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Large-time asymptotics of moving-reaction interfaces involving nonlinear Henry's law and time-dependent Dirichlet data

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Abstract

We study the large-time behavior of the free boundary position capturing the one-dimensional motion of the carbonation reaction front in concrete-based materials. We extend here our rigorous justification of the $\sqrt{t}$-behavior of reaction penetration depths by including non-linear effects due to deviations from the classical Henry's law and time-dependent Dirichlet data.

Keywords: Free boundary problem, concrete carbonation, Henry's law, large-time behavior, time-dependent Dirichlet data
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1. Introduction

In this paper, we deal with the following initial free-boundary value problem: Find $\{s, u, v\}$ such that

$$Q_s(T) = \{(t, x)|0 < x < s(t), 0 < t < T\},$$

$$u_t - (\kappa_1 u_x)_x = f(u, v) \quad \text{in } Q_s(T), \tag{1}$$

$$v_t - (\kappa_2 v_x)_x = -f(u, v) \quad \text{in } Q_s(T), \tag{2}$$

$$u(t, 0) = g(t), v(t, 0) = h(t) \quad \text{for } 0 < t < T, \tag{3}$$

$$u(0, x) = u_0(x), v(0, x) = v_0(x) \quad \text{for } 0 < x < s_0, \tag{4}$$

$$s'(t)(= \frac{d}{dt}s(t)) = \psi(u(t, s(t))) \quad \text{for } 0 < t < T, \tag{5}$$

$$\kappa_1 u_x(t, s(t)) = -\psi(u(t, s(t))) - s'(t)u(t, s(t)) \quad \text{for } 0 < t < T, \quad (6)$$

$$\kappa_2 v_x(t, s(t)) = -s'(t)v(t, s(t)) \quad \text{for } 0 < t < T, \quad (7)$$

$$s(0) = s_0, \quad (8)$$

where $T > 0$, κ_1 and κ_2 are positive constants, f is a given continuous function on $\mathbb{R}^2$, g and h are boundary data, u_0, v_0 and s_0 are initial data and $\psi(r) = \kappa_0 |[r]^+|^p$ where $\kappa_0 > 0$ and $p \geq 1$ are given constants. Here u and v represent the mass concentration of carbon dioxide dissolved in water and in air, respectively, while $s(t)$ denotes the position of the penetration reaction front in concrete at time $t > 0$. The interface s separates the carbonated from the non-carbonated regions.

We denote by $P(f)$ the above system $(1) \sim (8)$. $P(f)$ describes to so called *concrete carbonation process*, one of the most important physico-chemical mechanisms responsible for the durability of concrete structures; see [1, 2] for more details of the civil engineering problem.

The target here is to study the *large-time behavior of weak solutions*[1] in the presence of macroscopic nonlinear Henry's law and time-dependent Dirichlet boundary conditions. To get a bit the flavor of mathematical investigations of the effects by Henry's law for this or closely related reaction-diffusion systems, we refer the reader to [3] (linear Henry's law) and [4, 5] (micro- and micro-macro Henry-like laws). Essentially, we are able to present refined estimates that extend the proof of the rigorous large-time asymptotics beyond the settings that we have elucidated in [6, 7]. In practical terms, we show that there exist two positive constants c_* and C_*, depending on all material parameters and initial and boundary data, such that

$$c_* \sqrt{t} \leq s(t) \leq C_* \sqrt{t + 1} \quad \text{for } t \geq 0. \quad (9)$$

Based on (9), we can now explain that the deviations of carbonation fronts from the $\sqrt{t}$-law emphasized in [8] are certainly not due to eventual nonlinearities arising in the productions by Henry's law nor due to the time-changing (local) atmospheric dioxide concentrations. Therefore, there must be other reasons for this to happen. However, we prefer to not give rise here to many discussions in this direction. We just want to mention a first plausible reason: Depending on the cement chemistry, the carbonation reaction might

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[1]This is the way we translate the concept of "material durability" in mathematical terms.

not be sufficiently fast to justify a free-boundary formulation. This fact may naturally lead to a variety of different large-time asymptotics.

The reminder of the paper focuses on justifying rigorously the upper and lower bounds on the interface position $s(t)$ as indicated in (9).

2. Technical preliminaries. Statement of the main theorem

We consider $P(f)$ in the cylindrical domain $Q(T) := (0,T) \times (0,1)$ by using change of variables in order to define a solution with usual notations: Let

$$\bar{u}(t,y) = u(t, s(t)y) \text{ and } \bar{v}(t,y) = v(t, s(t)y) \text{ for } (t,y) \in Q(T). \qquad (10)$$

Then, it holds that

$$\bar{u}_t - \frac{\kappa_1}{s^2}\bar{u}_{yy} - \frac{s'}{s}y\bar{u}_y = f(\bar{u}, \bar{v}) \quad \text{in } Q(T),$$

$$\bar{v}_t - \frac{\kappa_2}{s^2}\bar{v}_{yy} - \frac{s'}{s}y\bar{v}_y = -f(\bar{u}, \bar{v}) \quad \text{in } Q(T),$$

$$\bar{u}(t,0) = g(t), \bar{v}(t,0) = h(t) \quad \text{for } 0 < t < T,$$

$$s'(t) = \psi(\bar{u}(t,1)) \quad \text{for } 0 < t < T,$$

$$-\frac{\kappa_1}{s(t)}\bar{u}_y(t,1) = s'(t)\bar{u}(t,1) + s'(t) \quad \text{for } 0 < t < T,$$

$$-\frac{\kappa_2}{s(t)}\bar{v}_y(t,1) = s'(t)\bar{v}(t,1) \quad \text{for } 0 < t < T,$$

$$s(0) = s_0, \bar{u}(0,y) = \bar{u}_0(y), \bar{v}(0,y) = \bar{v}_0(y) \quad \text{for } 0 < y < 1,$$

where $\bar{u}_0(y) = u_0(s_0 y)$ and $\bar{v}_0(y) = v_0(s_0 y)$ for $y \in [0,1]$.

For simplicity, throughout this paper we introduce the following notations related to some function spaces: We put $H := L^2(0,1)$, $X := \{z \in H^1(0,1) : z(0) = 0\}$, $|z|_X = |z_x|_H$ for $z \in X$, $V(T) = L^\infty(0,T;H) \cap L^2(0,T;H^1(0,1))$, $V_0(T) = V(T) \cap L^2(0,T;X)$ and $|z|_{V(T)} = |z|_{L^\infty(0,T;H)} + |z|_{L^2(0,T;X)}$ for $z \in V(T)$. Also, we denote by X^* and $\langle \cdot, \cdot \rangle_X$ the dual space of X and the duality pairing between X and X^*, respectively.

By using these notations we define a weak solution of $P(f)$ in the following way:

Definition 2.1. *Let s be a function on $[0,T]$ and u, v be functions on $Q_s(T)$ for $0 < T < \infty$. We call that a triplet $\{s, u, v\}$ is a weak solution of $P(f)$ on*

$[0, T]$ *if the conditions (S1) $\sim$ (S5) hold:*

(S1) $s \in W^{1,\infty}(0,T)$ with $s > 0$ on $[0,T]$, $(\bar{u}, \bar{v}) \in (W^{1,2}(0,T;X^*) \cap V(T) \cap L^\infty(Q(T)))^2$.

(S2) $\bar{u} - g, \bar{v} - h \in L^2(0,T;X)$, $s(0) = s_0$, $u(0) = u_0$ and $v(0) = v_0$ on $[0, s_0]$.

(S3) $s'(t) = \psi(u(t, s(t)))$ for a.e. $t \in [0,T]$.

