathbf{r}') \left( \frac{e^{-jk|r-r'|}}{|r-r'|} - \frac{e^{-jk|r-r'+2r'\hat{z}|}}{|r-r'+2r'\hat{z}|} \right) dS',
\]
(3.34)

\[
A_z(\mathbf{r}) = \frac{\mu}{4\pi} \int_S J_z(\mathbf{r}') \left( \frac{e^{-jk|r-r'|}}{|r-r'|} + \frac{e^{-jk|r-r'+2r'\hat{z}|}}{|r-r'+2r'\hat{z}|} \right) dS',
\]
(3.35)

\[
\Phi(\mathbf{r}) = -\frac{1}{4\pi j\omega\epsilon} \int_S \nabla S \cdot \mathbf{J}(\mathbf{r}') \left( \frac{e^{-jk|r-r'|}}{|r-r'|} - \frac{e^{-jk|r-r'+2r'\hat{z}|}}{|r-r'+2r'\hat{z}|} \right) dS'.
\]
(3.36)

Therefore, the impedance matrix is

\[
Z_{mn} = \eta_0 \int_S \int_S \left( jk \psi_m(\mathbf{r}_1) \cdot \mathbf{C} \psi_n(\mathbf{r}_2) + \frac{C^-}{jk} \nabla_1 \cdot \psi_m(\mathbf{r}_1) \nabla_2 \cdot \psi_n(\mathbf{r}_2) \right) e^{-jkR_{21}} \frac{4\pi R_{21}}{R_{21}^i} dS_1 dS_2,
\]
(3.37)

where

\[
C_- = 1 - \frac{R_{21}}{R_{21}^i} e^{-jk(R_{21}^i-R_{21})}, \quad C_+ = 1 + \frac{R_{21}}{R_{21}^i} e^{-jk(R_{21}^i-R_{21})},
\]
(3.38)

\[
\mathbf{C} = \begin{pmatrix} C_- & 0 & 0 \\ 0 & C_- & 0 \\ 0 & 0 & C_+ \end{pmatrix},
\]
(3.39)

\[
R_{21} = |\mathbf{r}_2 - \mathbf{r}_1|, \quad R_{21}^i = |\mathbf{r}_2 - \mathbf{r}_1 + 2(\mathbf{r}_1 \cdot \hat{z})\hat{z}|.
\]
(3.40)

The modification of the impedance matrix is tested for an input impedance of a half wavelength dipole placed above a ground plane horizontally and vertically, see Fig.3.10. The half wavelength dipole is excited at the center by the delta gap excitation[17] and the input resistance and the input reactance are calculated. The results are shown in Fig.3.11. It can be seen that the input resistance and input reactance are fluctuating when the height \(h\) is small in both horizontal and vertical cases, because of the strong coupling between the source currents and the image currents. As the height becomes larger, the input resistance and the input reactance approach to 88 \(\Omega\) and 44 \(\Omega\) respectively. These values agree with those of the half wavelength dipole in free space.
CHAPTER 3. IMPLEMENTATION OF MOM AND CVX CODES

Figure 3.10: The dipole above an infinite ground plane

Figure 3.11: Input impedance of the strip dipole above a ground plane
3.4.2 Modification of the stored energy

In Section 3.2.2, it has been mentioned that the electric reactance matrix \( \mathbf{X}^e \) and the magnetic reactance matrix \( \mathbf{X}^m \) can be derived from the derivative of the reactance matrix \( \mathbf{X} \) and these expressions are identical to the expressions by Vandenbosch. With this method, the expressions for the stored energy are extended to a ground plane case from the modified impedance matrix (3.37).

