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Computational Ice Sheet Dynamics

Error control and efficiency

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Abstract

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Ice sheets, such as the Greenland Ice Sheet or Antarctic Ice Sheet, have a fundamental impact on landscape formation, the global climate system, and on sea level rise. The slow, creeping flow of ice can be represented by a non-linear version of the Stokes equations, which treat ice as a non-Newtonian, viscous fluid. Large spatial domains combined with long time spans and complexities such as a non-linear rheology, make ice sheet simulations computationally challenging. The topic of this thesis is the efficiency and error control of large simulations, both in the sense of mathematical modelling and numerical algorithms. In the first part of the thesis, approximative models based on perturbation expansions are studied. Due to a thick boundary layer near the ice surface, some classical assumptions are inaccurate and the higher order model called the Second Order Shallow Ice Approximation (SOSIA) yields large errors. In the second part of the thesis, the Ice Sheet Coupled Approximation Level (ISCAL) method is developed and implemented into the finite element ice sheet model Elmer/Ice. The ISCAL method combines the Shallow Ice Approximation (SIA) and Shelfy Stream Approximation (SSA) with the full Stokes model, such that the Stokes equations are only solved in areas where both the SIA and SSA is inaccurate. Where and when the SIA and SSA is applicable is decided automatically and dynamically based on estimates of the modeling error. The ISCAL method provides a significant speed-up compared to the Stokes model. The third contribution of this thesis is the introduction of Radial Basis Function (RBF) methods in glaciology. Advantages of RBF methods in comparison to finite element methods or finite difference methods are demonstrated.

Keywords: ice sheet modelling, stokes equations, shallow ice approximation, finite element method, perturbation expansions, non-newtonian fluids, free surface flow

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Dedicated to my brave, curious, ambitious, and intelligent parents.

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I   J. Ahlkrona, N. Kirchner, and P. Lötstedt. A Numerical Study of Scaling Relations for Non-Newtonian Thin-film Flows with Applications in Ice Sheet Modelling. In *Quarterly Journal Of Mechanics And Applied Mathematics*, Vol. 66: pp. 417–435, 2013.

II   J. Ahlkrona, N. Kirchner and P. Lötstedt. Accuracy of the zeroth and second order shallow ice approximation - numerical and theoretical results. In *Geoscientific Model Development*, Vol. 6: pp. 2135–2152, 2013.

III   J. Ahlkrona, P. Lötstedt, N. Kirchner, and T. Zwinger. Dynamically coupling the non-linear Stokes equations with the Shallow Ice Approximation in glaciology: Description and first applications of the ISCAL method. In *Journal of Computational Physics*, Vol. 308: pp. 1–19, 2016.

IV   J. Ahlkrona. The ISCAL method and the Grounding Line – Combining the Stokes equations with the Shallow Ice Approximation and Shelfy Stream Approximation. Technical Report 2016-006, Department of Information Technology, Uppsala University, 2016.

V   J. Ahlkrona and V. Shcherbakov. A Meshfree Approach to Non-Newtonian Free Surface Ice Flow: Application to the Haut Glacier d'Arolla. Technical report 2016-005, Department of Information Technology, Uppsala University, 2016 (submitted).

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Related Work

Although not explicitly discussed in the comprehensive summary, the following paper is related to the contents of this thesis

- N. Kirchner, J. Ahlkrona, P. Lötstedt, E. Gowan, J. Lea, R. Noormets, L. von Sydow, J. Dowdeswell. Shallow Ice Approximation, Second Order Shallow Ice Approximation, and Full Stokes models: a discussion of their roles in palaeo-ice sheet modelling and development. In *Quaternary Science Reviews*, Vol. 135: pp. 103–114, 2016.

Contents

1. Introduction

Ice sheets are enormous ice masses covering vast land areas (by definition at least 50 000 km^2). We are currently living in a warm period - called the Holocene - and we only have two ice sheets on earth: the Greenland Ice Sheet and the Antarctic Ice Sheet. During the most recent ice age, about 110 000 to 12 000 years ago, there were several other ice sheets. Examples are the Laurentide ice sheet in North America and the Weichselian ice sheet covering Scandinavia and northern Europe.

The ice sheets of the past have formed many of the landscapes of today, for instance the landscape upon which Uppsala University is built, see Fig. 1.1. Ice sheets also play an important role in the global climate system, and contributes to sea level rise. A recent analysis of satellite data by NASA showed that the sea level has risen about 8 cm since the beginning of the measurements in early 1990's. Another study suggests that the effects of a warming climate on the West Antarctic Ice Sheet alone have a potential to raise sea-level with up to a meter by year 2100 [18]. These are some of the reasons why there is an increasing interest to understand the nature and dynamics of ice sheets.

Like many other systems in nature, industry, or society, the state and evolution of ice sheets can be described mathematically by a set of *Partial Differential Equations* (PDEs). It is often very complicated or time-consuming - even impossible - to solve PDEs analytically with only pen and paper. Instead, the PDEs are usually solved approximately in a computer by discretizing space and time. This is what *scientific computing* and *numerical analysis* is about - techniques for solving PDEs and other mathematical problems using computational tools and discretization techniques such as the *finite difference method* or *finite element method*. Once a PDE solver has been implemented into a computer, it is possible to use available observational data and run a computer simulation to obtain new information, for example to predict the future states of our ice sheets, or to understand past ice configurations. Both the choice of mathematical model (PDE) and numerical method are crucial to the accuracy and efficiency of the computer simulation.

Before the computer era, mathematical techniques such as *perturbation expansions* were a common way of solving PDEs approximatively, and to gain a better understanding of the problem. Later, approximations based on perturbation expansions were used in numerical simulations to reduce computational complexity. Early ice sheet models were typically based on models obtained through perturbation expansions, and were discretized by the finite difference (FD) method [14, 29, 38]. These models neglected some of the stress components in the ice. Today, state-of-the-art codes are being developed all over the
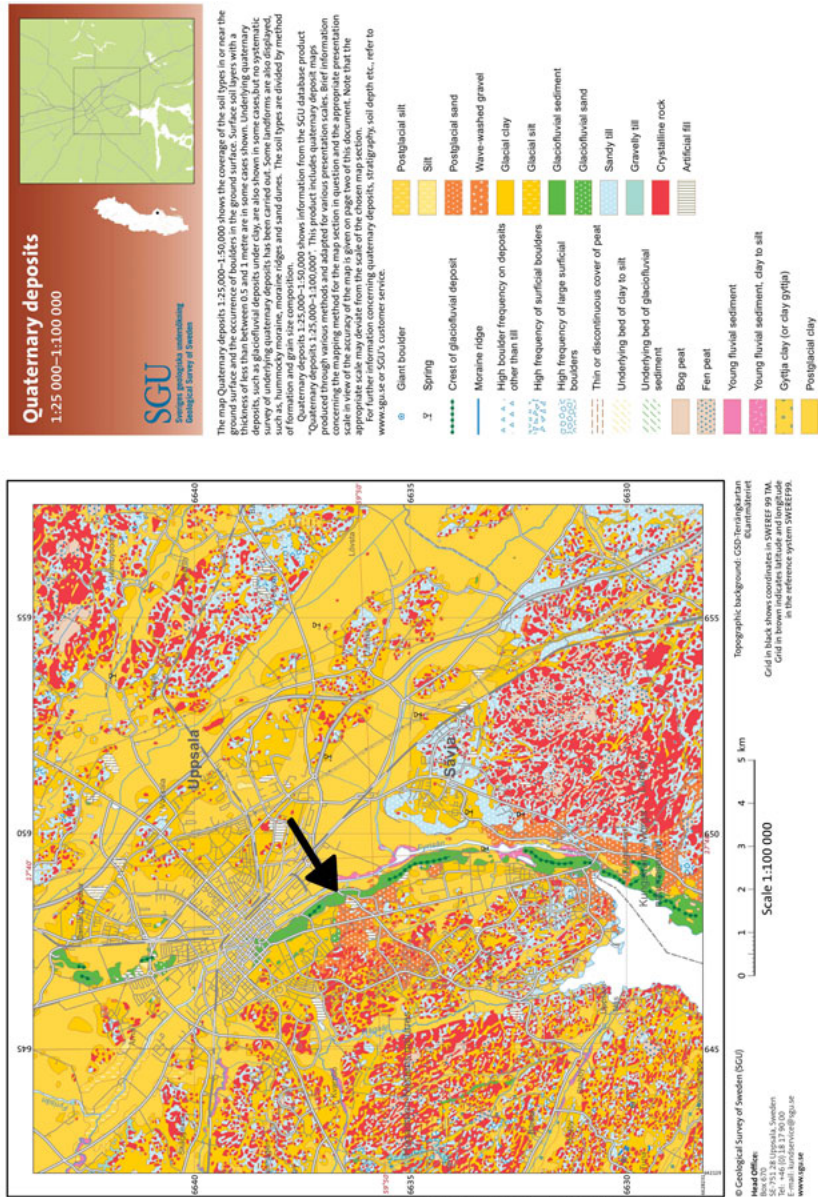
Figure 1.1. The soil types in or near the surface in Uppsala, Sweden. The location of the department where this thesis was written (indicated by the arrow) is situated on the esker Uppsalaåsen (in green), which is a memory of the Weichselian ice sheet covering Uppsala during the last ice age.