(S4)
$$\int_0^T \langle \bar{u}_t(t), z(t) \rangle_X dt + \int_{Q(T)} \frac{\kappa_1}{s^2(t)} \bar{u}_y(t) z_y(t) dy dt + \int_0^T (\frac{s'(t)}{s(t)} \bar{u}(t,1) + \frac{s'(t)}{s(t)}) z(t,1) dt$$
$$= \int_{Q(T)} (f(\bar{u}(t), \bar{v}(t)) + \frac{s'(t)}{s(t)} y \bar{u}_y(t)) z(t) dy dt \quad \text{for } z \in V_0(T).$$

(S5) $\int_0^T \langle \bar{v}_t(t), z(t) \rangle_X dt + \int_{Q(T)} \frac{\kappa_2}{s^2(t)} \bar{v}_y(t) z_y(t) dy dt + \int_0^T \frac{s'(t)}{s(t)} \bar{v}(t,1) z(t,1) dt$
$$= \int_{Q(T)} (-f(\bar{u}(t), \bar{v}(t)) + \frac{s'(t)}{s(t)} y \bar{v}_y(t)) z(t) dy dt \quad \text{for } z \in V_0(T).$$

Moreover, let s be a function on $(0, \infty)$, and u and v be functions on $Q_s := \{(t,x) | t > 0, 0 < x < s(t)\}$. We say that $\{s, u, v\}$ is a weak solution of $P(f)$ on $[0, \infty)$ if for any $T > 0$ the triplet $\{s, u, v\}$ is a weak solution of $P(f)$ on $[0, T]$.

Next, we give a list of assumptions for data as follows:

(A1) $f(u, v) = \phi(\gamma v - u)$ and ϕ is a locally Lipschitz continuous and increasing function on $\mathbb{R}$ with $\phi(0) = 0$ and

$$\phi(r) r \geq C_\phi |r|^{1+q} \text{ for } r \in \mathbb{R},$$

where $q \geq 1$ and C_ϕ is a positive constant.

(A2) $g, h \in W_{loc}^{1,2}([0, \infty)) \cap L^\infty(0, \infty)$, $0 < g_0 \leq g$, $h \geq 0$ on $[0, \infty)$, and $g - g_*, h - h_* \in W^{1,1}(0, \infty)$, where g_0, g_* and h_* are positive constants with $\gamma h_* = g_*$.

(A3) $s_0 > 0$ and $u_0, v_0 \in L^\infty(0, s_0)$, $u_0, v_0 \geq 0$ a.e. on $(0, s_0)$.

Our main result is as follows:

Theorem 2.2. *If (A1), (A2) and (A3) hold, then the problem $P(f)$ has a weak solution $\{s, u, v\}$ on $[0, \infty)$. Moreover, there exist two positive constants c_* and C_* such that*

$$c_* \sqrt{t} \leq s(t) \leq C_* \sqrt{t+1} \quad \text{for } t \geq 0.$$

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In order to prove Theorem 2.2 we introduce the following notations: For $m > 0$ we put

$$\phi_m(r) = \begin{cases} \phi(m) & \text{for } r > m, \\ \phi(r) & \text{for } |r| \le m, \\ \phi(-m) & \text{for } r < -m, \end{cases}$$

and $f_m(u, v) = \phi_m(\gamma v - u)$ for $(u, v) \in \mathbb{R}^2$. Obviously, for each $m > 0$ ϕ_m and f_m are Lipschitz continuous. Then, we can denote by C_m the common Lipschitz constant of ϕ_m and f_m.

Let $s \in W^{1,2}(0, T)$ and $m > 0$. By using these notations we consider the auxiliary problem $\text{SP}_m(s, \bar{u}_0, \bar{v}_0) := \{(11) \sim (16)\}$.

$$\bar{u}_t - \frac{\kappa_1}{s^2}\bar{u}_{yy} - \frac{s'}{s}y\bar{u}_y = f_m(\bar{u}, \bar{v}) \quad \text{in } Q(T), \tag{11}$$

$$\bar{v}_t - \frac{\kappa_2}{s^2}\bar{v}_{yy} - \frac{s'}{s}y\bar{v}_y = -f_m(\bar{u}, \bar{v}) \quad \text{in } Q(T), \tag{12}$$

$$\bar{u}(t, 0) = g(t), \bar{v}(t, 0) = h(t) \quad \text{for } 0 < t < T, \tag{13}$$

$$-\frac{\kappa_1}{s(t)}\bar{u}_y(t, 1) = s'(t)\bar{u}(t, 1) + \psi(\bar{u}(t, 1)) \quad \text{for } 0 < t < T, \tag{14}$$

$$-\frac{\kappa_2}{s(t)}\bar{v}_y(t, 1) = s'(t)\bar{v}(t, 1) \quad \text{for } 0 < t < T, \tag{15}$$

$$\bar{u}(0, y) = \bar{u}_0(y), \bar{v}(0, y) = \bar{v}_0(y) \quad \text{for } 0 < y < 1, \tag{16}$$

where $\bar{u}_0$ and $\bar{v}_0$ are given functions on the interval $[0, 1]$.

Relying on the basic properties of the solutions to $\text{SP}_m(s, \bar{u}_0, \bar{v}_0)$ (as indicated in the next section), we will be able prove our main result, that is Theorem 2.2, in the last section of the paper.

3. Basic results for the auxiliary problem $\text{SP}_m(s, \bar{u}_0, \bar{v}_0)$

We begin the section by showing a first result concerned with the solvability for the problem $\text{SP}_m(s, \bar{u}_0, \bar{v}_0)$.

Proposition 3.1. *Let $m > 0$, $T > 0$, $s \in W^{2,1}(0, T)$ with $s(0) > 0$ and $s' \ge 0$ on $[0, T]$, $g, h \in W^{1,2}(0, T)$, $\bar{u}_0 - g(0) \in X$ and $\bar{v}_0 - h(0) \in X$. Then there exist one and only one pair $(\bar{u}, \bar{v}) \in (W^{1,2}(0, T; H) \cap L^\infty(0, T; H^1(0, 1)) \cap L^2(0, T; H^2(0, 1)))^2$ satisfying (11) $\sim$ (16) in the usual sense, that is, $(\bar{u}, \bar{v})$ is a unique solution of $SP_m(s, \bar{u}_0, \bar{v}_0)$ on $[0, T]$.*

We can prove this proposition in a way quite similar to the working strategy illustrated in the proofs from Section 2 in [3]. Essentially, we rely on a Banach's fixed point argument. We omit here the proof and refer the reader to [3].

As next step, we establish the positivity and the existence of upper bounds for a solution of $SP_m(s, \bar{u}_0, \bar{v}_0)$.

