Experimental Investigations of Wave Motion and Electric Resistance in Collisionfree Plasmas

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Abstract
The work presented in this thesis concerns processes in collisionfree laboratory plasma when high voltage drops give particle acceleration. The three processes investigated are electron beam-plasma interaction in the presence of density gradients, the electrical resistance of an inhomogeneous plasma after the application of a step voltage, and non-linear current oscillations caused by electric double layers.

An experimental investigation of the electron beam-plasma interaction in an inhomogeneous plasma diode is presented. Even though the spread of the observed slow space charge waves follows linear growth and damping, experimental evidence and particle-in-cell simulations show that the oscillation frequencies are determined by standing waves trapped in the low density region at the diode boundary. The standing waves are identified as eigenmodes of the inhomogeneous plasma diode with the help of a fluid model.

Particle-in-cell simulations have shown that the application of a step voltage to a plasma with a density minimum may cause high potential drops which extend over the region where the electrons move towards decreasing density. The transient current pulses and the current limiting potential minimum are investigated experimentally. Their levels show good agreement with a one-dimensional model assuming fixed ions and steady electron motion under quasi-neutral conditions. The decrease of the current after its maximum is shown to be partly caused by radial ion losses due to the transient radial electric field. However, strong electric field fluctuations may also contribute to the resistance during the current decrease.

A steady double layer has formed after an ion transit time. Its negative differential resistance can give rise to nonlinear, large amplitude oscillations of potential and current. The circuit equation which has the form of a modified van der Pol equation, is shown to give excellent agreement with experiments and to explain the hysteresis and the jumps of the oscillation amplitude observed experimentally.

Keywords: Beam-plasma, Buneman instability, Density gradient, Double layer, Electric field measurements, Electron beam, Electrostatic eigenmode, Finite elements, HF probe, PIC simulation, van der Pol oscillator

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List of Papers

The thesis presented here is based on the following Papers:


The material has been presented by the first author as a poster at *The 25th International Congress on Plasma Physics*, ICPP 98, Prague, Czech Republic (1998) [1]


The material has been presented orally by the first author at *The 2nd EGS Alfvén Conference*, Stockholm, Sweden, 1999.


P1 to P3 and manuscript P4 are reprinted at the end of this thesis. The author’s contribution to the Papers is commented in Chapter 1.
Chapter 1

Introduction

“Plasma physics” is the physics of ionized gases. As the behavior of ionized gas, i.e., “plasma”, is determined by the charged particles in it, elements from both fluid mechanics and electrodynamics are needed to describe it. This mixture makes plasma physics to one of the more complicated disciplines of physics. A big fraction of the matter in the universe is in the “plasma state”, for example in the form of stars, planetary nebulae or the interstellar medium. Even though these seem to be far away, plasma related examples can be found in our everyday life. The classical examples are auroral lights, lightning and neon tubes. Many industrial applications like the hardening of metallic surfaces and the production of highly integrated circuits depend on plasma technology.

Plasmas can carry various types of waves. The simplest of these waves are so called “Langmuir waves”. Langmuir waves can propagate in regions where their oscillation frequency is above the local plasma frequency. Regions with low plasma frequency, which is equivalent to low density, can accordingly trap Langmuir waves with frequencies between the plasma frequency in the density minimum and the plasma frequency in the surrounding plasma. The trapping of Langmuir waves in “density depletions” or “cavities” in otherwise homogeneous plasmas has been shown experimentally by Tanikawa and Wong [2]. If the amplitude of the wave field is large enough, the minimum density decreases further, due to a nonlinear force, the “ponderomotive force”, acting on the plasma electrons. The resulting nonlinear development of the wave field has been extensively investigated experimentally by Wong, Cheung, MCFARLAND and coauthors [3-6]. In bounded plasmas regions of low density form at the limiting electrodes. The concentration of Langmuir waves to these pre-sheaths in an externally driven, bounded plasma has been investigated by Crawford and Harler [7]. Bounded plasmas, also called “plasma diodes”, have well defined eigenmodes with corresponding frequencies, similar to the oscillation modes of the strings on a violin. These eigenmodes can be excited with electron beams. In recent experiments, the high frequency field, in the inhomogeneous plasma in front of a hot cathode, has been interpreted as a combination of such excited eigenmodes.
Chapter 1. Introduction

[8]. A kinetic model of beam driven eigenmodes of inhomogeneous plasma diodes has been developed by Leavens and Love [9,10] and more recently by Löfgren [11]. Paper 2 of this thesis deals with a similar experiment in a double plasma device. The system investigated in Paper 2 has fewer eigenmodes with frequencies below the maximal plasma frequency than the system in reference [8]. Here, the amplitude of the trapped Langmuir waves is too low to spread the electron beam, as observed in [8], but high enough to select the frequencies of the beam-plasma waves. The knowledge of this kind of eigenmodes has industrial applications. A design of a plasma reactor, based on the excitation of eigenmodes has been proposed by Ku et al. [12,13].

Electrostatic double layers in plasma physics are steady state potential structures which can sustain large potential differences between two plasma regions over a relatively small spatial extent. The name “double layer” refers to the two space-charge layers between the quasi-neutral plasma regions which establish the strong localization of the DC electric field. Double layers have been extensively investigated analytically, by simulations, and by experiments. Their physics is described in several review articles [14–16]. Double layers play an important role in space related plasma physics and have been proposed to contribute to the acceleration of auroral electrons [17,18]. They have also been proposed to play a role in astrophysical phenomena like solar flares and X-ray pulsars [16]. If the differential conductance of a double layer is negative, the feedback of an inductive outer circuit can give rise to nonlinear, large amplitude oscillations in potential and current. These double layer oscillations, which take the form of “current disruptions” for strong nonlinearity, have been observed experimentally by Tovén and Carpenter in a triple plasma device [19,20]. For oscillation periods below the ion transit time the circuit equation has the form of a generalized van der Pol equation [21]. In Paper 1 of this thesis a theory of these oscillations is given and shown to agree with experiments. In particular, the theory can explain the hysteresis and the jumps of the oscillation amplitude (amplitude collapse) when the DC voltage over the device is changed slowly. When these double-layer oscillations are periodically driven, the bifurcation structure of the van der Pol oscillator can be found. This has been shown by Klinger et al. [22].

One of the most prominent plasma phenomena are the auroral lights in the polar regions of the Earth. These appear when electrons are accelerated along the magnetic field lines towards the Earth and excite atoms in the atmosphere at high altitudes. Satellite measurements show that above the auroral lights, regions with downward accelerated electrons lie next to regions where electrons are accelerated upward [23]. In these regions there is a close agreement between the potential drop transverse to the magnetic field and the energy of the upgoing electrons. This gives strong evidence of potential drops parallel to the magnetic field that can be described by U-shaped potential contours. Models of “quasi-steady” potential drops along the magnetic field lines have been proposed by Temerin and Carlson [24] and Römmark [25]. The authors assume that the electrons establish a steady state motion in agreement with the potential distribution, but use a fixed ion density
profile. Papers 3 and 4 in this thesis deal with investigations of similar potential drops, which appear when a step voltage is applied to an inhomogeneous plasma column in a triple plasma device. If the initial ion density profile has a minimum, the potential drop extends over the region where the electrons are accelerated towards decreasing ion density. The current during these "extended potential drops" is limited by a potential minimum, which forms near the low potential side of the device.

The author’s contribution to the Papers is the following. The hysteresis in the oscillation amplitudes presented in Paper 1 has been discovered by I. Axnäs and S. Torvén. The averaging method was also first used by them to explain its appearance. All measurements presented in Paper 1 and the quantitative comparison with the model were done by M. Wendt. In Paper 2, all measurements, their evaluation, the comparison with theory and the particle-in-cell simulations were done by M. Wendt. C. Franck assisted during the measurements which were done at the Christian-Albrechts-University in Kiel, Germany. T. Klinger and A. Piel contributed to the revision process of the manuscript. In Paper 3, which is a review article, the contribution of M. Wendt is limited to experiments presented in Section 5. In Paper 4, M. Wendt carries the responsibility for all measurements and their evaluation. The particle-in-cell simulations in Section 4.1 were made by M. Bohm. The manuscript was written in close cooperation between M. Wendt and S. Torvén.

This thesis consists of nine chapters. Chapter 2 gives a description of the experimental techniques. Chapter 3 describes the models for the quasi-neutral potential drops used in Papers 3 and 4, and a model of the steady state current voltage characteristic of the double layer in the triple plasma device explaining the appearance of a negative differential resistance. The calculation of beam-plasma dispersion relations for homogeneous plasmas is described in Chapter 4. The results are used in Paper 2 to identify the observed waves. The fluid-model giving the eigenmodes of inhomogeneous plasma diodes without a beam is described in Chapter 5. It was developed by Löfgren and Gunell [26] and is used here to interpret the particle-in-cell simulations which are presented in Chapter 6 and Paper 2. Chapter 7 presents high frequency measurements made during the transient current pulse investigated in Paper 4. Chapter 8 sums up the Papers which are reprinted in the end of this thesis. Conclusions are given in Chapter 9.
Chapter 2

Experimental Techniques

2.1 Langmuir Probes

One of the most important experimental techniques to determine the plasma parameters is the use of “electrostatic” or “Langmuir probes”. Their use and name goes back to Langmuir and Mott-Smith, who used electrostatic probes for the first time in the 1920s [27]. The simplest form of an electrostatic probe is a bare wire exposed to the plasma under investigation. The current $I(U)$ to the probe depends in a characteristic way on the probe bias voltage $U$ and the local plasma parameters, like the plasma potential $\phi_p$, the plasma density $n$ and its temperature $T_e$. Because of their simple structure they are used not only in low temperature laboratory and industrial plasmas, but also in the extremely thin ionospheric plasma ($n \approx 10^5 \text{m}^{-3}$) in order to measure electric fields and in the boundary layer of the dense and hot plasmas in controlled nuclear fusion experiments ($n \approx 10^{19} \text{m}^{-3}$) [28]. The analysis of the current-voltage characteristics of Langmuir probes is complicated by e.g. the presence of collisions, secondary emission, ionization, high frequency fields or DC magnetic fields. High frequency fields lead to an overestimation of the plasma temperature and an underestimation of the density when, time averaged characteristics are interpreted. In the presence of a DC magnetic field, such that the electron gyro radius is of the order of the probe dimension, the probe tends to empty the flux tube it occupies. Therefore a theory of Langmuir probes in strong magnetic fields must always include diffusion across the magnetic field. Theories of Langmuir probes can be found in several textbooks [29–31]. In the following, first, a simplified theory for the interpretation of Langmuir probe characteristics in collisionless, non-magnetized plasmas is described. Since the plasma and beam parameters in Paper 2 were determined with plane electrostatic probes, the expressions needed to estimate the distribution function normal to plane probes, are given then [31].
Figure 2.1. Theoretical form of Langmuir characteristics. a) ion-saturation region, b) exponential region, c) electron-saturation region, $\phi_f$ floating potential, $\phi_p$ plasma potential. The increase of the probe current in the electron-saturation region is due to sheath expansion effects and depends on the probe shape.