First, we can get the resistance matrix \( \mathbf{R} \) and the reactance matrix \( \mathbf{X} \) by simply decomposing the impedance matrix into its real part and imaginary part as follows.

\[
R_{mn} = \text{Re} [Z_{mn}] = \eta_0 \int_S \int_S (k^2 \psi_m(r_1) \cdot C_s \psi_n(r_2)) \sin(kR_{21}) \frac{dS_1 dS_2}{4\pi k R_{21}}, \tag{3.41}
\]

\[
X_{mn} = \text{Im} [Z_{mn}] = \eta_0 \int_S \int_S (k^2 \psi_m(r_1) \cdot C_c \psi_n(r_2)) \cos(kR_{21}) \frac{dS_1 dS_2}{4\pi k R_{21}}, \tag{3.42}
\]

where

\[
C_{\sin^{-}} = 1 - \frac{R_{21}}{R_{21}^i} \frac{\sin(kR_{21})}{\sin(kR_{21})}, \quad C_{\sin^{+}} = 1 + \frac{R_{21}}{R_{21}^i} \frac{\sin(kR_{21})}{\sin(kR_{21})}, \tag{3.43}
\]

\[
C_{\cos^{-}} = 1 - \frac{R_{21}}{R_{21}^i} \frac{\cos(kR_{21})}{\cos(kR_{21})}, \quad C_{\cos^{+}} = 1 + \frac{R_{21}}{R_{21}^i} \frac{\cos(kR_{21})}{\cos(kR_{21})}, \tag{3.44}
\]

\[
C_s = \begin{pmatrix} C_{\sin^{-}} & 0 & 0 \\ 0 & C_{\sin^{-}} & 0 \\ 0 & 0 & C_{\sin^{+}} \end{pmatrix}, \quad C_c = \begin{pmatrix} C_{\cos^{-}} & 0 & 0 \\ 0 & C_{\cos^{-}} & 0 \\ 0 & 0 & C_{\cos^{+}} \end{pmatrix}. \tag{3.45}
\]

\[
R_{21} = |r_2 - r_1|, \quad R_{21}^i = |r_2 - r_1 + 2(r_1 \cdot \hat{z}) \hat{z}|. \tag{3.46}
\]

The derivative of the reactance matrix is given by

\[
\frac{\partial X_{mn}}{\partial \omega} = \sqrt{\epsilon_0 \mu_0} \frac{\partial X_{mn}}{\partial k} = \epsilon_0 \int_S \int_S \left( k \psi_m(r_1) \cdot C_c \psi_n(r_2) + \frac{C_{\cos^{-}}}{k} \nabla_1 \cdot \psi_m(r_1) \psi_n(r_2) \right) \frac{\cos(kR_{21})}{4\pi k R_{21}} dS_1 dS_2
\]

\[
- \epsilon_0 \int_S \int_S \left( k^2 \psi_m(r_1) \cdot C_s \psi_n(r_2) - C_{\sin^{-}} \nabla_1 \cdot \psi_m(r_1) \psi_n(r_2) \right) \frac{\sin(kR_{21})}{4\pi k} dS_1 dS_2, \tag{3.47}
\]

where

\[
C_{\sin^{-}}' = 1 - \frac{\sin(kR_{21}^i)}{\sin(kR_{21})}, \quad C_{\sin^{+}}' = 1 + \frac{\sin(kR_{21}^i)}{\sin(kR_{21})}. \tag{3.48}
\]
\[
C_s' = \begin{pmatrix}
C_{s
} \sin & 0 & 0 \\
0 & C_{s
} \sin & 0 \\
0 & 0 & C_{s
} +
\end{pmatrix}.
\tag{3.49}
\]