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world, making use of modern supercomputers and refined algorithms. More complex mathematical models are implemented, often discretized by the finite element method (FEM). However, ice sheet simulations are still challenging, because of e.g. the complexity of the PDEs, large computational areas (Greenland or Antarctica), and time spans that can reach 100 000 years or more. Until recently, it was not feasible to perform simulations for a whole ice sheet using a mathematical model which includes all components of the stresses in the ice (i.e. the *full Stokes* model). Even today, such simulations are limited to a couple of hundred years for Greenland, and even shorter for Antarctica [11, 26, 48, 57, 58, 67, 75].

In my opinion, the old techniques for simplifying, solving and analysing the PDEs describing ice should not be forgotten and fully replaced by brute force computer power and complicated numerical algorithms. Perturbation expansions give a valuable insight into the problem, and can be combined with modern scientific computing. Also, as there now is - at least for small setups - enough computer power to solve problems very accurately, we have the opportunity to go back to the assumptions and derivations made in traditional perturbation expansions and check their validity.

This thesis demonstrates how perturbation expansions can be evaluated using numerical algorithms, and how numerical algorithms can be improved using perturbation expansions. It is also presents new numerical techniques that can benefit ice sheet simulations. The main contributions are

- Analysis of the asymptotic behaviour of approximations to the Stokes equations using a numerical solution to the Stokes solution as a reference (Paper I and Paper II).
- The ISCAL (Ice Sheet Coupled Approximation Level) method, coupling the SIA, SSA and the Stokes equations based on an automatic error estimation (Paper III and Paper IV).
- Introduction of radial basis function methods for solving PDEs in glaciology (Paper V).

During the work with this thesis, some insights on how the finite element method performs in glaciological applications were also gained. The methods in the papers were developed mainly with long term simulations in mind, and some of the results have been used to simulate an ice sheet covering Svalbard during the last glaciation [46]. However, some of the techniques can also be valuable for shorter term simulations.

The structure of this thesis is as follows: Chapter 2 introduces the full Stokes model and gives a general overview on the challenges associated to solving these equations numerically. Chapter 3 gives an introduction to approximate models which are easier to solve. The validity of these models is discussed in the context of perturbation theory. In Chapter 4 the specific numerical methods that were used throughout this thesis are discussed, i.e. the finite difference method, the finite element method, and the radial basis function method, focusing mainly on the last two.

2. The Full Stokes Model

2.1 Ice as a Fluid

Ice sheets, also called continental glaciers, rest on land but can be attached to
ice shelves that float in the ocean, see Fig. 2.1 and Fig. 2.2a. The ice-crystals
within the ice move relation to each other if forces are applied, causing a slow,
creeping flow, downwards and outwards. This flow measures a few meters per
year in the interior, but may be hundreds or even thousands of meters per year
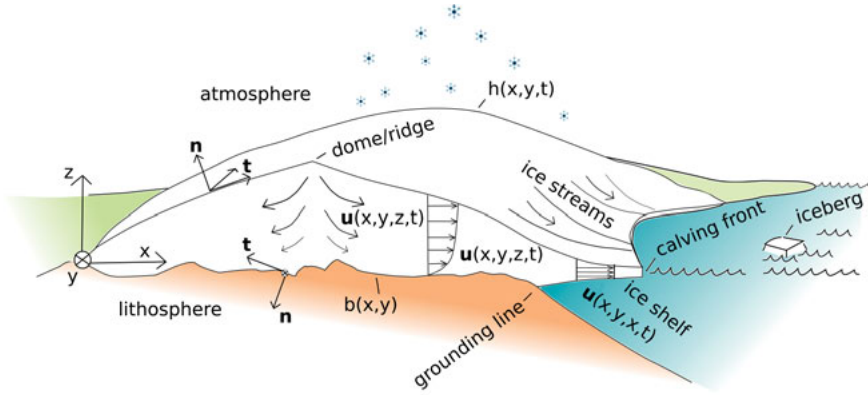in fast flowing ice streams or ice shelves, see Fig. 2.1 and Fig. 2.2.

Figure 2.1. Ice flow over the basal topography *b*. At the grounding line the ice be-
comes afloat, forming an ice shelf which breaks into icebergs at the calving front.
Inland, where the basal friction is high, the ice flows faster at the ice surface, *h*, than
at the base. In the fast flowing ice streams or the ice shelf, the friction is low and the
ice moves at approximately the same speed at the base and the surface. The normal
vector **n** points outwards from the ice body, and two tangential vectors *t* span a plane
parallel to the boundary. For aesthetic reasons, the ice sheet in the figure is about 100
times thicker than a real ice sheet.

Indeed, ice can in this context be described as a non-Newtonian, highly
viscous, incompressible, power-law fluid. It is this type of flow that is the focus
of this thesis. Laboratory experiments, field measurements and analysis by
Glen and Nye in the early 1950's determined a constitutive law characterizing
the material by describing how it deforms under stress [27, 55],

$$\mathbf{T} = \underbrace{A(T,P)^{-\frac{1}{n}} |\mathbf{D}|^{1/n-1}}_{2\eta} \mathbf{D}. \tag{2.1}$$
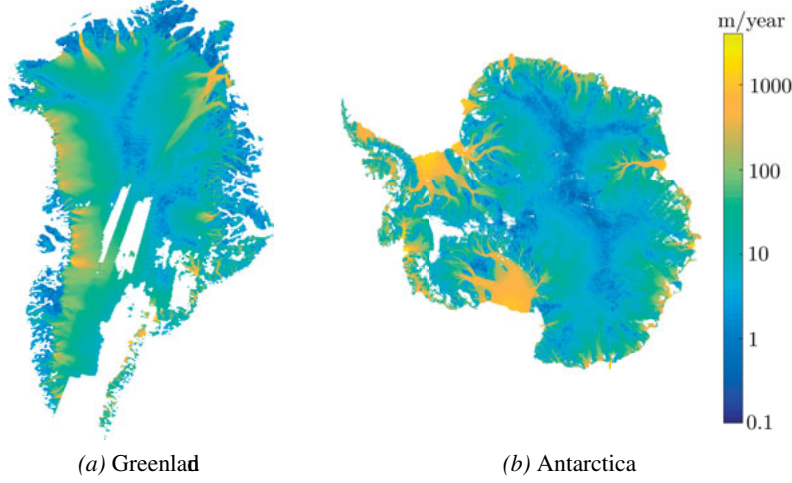
(a) Greenland   *(b)* Antarctica

Figure 2.2. Greenlandic and Antarctic surface velocity observations based on Interferometric Synthetic Aperture Radar (InSAR) [42, 61, 54]. The flow velocity is low in the interior and high in the coastal areas. The large regions with fast flowing ice in the West Antarctic Ice Sheet are the Ross and Ronne ice shelves. White areas are due to missing data. Note that the relative size of the two ice sheets are not realistic (the area of the Antarctic Ice Sheet is more than six times the area of the Greenland Ice Sheet). The datasets are freely available at `http://websrv.cs.umt.edu` and `https://nsidc.org/data`.

Here $\mathbf{T}$ is the deviatoric stress tensor, and $\mathbf{D}$ the strain rate tensor, $\mathbf{D} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$, where $\mathbf{u}$ is the velocity. The viscosity, η, is dependent on the velocity via the effective strain rate $|\mathbf{D}| = \sqrt{\frac{1}{2}tr(\mathbf{D}^2)}$, and on the temperature T and pressure P through the rate factor $A(T,P)$.