**Simplified Estimation of the Plasma Parameters**

Langmuir probes typically consist of small spheres, plane plates or cylindrical wires inserted into the plasma. Independent of the shape of the probe, its current voltage characteristic can be divided into three regions (Fig. 2.1):

**a) ion saturation region**

For strong negative probe bias, say $U < \phi_p - 10 k_B T_e/e$, an electron depleted sheath forms near the probe surface and the current to the probe is carried only by ions. The velocity $v_i$ with which the ions enter the electron depleted sheath must fulfill the “Bohm-criterion”, in order to guarantee a sheath with a monotonically decreasing potential

$$v_i \geq \sqrt{k_B T_e/m_i} = c_s. \quad (2.1)$$

The ions gain this velocity in a presheath, separating the quasi neutral plasma from the electron-depleted sheath. In the case of an equal sign in eq. (2.1) and vanishing
2.1. Langmuir Probes

![Diagram of Langmuir probe setup]

**Figure 2.2.** (a) Experimental setup for the measurement of time-averaged Langmuir probe characteristics (after [32]). (b) Example of the estimation of the distribution function and plasma parameters from plane Langmuir probe measurements. The model function \( f_x(v_x) \) (full line) consists of two Maxwellian background components (dashed lines) and a Maxwellian beam and is least square fitted to the experimental distribution function \( f_x^{exp}(v_x) \) (noisy full line). \( f_x^{exp}(v_x) \) is constructed by differentiation of the measured current voltage characteristic (eq. (2.9)). (I) and (II) indicate the regions for estimation of start values for the parameters of the background components.

Ion velocity in the plasma the potential difference between plasma and sheath-edge becomes

\[
\Delta \phi = \frac{1}{2} \frac{k_B T_e}{e}
\]

The ion current to the probe equals then

\[
I(U) = I_i^{\text{sat}} = -eA n_s c_s = -eA n \exp \left(-\frac{1}{2}\right) \sqrt{\frac{k_B T_e}{m_i}}.
\]

Here \( n \) and \( n_s \) are the densities of the quasi neutral plasma, undisturbed by the probe, and the plasma density at the sheath edge, respectively. \( I_i^{\text{sat}} \) is the product of the probe area \( A \) and the “Bohm current density”. In general, the ion current is not constant in the ion saturation region because of sheath expansion into the plasma.

**b) Exponential region**

At less negative probe voltages electrons with high kinetic energy reach the probe surface and cause an electron current, which adds to the ion current. In the case of a Maxwellian distribution the electron current is given by

\[
I_e(U) = eA \frac{nv_e}{2\sqrt{\pi}} \exp \left(\frac{e(U - \phi_p)}{k_B T_e}\right),
\]

\[
\Delta \phi = \frac{1}{2} \frac{k_B T_e}{e}
\]
which is the probe surface times the random flux through a unit surface times the Boltzmann-factor. Here \( v_e \equiv \sqrt{2k_B T_e/m} \) is the thermal velocity of the electrons. The probe assumes the “floating potential” \( \phi_f \) when it is disconnected from current and voltage sources, so that the total probe current is zero. At the “plasma potential” \( \phi_p \) the Boltzmann-factor equals unity and the electron current reaches its saturation value

\[
I_e(\phi_p) = I_e^{\text{sat}} = eA \frac{n v_e}{2\sqrt{\pi}}. \tag{2.5}
\]

c) electron saturation region

At probe voltages \( U \) above the plasma potential, an ion depleted sheath forms near the probe surface. All electrons passing its edge towards the probe are collected. In the “orbital theory”, in which the probe dimension is small compared to the sheath thickness, the electron current is given by

\[
I_e(U) = I_e^{\text{sat}} K(U), \tag{2.6}
\]

where \( K \) is the numerical factor

\[
K(U) = \begin{cases} 
1, & \text{for plane probes}, \\
\sqrt{\frac{1 + e (U - \phi_p) / (k_B T_e)}{1 + e (U - \phi_p) / (k_B T_e)}}, & \text{for cylindrical probes}, \\
1 + e (U - \phi_p) / (k_B T_e), & \text{for spherical probes}.
\end{cases}
\]

In a fast evaluation of plasma parameters the plasma potential \( \phi_p \) is approximated by the position of the knee in the probe current, which forms for all but spherical probes. After subtraction of the ion saturation current the plasma temperature can be extracted from the slope of the exponential region by

\[
\frac{e}{k_B T_e} = \frac{d \ln (I_e(U) - I_e^{\text{sat}})}{d U}. \tag{2.7}
\]

With known temperature \( T_e \), the density \( n \) can then be calculated from the electron saturation current \( I_e^{\text{sat}} \) (eq. (2.5))

**Evaluation of the Distribution Function normal to Plane Probes**

For plane probes with a diameter bigger than a few “Debye lengths” \( \lambda_D = \sqrt{e_0 k_B T_e / e^2 n} \) the exponential region of the probe characteristic can be treated one-dimensionally and the first derivative of the probe current in the exponential region can be used to estimate the electron distribution function \( f_x(v_x) \) in probe normal direction \([31]\). With \( f(v_x, v_y, v_z) \) being the electron distribution function and \( U < \phi_p \) the electron current \( I_e \) to the probe (prob e normal points to \( x = -\infty \)) is

\[
I_e(U) = eA \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_x f(v_x, v_y, v_z) dv_x dv_y dv_z
= eA \int_{-\infty}^{\infty} v_x f_x(v_x) dv_x \tag{2.8}
\]
Figure 2.3. Typical current-voltage characteristics of an emissive probe for different emission currents $I_{em}$ (from [31]). The floating potential of the emissive probe approximates the plasma potential as the wire temperature $T_w$ and by that its emission current increases (A to D).

and vice versa

$$f_x(v) = \frac{m}{e^2 A} \frac{d}{dU} I_e(U).$$

Here

$$v = \sqrt{\frac{2e}{m} (\phi_p - U)}$$

is the lower boundary of the integration. Measurements of time averaged Langmuir probe characteristics in the experiment in Paper 2 showed that the electron component, is characterized by a distribution consisting of the Maxwellian background components and a Maxwellian beam

$$f(v_x, v_y, v_z) = \frac{n_b}{\sqrt{\pi} v_b (\sqrt{\pi} v_{b\perp})} \exp\left(-\frac{v_x - v_{0x}}{v_b}\right) \exp\left(-\frac{v_y^2 + v_{0y}^2}{v_{b\perp}^2}\right)$$

$$+ \frac{n_c}{(\sqrt{\pi} v_c)^3} \exp\left(-\frac{v_x^2 + v_{0x}^2}{v_c^2}\right) + \frac{n_b}{\sqrt{\pi} v_b} \exp\left(-\frac{v_x^2 + v_{0y}^2 + v_{0z}^2}{v_b^2}\right)$$

Integration over $v_y$ and $v_z$ gives

$$f_x(v_x) \frac{n_c}{\sqrt{\pi} v_c} \exp\left(-\frac{v_x^2}{v_c^2}\right) + \frac{n_b}{\sqrt{\pi} v_b} \exp\left(-\frac{v_x^2}{v_b^2}\right) + \frac{n_b}{\sqrt{\pi} v_b} \exp\left(-\frac{(v_x - v_{0x})^2}{v_b^2}\right)$$ (2.12)
which leads with eq. (2.8) to

$$I_e(U) = eA \left[ \frac{n_e v_e}{2\sqrt{\pi}} \exp \left( -\frac{(U)^2}{v_e^2} \right) + \frac{n_h v_h}{2\sqrt{\pi}} \exp \left( -\frac{(U)^2}{v_h^2} \right) \right]$$

$$+ \frac{n_b v_b}{2\sqrt{\pi}} \exp \left( -\frac{(U - v_0)^2}{v_b^2} \right) \frac{e^2}{v_0^2} \frac{1}{e} \left( \frac{v_0 (U - v_0)}{v_b} \right) \right]. \quad (2.13)$$

The evaluation of the plasma parameters is done in the following way: First, the ion current is determined by fitting a linear function to the characteristic in the interval $U \in [-50, -40]$ V. After subtracting the ion current, the experimental distribution function $f_{exp}^e(v)$ is calculated by numerical differentiation. The voltage with maximum $dI/dU$ serves as a first approximation of the plasma potential. For a Maxwellian distribution $f_M(v) = n \exp(-v^2/v_i^2)/(\sqrt{\pi} v_i)$ it holds at any $v$

$$v_1 = \sqrt{-\frac{2v f_M(v)}{d f_M(v)/dv}}, \quad \text{and} \quad n = \sqrt{\pi} v_1 \exp \left( \frac{v^2}{v_1^2} \right) f_M(v). \quad (2.14)$$

Using these relations, approximate values for $n_b$, $v_b$, $n_c$ and $v_c$ are obtained from $f_{exp}^e(v)$ by applying them to different velocity intervals (Fig. 2.2(b)). The contribution of the two background components is then subtracted from $f_{exp}^e(v)$. The beam parameters $n_b$, $v_b$ and $v_0$ are estimated from the first three moments of the beam part of the distribution. The estimates of plasma and beam parameters are finally least square fitted in the region $U < \phi_b$. An example of the result of this procedure is shown in Figure (2.2).

**Emissive Probes**

“Emissive probes” are commonly used to measure the local plasma potential $\phi_p$. They resemble cylindrical Langmuir probes where the probe wire can be ohmically heated so that it thermally emits electrons. Typical characteristics of an emissive probe for different wire temperatures $T_w$ are shown in Fig. 2.3. The current-voltage characteristic of emissive probes can be thought of as the sum of the unheated probe characteristic and the thermal emission current. The temperature of the emitted electron population is of the order of the the wire temperature $T_w$. When the probe bias $U$ is a few $k_B T_w$ above the plasma potential, only very few emitted electrons have enough energy to leave the ion depleted sheath which forms around the probe. Consequently the emitted electrons cause no net current and the current-voltage characteristic in this voltage region is that of the unheated probe. When the probe bias is below the plasma potential, the emitted electrons flow freely into the surrounding plasma, so that the emitted electrons contribute to the probe current. The additional emission current in the exponential region of the cold probe causes the floating potential of the emissive probe to increase and approach the plasma potential. When the thermal emission current $I_{em}$ exceeds the electron saturation current $I_{sat}$ the floating potential may even be above the plasma potential. For
emission currents of this order, space charge effects of the emitted electrons become important [31]. A good compromise between accuracy and perturbation of the plasma by the emitted electrons, is obtain by a heating such that the characteristic is roughly symmetric [33].