From (3.9) and (3.10), the electric reactance matrix and the magnetic reactance matrix for a ground plane case are expressed as

\[
X_e^{mn} = \frac{\omega^2}{2} \left( \frac{\partial X_{mn}}{\partial \omega} - \frac{X_{mn}}{\omega} \right) = \eta_0 \int_S \int_S (C \cos \nabla_1 \cdot \psi_m(r_1) \nabla_2 \cdot \psi_n(r_2)) \frac{\cos(kR_{21})}{4\pi kR_{21}} dS_1 dS_2
\]

\[\quad - \eta_0 \int_S \int_S (k^2 \psi_m(r_1) \cdot C_s' \psi_n(r_2) - C_{s
} \sin \nabla_1 \cdot \psi_m(r_1) \nabla_2 \cdot \psi_n(r_2)) \frac{\sin(kR_{21})}{8\pi} dS_1 dS_2,
\tag{3.50}
\]

\[
X_m^{mn} = \frac{\omega^2}{2} \left( \frac{\partial X_{mn}}{\partial \omega} + \frac{X_{mn}}{\omega} \right) = \eta_0 \int_S \int_S (k^2 \psi_m(r_1) \cdot C_c \psi_n(r_2)) \frac{\cos(kR_{21})}{4\pi kR_{21}} dS_1 dS_2
\]

\[\quad - \eta_0 \int_S \int_S (k^2 \psi_m(r_1) \cdot C_s' \psi_n(r_2) - C_{s
} \sin \nabla_1 \cdot \psi_m(r_1) \nabla_2 \cdot \psi_n(r_2)) \frac{\sin(kR_{21})}{8\pi} dS_1 dS_2.
\tag{3.51}
\]

### 3.4.3 Modification of the radiation intensity

The last step in this section is the modification of the radiation intensity. Since the image currents affect the radiation pattern, one need to also modify the expression of the radiation intensity. This modification can be easily done by adding the image current to the expression (3.27) as follows.

\[
F_n = -\frac{jk\eta_0}{4\pi} \int_S \tilde{e}^\ast \cdot C_f \psi_n(r_1) e^{jk\hat{r} \cdot r_1} dS_1,
\tag{3.52}
\]

\[
C_f^- = 1 - e^{-jk\hat{r} \cdot (2r_1 \cdot \hat{z})}, \quad C_f^+ = 1 + e^{-jk\hat{r} \cdot (2r_1 \cdot \hat{z})},
\tag{3.53}
\]

\[
C_f = \begin{pmatrix}
C_f^- & 0 & 0 \\
0 & C_f^- & 0 \\
0 & 0 & C_f^+
\end{pmatrix}.
\tag{3.54}
\]
3.5 Validation of the implemented codes

In this section the implemented codes are validated by comparing three cases in published papers. The first case is $D/Q$ quotient for a planar rectangular structure in free space and the second case is similar to the first case but placed above an infinite ground plane. These two cases are capacitive ($W_e > W_m$), thus one need to check if the codes for the magnetic stored energy are implemented correctly. In the last case, the codes for the magnetic stored energy are validated by dual problems [3].

The first case is a planar rectangular structure in free space[8], see Fig.3.12. Consider that a planar rectangle is placed in the $xy$-plane, the polarization is $\hat{y}$ and the direction of the radiation is $\hat{r} = \hat{z}$. The structure is discretised such that $N_x = 20, N_y = 20$. The results of the $D/Q$ quotient and the $Q$-factor for $kL = \{0.1, 1\}$ are shown in Fig.3.13 and Fig.3.14 respectively. Here the $D/Q$ quotient is normalized with $(ka)^3$ to decrease the size dependence, where $a$ is the smallest radius of a sphere which circumscribes the whole structure. It can be observed that the $Q$-factor is considerably larger for electrically small structure ($kL = 0.1$). These results agree well with the results in [8].

![Figure 3.12: planar rectangle](image-url)
Figure 3.13: The $D/Q$ quotient of a planar rectangle

Figure 3.14: The $Q$-factor of a planar rectangle
The second case is a planar rectangular structure above an infinite ground plane[23], see Fig. 3.15. The planar rectangle is placed above an infinite ground plane. The width $W = 0.77L$, the height from the ground plane is $d$, the polarization is $\hat{e} = \hat{y}$ and the radiation vector is $\hat{r} = \hat{z}$. The structure is discretised with $N_x = 15$, $N_y = 25$ mesh. The $D/Q$ quotient and the $Q$-factor are calculated for three different heights $d = \{0.01L, 0.05L, 0.1L\}$. The results of the $D/Q$ quotient and the $Q$-factor are shown in Fig.3.16 and Fig.3.17 respectively. Note that in [23] the $D/Q$ quotient is not normalized. These results also agree well with the results in [23].