Lemma 3.2. *Under the same assumptions as in Proposition 3.1 let $(\bar{u}, \bar{v})$ be a solution of $SP_m(s, \bar{u}_0, \bar{v}_0)$ on $[0, T]$. If $0 \leq \bar{u}_0 \leq u_*$ and $0 \leq \bar{v}_0 \leq v_*$ on $[0, 1]$, $0 \leq g \leq u_*$ and $0 \leq h \leq v_*$ on $[0, T]$ and $u_* = \gamma v_*$, where u_* and v_* are positive constants, then*

$$0 \leq \bar{u} \leq u_*, 0 \leq \bar{v} \leq v_* \ on \ Q(T).$$

Proof. We multiply (11) by $-[-\bar{u}]^+$ to obtain

$$\frac{1}{2}\frac{d}{dt}|[-\bar{u}]^+|_H^2 + \frac{\kappa_1}{s^2}\int_0^1 |[-\bar{u}]_y^+|^2 dy - \frac{s'}{s}\bar{u}(\cdot, 1)[-\bar{u}(\cdot, 1)]^+$$
$$-\frac{1}{s}\psi(\bar{u}(\cdot, 1))[-\bar{u}(\cdot, 1)]^+$$
$$= -\int_0^1 \phi_m(\gamma\bar{v} - \bar{u})[-\bar{u}]^+ dy - \frac{s'}{s}\int_0^1 y\bar{u}_y[-\bar{u}]^+ dy \quad \text{a.e. on } [0, T].$$

Here, we note that

$$-\phi_m(\gamma\bar{v} - \bar{u})[-\bar{u}]^+ \leq -\phi_m(-\gamma[-\bar{v}]^+ - \bar{u})[-\bar{u}]^+$$
$$\leq C_m(\gamma[-\bar{v}]^+ + |\bar{u}|)[-\bar{u}]^+$$
$$\leq C_m(\gamma + 1)([-\bar{v}]^+[-\bar{u}]^+ + |[-\bar{u}]^+|^2) \quad \text{a.e. on } Q(T),$$

and
$$\psi(\bar{u}(\cdot, 1))[-\bar{u}(\cdot, 1)]^+ = 0 \quad \text{a.e. on } Q(T).$$

Then, it follows that

$$\frac{1}{2}\frac{d}{dt}|[-\bar{u}]^+|_H^2 + \frac{\kappa_1}{2s^2}|[-\bar{u}]_y^+|_H^2 \leq C_{1m}(|[-\bar{v}]^+|_H^2 + |[-\bar{u}]^+|_H^2) \text{ a.e. on } [0, T],$$

where $C_{1m} = 2C_m(\gamma + 1) + \frac{1}{2\kappa_1}|s'|_{L^\infty(0,T)}^2$.

Similarly, we can show that

$$\frac{1}{2}\frac{d}{dt}|[-\bar{v}]^+|_H^2 + \frac{\kappa_2}{2s^2}|[-\bar{v}]_y^+|_H^2 \le C_{2m}(|[-\bar{v}]^+|_H^2 + |[-\bar{u}]^+|_H^2) \text{ a.e. on } [0,T],$$

where $C_{2m} = 2C_m(\gamma+1) + \frac{1}{2\kappa_2}|s'|_{L^\infty(0,T)}^2$. From the above inequalities Gronwall's lemma implies that $[-\bar{u}]^+ = 0$ and $[-\bar{v}]^+ = 0$ a.e on $Q(T)$, that is, $\bar{u} \ge 0$ and $\bar{v} \ge 0$ a.e. on $Q(T)$.

From now on we shall show the boundedness of the solutions. First, by (A2) and (A3) we can take positive constants u_* and v_* satisfying the inequality in the assumption of this Lemma.

Next, we multiply (11) by $[\bar{u} - u_*]^+$ and have

$$\frac{1}{2}\frac{d}{dt}|[\bar{u} - u_*]^+|_H^2 + \frac{\kappa_1}{s^2}|[\bar{u} - u_*]_y^+|_H^2 + \frac{s'}{s}\bar{u}(\cdot,1)[\bar{u}(\cdot,1) - u_*]^+$$
$$+\frac{1}{s}\psi(\bar{u}(\cdot,1))[\bar{u}(\cdot,1) - u_*]^+$$
$$= \int_0^1 \phi_m(\gamma\bar{v} - \bar{u})[\bar{u} - u_*]^+ dy + \frac{s'}{s}\int_0^1 y\bar{u}_y[\bar{u} - u_*]^+ dy \quad \text{a.e. on } [0,T].$$

Similarly, we see that

$$\frac{1}{2}\frac{d}{dt}|[\bar{v} - v_*]^+|_H^2 + \frac{\kappa_2}{s^2}|[\bar{v} - v_*]_y^+|_H^2 + \frac{s'}{s}\bar{v}(\cdot,1)[\bar{v}(\cdot,1) - v_*]^+$$
$$= -\int_0^1 \phi_m(\gamma\bar{v} - \bar{v})[\bar{v} - v_*]^+ dy + \frac{s'}{s}\int_0^1 y\bar{v}_y[\bar{v} - v_*]^+ dy \quad \text{a.e. on } [0,T].$$

Here, elementary calculations lead to

$$\phi_m(\gamma\hat{v} - \bar{u})([\bar{u} - u_*]^+ - [\bar{v} - v_*]^+)$$
$$= \phi_m(\gamma(\hat{v} - v_*) - (\bar{u} - u_*))([\bar{u} - u_*]^+ - [\bar{v} - v_*]^+)$$
$$\le \phi_m(\gamma([\hat{v} - v_*]^+) - (\bar{u} - u_*))[\bar{u} - u_*]^+ - \phi_m(\gamma(\hat{v} - v_*) - [\bar{u} - u_*]^+)[\bar{v} - v_*]^+$$
$$\le C_{3m}(|[\bar{u} - u_*]^+|^2 + |[\hat{v} - v_*]^+|^2) \quad \text{a.e. on } Q(T),$$

where $C_{3m} = 2C_m\gamma + C_m(\gamma+1)$.

From the above inequalities it follows that

$$\frac{1}{2}\frac{d}{dt}(|[\bar{u} - u_*]^+|_H^2 + |[\bar{v} - v_*]^+|_H^2) + \frac{\kappa_1}{s^2}|[\bar{u} - u_*]_y^+|_H^2 + \frac{\kappa_2}{s^2}|[\bar{v} - v_*]_y^+|_H^2$$
$$\le C_{3m}(|[\bar{u} - u_*]^+|_H^2 + |[\hat{v} - v_*]^+|_H^2)$$

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$$+\frac{\kappa_1}{2s^2}|[\bar{u}-u_*]_y^+|_H^2 + \frac{1}{2\kappa_1}|s'|_{L^\infty(0,T)}^2|[\bar{u}-u_*]^+|_H^2$$

$$+\frac{\kappa_2}{2s^2}|[\bar{v}-v_*]_y^+|_H^2 + \frac{1}{2\kappa_2}|s'|_{L^\infty(0,T)}^2|[\bar{v}-v_*]^+|_H^2 \quad \text{a.e. on } [0,T]$$

so that

$$\frac{1}{2}\frac{d}{dt}(|[\bar{u}-u_*]^+|_H^2 + |[\bar{v}-v_*]^+|_H^2)$$

$$\leq \quad (C_{3m}+\frac{1}{2}|s'|_{L^\infty(0,T)}^2(\frac{1}{2\kappa_1}+\frac{1}{2\kappa_2}))(|[\bar{u}-u_*]^+|_H^2 + |[\bar{v}-v_*]^+|_H^2) \quad \text{a.e. on } [0,T].$$

Hence, by applying Gronwall's lemma we conclude that $\bar{u} \leq u_*$ and $\bar{v} \leq v_*$ a.e. on $Q(T)$. Thus we have proved this lemma. $\qquad\square$