The relation (2.1) is known as Glen's flow law in glaciology, and the characterizing parameter n is called the Glen parameter. Constitutive laws on the same power law form are used under different names for describing other materials, such as metal near its melting point, warm asphalt, or polymers. In these more general contexts, the power law parameter p is typically used instead of the Glen parameter n, where $p = 1/n + 1$. For $p = 2$, the viscosity is constant and the fluid is Newtonian, such as air and water. For $p > 2$ the viscosity is increasing for increasing shear rates and the fluid is non-Newtonian and *shear-thickening*, such as quicksand. For $p < 2$ the viscosity is decreasing for increasing shear rates and the fluid is non-Newtonian and *shear-thinning*, such as ketchup.

Glen and Nye suggested that $n = 3$ for ice, so that $p = 4/3 < 2$. Note that for $n = 3$, the viscosity is singular when the effective strain rate, $|\mathbf{D}|$ is zero in (2.1). Additionally, the rate factor A is an exponential function in temperature and pressure. The viscosity of ice is thus a highly varying function which may differ in order of magnitudes. As material laws often are, Glen's flow law is

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however a simplification of real glacial ice. For instance, dust and impurities may soften the ice, n is in fact not always 3, and contrary to what Glen's flow law describes, ice is not an isotropic material (i.e. reacting similarly to stresses in each direction) [17, 30, 52, 66]. However, the vast majority of ice sheet simulations today still employ Glen's flow law with $n = 3$, and this is what will be used throughout this thesis.

2.2 Governing Equations

Combining (2.1) with the fundamental physical principles of conservation of momentum and mass we arrive at the Stokes equations, which determine the velocity $\mathbf{u} = (u_x, u_y, u_z)$ and pressure P

$$-\nabla P + \nabla \cdot \left(\eta (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \right) + \rho \mathbf{g} = \mathbf{0}, \tag{2.2a}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{2.2b}$$

The density is denoted by ρ and $\rho \mathbf{g}$ is the force of gravity. There is an additional equation for the temperature T. However, in this thesis the focus is on the solution of the Stokes equations and how this solution determines the movement of the ice surface, as this is typically the most challenging problem in ice sheet simulations. The temperature is thus assumed to be constant, such that also the rate factor A is constant.

As a remark, the term 'Stokes equations' usually refers to the linear Stokes equations. In the case of a power law fluid, (2.2) together with (2.1) are sometimes called the *p-Stokes equations*, *p* referring to the power law parameter $p = 1/n + 1$. In glaciology, (2.2) are usually called the full Stokes equations, since approximative models neglecting some stress components are so common. In this thesis the terms 'the full Stokes equations', 'the Stokes equations', or 'the non-linear Stokes equations' will be used.

Ice sheet flow is not only a non linear flow, it is also a free surface flow. The surface position, h, of the ice mass is given by the surface evolution equation

$$\frac{\partial h}{\partial t} + u_x \frac{\partial h}{\partial x} + u_y \frac{\partial h}{\partial y} = u_z + a_s. \tag{2.3}$$

Here, a_s prescribes the net accumulation of ice (snow) at the ice surface, which depends on climate data such as precipitation and surface air temperature. The velocity thus determines the ice surface, and the surface shape in turn influences the velocity. As the Stokes equations are stationary, the time evolution of the (isothermal) ice sheet is only determined by the evolution of the surface. The basal topography underneath the ice, b, may also move due to isostatic adjustment. In this thesis, the base is considered rigid, while underneath floating ice shelves, an equation similar to (2.3) is solved for the basal surface.

2.3 Boundary Conditions

At the ice surface, atmospheric pressure and wind stresses are neglected, resulting in a stress free boundary condition,

$$(-P\mathbf{I} + \mathbf{T}) \cdot \mathbf{n} = 0, \tag{2.4}$$

where $\mathbf{I}$ is the identity matrix.

The boundary conditions at the base have a fundamental impact on the overall flow. If the ice is grounded and frozen to the base, no slip condition applies,

$$\mathbf{u} = 0. \tag{2.5}$$

If on the other hand the ice is floating (i. e. in an ice shelf), friction is negligible and free slip conditions apply.

$$(\mathbf{t}_i^T \cdot (-P\mathbf{I} + \mathbf{T}) \cdot \mathbf{n}) = 0, \quad i = 1,2. \tag{2.6}$$

In this case there is also an additional condition enforcing the sea pressure on the floating ice. The slip rate in temperate grounded areas is more complicated. It depends on topographical and hydrological conditions at the base as well as if there are sediments or other soft materials present, and on the type of these materials. Naturally, it is difficult to directly observe these basal conditions underneath a thick ice sheet. Some understanding may be obtained through e.g. radar data, inverse modelling or seismology [24, 50, 63]. In Paper III and Paper V, a linear sliding law is used,

$$(\mathbf{u} \cdot \mathbf{t}_i) = -(\mathbf{t}_i^T \cdot (-P\mathbf{I} + \mathbf{T}) \cdot \mathbf{n})/\beta, \quad i = 1,2, \tag{2.7}$$
$$(\mathbf{u} \cdot \mathbf{n}) = 0, \tag{2.8}$$

where the sliding parameter β is obtained from the inverse model in [26] or as a function of the geometry. Another common sliding law is the Weertman sliding law applied in Paper IV, in which β is a function of the magnitude of the basal velocity [72]. Note that processes entailing a non-zero velocity normal to the basal surface (e.g. basal melting) is not considered in (2.8).

There is an interplay between the ice dynamics and the conditions at the base. For instance, the ice flow may alter its underlying sediments or melt water from the ice sheet may lubricate the base. One phenomena that have gained increasing attention recently is Marine Ice Sheet Instability. The point where the basal boundary condition changes from a sliding condition to free slip (i.e. the grounding line) is dependent on the bedrock topography and the previous state of the ice sheet. If the ice sheet is grounded below sea level (i.e. if it is a marine ice sheet) and the grounding line is situated on a retrograde slope, a positive feedback mechanism may thereby accelerate a retreat [31, 41, 62, 65, 73].

Together with the above described boundary conditions and appropriate initial conditions, the system of equations (2.2)-(2.3) is on closed form. The

well-posedness of the Stokes equations was proved in [43] for sliding basal conditions. Initial conditions may be based on available data, but often a prior spin-up simulation is needed to initialize all field variables in a consistent manner.

2.4 Numerical Solution Procedure

This section briefly describes the numerical solution procedure independent of the numerical method. A more detailed description of the specific numerical methods are found in Chapter 4. The Stokes equations are a simplified version of the Navier-Stokes equations, which are the standard equations in computational fluid dynamics when simulating fluids like air or water. In the Navier-Stokes equations, there are two additional terms in equation (2.2a), an acceleration term $\mathbf{u}_t$, and a non-linear convective term $\mathbf{u} \cdot \nabla \mathbf{u}$, where this convective term would be the most challenging to treat numerically. For fluids with high viscosity such as glacial ice, these terms are negligible. However, as the viscosity is varying, singular and velocity dependent, the term $\nabla \cdot \left(\eta(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \right)$ is complicated instead. The standard way of resolving this non-linearity is to solve the Stokes equations repeatedly in a fixed point or Newton iteration, updating the viscosity in each iteration. For each update, the discretized systems must be solved and - in finite elements - assembled. The finite element assembly is costly, and moreover, the singularity in the constitutive law (2.1) may lead to iterative solvers converging slowly. Furthermore, both the Navier-Stokes equations and the Stokes equations constitutes saddle point problems, which require extra care when solving numerically.

The free surface problem (2.3) is solved only on the ice surface, and is in itself not computationally demanding in comparison to the Stokes equations. The feedback between velocity and surface height h may however render numerical simulations unstable unless short time steps of weeks, months or years are employed. Indeed, although (2.3) is on the form of a convection equation, free surface height equations for very viscous flows typically suffer from a parabolic time step constraint. By such a constraint, the maximum time step allowed for stable simulations, Δt, is related to the mesh size, Δx, via $\Delta t < C \Delta x^2$. The parameter C is independent of Δt and Δx [16, 34]. The constant deformation of the ice body also requires the computational mesh updated repeatedly throughout the simulation, adding to the computational cost. A more efficient approach to this is presented in Paper V.

Regardless of discretization method, the general algorithm applied in most ice sheet models follows Algorithm 1 to solve for the evolution of velocity field, pressure and ice surface position. In all papers, the community ice sheet model Elmer/Ice is either used as a reference (Paper I, II and V) or been developed (Paper III and IV). Elmer/Ice is based on the finite element multi-physics software Elmer [25].