The probe circuit used to measure the plasma potential oscillations in Paper 1 is shown in Figure 2.4. The probe is periodically heated through the transformer. During potential measurements the heating is interrupted, to avoid a voltage drop across the probe wire.

2.2 High Frequency Probes

The high frequency probes, used in Paper 2 to measure the high frequency electric field, have been developed by S. Torvén, H. Gunell and N. Brenning [34]. Low frequency or DC fields can be determined by measuring the floating potentials of two probes and dividing their difference by the spatial probe separation. To be able to measure the fluctuating plasma or floating potential, the probe tip has to maintain a high impedance to ground. For frequencies in the 100MHz to 1GHz range stray capacitances between the probe tip and ground make such a potential
Chapter 2. Experimental Techniques

Figure 2.5. High frequency probes: (a) Details of the probe tip, A hf sensitive part, B spiral, C ceramic tube ('×'), D coaxial cable, E torr seal [hatched] and shield, (b) Experimental setup for the measurement of time series. Each probe wire is connected to the hybrid-T with an individual rigid coaxial cable. The hf current flowing between them is obtained in the hybrid-T which short circuits them over its 50Ω input impedance. The signal is then amplified and measured. All connections are done with double shielded coaxial cables.

measurement difficult. The double probe technique, proposed in [34] measures instead the ac current flowing between two short circuited wires immersed in the high frequency electric field.

The high frequency probe used in Paper 2 consists of two parallel, cylindrical tungsten wires with radius \( r = 0.07 \)mm and a distance \( b = 0.5 \)mm. When the short circuited wires are immersed into a field of strength \( E_0 \), first, charges \( q \) and \(-q\) build up on the wires in order to create an electric field compensating \( E_0 \), so that the probe tips remain at the same potential. Secondly, the charge on each wire redistributes in angular direction, so that the surface of each wire is an equipotential surface. Treating the problem of two infinitely long parallel, cylindrical wires in vacuum electrostatically, a relation between the wire charge \( q \) and the electric field \( E_0 \) was derived in [34]. Writing the measured current \( I \) as \( I = i\omega q \), it follows

\[
|E_0| = \frac{I}{\omega \ell C_0 \sqrt{b^2 - 4r^2}},
\]

where \( \ell \) is the length of the probe wires and \( C_0 = \pi e_0 / \cosh^{-1}(b/2r) \) the vacuum capacitance per meter between them.

Details of the hf sensitive part of the probe, which has been manufactured at the Alfvén Laboratory are shown in Figure 2.5(a). Two miniature coaxial cables are led through a small ceramic tube, protecting them from the plasma. To ensure a common potential of the outer conductors at the probe wires they are soldered together close to the probe wires. The welding points, connecting the tungsten probe wires to the inner conductors, are shielded to avoid a pickup of hf fields, and then fixed within the ceramic tube with torr seal. To reduce the disturbance of
2.2. **High Frequency Probes**

the plasma by the probe shaft, the probe wires are spirally wound over a distance of 3cm. This part of the probe should not pick up hf signals, because they cancel each half turn. The hf sensitive part, where the wires are parallel, has a length of 10 mm. The measuring circuit is shown in Figure 2.5(b). All electrical connections are done with double shielded 50 Ω coaxial cables. To enable measurements of the absolute field strengths, the equipment was calibrated, in the hf electric field within a driven plane plate capacitor.

In the treatment above the electric current is assumed to be entirely caused by the displacement current due to surface charges on the probe wires. To check the contribution of the plasma electron, the output signal of a similar probe with the same dimensions has been measured with variation of the DC probe biases in ref. [34]. The contribution of the plasma electrons was typically 20% of the total output signal.
Chapter 3

Electrostatic Double Layers

3.1 Introduction

Electrostatic double layers are self-consistent, localized potential structures connecting plasma with different plasma potential, density or temperature. While the plasmas on both sides of the double layer are normally considered to be quasi neutral, the double layer itself deviates from quasi neutrality and shows one layer of negative and one layer of positive space charge, giving rise to a localized electric field and an associated potential drop $\phi_{DL}$ (Fig. 3.1). A double layer dissipates the power $I_{DL}\phi_{DL}$, by acceleration of electrons from its low potential and ions from its high potential side. $I_{DL}$ is the net current flowing across the double layer. The electron beam on the high potential side and the ion beam on the low potential side give rise to various beam related instabilities like the beam-plasma and the two-stream instability.

Electrostatic double layers were discussed first by Langmuir in 1929 [35]. A simple fluid model description of double layers [36], based on continuity and momentum equations for a cold electron and a cold ion beam, injected from the low and high potential side, respectively, and the Poisson equation gives two existence criteria for self-consistent solutions. The condition of vanishing electric fields at the double layer boundaries gives the “Langmuir condition”. In cold plasmas or when $\phi_{DL} \gg k_BT_e/e$ the Langmuir condition gets

$$\frac{j_e}{j_i} = \sqrt{\frac{m_i}{m_e}}. \tag{3.1}$$

It says that most of the double layer current $I_{DL}$ is carried by the electron current density $j_e$. The requirement that the net charge at the edges of the double layer has the correct sign results in the “Bohm condition” for the injection velocities $v_{e0}$ and $v_{i0}$ of the electron and ion beam:

$$m_e v_{e0}^2 \geq k_B T_e,$$

$$m_i v_{i0}^2 \geq k_B T_e. \tag{3.2}$$
Chapter 3. Electrostatic Double Layers

Figure 3.1. Phasespace and potential structure of a strong double layer (from [16]). The hatched regions indicate accelerated particles. The particles within the separatrix (dashed) get reflected at the double layer.

A more realistic picture of double layers, including reflected electrons on the high potential side and ions on the low potential side, is given within a kinetic description [14, 16]. Depending on the height of the potential drop \( \phi_{DL} \) one discriminates between strong \( (k_B T_e \gg e \phi_{DL}) \) and weak \( (k_B T_e \leq e \phi_{DL}) \) double layers. Weak double layers have been observed by satellites above the auroral zone [37]. Even though the magnetic mirror force contributes to the potential drop along the magnetic field lines, several weak double layers following each other were suggested to play a role in the acceleration process of auroral electrons [17, 18]. Strong and relativistically strong double layers have been proposed to play a role in astrophysical phenomena, e.g., solar flares and X-ray pulsars [16].

3.2 Quasi-Neutral Model of the Potential Drop for Fixed Ions

In Paper 4 the transient current pulses which form when a step voltage is applied to the inhomogeneous plasma column in the triple plasma device are investigated. The model used to compare the measured values of the maximum current and the axial minimum potential is presented here. It is an extension of the model proposed by M. Bohm and S. Torvén in reference [38] to asymmetrically driven plasma sources and includes a beam consisting of the primary electrons in the electron distribution function.
Due to radial ion losses, the ion density profile along the symmetry axis of the device has a minimum situated approximately in the center of the middle chamber. As the plasma has to maintain approximately quasi-neutrality all the time, electron reflecting potential drops build up near the apertures connecting the middle chamber to the sources. This is called “electron-rich” injection. Due to the potential drops near the apertures the plasma potential has also a minimum in the middle chamber. On the timescale of the electron transit time, the electrons establish a steady state, whereas the ions can be considered as fixed. The problem of calculating the current flowing between the sources can be treated by solving the Vlasov-equation for the electrons and by assuming a prescribed and fixed ion density profile. Let the ion density \( N_i(z) \) decrease from the values \( N_{ic} \) and \( N_{ia} \) in low and high potential source to the minimum value \( N_{im} \) at some \( z = z_m \). The potential profile drops similarly from the values \( \phi_c \) and \( \phi_c + \phi_a \) to the minimum value \( \phi_0 = 0 \) at \( z = z_0 \). Let the potential at the ion density minimum be \( \phi_m \). Note that the potential minimum and the density minimum do not coincide. \( \phi_m > 0 \) corresponds to the externally applied potential drop between the source anodes. The potential minimum reflects the fraction of the electrons injected from the sources, that is necessary to maintain quasi-neutrality, and thereby it controls the current density. Using the distribution functions of the injected electrons the electron density \( n_1 \) at the left and \( n_2 \) at the right of the potential minimum can be derived. The low potential source is at the left hand side. \( n_1 \) and \( n_2 \) are functions of the local potential \( \phi \), the potentials \( \phi_c \) and \( \phi_a \), and the densities of the background electrons \( n_c \) and \( n_a \) injected from the low and the high potential side. To find the current voltage characteristic \( j_e(\phi_a) \), the unknown \( \phi_c \), \( \phi_m \), \( n_c \) and \( n_a \) will be determined as functions of \( \phi_a \). This is done by assuming neutrality in the sources and at the ion density minimum and assuming that the electron density minimum coincides with the ion density minimum. These assumptions give the system of nonlinear equations

\[
\begin{align*}
N_{ic} &= n_1(\phi_c) \\
N_{ia} &= n_1(\phi_c + \phi_a) \\
N_{im} &= n_2(\phi_m) \\
0 &= \frac{dn_2}{d\phi}|_{\phi=\phi_m}
\end{align*}
\]

The cylindrical Langmuir probe measurements in the low potential source presented in Paper 4 show that next to the background electrons there is a strong beam of primary electrons. Because of its high velocity, which corresponds to the discharge voltage of the plasma source, and the potential drop of a few \( k_B T_e/e \) from the low potential source to the potential minimum, the electron density in the middle chamber is dominated by these beam electrons. Here \( T_e \) is the temperature of the background electrons.