Lemma 3.3. *Under the same assumptions as in Proposition 3.1 let $(\bar{u},\bar{v})$ be a solution of $SP_m(s,\bar{u}_0,\bar{v}_0)$ on $[0,T]$. If $u(t,x) = \bar{u}(t,\frac{x}{s(t)})$ and $v(t,x) = \bar{v}(t,\frac{x}{s(t)})$ for $(t,x) \in Q_s(T)$, then the following inequality holds:*

$$\frac{1}{2}\frac{d}{dt}\int_0^s |u-g|^2dx + \frac{\gamma}{2}\frac{d}{dt}\int_0^s |v-h|^2dx$$

$$+\kappa_1\int_0^s |u_x|^2dx + \kappa_2\gamma\int_0^s |v_x|^2dx + \psi(u(\cdot,s))(u(\cdot,s)-g)$$

$$+\frac{1}{2}s'(|u(\cdot,s)-g|^2 + \gamma|v(\cdot,s)-h|^2) + C_\phi\int_0^s |\gamma v-u|^{q+1}dx$$

$$\leq \quad -g'\int_0^s (u-g)dx - \gamma h'\int_0^s (v-h)dx \qquad (17)$$

$$-\int_0^s \phi_m(\gamma v-u)(g-g_*)dx + \gamma\int_0^s \phi_m(\gamma v-u)(h-h_*)dx$$

$$-s'g(u(\cdot,s)-g) - \gamma s'h(v(\cdot,s)-h) \quad \text{a.e. on } [0,T].$$

Proof. Since $(\bar{u},\bar{v})$ is a strong solution of $SP_m(s,\bar{u}_0,\bar{v}_0)$, it holds that

$$u_t - \kappa_1 u_{xx} = f_m(u,v) \quad \text{in } Q(T), \qquad (18)$$

$$v_t - \kappa_2 v_{xx} = -f_m(u,v) \quad \text{in } Q(T), \qquad (19)$$

$$u(0,t) = g(t), v(0,t) = h(t) \quad \text{for } 0 < t < T,$$

$$-\kappa_1 u_x(t,s(t)) = s'(t)u(t,s(t)) + \psi(u(t,s(t))) \quad \text{for } 0 < t < T,$$

$$-\kappa_2 v_x(t,s(t)) = s'(t)v(t,s(t)) \quad \text{for } 0 < t < T,$$

$$u(0,x) = u_0(x), v(0,x) = v_0(x) \quad \text{for } 0 < x < s_0.$$

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Here, we multiply (18) by $u - g$ and (19) by $\gamma(v - h)$, and by using integration by parts and the boundary conditions we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^{s(t)} |u(t) - g(t)|^2 dx + \kappa_1 \int_0^{s(t)} |u_x(t)|^2 dx$$
$$+ s'(t)|u(t, s(t)) - g(t)|^2 + \psi(u(t, s(t)))(u(t, s(t)) - g(t))$$
$$= -s'(t)g(t)(u(t, s(t)) - g(t)) - g'(t)\int_0^{s(t)} (u(t) - g(t))dx$$
$$+ \int_0^{s(t)} f_m(u(t), v(t))(u(t) - g(t))dx \quad \text{for a.e. } t \in [0, T]$$

and

$$\frac{\gamma}{2}\frac{d}{dt}\int_0^{s(t)} |v(t) - g(t)|^2 dx + \gamma\kappa_2 \int_0^{s(t)} |v_x(t)|^2 dx$$
$$+ \gamma s'(t)|v(t, s(t)) - h(t)|^2$$
$$= -\gamma s'(t)h(t)(v(t, s(t)) - h(t)) - \gamma h'(t)\int_0^{s(t)} (v(t) - h(t))dx$$
$$- \gamma \int_0^{s(t)} f_m(u(t), v(t))(v(t) - h(t))dx \quad \text{for a.e. } t \in [0, T].$$

It is easy to see that

$$f_m(u, v)\{(u - g) - \gamma(v - h)\}$$
$$= -\phi_m(\gamma v - u)(\gamma v - u) - \phi_m(\gamma v - u)\{g - g_* + \gamma(h_* - h(t)\}$$
$$\leq -C_\phi|\gamma v - u|^{q+1} - \phi_m(\gamma v - u)(g - g_*) - \gamma\phi_m(\gamma v - u)(h_* - v) \text{ a.e. on } Q_s(T).$$

Combining these inequalities leads in a straightforward manner to the conclusion of this Lemma. $\quad\square$

The aim of this section is to establish the existence and the uniqueness of a weak solution of $SP_m(s, \bar{u}_0, \bar{v}_0)$ in case $s \in W^{1,4}(0, T)$. Here, we define a weak solution of $SP_m(s, \bar{u}_0, \bar{v}_0)$

Definition 3.4. *Let $\bar{u}$, $\bar{v}$ be functions on $Q(T)$ for $0 < T < \infty$. We call that a pair $\{\bar{u}, \bar{v}\}$ is a weak solution of $SP_m(s, \bar{u}_0, \bar{v}_0)$ on $[0, T]$ if the conditions (SS1) $\sim$ (SS4) hold:*
(SS1) $(\bar{u}, \bar{v}) \in (W^{1,2}(0, T; X^*) \cap V(T) \cap L^\infty(Q(T)))^2$.

(SS2) $\bar{u} - g, \bar{v} - h \in L^2(0, T; X)$, $\bar{u}(0) = \bar{u}_0$ and $\bar{v}(0) = \bar{v}_0$.

(SS3) $\displaystyle\int_0^T \langle \bar{u}_t, z \rangle_X dt + \int_{Q(T)} \frac{\kappa_1}{s^2} \bar{u}_y z_y dy dt + \int_0^T (\frac{s'}{s} \bar{u}(\cdot, 1) + \frac{1}{s} \psi(\bar{u}(\cdot, 1))) z(\cdot, 1) dt$

$$= \int_{Q(T)} (f_m(\bar{u}, \bar{v}) + \frac{s'}{s} y \bar{u}_y) z dy dt \quad \text{for } z \in V_0(T).$$

(SS4) $\displaystyle\int_0^T \langle \bar{v}_t, z \rangle_X dt + \int_{Q(T)} \frac{\kappa_2}{s^2} \bar{v}_y z_y dy dt + \int_0^T \frac{s'}{s} \bar{v}(\cdot, 1) z(\cdot, 1) dt$

$$= \int_{Q(T)} (-f_m(\bar{u}, \bar{v}) + \frac{s'}{s} y \bar{v}_y) z dy dt \quad \text{for } z \in V_0(T).$$

Proposition 3.5. *Let* $T > 0$, $m > 0$, $s \in W^{1,4}(0, T)$ *with* $s(0) > 0$, $s' \geq 0$ *a.e. on* $[0, T]$, $g, h \in W^{1,2}(0, T)$ *with* $g, h \geq 0$ *on* $[0, T]$ *and* $\hat{u}_0, \hat{v}_0 \in L^\infty(0, 1)$ *with* $\hat{u}_0, \hat{v}_0 \geq 0$ *a.e. on* $[0, 1]$. *Then* $SP_m(s, \bar{u}_0, \bar{v}_0)$ *has a unique weak solution* $\{\bar{u}, \bar{v}\}$ *on* $[0, T]$. *Moreover, (17) holds a.e. on* $[0, T]$ *with* $\{u, v\}$, *where* $u(t, x) = \bar{u}(t, \frac{x}{s(t)})$ *and* $v(t, x) = \bar{v}(t, \frac{x}{s(t)})$ *for* $(t, x) \in Q_s(T)$.