Algorithm 1 General Solution Procedure. For each time step k, and non-linear iteration n, a linear system is solved.

 1: Set initial condition for velocity $\mathbf{u}_0^0$, pressure P_0^0, and ice surface h^0.
 2: **for** each time step k **do**
 3:     **while** change > tol **do**
 4:         Compute viscosity $\eta_n^k = \eta(\mathbf{u}_n^k)$
 5:         Assemble Stokes model using η_n^k
 6:         Solve for velocity $\mathbf{u}_{n+1}^k$ and pressure P_{n+1}^k.
 7:         change = function of $\mathbf{u}_{n+1}^k - \mathbf{u}_n^k$
 8:         $n = n + 1$
 9:     **end while**
10:     Insert $\mathbf{u}_n^k$ into (2.3)
11:     Update the computational mesh according to the new h
12:     $k = k + 1$
13: **end for**

3. Approximations to the Stokes equations

3.1 An Overview of the SIA, SSA and Blatter-Pattyn Model

In this section, the three most common approximations to the Stokes equations are described, namely the Shallow Ice Approximation (SIA), Shallow Shelf or Shelfy Stream Approximation (SSA), and the Blatter-Pattyn model. Also a higher extension of the SIA, the Second Order SIA (SOSIA) is discussed.

The SIA is a low order model constructed for grounded ice sheet flow in areas with high friction. The SSA is also a low order model, but constructed for low friction areas such as ice streams or ice shelves. The Blatter-Pattyn model is a higher order model, which is designed to be applied in both high friction and low friction areas. To give an overview of the main characteristics of these models, the Stokes equations (2.2) are written in component form in (3.1)-(3.4) with the terms included in the SIA underlined red, the terms included in the SSA underlined blue, and the terms included in the Blatter-Pattyn model underlined in purple.

$$-\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\eta\frac{\partial u_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(\eta\frac{\partial u_x}{\partial y} + \eta\frac{\partial u_y}{\partial x}\right) + \frac{\partial}{\partial z}\left(\eta\frac{\partial u_x}{\partial z} + \eta\frac{\partial u_z}{\partial x}\right) = 0 \qquad (3.1)$$

$$-\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\eta\frac{\partial u_y}{\partial x} + \eta\frac{\partial u_x}{\partial y}\right) + \frac{\partial}{\partial y}\left(2\eta\frac{\partial u_y}{\partial y}\right) + \frac{\partial}{\partial z}\left(\eta\frac{\partial u_y}{\partial z} + \eta\frac{\partial u_z}{\partial y}\right) = 0 \qquad (3.2)$$

$$-\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\eta\frac{\partial u_z}{\partial x} + \eta\frac{\partial u_x}{\partial z}\right) + \frac{\partial}{\partial y}\left(\eta\frac{\partial u_z}{\partial y} + \eta\frac{\partial u_y}{\partial z}\right) + \frac{\partial}{\partial z}\left(2\eta\frac{\partial u_z}{\partial z}\right) = \rho g \qquad (3.3)$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \qquad (3.4)$$

The viscosity η is approximated neglecting similar terms in the effective strain rate. It should be mentioned that the above illustration is a simplification, as also other aspects may be approximated in these approximations, such as the boundary conditions and the temperature equation. Further rearrangement is needed before the SIA, SSA and Blatter-Pattyn model are reached in

their final form. The presentation in equations (3.1)-(3.4) is however sufficient for discussing accuracy and efficiency.

The terms neglected in the Blatter-Pattyn model allows for decoupling (3.3) from the other components of the balance of momentum. The Stokes equations are thereby reduced from a system of equations in four variables (u_x, u_y, u_z, p) in three dimensions (x, y, z), to a system of equations in two variables (u_x, u_y) in three dimensions. The vertical velocity u_z and p are obtained through (3.4) and (3.3) after this system of equations has been solved. Obviously, this is less computationally costly than solving the Stokes equations. In addition to reducing the size of the systems of equations, any issues related to the saddle point nature of the problem is eliminated. The Blatter-Pattyn model was derived by Blatter in the 1990's [8] and later refined by Pattyn in [56]. It was shown theoretically to be second order accurate in the aspect ratio $\epsilon = [H]/[L]$ in [64]. It was also compared to the exact Stokes equations in numerical experiments on the Greenland Ice Sheet in [48], proving it to be highly accurate.

As the SSA neglects also the vertical derivatives in (3.1) and (3.2), it reduces the problem to a pure two dimensional problem which is solved for $u_x(x, y)$ and $u_y(x, y)$. Indeed, all quantities are assumed to be constant in the vertical direction. The viscosity is integrated over the ice column, so that singularities close to the ice surface are avoided. The SSA was derived in the end of the 1980's [51].

The SIA model is the most simplistic model but historically maybe also the most widely used model, not only for simulations but also to gain a more intuitive understanding of the behaviour of ice sheets. Once all but the red terms are removed, it is possible to integrate the remaining equations vertically, and obtain four algebraic formulas for velocity and pressure. Hence no equation system is solved, and no non-linear iteration is needed. The SIA was derived in the end of the 1970s by Fowler and Larson [22], Hutter [37] and Morland [53]. In the 1990's, the SIA made it possible to simulate continental scale ice sheets during entire glacial cycles by finite difference models as for instance in SICOPOLIS (SImulation COde for POLythermal Ice Sheets) [28]. Due to its computational efficiency and high accuracy in many parts of continental ice sheets, the SIA is still widely applied in the glaciological community.

During the 1990's, also a higher order extension of the SIA was developed, the SOSIA, and an extensive theoretical analysis of the errors of the SIA was performed [5, 6]. The SOSIA was believed to significantly reduce the errors in the SIA and provide a sound link to the SSA equations, while being almost as cheap as the SIA [5, 6, 47]. It was implemented by the author into SICOPOLIS in [2]. An integrated, more elaborate version based on the SOSIA, the iSOSIA, was developed in 2011 [20].

So-called hybrid models, that couple SIA to the SSA in various ways, have grown increasingly popular in recent years. Two of the most successful continental scale paleo ice sheet models, the PISM (Parallel Ice Sheet Model) and the Pollard & Deconto model, rely on such an approach [13, 59]. In PISM,

the SSA is simply set as a basal sliding condition for the SIA, so that the SSA accounts for sliding effects, and the SIA for shearing effects.

The theory behind the SIA, SSA and the SOSIA relies on perturbation expansions. Also the Blatter-Pattyn equations and SSA equations can be derived and analysed in terms of perturbation theory. The accuracy and validity of the perturbation expansions leading to the SIA and SOSIA are the subject of Paper I and Paper II, and the following sections therefore describes perturbation theory and its application in glaciology.

3.2 Perturbation Expansions

3.2.1 Regular Perturbation Theory and Asymptotic Expansions

In perturbation theory, the solution, f, to a given problem is described as a superposition of several components with varying character. The relative significance of each component is dependent on a small parameter inherent to the problem, ϵ.

$$f(x) = \sum_{i=0}^{N} \epsilon^{(i)} f(x)_{(i)} + \epsilon^{N+1} f(x, \epsilon)_{(N+1)}. \qquad (3.5)$$

The smaller ϵ is, the less important the terms multiplied by high powers in ϵ are, such that $\lim_{\epsilon \to 0} f = f_{(0)}$ in the asymptotic limit. Taylor series are an example of a perturbation series. In Taylor series, ϵ is the distance from some point in which the expansion is exact, and the components $f_{(i)}$ can be calculated by differentiating the function f.

Perturbation theory is useful both in order to understand what the main behaviour of a system is, and in order to construct approximations. To construct approximations, each variable is expanded into a sum, inserted into the original equation, and equal powers of ϵ are collected, see the example in Section 3.2.3. This results in a series of problems that are easier to solve individually than the original problem. Higher order terms are neglected by truncating the expansions. Such approximations are exact in the asymptotic limit $\epsilon \to 0$ [36].

3.2.2 Singular expansions, boundary layers, and matched asymptotics

The above section described a type of perturbation expansion called a *regular* perturbation expansion. Another, more complex, type of expansion is the *singular* perturbation expansion. Contrary to a regular expansion, a singular expansion typically requires the problem to be transformed before variables are be expanded. This transformation changes the nature of the problem in a way that is only valid for certain regions or cases, e.g. by linearising a non-linear problem. As a consequence, for certain situations, some higher order

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terms may grow instead of decrease as $\epsilon \to 0$, such that $\lim_{\epsilon \to 0} f = f_{(0)}$ is not uniformly valid [36].