In the following the potentials are normalized by \( k_B T_e/e \) and the velocities by
thermal velocity of the background electrons \( v_{th} = \sqrt{2k_B T_e/m} \). The phase space is divided along the separatrix

\[ v_b = \text{sign}(z - z_0)\sqrt{\phi} \]  

(3.7)

into a region with electrons coming at the low potential source \((v_z > v_x)\) at the left hand side and electrons originating from the high potential source \((v_z < v_x)\) at the right hand side. Here, \(v_x\) is the velocity along the z-axis. The total distribution function becomes

\[
f(v_z, \phi) = \frac{2n_e}{\sqrt{\pi}v_{th}} e^{\phi - \phi_e - \phi_0} \Theta(v_x - v_z) \\
+ \left[ \frac{2n_c}{\sqrt{\pi}v_{th}} e^{\phi - \phi_e - \phi_0} + f_{Bz} \left( \sqrt{v_z^2 - \phi + \phi_e} \right) \right] \Theta(v_x - v_z)
\] 

(3.8)

The first two terms are the partially reflected background electrons which are assumed to have Maxwellian distribution functions with the same temperature \(T_e\). The last term is the distribution function of the electron beam from the low potential side. The Heaviside function \(\Theta\) selects the proper distribution in velocity space. As the sign of \(v_x\) is different on each side of the potential minimum, \(f\) is a function of \(z\). Integration of \(f\) over \(v_x\) gives the electron density.

\[
n_{1,2}(\phi) = n_e e^{\phi - \phi_e} \left( 1 \pm \text{erf}\sqrt{\phi} \right) + n_e e^{\phi - \phi_e} \left( 1 \mp \text{erf}\sqrt{\phi} \right) \\
+ \int_{\mp\sqrt{\phi}}^{\infty} f_{Bz} \left( \sqrt{v_z^2 - \phi + \phi_e} \right) v_{th} dv_z.
\] 

(3.9)

Here, the upper sign holds for \(z \leq z_0\) and lower for \(z > z_0\). The error function is normalized so that it reaches unity of large arguments. The derivative of the electron density with respect to the potential reads

\[
\frac{d}{d\phi} n_{1,2} = n_e e^{\phi - \phi_e} \left( 1 \pm g(\phi) \right) + n_e e^{\phi - \phi_e} \left( 1 \mp g(\phi) \right) \\
+ \frac{d}{d\phi} \int_{\mp\sqrt{\phi}}^{\infty} f_{Bz} \left( \sqrt{v_z^2 - \phi + \phi_e} \right) v_{th} dv_z.
\] 

(3.10)

where \(g\) is defined as

\[
g(\phi) = \text{erf}\sqrt{\phi} + e^{-\phi}/\sqrt{\pi\phi}.
\] 

(3.11)

When the potential minimum is known, the unnormalized current density is

\[
J_e = -\frac{en_e v_{th}}{\sqrt{\pi}} e^{-\phi_e} + \frac{en_e v_{th}}{\sqrt{\pi}} e^{-\phi_e - \phi_0 - \phi_c} - e \int_{\sqrt{\phi_c}}^{\infty} v_x f_{Bz}(v_x) v_{th} dv_x
\] 

(3.12)
The beam distribution function in the low potential source is measured (see Paper 4). It is fitted with an isotropic, Maxwellian beam. It is reasonable to assume that the true beam distribution is isotropic as well, because the primary electrons are reflected many times between the floating end plates of the plasma source before drifting across the magnetic field to the anode. It may be noted that the method of deriving the distribution function from the second derivative of the current-voltage characteristic of a cylindrical Langmuir probe, is strictly valid only for isotropic distribution. This condition is violated in the middle chamber, because of the discontinuities along the separatrix \( v_b \). The distribution function of the beam in the source \( f_{Bxyz} \) is

\[
f_{Bxyz}(v_x, v_y, v_z) = \frac{2\alpha N_{ic}}{C v_{th}^3} \exp \left( - \left( \sqrt{\frac{v_x^2 + v_y^2 + v_z^2}{v_{th}^2}} - \frac{v_b}{v_{th}} \right)^2 \right). \tag{3.13}
\]

Here

\[
C = 2\pi v_{th} \left( w_{th} \nu_b e^{-v_b^2/w_{th}^2} + \sqrt{\pi} \left( v_{th}^2 + w_{th}^2/2 \right) \left( 1 + \text{erf} \left( v_{th}/w_{th} \right) \right) \right)
\]

is the normalization constant so that

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{Bxyz}(v_x, v_y, v_z) v_{th}^3 dvx dv_y dv_z = 2\alpha N_{ic}.
\]

\( v_b \) and \( w_{th} \) are the beam’s normalized mean and thermal velocity. \( \alpha N_{ic} \) is the density of the beam electrons injected into the middle chamber \((v_z \geq 0)\). The appropriate distribution function \( f_{Bz} \) to be used in the one-dimensional model is the integral of \( f_{Bxyz} \) over \( v_x \) and \( v_y \). It reads

\[
f_{Bz}(v_x) = \frac{2\alpha N_{ic} w_{th}^2 \pi}{C} \left[ \exp \left( - \frac{(|v_x| - v_b)^2}{w_{th}^2} \right) \right.
\]

\[
+ \sqrt{\pi} \frac{v_{th}}{w_{th}} \left( 1 - \text{sign}(|v_x| - v_b) \text{erf} \left( \frac{|v_x| - v_b}{w_{th}} \right) \right) \right]. \tag{3.14}
\]

In the limit of a cold isotropic beam the distribution function \( f_{Bxyz} \) goes to a delta function and \( f_{Bz} \) to a waterbag distribution. All integrals can be handled analytically. The distribution functions are in this limit

\[
f_{Bxyz}(v_x, v_y, v_z) = \frac{2\alpha N_{ic}}{v_{th}^4 4\pi v_{th}^2} \delta \left( \sqrt{\frac{v_x^2 + v_y^2 + v_z^2}{v_{th}^2}} - v_b \right), \tag{3.15}
\]

\[
f_{Bz}(v) = \frac{2\alpha N_{ic}}{v_{th}^2 2v_{th}} \Theta(v - v_b). \tag{3.16}
\]

In the case \( v_b < \sqrt{\phi_c} \) no beam electrons can pass the potential minimum and \( \phi_m \) can be treated as in the model without beam. For the case \( v_b \geq \sqrt{\phi_c} \) it follows

\[
n_{1,2} = n_e e^{\phi - \phi_c} \left( 1 \pm \text{erf} \sqrt{\phi} \right) + n_a e^{\phi - \phi_a} \left( 1 \mp \text{erf} \sqrt{\phi} \right)
\]
\[ + \frac{\alpha N_{ic}}{n_b} \left( \sqrt{v_b^2 - \phi_c + \phi} \pm \sqrt{\phi} \right), \quad (3.17) \]
\[
\frac{d}{d \phi} n_{1,2} = e^{\phi - \phi_c} \left[ n_c + n_a e^{-\phi_a} \pm g(\phi) (n_c - n_a e^{-\phi_a}) \right]
+ \frac{\alpha N_{ic}}{2v_b} \left( \sqrt{v_b^2 - \phi_c + \phi} \pm \frac{1}{\sqrt{\phi}} \right), \quad (3.18) \]
\[
je = -\frac{en_c v_{ih} e^{-\phi_c}}{\sqrt{\pi}} - \frac{en_a v_{ih} e^{-\phi_a} - e\alpha N_{ic} v_{ih}}{2v_b} (v_b^2 - \phi_c) \quad (3.19) \]

As \( g(\phi) \) is positive, \( dn_1/d\phi \) is monotonically increasing for \( n_c > n_a \exp(-\phi_a) \). As \( g(\phi) \) goes to \(+\infty\) when \( \phi \) goes to zero, \( dn_2/d\phi \) is negative for small \( \phi \) but positive for large \( \phi \). This means that \( n_2 \) has a minimum. This minimum can coincide with the ion density minimum only if \( z_0 < z_m \). The choice of \( n_2 \) is eqs. (3.5) and (3.6) is accordingly justified.

System (3.3)-(3.6) is solved numerically by minimizing the deviation from neutrality at the ion density minimum
\[
\varepsilon = N_{im} - n_c(\phi_m) \quad (3.20) \]

with respect to \( \phi_c \), with the Newton-method in the following way. For a given applied potential \( \phi_a \) and a start guess \( \phi_c, \varepsilon(\phi_c) \) is determined with two intermediate steps. First eqs. (3.3) and (3.4), which form a linear system independent of \( \phi_m \), is solved for \( n_c \) and \( n_a \). Using these values for \( n_c \) and \( n_a \), the root \( \phi_m \) of eq. (3.6) is determined as well as the Newton-method. A solution of eq. (3.6) exists always, as argued in the previous paragraph, but a solution of eq. (3.5) may not, depending on the start guess of \( \phi_c \). Evaluating now eq. (3.5) at \( \phi_m \) gives the deviation from exact neutrality as a function of the potential \( \phi_c \).

In Figure 3.2 the solutions of the quasi-neutrality model with a beam and without beam are compared. The plasma parameter with beam ('\( \Box \)') \( k_B T_c/e = 5.4\text{eV}, N_{ic} = 3.0 \times 10^{16} \text{m}^{-3}, N_{im} = 4.5 \times 10^{14} \text{m}^{-3}, N_{ia} = 1.05 \times 10^{16} \text{m}^{-3}, \alpha = 0.04, v_b = 3.51 \times 10^6 \text{ms}^{-1}/v_{ih} \), correspond to the experimental values for Figure 7 in Paper 4. Curves labeled 'o' have no beam. \( \phi_a = 60 \) corresponds approximately to 325V. The general behavior of the solution is that the depth of the virtual cathode \( \phi_c \) decreases with increasing applied potential \( \phi_a \), so that \( j_e \) increases with increasing \( \phi_a \). The depth of the virtual cathode is lower with than without beam, because in the case with beam more of the background electrons have to be reflected to maintain neutrality at the ion density minimum. \( \phi_m \) itself does not differ significantly in the two both cases. \( \phi_m \) follows \( \phi_a \) approximately linearly, so that most of the potential drop is concentrated between the potential and the ion density minimum. Even though the normalized depth of the potential minimum differs by 0.5–1.0, the current densities are approximately the same. The current density with beam exceeds that one without beam by more than 10%, only for \( \phi_a \leq 9 \). The contribution of the beam to the \( j_e \) decreases from approximately 70% at \( \phi_a = 10 \)
3.2. Quasi-Neutral Model of the Potential Drop for Fixed Ions

Figure 3.2. Comparison of the solution of the quasi-neutral model with fixed ions, with electron beam ["[]"] and without beam ["•••"]. a) depth of the potential minimum $\phi_c$, b) potential $\phi_m$ at the ion density minimum $N_{im}$, c) current density $j_e$, d) injection densities, e) exponents $q$ fitted from the relation $j_e = a_0 \phi_a^q$ and f) effective temperature $T_{eff} = -\Delta \log(j_e)/\Delta \phi_e$. 
to 35% at $\phi_a = 60$. Fitting a relationship $j_e = a\phi_b^{2}$ to the model results gives exponents displayed in Fig. 3.2(e). The value without beam decrease from maximal 0.7 at $\phi_a \approx 5$ to 0.55 at $\phi_a > 10$. Without beam it can be shown that $j_e$ is proportional to $\phi_m^{1/2}$ (see Paper 4). Asymptotically $\phi_m$ approaches $\phi_a$, for $\phi_a \gg 1$. However, Figure 3.2(e) shows that their difference can not be disregarded for typical voltages in the experiment. The exponents with beam are lower and lie close to 0.55 for $\phi_a \geq 10$. This results from the fact that $j_e$ with beam exceeds $j_e$ without, but their difference decreases with increasing $\phi_a$. Corresponding exponents are determined for experimental current values in Paper 4 but for different plasma parameters. They were in the interval between 0.55 and 0.83. From the model one can conclude that the expected experimental values depend strongly on the true beam density and the fit interval, even if two-dimensional effects and collisions are disregarded.