Figure 3.15: planar rectangle above an infinite ground plane

Figure 3.16: The $D/Q$ quotient of a planar rectangle above a ground plane
These two cases mentioned above are capacitive cases and the implemented codes for the magnetic stored energy have not been validated yet. In the last case, the codes for the magnetic stored energy are validated by solving the dual problems. Brief explanations of the dual problem are mentioned below, see [3] for the detailed explanations.

The minimization problem (3.29) can be simplified by introducing the dual function $d(\alpha)$, where $\alpha$ is the scaling factor $0 \leq \alpha \leq 1$ and the dual function is defined as the solution of the following minimization problem [3].

$$
\begin{align*}
\text{minimize} & \quad \mathbf{I}_\alpha^H (\alpha \mathbf{X}_e + (1-\alpha)\mathbf{X}_m) \mathbf{I}_\alpha, \\
\text{subject to} & \quad \mathbf{F} \mathbf{I}_\alpha = -j.
\end{align*}
$$

This minimization problem can be solved explicitly and given by [3]

$$d(\alpha) = \frac{1}{\mathbf{F} (\alpha \mathbf{X}_e + (1-\alpha)\mathbf{X}_m)^{-1} \mathbf{F}^H d(\alpha)}. \quad \text{(3.56)}$$

$$\mathbf{I}_\alpha = -j (\alpha \mathbf{X}_e + (1-\alpha)\mathbf{X}_m)^{-1} \mathbf{F}^H d(\alpha). \quad \text{(3.57)}$$

Here, the associated electric and magnetic $Q$-factor, $Q_{ea}$ and $Q_{ma}$ respectively and the associated partial directivity $D_{\alpha}$ can be calculated from the dual current $\mathbf{I}_\alpha$. In dual problems, the dual function is maximized for $0 \leq \alpha \leq 1$ and the optimal $D/Q$ quotient is given by [3]

$$D/Q_{\text{opt}} = \frac{4\pi}{\eta_0 \max_{0 \leq \alpha \leq 1} d(\alpha)}. \quad \text{(3.58)}$$
In [3], the associated electric and magnetic $Q$-factor are calculated for a planar rectangle. Here we validate the implemented codes by comparing these results in [3]. The calculation model is identical to Fig.(3.12) with the length $L = \lambda/10$ and the width $W = \lambda/20$. The structure is discretised with $N_x = 48$, $N_y = 24$. The results of the associated electric and magnetic energy are shown in Fig.3.18 for the polarization $\hat{e} = \hat{y}$ and the direction of the radiation $\hat{r} = \hat{x}$. These results agree well with the results in [3].

From the three cases mentioned above, it is confirmed that the codes for the radiated power, the stored electric energy, the stored magnetic energy and CVX are implemented correctly.

Figure 3.18: $D_{\alpha}/Q_{ea}$ and $D_{\alpha}/Q_{ma}$ by the dual problem
Chapter 4

Calculation results

The calculation models are based on the array antenna designed for base station applications with the frequency range from 700MHz to 4.2GHz[14], see Fig.4.1. Although this antenna can cover a wide range of frequency, several problems occur at the low frequency limit. The electrical size of the array antenna becomes smaller at the low frequency and it becomes harder to obtain high directivity. Another issue is that the Q-factor becomes higher as the electrical size becomes smaller. Therefore, it is important to know the trade-off between the directivity and the bandwidth, or the Q-factor. In this chapter, we investigate the relation between the directivity and the Q-factor for the array antenna at 745MHz, which is the center frequency of GSM 700MHz bands. We assume that the array antenna is placed above an infinite ground plane. The electrical size of the array antenna is \( ka \approx 2 \) at 745MHz, where \( k \) is the wavenumber and \( a \) is the smallest radius of a sphere which circumscribes all current distributions including the mirror currents. The results are depicted in two cases, one is the vertical case, where each array elements is placed vertically above an infinite ground plane, and the other is the horizontal case, where each array elements is placed horizontally. In both cases, the array antenna radiates in the broadside direction. The effects of the negative stored energy and mesh size are also discussed.