Boundary layer problems are examples where regular expansions are not valid, and singular expansions must be used instead. A boundary layer is a region close to some boundary of the domain, in which the solution, or *inner* solution ,exhibits different properties than in the bulk of the domain, where the *outer* solution is valid. The inner solution can be expanded in a singular perturbation expansion that is valid inside the boundary layer, and the outer solution can be expanded in another singular expansion valid in the bulk of the domain. In order to regain a solution valid in the entire domain, these two solutions are matched by a technique called *matched asymptotics*. The boundary layer thickness typically depends on the small expansion parameter ϵ, such that the boundary layer becomes thinner as ϵ decreases. As a consequence, the accuracy of the matched solution increases as $\epsilon \to 0$ [36].

3.2.3 Perturbation Expansions in Glaciology

Perturbation expansions have been widely applied in glaciology since the 1980's [22, 37]. The small parameter ϵ is the aspect ratio of the ice sheet, $\epsilon = [H]/[L]$, where $[H]$ is the some typical vertical length scale of an ice sheet, and $[L]$ is some typical horizontal length scale. These typical scales are often taken as the approximate thickness and horizontal extent of an entire ice sheet or glacier. The Greenland Ice Sheet is more than 2 km thick in most places, and is 1000-2000 km wide, corresponding to an aspect ratio $\epsilon \approx 0.001$. The Antarctic Ice Sheet is larger, but the aspect ratio is of the same order of magnitude. The aspect ratio of a glacier is usually 0.01-0.1. However, it is important to realize that the typical length scales are dependent on how input data such as geometry varies. In this way the frequency of e.g. the bedrock topography underneath the ice sheet increases ϵ locally, see Fig. 3.1. It also means that there is an upper limit to the resolution of the computational grids possible when using the SIA. If too rapid variations in data are resolved, the SIA will yield high errors.

3.2.4 The SIA and SOSIA Revisited

The shallowness of ice sheets influences the magnitude of different stress components and velocity components. This is exploited together with perturbation expansions in order to construct the SIA. In this section follows summary of a classical derivation of the SIA, described in e.g. [5].

Each field variable is non-dimensionalized in terms of the aspect ratio ϵ, typical thickness $[H]$, typical length $[L]$, density ρ and constant of gravity g, in order to assess their relative importance. The non-dimensionalised variables
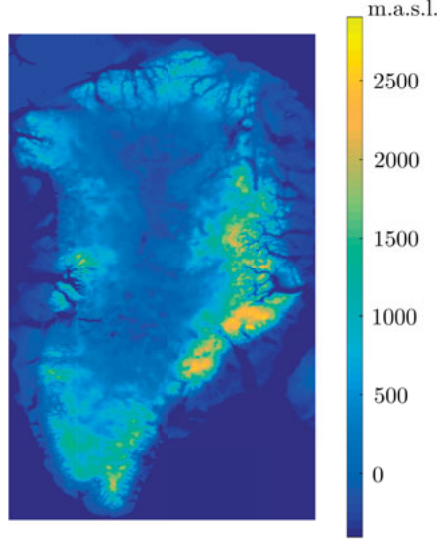
Figure 3.1. Bedrock elevation under and around the Greenland Ice Sheet, in meters above sea level [4, 39, 49]. The frequency of surface variations is especially high at the margins, which lowers the local aspect ratio ϵ there. The data-set is freely available at `http://websrv.cs.umt.edu`.

are denoted by $\sim$.

$$
\begin{aligned}
(x,y) &= [L](\tilde{x},\tilde{y}), & P &= \rho g[H]\tilde{P}, \\
z &= [H]\tilde{z}, & (t^D{}_{xz},t^D{}_{yz},\sigma) &= \epsilon\rho g[H](\tilde{t}^E_{xz},\tilde{t}^D_{yz},\tilde{\sigma}), \\
(u_x,u_y) &= [V_L](\tilde{u_x},\tilde{u_y}), & (t^D{}_{xx},t^D{}_{yy},t^D{}_{xy},t^D{}_{zz}) &= \epsilon^2\rho g[H](\tilde{t}^D_{xx},\tilde{t}^D_{yy},\tilde{t}^D_{xy},\tilde{t}^D_{zz}), \\
u_z &= [V_H]\tilde{u_z}, & t &= ([L]/[V_L])\tilde{t}, \\
\epsilon &= [H]/[L] = [V_H]/[V_L], & F &= [V_L]^2/g[L],
\end{aligned}
$$

$$(3.6)$$

The components of the deviatoric stress tensor **T** are denoted by t_{ij} ($i,j = x,y,z$). Equation (3.6), expresses that vertical shear stress, t_{xz} and t_{yz} are the dominant stresses in grounded ice sheet flow. This is because the ice is frozen to the bedrock, while the surface flow is non-zero, causing a shearing motion. Note also that the magnitude of the vertical velocity, $[V_H]$, is much smaller than the magnitude of the horizontal velocity, $[V_L]$, reflecting the shallowness of the ice sheet. The non-dimensionalized variables are now expanded in a power series like (3.5), and inserted in the original equation (2.2). If only the lowest order terms (pre-multiplied with ϵ^0) are collected, the SIA equations are obtained, which are the equations corresponding to the highlighted terms in (3.1)-(3.4). As already mentioned, these equations can be solved analytically

by rearranging and integrating in the z-direction, so that

$$u_{x(0)} = u_{b,x(0)} - 2(\rho g)^n \frac{\partial h_{(0)}}{\partial x} ||\nabla_{x,y} h_{(0)}||_2^{n-1} \int_b^z A(T')(h_{(0)} - z')^n dz',$$

(3.7a)

$$u_{y(0)} = u_{b,y(0)} - 2(\rho g)^n \frac{\partial h_{(0)}}{\partial y} ||\nabla_{x,y} h_{(0)}||_2^{n-1} \int_b^z A(T')(h_{(0)} - z')^n dz',$$

(3.7b)

$$u_{z,(0)} = u_{b,z(0)} - \int_b^z \left(\frac{\partial v_{x(0)}}{\partial x} + \frac{\partial v_{y(0)}}{\partial y} \right) dz',$$

(3.7c)

$$p_{(0)} = \rho g (h_{(0)} - z) \qquad ,$$

(3.7d)

where $\mathbf{u}_b$ denotes the velocity at the ice base. If not only the zeroth order terms $(\cdot)_0$ are kept in the expansions, but also the first order and second order terms, the SOSIA equations are obtained. The solution to the SOSIA can also be obtained analytically, if the zeroth SIA and the first order SIA are solved first.

This regular perturbation expansion is limited, not only because ϵ is not always sufficiently small, but because the expansions break down in certain regions, namely at domes, at the ice margins, and in some areas near the ice surface. Due to the non-linear rheology, a boundary layer develops near the surface, such that a regular expansion is not appropriate. The boundary layer was predicted by theory in [40] and [64] along with recommendations for singular perturbation expansions. Since exact Stokes models are available today, it is possible to numerically test the validity of the scaling relations (3.6) and other proposed scalings in literature, such as the ones in [64], and to observe the asymptotic behaviour of the SIA and SOSIA as $\epsilon \to 0$. This is the topic of Paper I and Paper II. These papers show that indeed the presence of a surface boundary layer is inconsistent with the classical scalings in (3.6) and the associated error estimates of the SIA and SOSIA. It should be mentioned that the boundary layer was recognized even in the classical derivation of both the SIA and the SOSIA [5, 6], but it was believed that the introduction of a regularization parameter in Glens flow law could circumvent the need for singular expansions. As shown in Paper II, this does however render the SOSIA so sensitive to the introduced regularization parameters that it is impractical. The scalings introduced in [64], together with a recommendation for singular expansions are in agreement with the numerical scaling relations found in Paper I. These predict a slightly lower order or accuracy of the SIA, which is confirmed in Paper II. It is however unclear whether singular expansions and matched asymptotics are recommendable since - despite the term 'boundary layer' suggesting a very thin layer - the near surface layer is found in Paper I to be thick and diffuse.