Proof. First, we take sequences $\{s_n\} \subset W^{2,1}(0, T)$, $\{\bar{u}_{0n}\} \subset H^1(0, 1)$ and $\{\bar{v}_{0n}\} \subset H^1(0, 1)$ such that $s_n \to s$ in $W^{1,4}(0, T)$ as $n \to \infty$, $s_n(0) = s(0)$, $s'_n \geq 0$ on $[0, T]$ for n, $\bar{u}_{0n} \to \bar{u}_0$ and $\bar{v}_{0n} \to \bar{v}_0$ in H as $n \to \infty$, $0 \leq \bar{u}_{0n} \leq |\bar{u}_0|_{L^\infty(0,1)} + 1$, $0 \leq \bar{v}_{0n} \leq |\bar{v}_0|_{L^\infty(0,1)} + 1$ on $[0.1]$ and $\bar{u}_{0n} - g(0), \bar{v}_{0n} - h(0) \in X$ for n. Obviously, there exists a positive constant L such that $0 < s(0) \leq s_n \leq L$ on $[0, T]$ for n. Then, Proposition 3.1 and Lemma 3.2 imply that $SP_m(s_n, \bar{u}_{0n}, \bar{v}_{0n})$ has a solution $(\bar{u}_n, \bar{v}_n)$ on $[0, T]$ and $0 \leq \bar{u}_n \leq u_*$ and $0 \leq \bar{v}_n \leq v_*$ on $Q(T)$ for each n, where u_* and v_* are positive constants satisfying $u_* \geq \max\{|\bar{u}_0|_{L^\infty(0,1)} + 1, |g|_{L^\infty(0,T)}\}$, $v_* \geq \max\{|\bar{v}_0|_{L^\infty(0,1)} + 1, |h|_{L^\infty(0,T)}\}$ and $u_* = \gamma v_*$. Moreover, by Lemma 3.3 and putting $u_n(t, x) = \bar{u}_n(t, \frac{x}{s_n(t)})$ and $v_n(t, x) = \bar{v}_n(t, \frac{x}{s_n(t)})$ for $(t, x) \in Q_{s_n}(T)$, we see that

$$\frac{1}{2} \frac{d}{dt} \int_0^{s_n} |u_n - g|^2 dx + \frac{\gamma}{2} \frac{d}{dt} \int_0^{s_n} |v_n - h|^2 dx$$
$$+ \kappa_1 \int_0^{s_n} |u_{nx}|^2 dx + \kappa_2 \gamma \int_0^{s_n} |v_{nx}|^2 dx + \psi(u_n(\cdot, s_n))(u_n(\cdot, s_n) - g)$$

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$$+\frac{1}{2}s'_n(|u_n(\cdot,s_n))-g|^2+\gamma|v_n(\cdot,s_n)-h|^2)+C_\phi\int_0^{s_n}|\gamma v_n-u_n|^{q+1}dx$$

$$\leq\ -g'\int_0^{s_n}(u_n-g)dx-\gamma h'\int_0^{s_n}(v_n-h)dx$$

$$-\int_0^{s_n}\phi_m(\gamma v_n-u_n)(g-g_*)dx+\gamma\int_0^{s_n}\phi_m(\gamma v_n-u_n)(h-h_*)dx$$

$$-s'_ng(u_n(\cdot,s_n)-g)-\gamma s'_nh(v_n(\cdot,s_n)-h)\quad\text{a.e. on }[0,T].$$

Here, we note that

$$|\phi_m(\gamma v_n-u_n)|\leq\phi(\gamma v_*)-\phi(-u_*)=:C_4\quad\text{on }Q_{s_n}(T),$$

$$\psi(u_n(\cdot,s_n))(u_n(\cdot,s_n)-g)\geq\hat\psi(u_n(\cdot,s_n))-\hat\psi(g)\quad\text{a.e. on }[0,T],$$

where $\hat\psi(r)=\int_0^r\psi(\xi)d\xi$ for $r\in\mathbb{R}$. Then by using Young's inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^{s_n}|u_n-g|^2dx+\frac{\gamma}{2}\frac{d}{dt}\int_0^{s_n}|v_n-h|^2dx$$

$$+\kappa_1\int_0^{s_n}|u_{nx}|^2dx+\kappa_2\int_0^{s_n}|v_{nx}|^2dx+\hat\psi(u_n(\cdot,s_n))$$

$$+\frac{1}{4}s'_n(|u_n(\cdot,s_n)-g|^2+\gamma|v_n(\cdot,s_n)-h|^2)+C_\phi\int_0^{s_n}|\gamma v_n-u_n|^{q+1}dx$$

$$\leq\ \hat\psi(g)+\frac{1}{2}(|g'|^2+\gamma|h'|^2)+\frac{1}{2}(\int_0^{s_n}|u_n-g|^2dx+\gamma\int_0^{s_n}|v_n-h|^2dx)$$

$$+C_4\int_0^{s_n}(|g-g_*|+\gamma|h-h_*|)dx+s'_n(|g|^2+\gamma|h|^2)\quad\text{a.e. on }[0,T].$$

Hence, by applying Gronwall's lemma we observe that

$$\int_0^T\int_0^{s_n}(|u_{nx}|^2+|v_{nx}|^2)dxdt\leq C\text{ for }n,$$

where C is a positive constant independent of n. This implies that $\{\bar u_n\}$ and $\{\bar v_n\}$ are bounded in $L^2(0,T;H^1(0,1))$, since $|\bar u_{ny}(t)|_H^2=s_n(t)\int_0^{s_n(t)}|u_{nx}(t)|^2dx$ for $t\in[0,T]$.

Next, we provide the boundedness of $\bar u_{nt}$ and $\bar v_{nt}$. Let $\eta\in X$. Then it is easy to see that

$$|\int_0^1\bar u_{nt}\eta dy|$$

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$$= |\int_0^1 (\frac{\kappa_1}{s_n^2}\bar{u}_{nyy} + \frac{s_n'}{s_n}y\bar{u}_{ny} + f_m(\bar{u}_n, \bar{v}_n))\eta dy|$$

$$\leq \frac{\kappa_1}{s_n^2}\int_0^1 |\bar{u}_{ny}||\eta_y|dy + \frac{s_n'}{s_n}|\bar{u}_n(\cdot, 1)\eta(1)| + |\psi(\bar{u}_n(\cdot, 1))\eta(1)|$$

$$+ \frac{s_n'}{s_n}|\int_0^1 \bar{u}_n(\eta + y\eta_y)dy| + \frac{s_n'}{s_n}|\bar{u}_n(\cdot, 1)\eta(1)| + |\int_0^1 f_m(\bar{u}_n, \bar{v}_n))\eta dy|$$

$$\leq \frac{\kappa_1}{s_n^2}|\bar{u}_{ny}|_H|\eta_y|_H + \frac{s_n'}{s_n}u_*|\eta(1)| + |\psi(u_*)||\eta(1)| \qquad (20)$$

$$+ \frac{s_n'}{s_n}u_*(|\eta|_H + |\eta_y|_H) + \frac{s_n'}{s_n}u_*|\eta(1)| + C_4|\eta|_H \quad \text{a.e on } [0, T].$$

On account of the boundedness of $\{\bar{u}_n\}$ in $L^2(0, T; H^1(0, 1))$ we infer that $\{\bar{u}_{nt}\}$ is bounded in $L^2(0, T; X^*)$. Similarly, $\{\bar{v}_{nt}\}$ is also bounded in $L^2(0, T; X^*)$.

From the above uniform estimates there exist a subsequence $\{n_j\} \subset \{n\}$ and $(\bar{u}, \bar{v})$ such that $(\bar{u}, \bar{v})$ satisfies (SS1), $\bar{u}_{n_j} \to \bar{u}$ and $\bar{v}_{n_j} \to \bar{v}$ weakly* in $L^\infty(Q(T))$, weakly in $L^2(0, T; H^1(0, 1))$ and weakly in $W^{1,2}(0, T; X^*)$ as $j \to \infty$. Accordingly, Aubin's compactness theorem (see [9]) implies that $\bar{u}_{n_j} \to \bar{u}$ and $\bar{v}_{n_j} \to \bar{v}$ in $L^2(0, T; H)$ as $j \to \infty$. Moreover, $\bar{u}_{n_j}(t) \to \bar{u}(t)$ and $\bar{v}_{n_j}(t) \to \bar{v}(t)$ weakly in H as $j \to \infty$ for any $t \in [0, T]$, (SS2) is valid, and $0 \leq \bar{u} \leq u_*$ and $0 \leq \bar{v} \leq v_*$ a.e. on $Q(T)$.