In Panel (f) an effective temperature $T_{\text{eff}} = -\Delta \log(j_e) / \Delta \phi_b$ is fitted to the model solution. Without beam and constant injected densities values of $T_{\text{eff}}$ close to $T_e$ would be expected according to eq. (3.12) for high applied voltages $\phi_a$. The values below $T_e$ for high $\phi_a$ are caused by the increase of the injected background density (see Panel (d)). This increase is a consequence of the quasi-neutrality demanded in the source regions. The values $T_{\text{eff}} < T_e$ for small $\phi_a$ are caused by background electrons from the high potential side passing the potential minimum. The same behavior of $T_{\text{eff}}$ can be seen with beam, however the fitted temperatures are higher, reaching here approximately $2T_e$. This is expected because the beam current density varies slower, namely linearly, with $\phi_c$ than the current density caused by the background electrons.

### 3.3 Quasi-Neutral Model of the Triple Plasma Double Layer

The self-excited oscillations investigated in Paper 1 represent the dominating low frequency instability caused by an electric double layer. They are a result of the combination of the impedance in the outer circuit and the negative differential resistivity $dI / dU_a$ of the double layer within the Triple plasma machine. A Figure and description of this device is given in Paper 1. A simple physical model of the current voltage characteristic $I(U_a)$ of the double layer has been presented in [22]. An extension of this model to asymmetrically driven plasma sources is presented here. Figure 3.3(b) shows the measured double layer potential profiles at different times during an oscillation period. The potential profiles clearly show a potential minimum $\phi_{\text{min}}$ between the left source region and the double layer, at $z < 10$ cm. In earlier investigations [19, 20] this minimum has been observed to first increase with increasing double layer potential drop $U_a$, but then, as a consequence of quasi-neutrality in the low potential region, decrease again. The increase and decrease of $\phi_{\text{min}}$ is connected to a corresponding increase and decrease of the current $I(U_a)$ due to the reflection of the current carrying electrons at the potential minimum. The quantitative model is the following.

The electrons are treated exactly as on Section 3.2 except that the electron beam
3.3. Quasi-Neutral Model of the Triple Plasma Double Layer

Figure 3.3. Experimental current and potential oscillations for $C = 2 \mu F$, $U_{dc} = 64.5 \text{V}$ (from Paper 1): (a) The oscillation of the voltage $U$ between the sources is closely harmonic ($f = 458 \text{Hz}$) and the current $I$ crossing the double layer follows the minimum potential $\phi_{\text{min}}$. [b] Plasma potential profiles $\phi(z)$ for four different times. $z$ is the distance along the plasma column, which connects to the low and high potential plasma sources through apertures, situated at $z = 0$ and $z = 60 \text{cm}$, respectively. (c) Time and space resolved potential oscillation. When the potential difference between the sources $U$ is high, a strong double layer forms close to the high potential aperture, while the current $I$ is limited by the potential minimum close to the low potential aperture ($z < 10 \text{cm}$).
Chapter 3. Electrostatic Double Layers

Figure 3.4. Illustration of the quasi-neutral model for the double layer current-voltage characteristic for movable ions. The sources are driven symmetrically ($\beta = 0$) and a typical experimental ion loss parameter $\kappa = 0.18$ is used. a) Potential drop $\phi_c$ and electron current density $j_e$ as a function of the applied potential between the sources $\phi_a$. b) Normalized conductance $g(\phi_a) = R_m \frac{d j_e}{d \phi_a}$, where $R_m = 1/\min(dj_e/d\phi_a)$.

from the low potential side is left out. The potential $\phi(z)$ increases monotonically from its minimum value $\phi_0 = 0$ at some position $z_0$ to $\phi = \phi_c$ at the left and $\phi_c + \phi_a$ at the right source region, respectively. Application of the quasi-neutrality condition at the potential minimum $\phi_0 = 0$ and in the sources regions will give the current voltage characteristic $j_e(\phi_a)$. The potential is normalized by $k_B T/e$ and the velocities by the thermal velocity $v_{th} = \sqrt{2k_B T_e/m}$. The electron densities $n_1(\phi)$ and $n_2(\phi)$, left and right of the potential minimum, become according to eq. (3.9)

$$n_{1,2}(\phi) = n_e e^{\phi_0 - \phi_c} \left( 1 \pm \text{erf}(\sqrt{\phi}) \right) + n_a e^{\phi_0 - \phi_a} \left( 1 \mp \text{erf}(\sqrt{\phi}) \right)$$  \hspace{1cm} (3.21)

The electron current $j_e$ flowing through the potential structure reads

$$j_e = - \frac{en_e v_{th}}{\pi} \exp(-\phi_c) \left[ 1 - \exp(\beta - \phi_a) \right].$$  \hspace{1cm} (3.22)

$j_e$ is limited by the potential minimum, which reflects a part of the electrons injected from the left source. $\beta = \ln(n_a/n_c)$ measures the asymmetry of the source plasmas. In the case of a strong double layer the term $\exp(\beta - \phi_a)$, which represents the electron current originating from the high potential side, becomes negligible.

It is assumed that the ions at the potential minimum can be represented by ion beams injected from the source regions with the Bohm velocity $v_B = \sqrt{k_B T_e/m}$. Their contribution to the total current is neglected. The ions coming from the high potential side pass the $z_0$ region once and get lost at the low potential aperture. The ions from the low potential side get reflected at the double layer, so that they
pass the potential minimum twice, before they get lost in the left aperture. The ion density at the minimum potential \( \phi_0 = 0 \) equals then

\[
N_{i0} = \kappa \left[ \frac{N_{ia}}{\sqrt{1 + 2\phi_a + 2\phi_c}} + \frac{2N_{ic}}{\sqrt{1 + 2\phi_c}} \right] \tag{3.23}
\]

Here, the ion loss parameter \( \kappa \in [0, 1] \) is introduced to account for the radial ion losses in the aperture regions.

To determine the current-voltage characteristic, \( \phi_c \) needs to be expressed as a function of \( \phi_a \). The resulting system of equations is

\[
\begin{align*}
N_{ic} &= n_c \left( 1 + \text{erf} \sqrt{\phi_c} \right) + n_h e^{-\phi_c} \left( 1 - \text{erf} \sqrt{\phi_c} \right), \tag{3.24} \\
N_{ia} &= n_c e^{\phi_a} \left( 1 - \text{erf} \sqrt{\phi_c + \phi_a} \right) + n_a \left( 1 + \text{erf} \sqrt{\phi_c + \phi_a} \right), \tag{3.25} \\
N_{i0} &= n_c e^{-\phi_c} \left( 1 + e^{\beta - \phi_a} \right). \tag{3.26}
\end{align*}
\]

Substituting now \( N_{i0}, N_{ia} \) and \( N_{ic} \) in eq. (3.23) with these equations a function \( f_{QN}(\phi_a, \phi_c, \kappa, \beta) \) proportional to \( n_e(0) - N_{i0} \) can be derived. Quasi-neutrality at the potential minimum is fulfilled if

\[
0 = f_{QN}(\phi_a, \phi_c, \kappa, \beta) \\
= +\sqrt{1 + 2\phi_c} \sqrt{1 + 2\phi_c + 2\phi_a} e^{-\phi_c} \left( 1 + e^{\beta - \phi_a} \right) \\
-2\kappa \sqrt{1 + 2\phi_c + 2\phi_a} \left[ 1 + \text{erf} \sqrt{\phi_c} + e^{\beta - \phi_a} \left( 1 - \text{erf} \sqrt{\phi_c} \right) \right] \tag{3.27}
\]
\[-\kappa \sqrt{1 + 2\phi_c} \left[ e^\beta \left( 1 + \text{erf} \sqrt{\phi_c + \phi_a} \right) + e^{\phi_a} \left( 1 - \text{erf} \sqrt{\phi_c + \phi_a} \right) \right].\]

This relation implicitly defines \( \phi_c \) as a function of \( \phi_a, \kappa \) and \( \beta \). Provided that \( \kappa \) is independent of the potentials, it can be determined from a single measurement of \( \phi_c \) at known \( \phi_a \). In [22] a typical value of \( \phi_c = 1.0 \) was measured for \( \phi_a = 0.5 \) in symmetric operation (\( \beta = 0 \)), which results in \( \kappa = 0.18 \). As an illustration the behavior of \( \phi_c(\phi_a) \) and \( j_e(\phi_a) \) for these values of \( \beta \) and \( \kappa \) is shown in Figure 3.4. The potential difference between left source and minimum first decreases, which leads to an increase of the current, but decreases again for higher \( \phi_a \). The increase of \( \phi_c \) can be understood in the following way. When \( \phi_a \gg 1 \) only the electron population emitted from the low potential side contributes to neutralize the ion beams at the potential minimum and sustain quasi-neutrality there. When \( \phi_a \) is increased the density of the ion beam coming from the high potential is decreased because of the stronger acceleration in the double layer. This decrease is compensated when \( \phi_c \) increases so that a bigger fraction of the electrons population emitted from the low potential side gets reflected.

The current maximum, and potential minimum are not exactly at the same applied potential \( \phi_a \). The normalized differential conductance \( g(\phi_a) = (dj_e/d\phi_a)R_m \) of the double layer is negative when \( j_e \) has gone through its maximum. Here, \( R_m = -1/(dj_e/d\phi_a) \) is used to normalize the minimum value of \( g(\phi_a) \) to \(-1\).