3.3 Coupling Approximations - the ISCAL method

The relative error in the SIA compared to the Stokes equations is shown in Fig. 3.2a. The error is high in areas where the regular perturbation expansions break down, especially in high sliding areas, at steep margins, and at the domes. However, the high accuracy in the interior of the ice sheet and the low
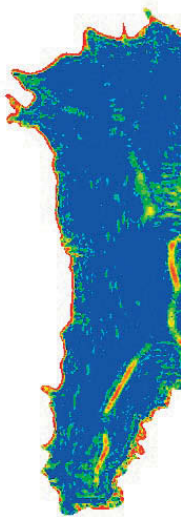
(a) SIA er...   *(b)* Stokes and SIA areas, Greenland

computational cost are clear advantages. In Paper III, the ISCAL (Ice Sheet Coupled Approximation Levels) method is introduced. The ISCAL method automatically and dynamically couples the Stokes equations to the SIA equations. It is implemented in Elmer/Ice and is based on automatic error estimates of the SIA error. This allows for the SIA to be applied in areas where it is sufficiently accurate, while the computationally expensive Stokes equations are only solved in areas where needed, such as in the high sliding areas, see Fig. 3.2b. The number of degrees of freedom in the finite element stiffness matrix is thus reduced, and the assembly and solution phase is accelerated in each non-linear iteration. The error estimation is constructed by assembling the stiffness matrix for the entire domain in the last iteration, providing a reference solution. The error may be estimated in terms of 1) the solution itself, 2) the residual of the Stokes equations, or 3) in a functional of the solution, e.g. flux over a line.

The ISCAL method provides a significant speed-up compared to solving the Stokes equations. In Paper IV, the ISCAL method is extended such that it couples a hybrid SIA+SSA model with the Stokes equations. The hybrid is constructed following the approach in PISM [13], i.e. the SSA is solved as a basal boundary condition for the SIA. Initial tests are made on a coupled ice sheet/ice shelf system with a moving grounding line.

The ISCAL method has been applied for paleo-simulations of the Svalbard Barents Sea Ice Sheet. First results are found in [46].

4. Numerical Methods

Three different discretizations techniques have been used in this thesis. The SIA and SOSIA equations in Paper I and Paper II are implemented using the finite difference method following the methods of SICOPOLIS. The Stokes solution used as a reference in Paper I and Paper II is computed using Elmer/Ice, i.e. by the finite element method. The ISCAL method in Paper III and Paper IV is implemented using finite elements in Elmer/Ice, and Paper V introduces the radial basis function method for glaciological applications. In this chapter the finite element method and radial basis function is briefly described. Information about the finite difference implementation of SIA and SOSIA can be found in [2].

4.1 The Finite Element Method

4.1.1 Variational Formulation and Discretization

The following description of the finite element method follows the procedure in Elmer/Ice. For brevity of presentation, no-slip conditions are assumed at the ice base here.

In variational form, the Stokes problem reads: find $\mathbf{u} \in V$ and $p \in Q$ such that

$$
\begin{aligned}
A(\mathbf{u}, \mathbf{v}) + B(\mathbf{v}, p) &= F(\mathbf{v}) \quad \forall \mathbf{v} \in V, & (4.1) \\
B(\mathbf{u}, q) &= 0 \quad \forall q \in Q. & (4.2)
\end{aligned}
$$

Here,

$$
\begin{aligned}
A(\mathbf{u}, \mathbf{v}) &= \int_\Omega \eta (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) : \nabla \mathbf{v} d\Omega, \\
B(\mathbf{v}, p) &= \int_\Omega p \nabla \cdot \mathbf{v} d\Omega, \\
F(\mathbf{v}) &= \int_\Omega f \cdot \mathbf{v} d\Omega,
\end{aligned}
$$

where $V := \left\{ \mathbf{v} \in [H^1(\Omega)]^3 : \mathbf{v}_{\Gamma_d} = \mathbf{0} \right\}$ and $Q := \left\{ q \in L^2(\Omega) \right\}$. The domain is denoted Ω and the basal boundary is denoted Γ_d. Next, the infinite-dimensional spaces V and Q are restricted to the finite-dimensional subspaces V_h and Q_h, consisting of piecewise linear polynomial functions. These functions are uniquely

defined by their values in the nodes of the finite element mesh described in Section 4.1.2. The discretized Galerkin mixed problem reads: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$A(\mathbf{u}_h, \mathbf{v}_h) + B(\mathbf{v}_h, p_h) \quad = \quad F(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h, \tag{4.3}$$

$$B(\mathbf{u}_h, q_h) \quad = \quad 0 \quad \forall q_h \in Q_h, \tag{4.4}$$

which can be written as a (non-)linear system

$$\begin{pmatrix} \mathbf{A} & B^T \\ B & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{4.5}$$

Note that the stiffness matrix in (4.5) depends on the viscosity η. As already mentioned in previous chapters, the assembly of the stiffness matrix may be expensive. Since the equations are stated in weak form and the viscosity enters inside the integral (4.3), it is not possible pre-assemble the main part of the equation and only reassemble the viscosity in each non-linear iteration. The solution phase may on the other hand be accelerated in non linear problems, since the solution from the previous non linear iteration can be used as a good initial guess for an iterative solver. In Paper III the assembly phase is measured to occupy about 85 % of the total simulation time, while the solution phase on only require 12 %. In Paper VI, an alternative discretization technique for glaciology applications is presented which is on strong form such that the assembly may be accelerated by pre-assembly, while still allowing for complex geometries.

The free suface equation (2.3) is discretized in space by linear elements on the surface of the domain, and in time by semi-implicit methods. Since the equation is on convection from it is standard to apply a Stream-line Upwind/ Petrov-Galerkin stabilization [12]. However, this seem to be unnecessary in many cases as the non-linear components of the free surface problem introduces strong diffusion. Once a new surface position is computed the mesh is moved in the vertical position. In reality, also the margins of the ice sheet moves when it advances or retreats, requiring remeshing of the domain, which is computationally expensive. Consequently, moving margins are usually omitted for short simulations.

4.1.2 Mesh Generation

Finite element meshes for glaciological applications are usually constructed from a two dimensional footprint mesh consisting of triangles. The footprint mesh is extruded in the z-direction such that it aligns with the bedrock topography and surface elevation. The resulting three dimensional grid consists of prismatic elements, see Fig. 4.1. This type of mesh is favourable for ice sheet modelling as it allows for moving mesh points vertically when the free surface moves, and since it facilitates approximations that are vertically integrated,
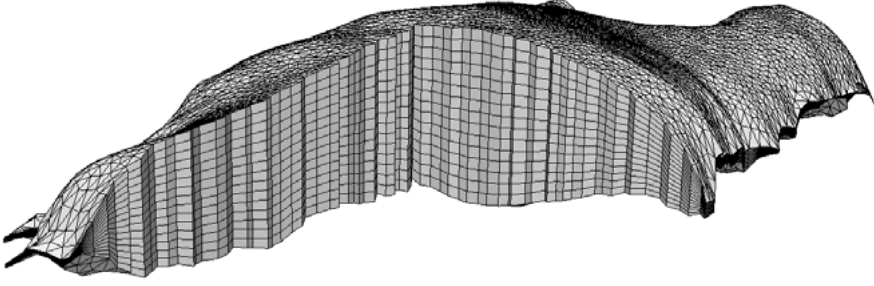
Figure 4.1. A cross section of an extruded mesh on the Greenland Ice Sheet consisting of 162 000 prismatic elements, created by extruding a triangular footprint mesh into 15 layers. The vertical component is scaled by a factor 100 in the figure. In reality the element aspect ratio is small.

such as the SIA. The ice has a positive height at the margin, thus avoiding degenerated elements. As ice sheets are thin, the elements will have a high aspect ratio. The number of vertical layers is usually 15-20 and the ice is a couple of kilometres thick, so that a typical vertical edge size is about 200 meters. The resolution in the horizontal plane can vary from 50 km down to 500 m if static mesh adaptation is used, [26, 48, 67]. This renders an element aspect ratio of about 2 - 200.