Now, I shall prove (SS3). Let $z \in V_0(T)$. Then it holds that

$$\int_0^T\int_0^1 \bar{u}_{nt}zdxdt + \int_{Q(T)} \frac{\kappa_1}{s_n^2}\bar{u}_{ny}z_y dydt + \int_0^T (\frac{s_n'}{s_n}\bar{u}_n(\cdot, 1) + \frac{1}{s_n}\psi(\bar{u}_n(\cdot, 1)))z(\cdot, 1)dt$$

$$= \int_{Q(T)} (f_m(\bar{u}_n, \bar{v}_n) + \frac{s_n'}{s_n}y\bar{u}_{ny})zdydt \quad \text{for } n. \qquad (21)$$

Since $s_n \to s$ in $C([0, T])$, from the above convergences it is clear that

$$\int_0^T\int_0^1 \bar{u}_{n_j t}zdxdt \to \int_0^T \langle\bar{u}_t, z\rangle_X dt, \quad \int_{Q(T)} \frac{\kappa_1}{s_{n_j}^2}\bar{u}_{n_j y}z_y dydt \to \int_{Q(T)} \frac{\kappa_1}{s^2}\bar{u}_y z_y dydt,$$

$$\int_{Q(T)} (f_m(\bar{u}_{n_j}, \bar{v}_{n_j}) + \frac{s_{n_j}'}{s_{n_j}}y\bar{u}_{n_j y})zdydt \to \int_{Q(T)} (f_m(\bar{u}, \bar{v}) + \frac{s'}{s}y\bar{u}_y)zdydt \text{ as } j \to \infty.$$

We show convergences of the third and fourth terms in the left hand side of (21) in the following way:

$$|\int_0^T (\frac{s_{n_j}'}{s_{n_j}}\bar{u}_{n_j}(\cdot, 1) - \frac{s'}{s}\bar{u}(\cdot, 1))z(\cdot, 1)dt|$$

$$\leq \int_0^T |\frac{s'_{n_j}}{s_{n_j}} - \frac{s'}{s}||\bar{u}_{n_j}(\cdot, 1)||z(\cdot, 1)|dt + \int_0^T \frac{s'}{s}|\bar{u}_{n_j}(\cdot, 1) - \bar{u}(\cdot, 1))||z(\cdot, 1)|dt$$

$$\leq u_* \int_0^T |\frac{s'_{n_j}}{s_{n_j}} - \frac{s'}{s}||z|_X dt + \frac{\sqrt{2}}{s(0)} \int_0^T |s'||\bar{u}_{n_j} - \bar{u}|_H^{1/2}|\bar{u}_{n_j y} - \bar{u}_y|_H^{1/2}|z|_H^{1/2}|z_y|_H^{1/2} dt,$$

and

$$|\int_0^T (\frac{1}{s_{n_j}}\psi(\bar{u}_{n_j}(\cdot, 1)) - \frac{1}{s}\psi(\bar{u}(\cdot, 1)))z(\cdot, 1)dt|$$

$$\leq \int_0^T |\frac{1}{s_{n_j}} - \frac{1}{s}||\psi(\bar{u}_{n_j}(\cdot, 1))||z(\cdot, 1)|dt + \int_0^T \frac{1}{s}|\psi(\bar{u}_{n_j}(\cdot, 1)) - \psi(\bar{u}(\cdot, 1))||z(\cdot, 1)|dt$$

$$\leq \psi(u_*) \int_0^T |\frac{1}{s_{n_j}} - \frac{1}{s}||z|_X dt + \frac{C_5}{s(0)} \int_0^T |\bar{u}_{n_j}(\cdot, 1)) - (\bar{u}(\cdot, 1))||z(\cdot, 1)|dt$$

$$\leq \psi(u_*) \int_0^T |\frac{1}{s_{n_j}} - \frac{1}{s}||z|_X dt + \frac{\sqrt{2}C_5}{s(0)} \int_0^T |\bar{u}_{n_j} - \bar{u}|_H^{1/2}|\bar{u}_{n_j y} - \bar{u}_y|_H^{1/2}|z|_X dt \text{ for } j,$$

where C_5 is a positive constant satisfying $|\psi(r) - \psi(r')| \leq C_5|r - r'|$ for $0 \leq r, r' \leq u_*$. Hence, we conclude that (SS3) holds. Note that we can get (SS4) in a similar fashion.

As next step, we prove the uniqueness of a weak solution to $\mathrm{SP}_m(s, \bar{u}_0, \bar{v}_0)$ on $[0, T]$. Let $(\bar{u}_1, \bar{v}_1)$ and $(\bar{u}_2, \bar{v}_2)$ be weak solutions of $\mathrm{SP}_m(s, \bar{u}_0, \bar{v}_0)$ on $[0, T]$ and put $\bar{u} = \bar{u}_1 - \bar{u}_2$ and $\bar{v} = \bar{v}_1 - \bar{v}_2$ on $Q(T)$. Then (SS3) implies that

$$\langle \bar{u}_t, z \rangle_X + \frac{\kappa_1}{s^2} \int_0^1 \bar{u}_y z_y dy + \frac{s'}{s}\bar{u}(\cdot, 1)z(\cdot, 1) + \frac{1}{s}(\psi(\bar{u}_1(\cdot, 1)) - \psi(\bar{u}_2(\cdot, 1)))z(\cdot, 1)$$

$$= \int_0^1 (f_m(\bar{u}_1, \bar{v}_1) - f_m(\bar{u}_2, \bar{v}_2))z dy + \frac{s'}{s} \int_0^1 y\bar{u}_y z dy \text{ for } z \in X \text{ a.e. on } [0, T]. (22)$$

By taking $z = \bar{u}$ in (22) we have

$$\frac{1}{2}\frac{d}{dt}|\bar{u}|_H^2 + \frac{\kappa_1}{s^2}|\bar{u}_y|_H^2 + \frac{s'}{s}|\bar{u}(\cdot, 1)|^2 + \frac{1}{s}(\psi(\bar{u}_1(\cdot, 1)) - \psi(\bar{u}_2(\cdot, 1)))\bar{u}(\cdot, 1)$$

$$\leq C_m(|\bar{u}|_H + |\bar{v}|_H)|\bar{u}|_H + \frac{\kappa_1}{2s^2}|\bar{u}_y|_H^2 + \frac{1}{2\kappa_1}|s'|^2|\bar{u}|_H^2 \text{ a.e. on } [0, T]$$

so that

$$\frac{1}{2}\frac{d}{dt}|\bar{u}|_H^2 + \frac{\kappa_1}{2s^2}|\bar{u}_y|_H^2 \leq C_m(|\bar{u}|_H + |\bar{v}|_H)|\bar{u}|_H + \frac{1}{2\kappa_1}|s'|^2|\bar{u}|_H^2 \text{ a.e. on } [0, T].$$

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We can also obtain the inequality for $\bar{v}$. Accordingly, by adding these two inequalities and Gronwall's inequality we show the uniqueness.