To find out for which sets of parameters \( \kappa \) and \( \beta \) it is possible to find a positive \( \phi_c \), I consider the behavior of \( f_{QN} \) at constant \( \phi_a \). For \( \phi_c \gg 1 \), the first term in eq. (3.27), which is proportional to \( \exp(-\phi_c) \), approaches zero, and the terms \( \text{erf}\sqrt{\phi_c} \) and \( \text{erf}\sqrt{\phi_c + \phi_a} \) approach unity. \( f_{QN} \) takes the form

\[ f_{QN}(\phi_a, \phi_c, \kappa, \beta) = -4\kappa \sqrt{1 + 2\phi_c + 2\phi_a + \exp(\beta)\sqrt{1 + 2\phi_c}}. \] (3.28)

which is negative for all \( \phi_a, \kappa \in (0, 1] \) and \( \beta \). The existence of a positive \( \phi_c \) is hence ensured if \( f_{QN}(\phi_a, 0, \kappa, \beta) > 0 \). The limiting ion loss parameter \( \kappa_1(\phi_a, \beta) \) ensuring a positive \( \phi_c \) at given \( \phi_a \) and \( \beta \), fulfills the condition \( f_{QN}(\phi_a, 0, \kappa_1, \beta) = 0 \). This condition can be resolved to

\[ \kappa_1(\phi_a, \beta) = \frac{1 + \exp(-\phi_a)\sqrt{2\phi_a + 1 + \sqrt{2\pi}}}{\left[ 2\sqrt{1 + 2\phi_a} (1 + \exp(-\phi_a)) + \exp(\beta) (1 - \text{erf}\sqrt{\phi_a}) + \exp(\beta) (1 + \text{erf}\sqrt{\phi_a}) \right]^2}. \] (3.29)

As the dominator equals twice the numerator and the last two terms in the dominator are always positive, it follows that \( 0 < \kappa_1 < 0.5 \). This shows that in this model ion losses are vitally needed to guarantee a solution \( \phi_c > 0 \). For \( \phi_a \gg 1 \), \( \kappa_1 \) approximates

\[ \kappa_1 \approx \frac{\phi_a}{2\phi_a + \exp(\beta)\sqrt{2\phi_a + 1}/\sqrt{2\pi}} \] (3.30)

so that \( \kappa_1 \) approaches \( 1/2 \) in the limit of big \( \phi_a \). This means that, for a given \( \kappa < 1/2 \) there is no principle upper limit for the applicable potential drop \( \phi_a \).
However, intermediate regions, can be found, where $f_{QN}(\phi_a, 0, \kappa, \beta) < 0$. Figure 3.5 shows $\kappa_i$ for a set of asymmetries $\beta$. While $\kappa$ increases monotonically with $\phi_a$, for symmetric operation ($\beta = 0$), it shows a local minimum for $\beta > 0$, which decreases with increasing $\beta$.

In experiments it has been observed that, when the applied potential drop is held constant and the asymmetry increased, the double layer disappears. The potential drop $\phi_a$ is then concentrated in a region around the aperture of the low potential source. The above model offers simple explanation. Solutions with a potential minimum exist only when the experimental loss parameter $\kappa$ is less the $\kappa_i$. When $\beta$ is increased, $\kappa_i$ decreases until $\kappa_i(\phi_a, \beta)$ reaches the $\kappa$ value caused by the actual ion losses near the apertures.
Chapter 4

Dispersion Relations for the Beam-plasma Interaction

The beam-plasma interaction has been investigated since the 60's. Characteristic of the beam-plasma instability is the conversion of the kinetic energy of an electron beam, streaming through a background plasma, to an electrostatic wave, propagating with a phase velocity close to the beam velocity. When the phase space density of the electrons can not be neglected in the region of the phase velocity of the wave the transfer of electrostatic wave energy to kinetic energy of the electrons, called Landau-damping, has an important influence on the growth rate of the wave. In low temperature laboratory plasmas this is usually the case. The first dispersion relations of the beam-plasma instability including Landau-damping, were given by O'Neil and Malmberg [39]. A bit later Self and coauthors gave expressions for the maximal spatial and temporal growth rates in the limits of a cold beam \( v_b/v_0 \ll (n_b/2n_e)^{1/3}(v_0/v_e)^{2/3} \) and the bump on tail distribution \( v_b/v_0 \gg (n_b/2n_e)^{1/3}(v_0/v_e)^{2/3} \) [40]. Here \( n_b \) and \( n_e \) are the densities of the beam and the plasma, \( v_b \) and \( v_e \) are their thermal velocities and \( v_0 \) is the drift velocity of the beam. All cases in between have to be treated numerically, e.g., by expanding the zeroth order distribution as done in [41] or [42]. The theoretical maximal growth rates used in Paper 2 were calculated with the use of the plasma dispersion function \( Z(\zeta) \) as defined by Fried and Conte [43]. The classical way of solving electrostatic dispersion relations for infinite, homogeneous, thermal plasmas is based on the first order perturbation of Vlasov and Poisson's equations. A description can be found in textbooks, for example [44,45]. For \( N \) particle species \( j \) with particle charge \( q_j \) and zeroth order distribution function \( f_{0j} \) in one dimension they read, respectively:

\[
\frac{\partial}{\partial t} f_{1j} + v \frac{\partial}{\partial x} f_{1j} + \frac{q_j}{m_j} E \frac{\partial}{\partial v} f_{0j} = 0,
\]  

(4.1)
\[ \epsilon_0 \frac{\partial}{\partial x} E_1 = \sum_j q_j n_{1j} \]  

(4.2)

\( E_1 \) is the wave field, \( f_{1j} \) and \( n_{1j} \) the perturbations of \( f_{0j} \) and the density of species \( j \). Assuming plane waves proportional to \( \exp(i(kx - \omega t)) \) this reads:

\[ f_{1j} = -i q_j E_1 \frac{1}{m_j} \frac{\partial}{\partial v} f_{0j}, \]  

(4.3)

\[ E_1 = \frac{1}{i \epsilon_0} \sum_j q_j n_{1j}. \]  

(4.4)

Integration of \( f_{1j} \) with respect to \( v \) gives \( n_{1j} \) in terms of \( E_1 \), so that the Poisson equation gives the dispersion equation

\[ 1 = \frac{1}{k^2} \sum_j \omega_{pj}^2 \int_{-\infty}^{+\infty} \frac{1}{v - \omega/k} \frac{\partial}{\partial v} f_{0j} dv \]  

(4.5)

for \( \text{Imag}(\omega) > 0 \) (growing modes). Landau’s analysis shows how the range of definition is extended to damped modes by analytical continuation, introducing Landau contours for the integration [46]. Here \( \omega_{pj} = \sqrt{q_j^2 n_{0j} / (\epsilon_0 m_j)} \) is the plasma frequency of species \( j \) and \( f_{0j} \) its normalized distribution function \( (\int_{-\infty}^{+\infty} f_{0j} dv = 1) \).

For one-dimensional Maxwellian distribution functions

\[ f_{0j} = \frac{1}{\sqrt{\pi v_j}} \exp\left(-\left(v - v_{0j}\right)^2/v_j^2\right), \]  

(4.6)

where \( v_{0j} \) is the drift velocity and \( v_j = \sqrt{2k_B T_j/m_j} \) the thermal velocity, the plasma dispersion function \( Z \) can be used to express the dispersion equation in a shorter form. Using the \( Z \)-function, the integral in eq. (4.5) can be expressed by the first derivative of \( Z \), so that the dispersion relation gets the form

\[ 1 = \frac{1}{k^2} \sum_j \frac{\omega_{pj}^2}{v_j^2} Z^{'}(\zeta_j), \]  

where \( \zeta_j = \frac{\omega/k - v_{0j}}{v_j} \).  

(4.7)

The prime denotes differentiation with respect to the argument.

In the experimental situation in Paper 2 time averaged Langmuir probe measurements showed that the plasma consisted of two Maxwellian background components and a warm beam. The beam was approximated by a drifting Maxwellian. When \( n_c, n_h \) and \( v_c, v_h, \) were the densities and thermal velocities of the background components and \( n_b, v_b, \) the density, thermal and drift velocity of the beam, respectively, the dispersion relation to be solved is

\[ 1 = \frac{1}{k^2} \left[ \frac{(1 - \alpha)(1 - \delta)}{v_c^2} Z^{'} \left( \frac{\omega}{v_c k} \right) + \frac{1 - \alpha}{v_h^2} Z^{'} \left( \frac{\omega}{v_h k} \right) + \frac{\alpha}{v_b^2} Z^{'} \left( \frac{1}{v_b} \left( \frac{\omega}{k} - 1 \right) \right) \right]. \]  

(4.8)
Here $\alpha = n_b/(n_c + n_h + n_b)$ is the relative beam density and $\delta = n_h/(n_b + n_c)$ the contribution of the warm component to the background density. The physical quantities are normalized by $v_0$ and the total plasma frequency $\omega_p = \sqrt{(n_c + n_h + n_b)e^2/(enm)}$ in the following way

$$\hat{\omega} = \omega/\omega_p, \quad \hat{k} = kv_0/\omega_p, \quad \hat{v}_c = v_c/v_0, \quad \hat{v}_h = v_h/v_0 \quad \text{and} \quad \hat{v}_b = v_b/v_0. \quad (4.9)$$

While the velocities in the arguments of the Z-functions are real, $\hat{\omega}$ and $\hat{k}$ can be complex. Instead of solving the dispersion equation for real $\hat{k}$ and complex $\hat{\omega}$, as it is done when the temporal response of the plasma to an initial distribution, which is sinusoidal and infinite in space, is investigated, I solve eq. (4.8) for real $\hat{\omega}$ and complex $\hat{k}$. In this way I get the spatial response of the plasma to a perturbation at a fixed location, being sinusoidal and infinite in time. Negative $\text{Im} (\hat{k})$ correspond to spatially growing and positive $\text{Im} (\hat{k})$ to spatially damped waves. Writing $1/k^2 = \hat{r}^2/\hat{\omega}^2$ with the complex phase velocity $\hat{r} = \hat{\omega}/\hat{k}$ the dispersion equation can be written as

$$0 = D(\hat{r}, \hat{\omega}) = \hat{\omega}^2 - \hat{r}^2 \left[ \left( \frac{1 - \alpha}{\hat{v}_c^2} \right) \mathcal{Z}' \left( \frac{\hat{r}}{\hat{v}_c} \right) + \left( \frac{1 - \delta}{\hat{v}_h^2} \right) \mathcal{Z}' \left( \frac{\hat{r}}{\hat{v}_h} \right) + \frac{\alpha}{\hat{v}_b^2} \mathcal{Z}' \left( \frac{\hat{r} - 1}{\hat{v}_b} \right) \right] \quad (4.10)$$

Equation (4.10) is solved numerically for real $\hat{\omega}$ in the complex $\hat{r}$-plane. To do so the plasma dispersion function has to be evaluated.