4.1.3 Stabilization Techniques

The stokes problem (4.3) is of saddle point nature. Only certain choices of $V_h \times Q_h$, that that fulfil the inf-sup condition, will lead to stable solutions [10]. The equal order linear elements described in Section 4.1.1, do not fulfil the inf-sup condition, but are commonly used in glaciology since they are easy to implement and provide a sufficiently high order of accuracy. To avoid pressure oscillations, stabilization techniques that allow for circumventing the inf-sup condition are necessary. Common stabilization techniques used in Elmer/Ice and many other codes are Galerkin Least Squares (GLS), Pressure Stabilized Petrov Galerkin (PSPG) or MINI elements [3, 23, 71]. In this thesis, PSPG or GLS stabilization is used. These techniques adds element-wise stabilization terms pre-multiplied by a stabilization parameter $\tau \sim h^2/\eta$ to the equations, as in [23]. The cell size h is a measure of the size of an element. Because of the high aspect ratio elements in ice sheets, it is often preferable to define the cell-size h as the minimum edge length for accuracy, but the choice is problem dependent. Several stabilization techniques for anisotropic elements exists [7, 9] but they do not seem to significantly increase accuracy or efficiency for ice sheet simulations. The stabilization parameter and the stabilization terms were developed with the Newtonian Navier-Stokes equations in mind and are not optimized for ice sheet modelling [23]. MINI elements

are often more robust for ice sheet simulations [25], but introduce extra degrees of freedom which significantly increases simulation time. There exist specialized stabilization techniques for the p-Stokes equations in the framework of localized projection stabilization [1], but these were not yet tested in ice sheet modelling. Localized projection stabilization techniques, also in the standard form, may be beneficial in ice sheet simulations as they are less sensitive to the stabilization parameter τ and avoid artificial boundary conditions. For glaciological applications, it is possible to construct simplified local projection stabilizations that avoid a wide discretization stencil, by integrating the computations in the non-linear iteration of Algorithm 1, and initializing the pressure projection with the SIA equations.

If the problem is over-stabilized, for instance by an unfortunate choice of h, or by not updating the viscosity η in the stabilization parameter τ in a consistent manner, it influences the vertical velocity u_z more than the horizontal velocity. This is more due to the body forces being directed in the vertical direction and the physical domain being thin, than due to the flat elements. Despite the vertical velocity being, in general, a factor ϵ smaller than the horizontal velocity, it is important to consider any errors introduced in this velocity component as it has as an important impact on the solution of the free surface equation (2.3) as the horizontal velocity does. This is because the larger horizontal velocity is pre-multiplied with the gradient of the ice surface position, which is proportional to ϵ. In fact, when surface gradients are small, the vertical velocity has a great impact on the stability of the free surface equations.

4.2 The Radial Basis Function Method

Radial basis functions (RBFs) were first used in the 1970's to interpolate scattered data points in cartography and digital terrain models [32, 33]. In glaciology radial basis functions have been used to interpolate e.g. radar data and surface elevation data [35, 70]. The interpolant $\mathcal{J}$ of data $\underline{u} = [u(\mathbf{x}_1), u(\mathbf{x}_2), \ldots, u(\mathbf{x}_N)]$ observed in N scattered nodes $\underline{x} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N]$ in a domain Ω, is given by

$$u(\mathbf{x}) \approx \mathcal{J}(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \phi(\|\mathbf{x} - \mathbf{x}_j\|), \quad \mathbf{x} \in \Omega. \tag{4.6}$$

Here α_j are unknown coefficients, ϕ is a real-valued radial basis function whose value depends only on the distance from its center, and $\|\cdot\|$ is the Euclidean norm. The coefficients α_j are determined by enforcing interpolation conditions

$$\mathcal{J}(\mathbf{x}_j) = u(\mathbf{x}_j), \quad j = 1, 2, \ldots N. \tag{4.7}$$

in the nodes $\underline{x}$. This can be expressed as a linear system

$$A\underline{\alpha} = \underline{u}, \tag{4.8}$$

(a) Gaussian radial basis functions   *(b)* Linear finite element basis functions
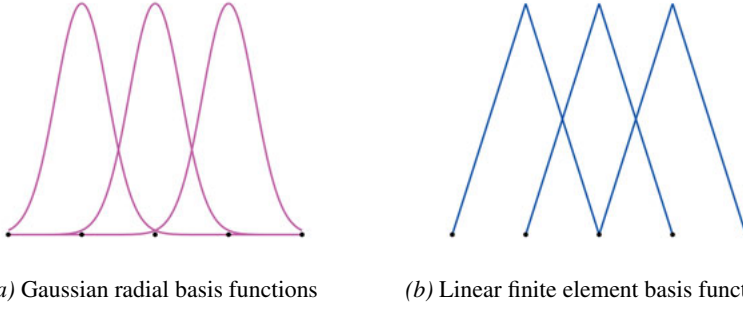
Figure 4.2. Gaussian RBFs and linear finite element basis functions. Note that, contrary to the finite element basis functions, the RBFs have support in the entire domain.

with $A_{ij} = \phi(\|\mathbf{x}_i - \mathbf{x}_j\|)$.

Common choices of radial basis functions are Gaussian functions, multiquadric, inverse multiquadric, and inverse quadratic functions, see Table 4.1.

Table 4.1. *Commonly used radial basis functions.*

| RBF	$\phi(r)$
Multiquadric (MQ)	$(1 + (\varepsilon r)^2)^{1/2}$
Inverse Multiquadric (IMQ)	$(1 + (\varepsilon r)^2)^{-1/2}$
Inverse Quadratic (IQ)	$(1 + (\varepsilon r)^2)^{-1}$
Gaussian (GS)	$e^{-(\varepsilon r)^2}$

In Table 4.1, r is the distance from a node, and ε is the *shape parameter*, determining how flat (small ε) or narrow (large ε) the RBF is. Fig. 4.2 shows the Gaussian radial basis functions in magenta and linear finite element basis functions in blue. The radial basis functions have support in the entire domain.

In the 1990's, a method for discretizing and solving PDEs by RBFs was developed [45]. Compared with the finite difference and the finite element method, it thus has a shorter history. However RBF methods of various forms have been used to solve problems in e.g. finance [21, 68], fluid dynamics [45, 74] and quantum mechanics [19]. Let us consider a PDE, where $\mathcal{L}$ is a differential operator, u is the solution and f is the right hand side,

$$\mathcal{L}u = f. \tag{4.9}$$

Collocating (4.9) based on the interpolant $\mathcal{J}$ and combining with (4.8) leads to the following system of equations, which determines and approximate solution to the PDE (4.9).

$$LA^{-1}\underline{u} = f, \tag{4.10}$$
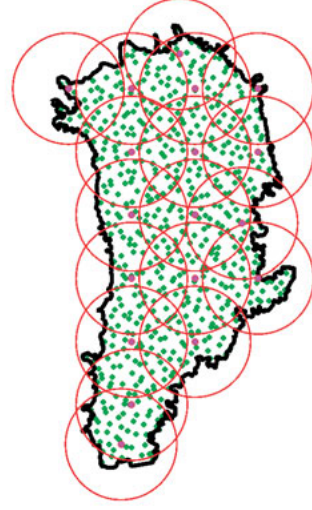
(b) RBF–PUM

Figure 4.3. The left panel show N scattered nodes $\underline{x}$ in a general domain Ω. The right panel illustrates a partition of the domain into $M = 17$ disks.

where $L_{ij} = \mathcal{L}\phi(\|\mathbf{x_i} - \mathbf{x}_j\|)$. This is the *global RBF method*

Global RBF methods exhibit exponential convergence for smooth problems [44, 60]. However, since the RBF ϕ has support in the entire domain Ω, the discretized matrix L is dense, which is of course computationally inefficient. To sparsify the matrix the RBF partition of unity method (RBF–PUM) can be used instead of the above described global RBF method [15, 68].

In the RBF–PUM setting, the domain Ω is partitioned into M overlapping patches

$$\Omega \subset \bigcup_{i=1}^{M} \Omega^i, \tag{4.11}$$

see Fig. 4.3b. In each patch a local interpolant is defined

$$\mathcal{J}_u^i(\mathbf{x}) = \sum_{j=1}^{N^i} \alpha_j^i \phi(\|\mathbf{x} - \mathbf{x}_j^i\|), \quad \mathbf{x} \in \Omega, \tag{4.12}$$

where N^i is the number of node points, which fall inside the i-th patch. The local interpolants are combined into a global interpolant

$$\mathcal{J}_u(\mathbf{x}) = \sum_{i=1}^{M} w^i(\mathbf{x}) \mathcal{J}_u^i(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{4.13}$$

where the partition of unity weights $w^i(\mathbf{x})$ can be constructed using Shepard's method [69]. The RBF–PUM is significantly faster than the global RBF method, while a high accuracy is still maintained.

In Paper V, both the global RBF method and RBF–PUM is used to simulate free surface glacier flow described by the Blatter-Pattyn model. Multiquadric RBFs are chosen,

$$\phi(r) = (1 + (\varepsilon r)^2)^{1/2}. \tag{4.14}$$

The accuracy of RBF methods is sensitive to the value of the shape parameter ε. Flat basis functions yields high accuracy but ill-conditioned systems, while narrow basis functions yields lower accuracy but better conditioned systems. In Paper VI a residual based approach is used to determine an appropriate shape parameter.