Finally, in order to prove (17), we put $u(t, x) = \bar{u}(t, \frac{x}{s(t)})$ and $v(t, x) = \bar{v}(t, \frac{x}{s(t)})$ for $(t, x) \in Q_s(T)$ and $u_n(t, x) = \bar{u}_n(t, \frac{x}{s_n(t)})$ and $v_n(t, x) = \bar{v}_n(t, \frac{x}{s_n(t)})$ for $(t, x) \in Q_{s_n}(T)$ and n. Then Lemma 3.3 guarantees the following inequality:

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int_0^{s_n} |u_n - g|^2 dx + \frac{\gamma}{2}\frac{d}{dt}\int_0^{s_n} |v_n - h|^2 dx \\
& +\kappa_1 \int_0^{s_n} |u_{nx}|^2 dx + \kappa_2\gamma \int_0^{s_n} |v_{nx}|^2 dx + \psi(u_n(\cdot, s_n))(u(\cdot, s_n) - g) \\
& +\frac{1}{2}s_n'(|u_n(\cdot, s_n) - g|^2 + \gamma|v_n(\cdot, s_n) - h|^2) + C_\phi \int_0^{s_n} |\gamma v_n - u_n|^{q+1} dx \\
\leq \ & -g'\int_0^{s_n} (u_n - g)dx - \gamma h'\int_0^{s_n} (v_n - h)dx \qquad\qquad (23)\\
& -\int_0^{s_n} \phi_m(\gamma v_n - u_n)(g - g_*)dx + \gamma\int_0^{s_n} \phi_m(\gamma v_n - u_n)(h - h_*)dx \\
& -s_n'g(u_n(\cdot, s_n) - g) - \gamma s_n'h(v_n(\cdot, s_n) - h) \quad \text{a.e. on } [0, T],
\end{aligned}
$$

We integrate (23) on $[0, t_1]$ with respect to t for $0 < t_1 \leq T$. Then on account of the lower semi continuity of integral, we obtain by letting $n \to \infty$

$$
\begin{aligned}
& \frac{1}{2}\int_0^{s(t_1)} |u(t_1) - g(t_1)|^2 dx + \frac{\gamma}{2}\int_0^{s(t_1)} |v(t_1) - h(t_1)|^2 dx \\
& +\kappa_1 \int_0^{t_1}\int_0^s |u_x|^2 dxdt + \kappa_2\gamma \int_0^{t_1}\int_0^s |v_x|^2 dxdt + \int_0^{t_1} \psi(u(\cdot, s))(u(\cdot, s) - g)dt \\
& +\frac{1}{2}\int_0^{t_1} s'(|u(\cdot, s) - g|^2 + \gamma|v(\cdot, s) - h|^2)dt + C_\phi \int_0^{t_1}\int_0^s |\gamma v - u|^{q+1}dxdt \\
\leq \ & -\int_0^{t_1} g'\int_0^s (u - g)dxdt - \int_0^{t_1} \gamma h'\int_0^s (v - h)dxdt \\
& -\int_0^{t_1}\int_0^s \phi_m(\gamma v - u)(g - g_*)dxdt + \gamma\int_0^{t_1}\int_0^s \phi_m(\gamma v - u)(h - h_*)dxdt \\
& -\int_0^{t_1} (s'g(u(\cdot, s) - g) + \gamma s'h(v(\cdot, s) - h))dt \quad \text{for } 0 < t_1 \leq T.
\end{aligned}
$$

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Relying on uniqueness, $(\bar{u}, \bar{v})$ is also a weak solution of the problem $\mathrm{SP}_m(s, \bar{u}(t_0), \bar{v}(t_0))$ on $[t_0, T]$ for $0 < t_0 \le T$. Hence, it holds that

$$
\begin{aligned}
& \frac{1}{2} \int_0^{s(t_1)} |u(t_1) - g(t_1)|^2 dx + \frac{\gamma}{2} \int_0^{s(t_1)} |v(t_1) - h(t_1)|^2 dx \\
& + \kappa_1 \int_{t_0}^{t_1} \int_0^s |u_x|^2 dx dt + \kappa_2 \gamma \int_{t_0}^{t_1} \int_0^s |v_x|^2 dx dt + \int_{t_0}^{t_1} \psi(u(\cdot, s))(u(\cdot, s) - g) dt \\
& + \frac{1}{2} \int_{t_0}^{t_1} s'(|u(\cdot, s) - g|^2 + \gamma |v(\cdot, s) - h|^2) dt + C_\phi \int_{t_0}^{t_1} \int_0^s |\gamma v - u|^{q+1} dx dt \\
\le \ & - \int_{t_0}^{t_1} g' \int_0^s (u - g) dx dt - \int_{t_0}^{t_1} \gamma h' \int_0^s (v - h) dx dt \qquad (24) \\
& - \int_{t_0}^{t_1} \int_0^s \phi_m(\gamma v - u)(g - g_*) dx dt + \gamma \int_{t_0}^{t_1} \int_0^s \phi_m(\gamma v - u)(h - h_*) dx dt \\
& - \int_{t_0}^{t_1} (s' g(u(\cdot, s) - g) + \gamma s' h(v(\cdot, s) - h)) dt \quad \text{for } 0 \le t_0 < t_1 \le T.
\end{aligned}
$$

Therefore, by dividing it by $t_1 - t_0$ and letting $t_1 \downarrow t_0$ we can obtain (24). Thus we have proved this Proposition. □

4. Interfaces propagate asymptotically like $\sqrt{t}$ as $t \to \infty$

In this section, we finally prove the main result – Theorem 2.2.

4.1. Proof of the existence of a weak solution

We suppose (A1), (A2) and (A3). Then, since f_m is Lipschitz continuous on $\mathbb{R}$ for each $m > 0$, by Theorem 1.1 of [3] $\mathrm{P}(f_m)$ has a unique weak solution $\{s, u, v\}$ on $[0, T_m]$ for some $T_m > 0$.

First, we show that $\mathrm{P}(f_m)$ has a weak solution on $[0, \infty)$. In fact, let $[0, T_m^*)$ be the maximal interval of existence of a weak solution of $\mathrm{P}(f_m)$. We assume that T_m^* is finite. Obviously, Lemma 3.2 implies that

$$
0 \le u \le u_* \text{ and } 0 \le v \le v_* \text{ on } Q_s(T_m^*)
$$

so that $s'(t) = \psi(u(t, s(t)) \le \psi(u_*)$ for $0 \le t < T_m^*$, where u_* and v_* are positive constants given in the proof of Lemma 3.2. Accordingly, there exists a number $s(T_m^*) > 0$ such that $s(t) \to s(T_m^*)$ as $t \uparrow T_m^*$. Therefore, on account of (17) we infer that $\bar{u}, \bar{v} \in L^2(0, T_m^*; H^1(0, 1))$, where $\bar{u}$ and $\bar{v}$ are

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functions defined by (10). Similarly to (20), $\bar{u}, \bar{v} \in W^{1,2}(0, T_m^*; X^*)$. This shows that there exist $\bar{u}(T_m^*), \bar{v}(T_m^*) \in L^\infty(0,1)$ such that $\bar{u}(t) \to \bar{u}(T_m^*)$ and $\bar{v}(t) \to \bar{v}(T_m^*)$ weakly in H as $t \uparrow T_m^*$. Hence, by applying Theorem 1.1 of [3], again, we can extend the solution beyond T_m^*. This is a contradiction, that is, $P(f_m)$ has a weak solution on $[0, \infty)$. Moreover, it is obvious the weak solution of $P(f_m)$ is also a weak solution to $P(f)$, in case $m \geq \gamma v_* + u_*$. Thus we have proved the existence of a weak solution to $P(f)$ on $[0, \infty)$. $\qquad \square$