Figure 4.1: Array antenna for base station applications
4.1 Calculation model

The calculation model is shown in Fig.4.2. Although each plate contains seven array elements in Fig.4.1, for simplicity, each plate is considered as a simple rectangular array element in this thesis. Each rectangular plate has the length $L = 235\text{mm}$ and the width $W = 12\text{mm}$ and they are vertically placed above an infinite ground plane with the height $h = 23\text{mm}$ and the distance between each element $d = 34\text{mm}$. We also calculate with the model shown in Fig.4.3 for comparison. In Fig.4.3, each element has the same size as the model in Fig.4.2, but they are placed horizontally above an infinite ground plane. We assume that the infinite ground plane has no losses in this thesis.

![Array antenna placed vertically above an infinite ground plane](image1)

Figure 4.2: Array antenna placed vertically above an infinite ground plane

![Array antenna placed horizontally above an infinite ground plane](image2)

Figure 4.3: Array antenna placed horizontally above an infinite ground plane
4.2 Calculation result

4.2.1 The directivity versus the $Q$-factor

Here, the relation between the directivity and the $Q$-factor is discussed. The results are calculated for the vertical case Fig.4.2 and the horizontal case Fig.4.3, and each array element is discretized into $N_t = 300$ triangles (5 by 60 mesh). Consider that the polarization is $\hat{e} = \hat{y}$ and the array antenna radiates in the broadside direction $\hat{r} = \hat{z}$ at 745MHz. The $Q$-factor for the number of array elements $N = \{1, 2, 3, 4, 5\}$ cases is calculated using the minimization problem (3.33). The electrical size $ka$ of each array antenna $N = \{1, 2, 3, 4, 5\}$ for the vertical and the horizontal cases are shown in Table 1 and Table 2 respectively, where $k$ is the wavenumber and $a$ is the smallest radius of a sphere which circumscribes all current distributions including the mirror currents.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$ka$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.916</td>
</tr>
<tr>
<td>2</td>
<td>1.933</td>
</tr>
<tr>
<td>3</td>
<td>1.987</td>
</tr>
<tr>
<td>4</td>
<td>2.074</td>
</tr>
<tr>
<td>5</td>
<td>2.189</td>
</tr>
</tbody>
</table>

Table 1: Electrical size of the array antenna: Vertical case

<table>
<thead>
<tr>
<th>$N$</th>
<th>$ka$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.837</td>
</tr>
<tr>
<td>2</td>
<td>1.870</td>
</tr>
<tr>
<td>3</td>
<td>1.938</td>
</tr>
<tr>
<td>4</td>
<td>2.039</td>
</tr>
<tr>
<td>5</td>
<td>2.168</td>
</tr>
</tbody>
</table>

Table 2: Electrical size of the array antenna: Horizontal case

An antenna is referred to as electrically small if its electrical size is $ka < 1[4]$. Here, we are looking at $ka \approx 2$ cases. The results of the directivity versus the $Q$-factor are shown in Fig.4.4 and Fig.4.5 for the vertical and the horizontal cases respectively. It can be observed that as the number of array elements increases the $Q$-factor decreases for both the vertical and the horizontal cases. Since the area where we can control the current distribution becomes larger, it becomes easier to obtain higher directivity. The $Q$-factor is slightly lower for the vertical cases than horizontal cases. The reasons for this difference are discussed in the next section by examining the current distribution.
CHAPTER 4. CALCULATION RESULTS