Except for a high accuracy, advantages typically associated with RBF methods are their meshfree nature. When working with RBF methods for glacier dynamics, another advantage became apparent – the strong formulation of equations allows for efficient handling of assembly in non-linear problems. In contrast to the finite element methods where the equations are stated in weak form, the viscosity is not intertwined with the divergence and symmetric gradient operator through integration, as in (4.3). Parts of the linear system assembly can therefore be moved outsite the non-linear iteration in Algorithm 1. This would of course also be the case for finite difference methods which are also stated on strong form, but then again finite difference methods do not allow for unstructured meshes in any practical way. In Paper VI part of the assembly is not only moved out from the non-linear iteration, but also outside the time integration loop in Algorithm 1. Also the advantage of a meshfree approach is demonstrated. Instead of repeated remeshing when the ice domain evolves, computational nodes are simply included or excluded from the domain as its boundaries moves. In this way the boundary conditions can be imposed directly at the boundary nodes, while the majority of the nodes are stationary so that most matrix elements L_{ij} keeps their previous value. The underlying model in Paper V is the Blatter-Pattyn model. In order to extend the approach to the Stokes model, an appropriate treatment of the saddle point problem for non-linear steady problems must be found within the framework of RBFs.

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5. Summary of Papers

5.1 Paper I

This paper evaluates scaling relations and assumptions used in perturbation expansions in glaciology. Elmer/Ice is employed to solve the Stokes equations for a two-dimensional ice sheet flowing down an inclined plane with sinusoidal bumps. The wavelength of the sinusoidal defines a typical length scale L and thereby the aspect ratio ε. By solving repeatedly for varying L we find how stresses, velocity and pressure depend on ε, i.e. we find the scaling relations. The results show that there is a layer near the ice surface in which the field variables have a different relation to ε than in the bulk of the ice, such that a regular perturbation technique is not appropriate and the uniform scaling relations often used to derive the SIA and SOSIA are inaccurate. The numerical scaling relations agree well with [64]. However, the near surface layer is found to be thick and diffuse such that singular perturbation expansions suggested in literature may be problematic.

Contribution: The author of this thesis developed the ideas in discussion with the last author, did the implementation and numerical experiments. The manuscript was written by the author of this thesis, in discussion with the other authors.

5.2 Paper II

The results in Paper I suggests that the very assumptions behind the SOSIA are inappropriate. This was recognized when SOSIA was developed, but it was believed that the introduction of a regularization parameter, σ_{res}, would be sufficient to remedy the problem . In this paper, we show that the SOSIA is inaccurate for most choices or σ_{res}, does not converge with ε as predicted, and is very sensitive to σ_{res}. We also show that the accuracy of SIA is predicted by singular expansions rather than the regular expansions, which slightly overestimated the order of accuracy. The results in this paper are shown both by comparing a numerical solution to the Stokes model with a numerical solution to the SIA and SOSIA for varying ϵ, and by solving SOSIA (and SIA) analytically. The Stokes equations are solved by Elmer/Ice. The numerical SOSIA (and SIA) solution is computed by a MATLAB version of the SICOPOLIS code implemented in [2].

Contribution: The author of this thesis developed the ideas in discussion with the last author, did the implementation, and numerical experiments. The manuscript was written by the author of this thesis, in discussion with the other authors.

5.3 Paper III

This paper presents the ISCAL method and demonstrates its efficiency and accuracy on conceptual model problems and on the Greenland Ice Sheet. The ISCAL method couples the SIA with the Stokes equations, such that the full Stokes equations are only solved where the SIA error is higher than a user defined tolerance. Three different automatic error estimations are developed to assess the SIA error. The ISCAL is capable of detecting and adjusting to rapid changes in the flow, and provides a significant speed-up compared to the Stokes equations for quasi-uniform meshes. The method is implemented in Elmer/Ice.

Contribution: The author of this thesis developed the ideas in discussion with the last author. The implementation was done by the first author with advise from the last author. Numerical experiments and the main part of writing the manuscript was done by the author of this thesis.

5.4 Paper IV

The ISCAL method is developed further such that the Stokes equations are coupled with a SSA+SIA hybrid model. In this way, the Stokes equations are not only avoided in high friction areas, but also in fast flowing regions. The method is demonstrated on the MISMIP set-up, i.e. on a grounded ice sheet connected to a floating ice shelf, with a moving grounding line. The ISCAL method adjusts such that the Stokes equations are solved around the grounding line. Efficient load balancing for parallel simulations using ISCAL is discussed.

Contribution: The author of this thesis is the sole author of this paper.

5.5 Paper V

This paper introduces a radial basis function (RBF) method for computing ice sheet flow and moving ice surface position. The method is meshfree, which is an advantage over traditional methods such as the finite element method and finite difference method when dealing with an evolving domain. Compared to

the finite element method, which most state of the art ice sheet models employ today, the assembly of a linear system (inside a non-linear solver) is accelerated. The results can be generalized to other non-Newtonian free surface flows on complex domains.

Contribution: This paper was made in close collaboration between the authors.

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7. Summary in Swedish

Inlandsisar har format landskap, interagerar med det globala klimatet, och är en av de största källorna till global havsnivåhöjning. Liksom många andra system eller fenomen, kan inlandsiar beskrivas av *partiella differential ekvationer*. Partiella differentialekvationer är ofta för komplicerade för att lösas analytiskt med penna och papper, och därför *diskretiserar* man istället ekvationerna och implementerar dem i datorer for att hitta numeriska lösningar. Innan kraftfulla datorer fanns tillhanda, användes approximativa tekniker så-som *perturbationsexpansioner*. Dessa tekniker är fortfarande populära inom glaciologi, eftersom de styrande partiella differential ekvationerna är mycket komplicerade och kräver mycket datorkraft. I denna avhandling används approximativa lösningar i kombination med med sofistikerade modeller för att accelerera datorsimuleringar. Numeriska lösningar används ocksa för att analysera noggrannheten och validiteteten hos perturbations expansioner. Slutligen introduceras ett nytt sätt att diskretisera ekvationerna som beskriver inlandsisar.

Is kan beskrivas som en icke-Newtonsk, inkompressibel fluid. Rörelsen av denna fluid bestäms av lösningen till p-Stokes ekvationer, där p indikerar olinjäriteten i materialet. Isytan deformeras enligt detta flöde av is, och dess position bestäms av ytterligare en ekvation. Anledningen till att problemet är mycket krävande att lösa är främst de stora beräkningsdomänerna (Grönland eller Antarktis), långa tidsintervallen som kan sträcka sig över 100 000 år, olinjäriteten i materialet och att ekvationen som styr deformationen av isytan är känslig for perturbationer.

Två vanliga approximationer som bygger på perturbationsexpansioner och reducerar den beräkningsmässiga komplexiteten är SIA (Shallow Ice Approximation) och SSA (Shelfy Stream Equation). Det finns även en högre ordningens utvidgning av SIA, den så kallade SOSIA (Second Order Shallow Ice Approximation). I Artikel I visar vi med hjälp av numeriska lösningar att antagandena bakom den traditionella härledningen av SIA och SOSIA inte på ett tillfredsställande sätt tar i beaktande närvaron av ett gränsskikt nära isytan. Detta gränsskikt visar sig vara relativt tjockt och diffust. Artikel II visar att gränsskiktet gör SOSIA obrukbar och att SIA konvergerar enligt en teori som inkluderar detta gränsskikt, snarare an den klassiska teorin som beskriver SIA.

I Artikel III och Artikel IV introduceras metoden ISCAL (Ice Sheet Coupled Approximation Levels). ISCAL kopplar p-Stokes ekvationerna med SIA och SSA, så att de approximativa modellerna endast används i områden där de är tillräckligt noggranna. De beräkningstunga p-Stokes ekvationer löses

därmed endast i mindre områden, vilket väsentligen accelererar simuleringstider. ISCAL är implementerad i finita element koden Elmer/Ice.

I Artikel V används en ny diskretiseringsmetod baserad på radiella basfunktioner, for att beräkna isflöde och ytdeformation. Metoden har fördelar jämfört med finita element metoden och finita differenser, i och med att den inte kräver ett beräkningsnät och således underlättar beräkningen av isdeformationen. Den öppnar också upp för effektivseringar av konstruktionen av systemmatrisen, som inte är möjliga med finita element.

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