4.2. Proof of the upper estimate for the free boundary position

Let $\{s, u, v\}$ be a weak solution of $P(f)$ on $[0, \infty)$ and $\bar{u}$ and $\bar{v}$ are functions defined by (10). Then (S4) leads to:

$$\langle \bar{u}_t(t), z \rangle_X + \frac{\kappa_1}{s^2(t)} \int_0^1 \bar{u}_y(t) z_y dy + \left(\frac{s'(t)}{s(t)} \bar{u}(t, 1) + \frac{s'(t)}{s(t)} \right) z(1) dt$$

$$= \int_0^1 \left(f(\bar{u}(t), \bar{v}(t)) + \frac{s'(t)}{s(t)} y \bar{u}_y(t) \right) z dy \quad \text{for } z \in X \text{ and a.e. } t \in [0, \infty).$$

Accordingly, by taking $z = s^2 y$, we have

$$\langle \bar{u}_t, s^2 y \rangle_X + \kappa_1 \int_0^1 \bar{u}_y dy + ss'(\bar{u}(\cdot, 1) + 1)$$

$$= \int_0^1 (f(\bar{u}, \bar{v}) s^2 y dy + ss' \int_0^1 \bar{u}_y y^2 dy \quad \text{a.e. on } [0, \infty).$$

It is clear that (see [10, Proposition 23.23])

$$\langle \bar{u}_t, s^2 y \rangle_X = \frac{d}{dt} \int_0^1 \bar{u} s^2 y dy - \int_0^1 2 \bar{u} ss' y dy, \quad \int_0^1 \bar{u} s^2 y dy = \int_0^s xu dx \text{ a.e. on } [0, \infty).$$

It follows that

$$\frac{d}{dt} \int_0^s xu dx + \kappa_1 \int_0^1 \bar{u}_y dy + ss' = \int_0^1 f(\bar{u}, \bar{v}) s^2 y dy \text{ a.e. on } [0, \infty). \quad (25)$$

We can obtain the similar equation for $\bar{v}$ to (25). Accordingly, we see that

$$\frac{d}{dt} \int_0^s x(u + v) dx + \kappa_1 \int_0^1 \bar{u}_y dy + \kappa_2 \int_0^1 \bar{v}_y dy + ss' = 0 \text{ a.e. on } [0, \infty). \quad (26)$$

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By integrating it, it holds that

$$\int_0^{s(t)} x(u(t)+v(t))dx + \kappa_1 \int_0^t u(\tau,s(\tau))d\tau + \kappa_2 \int_0^t v(\tau,s(\tau))d\tau + \frac{1}{2}s^2(t)$$

$$= \int_0^{s_0} x(u_0+v_0)dx + \kappa_1 \int_0^t g(\tau)d\tau + \kappa_2 \int_0^t h(\tau)d\tau + \frac{1}{2}s_0^2 \quad \text{for } t \in [0,\infty).$$

Making use of the positivity of u and v, we observe that

$$\frac{1}{2}s(t)^2 \le \frac{1}{2}s_0^2 + \int_0^{s_0} x(u_0+v_0)dx + (\kappa_1 g^* + \kappa_2 h^*)t \quad \text{for } t \in [0,\infty),$$

where $g^* = |g|_{L^\infty(0,\infty)}$ and $h^* = |h|_{L^\infty(0,\infty)}$. This inequality guarantees the existence of a positive constant C_* satisfying

$$s(t) \le C_*\sqrt{t+1} \text{ for } t \ge 0. \qquad \square$$

Proof of the lower estimate for the free boundary. First, we show

$$\int_0^t \int_0^s |v_x|^2 dxd\tau \le K_1(s(t)+1) \quad \text{for } t \ge 0, \tag{27}$$

where K_1 is a positive constant. In fact, Proposition 3.5 implies

$$\frac{1}{2}\int_0^{s(t)} |u(t)-g(t)|^2 dx + \frac{\gamma}{2}\int_0^{s(t)} |v(t)-h(t)|^2 dx$$

$$+\kappa_1 \int_0^t \int_0^s |u_x|^2 dxd\tau + \kappa_2\gamma \int_0^t \int_0^s |v_x|^2 dxd\tau + \int_0^t s'u(\cdot,s)d\tau$$

$$\le \frac{1}{2}\int_0^{s_0} |u_0-g(0)|^2 dx + \frac{\gamma}{2}\int_0^{s_0} |v_0-h(0)|^2 dx$$

$$+ \int_0^t s'gd\tau - \int_0^t g' \int_0^s (u-g)dxd\tau - \gamma \int_0^t h' \int_0^s (v-h)dxd\tau$$

$$- \int_0^t \int_0^s \phi(\gamma v - u)(g-g_*)dxd\tau + \gamma \int_0^t \int_0^s \phi(\gamma v - u)(h - h_*)dxd\tau$$

$$- \int_0^t (s'g(u(\cdot,s)-g) + \gamma s'h(v(\cdot,s)-h))d\tau$$

$$\le \frac{1}{2}\int_0^{s_0} |u_0-g(0)|^2 dx + \frac{\gamma}{2}\int_0^{s_0} |v_0-h(0)|^2 dx \tag{28}$$

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$$+ g^*(s(t) - s_0) + (u_* + g^*)s(t) \int_0^t |g'| d\tau + \gamma(v_* + h^*)s(t) \int_0^t |h'| d\tau$$

$$+ s(t)C_4 \Big(\int_0^t |g - g_*| dx d\tau + \gamma \int_0^t |h - h_*| dx d\tau \Big)$$

$$+ (g^*(u_* * + g^*) + \gamma h^*(v_* + h^*))(s(t) - s_0) \quad \text{for } t \geq 0.$$

Obviously, by (A2) we can take a positive number K_1 satisfying (27).

Recalling (26), we have

$$\int_0^{s(t)} x(u(t) + v(t)) dx + \kappa_1 \int_0^t u(\tau, s(\tau)) d\tau + \kappa_2 \int_{Q_s(t)} v_x dx d\tau + \frac{1}{2} s^2(t)$$

$$= \int_0^{s_0} x(u_0 + v_0) dx + \kappa_1 \int_0^t g(\tau) d\tau + \frac{1}{2} s_0^2$$

$$\geq \kappa_1 g_0 t \quad \text{for } t \geq 0$$

so that

$$\kappa_1 g_0 t \leq \kappa_2 \Big(\int_{Q_s(t)} |v_x|^2 dx d\tau \Big)^{1/2} (s(t)t)^{1/2} + (u_* + v_*) \int_0^{s(t)} x dx + \frac{1}{2} s^2(t)$$

$$+ \frac{\kappa_1}{\kappa_0^{1/p}} \int_0^t |s'|^{1/p} d\tau$$

$$\leq \kappa_2 (K_1 s(t) + K_1)^{1/2} (s(t)t)^{1/2} + \frac{1}{2}(u_* + v_*)s(t)^2 + \frac{1}{2} s^2(t)$$

$$+ \frac{\kappa_1}{\kappa_0^{1/p}} \Big(\int_0^t |s'| d\tau \Big)^{1/p} t^{1-1/p}$$

$$\leq K_2(s(t) + 1)s(t) + \frac{\kappa_1 g_0}{2} t \quad \text{for } t \geq 0,$$

where K_2 is a positive constant. Then it holds that

$$\frac{\kappa_1 g_0}{2} t \leq K_2 s(t)^2 + \frac{\kappa_1 g_0}{4} + \frac{1}{\kappa_1 g_0} K_2^2 s(t)^2 \text{ for } t \in [0, T].$$

Hence, it is easy to see that

$$\frac{\kappa_1 g_0}{4} t \leq (K_2 + \frac{1}{\kappa_1 g_0} K_2^2) s(t)^2 \text{ for } t \geq 1.$$

In case $0 \leq t \leq 1$, we have

$$s_0 \sqrt{t} \leq s_0 \leq s(t).$$

Thus we have now completed the proof of the Theorem. $\qquad \square$

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