Figure 4.4: Directivity versus $Q$-factor: Vertical case (broadside direction)

Figure 4.5: Directivity versus $Q$-factor: Horizontal case (broadside direction)
4.2.2 Optimal currents distributions and radiation patterns

In this section, we show the optimal current distributions and the radiation patterns. The optimal current distributions are solutions to the minimization problem that gives lower bounds for the $Q$-factor. The results illustrate the optimal current distributions for the case $N = 5$. The calculation conditions are same as the previous section: each array element is discretized into $N_t = 300$ triangles (5 by 60 mesh), the polarization is $\hat{\mathbf{e}} = \hat{\mathbf{y}}$ and the array antenna radiates in the broadside direction $\hat{\mathbf{r}} = \hat{\mathbf{z}}$ at 745MHz. Fig.4.6-Fig.4.9 illustrate the optimal current distributions for the vertical model with the lowest directivity $D = 9.54$ and $D = 15$ cases respectively. For $D = 9.54$ case (Fig.4.6), we have currents $J_y$ mainly on the upper edges of each element but also have weak currents $J_y$ on the lower edges of the outermost array elements. Note that $J_z$ is almost zero since a current directed in the $z$-direction does not radiate in the $z$-direction. For $D = 15$ case (Fig.4.6), currents $J_y$ distribute at the center and four edges of the aperture. It can also be observed that currents $J_y$ on the lower edge of the outermost array elements are stronger than the lower directivity case. For the horizontal model, we have the strongest currents $J_y$ on the outermost edges of the outermost elements. These currents become stronger for the higher directivity case.

One possible reason why the vertical model gives slightly lower $Q$-factor than the horizontal model is that the vertical model can create an end fire array when it radiates $+z$-direction. Yaghjian investigated the maximum directivity for electrically small isotropic arrays and revealed that end fire arrays can obtain higher maximum directivity than broadside arrays if the space between the array elements is less than a half wavelength[24]. This fact implies that the $Q$-factor is lower for end fire arrays than broad side arrays in a certain case. In our vertical model, currents can change along the $z$-direction and create an end fire array, however in the horizontal model, currents cannot change along the $z$-direction and cannot create an end fire array. This fact might be one reason why the vertical model gives slightly lower $Q$-factor than the horizontal model.

Note that the obtained optimal current distributions are not unique. There might exist other current distributions which give the same $Q$-factor.
Figure 4.6: Current Norm: Vertical case, $D = 9.54$, broadside direction

Figure 4.7: Current Phase: Vertical case, $D = 9.54$, broadside direction
Figure 4.8: Current Norm: Vertical case, $D = 15$, broadside direction

Figure 4.9: Current Phase: Vertical case, $D = 15$, broadside direction
CHAPTER 4. CALCULATION RESULTS

Figure 4.10: Current Norm: Horizontal case, \( D = 8.4 \), broadside direction

Figure 4.11: Current Phase: Horizontal case, \( D = 8.4 \), broadside direction
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Figure 4.12: Current Norm: Horizontal case, $D = 15$, broadside direction

Figure 4.13: Current Phase: Horizontal case, $D = 15$, broadside direction
CHAPTER 4. CALCULATION RESULTS