### 4.1 Numerical Evaluation of $Z(\zeta)$

The plasma dispersion function $Z(\zeta)$ was first defined by Fried and Conte [43] as

$$Z(\zeta) = \begin{cases} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-s^2) ds, & \text{for } \text{Im} (\zeta) > 0, \\
\quad i\sqrt{\pi} \exp(\zeta^2) - 2 \exp(-\zeta^2) \int_{0}^{\zeta} \exp(s^2) ds, & \text{for } \text{Im} (\zeta) = 0, \\
\quad 2i\sqrt{\pi} \exp(\zeta^2) + Z^*(\zeta^*), & \text{for } \text{Im} (\zeta) < 0. \end{cases} \quad (4.11)$$

The asterisk denotes the complex conjugate. For the numerical evaluation of $Z(\zeta)$ I follow Fried and Conte [43] and McCabe [47]. The following properties of $Z$ are used (see [47]). For all $\zeta$, $Z$ can be expressed as:

$$Z(\zeta) = i\sqrt{\pi}\sigma \exp(-\zeta^2) + \lim_{N \to \infty} z_N(\zeta), \quad \sigma = \begin{cases} 0, & \text{Im} (\zeta) > 0 \\
1, & \text{Im} (\zeta) = 0 \\
2, & \text{Im} (\zeta) < 0 \end{cases} \quad (4.12)$$
The imaginary part of \( Z \) is symmetric to the imaginary axis, and the real part antisymmetric: 
\[
\text{Imag}(Z(x + iy)) = \text{Imag}(Z(-x + iy)), \quad \text{Real}(Z(x + iy)) = -\text{Real}(Z(-x + iy)).
\]
The term \( i\sqrt{\pi} \exp(-\zeta^2) \) gives rise to an infinite number of lobes \( \text{Imag}(Z) = 0 \) and \( \text{Real}(Z) = 0 \) in the region \( \text{Imag}(\zeta) < 0 \). The corresponding lobes of \( Z \) have a similar structure and give rise to an infinite number of solutions of the dispersion equation.

where \( z_N(\zeta) \) is the continuous fraction
\[
z_N(\zeta) = -\frac{a_0}{b_0 - \frac{a_1}{b_1 - \frac{a_2}{b_2 - \cdots}}}.
\]
\( a_0 = 2, \quad a_n = 2n, \quad b_n = 2\zeta, \quad \text{for} \quad n \in [1, 2, \ldots, N], \quad b_0 = \frac{a_1}{a_2} \).

The second very useful property of \( Z \) is the differential equation
\[
Z'(\zeta) + 2\zeta Z(\zeta) = -2, \quad Z(0) = i\sqrt{\pi}.
\]

For the numerical calculation of \( Z \), eq. (4.13) is used preferably. \( N \) is chosen so that the error \( |z_{N+1}(\zeta) - z_N(\zeta)|/|z_N(\zeta)| < 10^{-10} \). For \( \text{Imag}(\zeta) > 5 \), \( N \) has a typical value less than 10. Close to the real axis the \( N \) value, necessary to keep this accuracy,
increases dramatically, so that \( Z(\zeta) \) is here calculated by numerical integration of eq. (4.14) instead. The integration is done in two steps. First \( Z(\text{Real}(\zeta)) \) is calculated by numerical evaluation of eq. (4.11) with Romberg’s method. Then, using \( Z(\text{Real}(\zeta)) \) as initial value, eq. (4.14) is integrated with the Runge-Kutta method of 4th order on the path \( \text{Real}(\zeta) + i[0,\text{Imag}(\zeta)] \) parallel to the imaginary axis. The derivative necessary in the dispersion equation is always calculated using eq. (4.14).

Figure 4.1 illustrates the behavior of \( Z(\zeta) \) in the region \( |\text{Imag}(\zeta)| \leq 5, |\text{Real}(\zeta)| \leq 5 \). While \( \text{Imag}(Z) \) is symmetric to the imaginary axis, is \( \text{Real}(Z) \) antisymmetric. In the region \( \text{Imag}(\zeta) < 0 \) the term \( i\sqrt{\pi}\sigma\exp(-\zeta^2) \) gives rise to an infinite number of lobes with \( \text{Imag}(Z(\zeta)) = 0 \) and \( \text{Real}(Z(\zeta)) = 0 \).

4.2 Determination of the Dispersion Roots

For a given \( \hat{\omega} \), the \( \hat{r} \) value solving the dispersion equation must fulfill simultaneously \( \text{Real}(D(\hat{r}, \hat{\omega})) = 0 \) and \( \text{Imag}(D(\hat{r}, \hat{\omega})) = 0 \). In a contour plot, showing the zero levels of the imaginary and real part of \( D \), the solutions appear as the crossing points of the two sets of contours. With \( D_0(\hat{r}) \) defined as \( D(\hat{r}, \hat{\omega} = 0) \) it holds \( D(\hat{r}, \hat{\omega}) = \hat{\omega}^2 + D_0(\hat{r}) \). As \( \hat{\omega} \) is real, the contours \( \text{Imag}(D(\hat{r}, \hat{\omega})) = 0 \) are independent of \( \hat{\omega} \). In addition, the contours \( \text{Real}(D(\hat{r}, \hat{\omega})) = 0 \) can easily be generated for all \( \hat{\omega} \) as soon as \( D_0(\hat{r}) \) is known. To get an overview of the dispersion roots, it is sufficient to calculate \( D_0(\hat{r}) \) on a course mesh of \( \hat{r} \)-values. Figure 4.2(a) shows the contours \( \text{Imag}(D_0(\hat{r})) = 0 \) and \( \text{Real}(D(\hat{r}, 0)) = 0 \) for the experimental parameters of Paper 2 in the case \( I = 0.1 \ m \)

\[
\delta = 0.22, \quad T_e = 1.5 \text{ eV}, \quad T_h = 30 \text{ eV}, \quad \alpha = 0.05, \quad T_h = 0.3 \text{ eV}, \quad v_0 = \sqrt{2e \times 40 \text{ V/m}}.
\]

The crossing points of the two sets of contours move along the contours \( \text{Imag}(D_0(\hat{r})) = 0 \) as \( \hat{\omega} \) is changed, while the latter themselves are independent of \( \hat{\omega} \). As \( Z \) is an analytic function the Newton method is applied to solve the dispersion equation for the slow space charge wave and the Langmuir wave. For given \( \hat{\omega} \) the \( k \)-th iteration step is given by

\[
\hat{r}_{k+1} = \hat{r}_k - \frac{D(\hat{r}_k, \hat{\omega})}{\partial D(\hat{r}_k, \hat{\omega})/\partial \hat{r}} = \hat{r}_k - \frac{2h D(\hat{r}_k, \hat{\omega})}{D(\hat{r}_k + h, \hat{\omega}) - D(\hat{r}_k - h, \hat{\omega})}. \tag{4.15}
\]

Here the derivative is approximated by the symmetric difference quotient. \( h \) is chosen to \( h = 10^{-6} \). The initial value for each dispersion relation is taken from an overview picture of \( D_0(\hat{r}) \), calculated on an equidistant mesh of only \( 30 \times 50 \) \( \hat{r} \)-values covering the region \([-0.4, 2.0] + i[-0.5, 0.5] \). The iteration is stopped at \( |r_{k+1} - r_k| \leq 10^{-5} \) or \( \text{Real}(\hat{w}/\hat{r}_k) > 10 \). The latter condition, corresponding to waves with a phase velocity bigger than 10 times the beam velocity, is fulfilled for Langmuir wave at frequencies \( \hat{\omega} \) close to 1. The dispersion relations of the slow space charge wave and the Langmuir wave for the experimental parameters are shown in figure 4.2(b).
Figure 4.2. Roots of the dispersion equation for the experimental parameters of Paper 2 \( t = 0.1 \text{ m} \). a) Contours \( \text{Imag}(D(r, \omega = 0)) = 0 \) (labeled 'I') and \( \text{Real}(D(r, \omega = 0)) = 0 \) (labeled 'r'). As \( \omega \) is changed the curves \( \text{Real}(D(r, \omega)) = 0 \) move across the contours \( \text{Imag}(D(r, \omega)) = 0 \), which are independent of \( \omega \). The crossing points of both contours form the dispersion relations. The least damped ones are the slow space charge wave (SSCW) and the Langmuir wave (LW). For \( \omega = 0 \) the crossing point on the LW-branch lies close to the real axis at \(+\infty\). b) Dispersion relations of slow space charge wave and Langmuir wave.
Chapter 5

Fluid Model of the Eigenmodes of Inhomogeneous Plasma Diodes

The fluid model for electrostatic eigenmodes of inhomogeneous plasma diodes, presented here and used in Paper 2 has been developed by Löfgren and Gunell [26]. The theoretical background for the finite-element-methods used to solve the model equations can for example be found in [48]. The system investigated in Paper 2 is a bounded plasma diode with an inhomogeneous density profile. The energy source of the excited waves is a freely streaming electron beam, which enters the diode through a grid after being preaccelerated between two differently biased grids.

5.1 Numerical Solution of the Steady State

The solution of the steady state in the inhomogeneous plasma diode is essential for the fluid model of the high frequency electric field, because the plasma inhomogeneity enters the high frequency model through inhomogeneous coefficient functions. Even though the fluid model of the high frequency behavior does not include the electron beam, the method used to solve the complete zeroth order problem is described here. In the simulation presented in Paper 2 the zeroth order densities and the potential were used to load the initial configuration space of the simulations in order to reduce initial transient fluctuations of the plasma parameters.

The electron phase space of the model diode is shown in Figure 5.1. The ion component of the plasma is represented as an imobile neutralizing background and is prescribed according to measurements on the real system. Its density has
a maximum $N_m$ at the center of the diode at $x = l/2$ and falls to $N_m/2$ at the bounding electrodes. The electron component consists of a Maxwellian background component and a thermal beam. The potential $\phi$ is fixed to $\phi_l$ and $\phi_r$ at the electrodes and assumed to rise to a single maximum $\phi_m$ at $x = l/2$, so that $\phi(x)$ obeys Poisson's equation and the boundary conditions (BC)

$$0 = \frac{\partial^2 \phi}{\partial x^2} + \frac{e}{\epsilon_0} \left[ n_l(x) - n_e(\phi) - n_b(\phi) \right]$$

BC: $\phi(x = 0) = \phi_l$, $\phi(x = l) = \phi_r$. (5.1)

As the potential has no local minima, the electron density can be expressed as a unique function of the potential, so that eq. (5.1) is a second order, nonlinear and inhomogeneous ordinary differential equation. The electron phase space is separated into three regions, characterized by the total energy $H = mv^2/2 - e\phi$

I) trapped electrons: $-e\phi_m < H < -e \max(\phi_l, \phi_r)$
II) reflected electrons: $-e \max(\phi_l, \phi_r) < H < -e \min(\phi_l, \phi_r)$
III) free electrons: $-e \min(\phi_l, \phi_r) < H$