The radiation patterns calculated from the optimal currents distribution are discussed here. The calculation conditions are same as the previous section: each array element is discretized into $N_t = 300$ triangles (5 by 60 mesh), the polarization is $\hat{e} = \hat{y}$ and the array antenna radiates in the broadside direction $\hat{r} = \hat{z}$ at 745MHz. The radiation patterns for the vertically placed cases are illustrated in Fig.4.14-Fig.4.15, and Fig.4.16-Fig.4.17 depict the radiation patterns for the horizontal cases. We calculate the radiation patterns with the lowest directivity case shown in a red line and $D = 15$ case shown in a black line. Solid lines show the radiation patterns in the $yz$-plane, and dashed lines depict the radiation patterns in the $xz$-plane. For a single element case (Fig.4.14 and Fig.4.16), it can be observed that the main beam in the $xz$-plane is getting narrower significantly, whereas the main beam in the $yz$-plane changes slightly to obtain higher directivity. This fact can be seen in both the vertical and the horizontal cases. It is known that for a planar rectangular array on the $xy$-plane, the array factor can be expressed as the product of the array factors in the $x$-direction and in the $y$-direction[13]. The array factor in the $x$-direction affects the beam only in the $xz$-plane, and the array factor in the $y$-direction affects the beam in the $yz$-plane. For our single element case, since the current cannot change along the $x$-direction it is harder to obtain the narrower beam in the $xz$-plane than the beam in the $yz$-plane, thus the beam only in the $yz$-plane is getting narrower significantly to achieve higher directivity and this fact gives high $Q$-factor. It can also be observed that the side lobe level is getting higher as the beam width becomes narrower. Increasing the number of array elements $N$ allows the current to change along the $x$-direction, then we can obtain the narrower beam in the $xz$-plane. Since we can make the beam narrower both in the $xz$-plane and the $yz$-plane for $N = 5$ case, the beam in the $yz$-plane can be wider for $N = 5$ case than $N = 1$ case, and consequently the $Q$-factor becomes lower for $N = 5$ case.
CHAPTER 4. CALCULATION RESULTS

Figure 4.14: Radiation patterns: Vertical case, N=1, broadside direction

Figure 4.15: Radiation patterns: Vertical case, N=5, broadside direction
Figure 4.16: Radiation patterns: Horizontal case, \( N=1 \), broadside direction

Figure 4.17: Radiation patterns: Horizontal case, \( N=5 \), broadside direction
4.3 Negative eigenvalues

As mentioned in Chapter 3, the expression for the stored energy by Vandenbosch can produce the negative stored energy and cause an uncertainty for the $Q$-factor of the order of $\kappa a$. Due to this fact, the reactance matrices are not positive semi-definite for all structures and might have negative eigenvalues. Eigenvalues of the $\mathbf{R}$, $\mathbf{X}^e$ and $\mathbf{X}^m$ matrices for the structure in Fig.4.2 are shown in Fig.4.18-Fig.4.20 respectively, here the negative eigenvalues are expressed in circles. Each element is divided into 5 by 60 mesh and creates 835 vector basis functions.

The radiated power must be positive and consequently the resistance matrix $\mathbf{R}$ must be positive semi-definite for all structures. However several negative eigenvalues of the resistance matrix $\mathbf{R}$ are observed in Fig.4.18. The numerical precision limits of the double type are the order of $10^{-15}$. Therefore, the eigenvalues whose order is less than $10^{-15}$ can be considered as the numerical error due to the finite numerical precision but still there are several negative eigenvalues whose order is larger than $10^{-15}$. One possible reason for these relatively large negative eigenvalues is because of the approximation used in Makarov codes. In Makarov codes, several approximations are used and the impedance matrix $\mathbf{Z}$ is not symmetric, see Chapter 3. These approximation might produce small negative eigenvalues. The electric reactance matrix $\mathbf{X}^e$ is positive definite for all $N$, whereas the magnetic reactance matrix $\mathbf{X}^m$ has negative eigenvalues for all $N$. These negative eigenvalues are considerably large compared to those of the resistance matrix, thus one cannot take these negative eigenvalues away due to the numerical error. These negative eigenvalues come from the negative stored energy produced by the expressions (3.7)-(3.8) since the structures we are looking at are not electrically small. However since we study the minimization of $\max(W_e, W_m)$, these are not affecting the minimization as long as $W_e > 0$.

In this thesis, we consider the negative stored energy as 0 and negative eigenvalues are replaced by 0. Due to the negative stored energy, the results shown in Fig.4.4 contain the uncertainty of the order of $\kappa a$. Since $\kappa a \approx 2.2$ for $N = 5$ case, the lower bands for the $Q$-factor can at max be 10% larger in the region where the $Q$-factor is less than 22. Note that the proposed $Q_p$ satisfies $\max(Q, 0) \leq Q_p \leq Q + \kappa a$, where $Q$ is the presented $Q$-factor in this thesis.
Figure 4.18: Eigenvalues of the $R$ matrix

Figure 4.19: Eigenvalues of the $X^e$ matrix
CHAPTER 4. CALCULATION RESULTS

4.4 Effect of mesh size

In this thesis, each array element is discretized into $N_t = 300$ triangles (5 by 60 mesh). Here, effects of mesh size are discussed and we show $N_t = 300$ mesh gives the sufficient convergence. Fig.4.21 shows the relation between mesh size and the $Q$-factor. The vertical model with the number of array element $N = 1$, the directivity $D = 9$ and the broadside radiation are used for this calculation. The results are depicted for $N_t = \{1 \times 12, 2 \times 24, 3 \times 36, 4 \times 48, 5 \times 60, 6 \times 72, 7 \times 84, 9 \times 108, 11 \times 132\}$. It can be observed that the $Q$-factor converges very fast and the mesh we are using (marked with a red circle) has only $1.84\%$ of the deviation from the case for $N_t = 11 \times 132$. The computation time and memory usage increase enormously as the mesh size increases, thus one need to chose the suitable mesh size. In order to calculate array cases up to $N = 5$, we chose $N_t = 5 \times 60$ mesh but still more than 24GB of memory are used for $N = 5$ cases.

Figure 4.20: Eigenvalues of the $X^m$ matrix
Figure 4.21: The effects of mesh size
Chapter 5

Conclusion and future work

5.1 Conclusion

Base station antennas for next generation will be required to have a wide bandwidth for compatibility. However, a wide-band antenna faces several problems at the low frequency limit, such as a high $Q$-factor and a low directivity. In this thesis, we calculate the trade-off between the directivity and the $Q$-factor at the low frequency limit and extend our analysis to a superdirective antenna array. The implemented codes are based on the codes in Makarov book. The expressions for the stored energy by Vandenbosch and the radiated power are formulated in a matrix form based on the Method of Moment using RWG basis functions. To account for the effects of an infinite ground plane, the Green’s function for the impedance matrix expression is modified using the image theory and the expressions of the stored energy for an infinite ground plane case are derived. The implemented codes are validated by comparing the result of three cases in published papers and we show these results agree well.

The calculation model is based on an antenna designed for base station applications and its electrical size is $ka \approx 2$. The results are compared with the two cases; one is the vertical model, where each array elements is placed vertically above an infinite ground plane, and the other is the horizontal model, where each array elements is placed horizontally. It is observed that the $Q$-factor is slightly lower for the vertical model than the horizontal model and the corresponding current distributions are different from each other. We also show that the $Q$-factor decreases as the number of array elements $N$ increases and it is easier to design a high directive antenna with a larger array as we expected.
5.2 Future work

We use CVX as the convex optimization solver in this thesis. However, solving a convex optimization problem with CVX codes is the most time-consuming calculation in the implemented codes especially for a large structure. One possible remedy to reduce computation time is to reformulate the convex optimization problem to a dual optimization problem. The dual optimization problem does not require the CVX codes implementation and it uses the Newton’s method bisection method, golden section search and parabolic interpolation[3]. These methods are much faster than CVX and we can reduce computation time. In this thesis, we assume an infinite ground plane and an antenna that have no losses. However, one cannot remove all losses in reality and it is important to include the effects of the losses and evaluate the gain rather than the directivity. Moreover, we derive optimal current distributions, which gives lower bounds on the $Q$-factor. However, these results do not say whether we can create such currents or not. From the practical viewpoint, it is important to know if the optimal current distribution can be created easily or not.
References