The beam electrons enter the diode after acceleration by the voltage drop $\phi_l - \phi_b$, and are free. In order to describe the cutoff in the background velocity distributions a cutoff potential $\phi_c$ can be introduced. The distribution functions $f_e$ and $f_b$ of background and beam electrons can then be written as

$$f_e(v, \phi) = \frac{n_{e0}}{\sqrt{\pi}v_e} \exp \left( -\frac{v^2}{v_e^2} \right) \exp \left( \frac{e(\phi - \phi_m)}{k_BT_e} \right) \Theta \left( e\phi - mv^2/2 - e\phi_c \right), \quad (5.2)$$
$$f_b(v, \phi) = \frac{n_{b0}}{\sqrt{\pi}v_b} \exp \left( -\frac{v^2}{v_b^2} \right) \exp \left( \frac{e(\phi - \phi_b)}{k_BT_b} \right) \Theta \left( e\phi_b - e\phi + mv^2/2 \right). \quad (5.3)$$

Here $v_e = \sqrt{2k_BT_e/m}$ is the thermal velocity of the background component and $n_{e0}$ its density at the reference potential $\phi_m$, if the cutoff in velocity space was disregarded. $v_b, n_{b0}$ and $\phi_b$ are the corresponding quantities for the beam. The Heaviside functions $\Theta$ cut off the background distribution for total energies $H > -e\phi_c$ and the beam distribution for energies less than $-e\phi_b$. Integration with respect to $v$ gives the densities as a function of the potential $\phi$

$$n_c(\phi) = n_{e0} \exp \left( \frac{e(\phi - \phi_m)}{k_BT_e} \right) \erf \left( \frac{e(\phi - \phi_c)}{k_BT_e} \right), \quad (5.4)$$
$$n_b(\phi) = n_{b0} \exp \left( \frac{e(\phi - \phi_b)}{k_BT_b} \right) \erfc \left( \frac{e(\phi - \phi_b)}{k_BT_b} \right). \quad (5.5)$$

The $\Theta$-function in the expression for $n_c(\phi)$ ensures $n_e(x) = 0$ in the sheath re-
5.1. Numerical Solution of the Steady State

Figure 5.1. Numerical solution of the DC problem: a) Phase space trajectories separating trapped [I], singly reflected [II] and free electrons [III]. The upper trajectory indicates the slowest beam electrons. b) densities and relative beam density, c) DC potential $\phi_0$ and electric field $E_0$, d) density gradient length $n_e/(dn_e/dx)$.

regions where $\phi(x) < \phi_c$. Quasi-neutrality at the maximum plasma potential $\phi_m$ determines $n_{e0}$ and $n_{b0}$ to be:

$$n_{e0} = (1 - \alpha) N_{m} / \text{erf} \left( \frac{\phi_m - \phi_c}{k_B T_e} \right), \quad \text{and} \quad n_{b0} = \alpha N_{m} / \text{erf} \left( \frac{\phi_m - \phi_c}{k_B T_b} \right),$$  

(5.6)

where $\alpha = n_b(\phi_m)/[n_e(\phi_m) + n_e(\phi_m)]$ is the relative beam density at the maximum plasma potential.

The cutoff potential $\phi_c$ is a free parameter, but should lie between $\phi_l$ and $\phi_r$. $\phi_c = \max(\phi_l, \phi_r)$ describes electrodes without particle emission. If $\phi_c < \max(\phi_l, \phi_r)$, then there is injection of thermal background electrons into the diode from the left electrode with a current density

$$j_l = -\frac{e n_e v_e}{2 \sqrt{\pi}} \exp \left( \frac{e (\phi_l - \phi_m)}{k_B T_e} \right) \left[ 1 - \exp \left( -\frac{e (\phi_l - \phi_c)}{k_B T_e} \Theta(\phi_l - \phi_c) \right) \right].$$  

(5.7)
Changing \( \phi_i \) to \( \phi_r \) gives the analogous expression for \( j_r \). The current density of the beam \( j_b = -en_0 \phi_b/(2 \sqrt{\pi}) \) is conserved, as expected.

The boundary value problem described by equations (5.1), (5.4) and (5.5) is solved with a finite element method (FEM). In the FEM-method the differential equation is first transformed into a variational formulation by multiplying the differential equation first with a test function \( \eta(x) \) and then integrating it once with respect to \( x \). Approximating the true solution by typically piecewise linear or polynomial functions and choosing a set of basis functions as test functions, a set of difference equations is derived and then numerically solved. Here the Poisson’s equation is solved for piecewise linear functions on an equidistant mesh \( x_i = i \cdot h \), \( i \in \{0, \ldots, N\} \), \( x_N = l \). With the usual hat-function basis for piecewise linear functions

\[
\Lambda_i(x) = \begin{cases} 
(x - x_{i-1})/(x_i - x_{i-1}), & x \in [x_{i-1}, x_i], \\
-(x - x_{i+1})/(x_{i+1} - x_i), & x \in [x_i, x_{i+1}], \\
0, & \text{else},
\end{cases}
\]

the FEM solution is expressed as \( \phi^{\text{FEM}}(x) = \sum_{i=0}^{N} \phi_i \Lambda_i(x) \). Approximating the electron densities by

\[
n_e(x) = \sum_i n_e(\phi_i) \Lambda_i(x), \quad \text{and} \quad n_b(x) = \sum_i n_b(\phi_i) \Lambda_i(x),
\]

the FEM representation of (5.1) with “strongly imposed boundary conditions” gives

\[
0 = \varepsilon_i(\phi_k) = A_{ik} \phi_k + eB_{ik} [n_e(\phi_k) + n_b(\phi_k)] / \epsilon_0 - eB_{ik} n_i(x_k) / \epsilon_0 - \phi_i^{\text{BC}}. \quad (5.9)
\]

\( A_{ik} \) and \( B_{ik} \) are the \( (N + 1) \times (N + 1) \) matrices

\[
A_{ik} = \begin{cases} 
\frac{1}{h} (-\delta_{i,k+1} + 2 \delta_{i,k} - \delta_{i,k-1}), & i \in \{1, \ldots, N - 1\}, \\
\delta_{ii}, & i \in \{0, N\},
\end{cases}
\]

\[
B_{ik} = \begin{cases} 
\frac{h}{6} (\delta_{i,k+1} + 4 \delta_{i,k} + \delta_{i,k-1}), & i \in \{1, \ldots, N - 1\}, \\
0, & i \in \{0, N\},
\end{cases}
\]

The first and last line of eq. (5.9) together with \( \phi_i^{\text{BC}} = \delta_{0,0} \phi_0 + \delta_{N,N} \phi_N \) implement the boundary conditions. As the finite element problem is nonlinear, some iterative method has to be used to minimize the error \( \varepsilon_i \). As the derivative of \( \varepsilon_i \)

\[
\frac{\partial}{\partial \phi_k} \varepsilon_i = A_{ik} + \frac{e}{\epsilon_0} B_{ik} \frac{\partial}{\partial \phi_k} [n_e(\phi_i) + n_b(\phi_i)]
\]

is known analytically, the Newton-method is chosen. In each iteration step \( l \) the new potential \( \phi_i^{l+1} \) is found by solving the linear system of equations

\[
0 = \frac{\partial \varepsilon_i}{\partial \phi_k} (\phi_i^{l+1} - \phi_i^l) + \varepsilon_i^l \quad (5.12)
\]


5.2. Solution of the High Frequency Eigenmodes

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximum ion density</td>
<td>$N_{im}$</td>
</tr>
<tr>
<td>relative beam density</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>electron temperature</td>
<td>$T_e$</td>
</tr>
<tr>
<td>beam temperature</td>
<td>$T_b$</td>
</tr>
<tr>
<td>potential (left)</td>
<td>$\phi_l$</td>
</tr>
<tr>
<td>potential (middle)</td>
<td>$\phi_m$</td>
</tr>
<tr>
<td>potential (right)</td>
<td>$\phi_r$</td>
</tr>
<tr>
<td>beam potential</td>
<td>$\phi_b$</td>
</tr>
<tr>
<td>cutoff potential</td>
<td>$\phi_c$</td>
</tr>
</tbody>
</table>

Table 5.1. Plasma parameters used for the fluid model shown in Figure 5.1.

As a start guess a potential profile $\phi_0^i$ fulfilling the quasi-neutrality condition $n_e(x_i) = n_e(\phi_i) + n_b(\phi_i)$ is used. The total electron density is first calculated for a set of potential values $\phi_k^N$ covering the interval $[\min(\phi_l, \phi_c), \phi_m]$. For each $x_i$, the $k$ fulfilling $n_e(\phi_k^N) + n_b(\phi_k^N) < n_i(x_i) \leq n_e(\phi_{k+1}^N) + n_b(\phi_{k+1}^N)$ is determined and the start potentials $\phi_i^0$ set to a linear interpolation of $\phi_k^N$ and $\phi_{k+1}^N$. In the regions closer than seven Debye lengths to the electrodes, the potential is chosen to be linear between the electrode potential and the first quasi-neutral potential. Using this start value, after smoothing it in order to round off the corners at the artificial sheath edges, the Newton method reduces the total error $(\sum \varepsilon_i \varepsilon_i)^{1/2}$ to less than $10^{-6}$ after typically 10 steps.

The plasma parameters and potentials chosen here to model the plasma in the experiment in Paper 2 are summed up in Table 5.1. The DC solution for these values is shown in Figure 5.1. The mid potential $\phi_m$ equals twice $k_B T_e/e$ and the potential of the left electrode $\phi_l$ is chosen to be negative, so that the thermal electrons are reflected. The beam potential $\phi_b$ is chosen, so that $\phi_m - \phi_b$ is the experimental beam acceleration voltage. The cutoff potential $\phi_c$ is set so that some electrons are injected from the right electrode, but none from the left.

5.2 Solution of the High Frequency Eigenmodes

The behavior of the high frequency electric field in an inhomogeneous plasma diode with an outer circuit can be treated with the fluid model described in [26]. The model gives the high frequency impedance $Z(\omega) = \int_0^L E_t(x, \omega)dx/j_T$ as a function of the oscillation frequency $\omega$. Together with Kirchhoff’s voltage law including the outer circuit

$$0 = Z(\omega)j_T + Z_\omega(\omega)j_T$$

(5.13)

the model gives a finite number of stationary eigenmode $E_{1,m}$ with eigenfrequencies $\omega_m$ below the maximum plasma frequency. Here, $Z_\omega(\omega)$ is the impedance of the passive outer circuit. While the fluid model takes account for the density inhomogeneity, it leaves out Landau-damping effects and the energy transfer from the
beam to the wave. Nevertheless it facilitates the physical understanding of the high frequency behavior of inhomogeneous plasma diodes. As noticed in Paper 2 a good quantitative agreement of the fluid eigenfrequencies and dominating high frequency structures in PIC simulations can be established.

The fluid model is based on the first order Poisson, momentum and continuity equation for the background density

\[
\frac{\partial}{\partial x} E_1 = \frac{e}{\varepsilon_0} n_1, \\
\frac{m n_0}{\partial t} v_1 = -e n_0 E_1 - e n_1 E_0 - \gamma k_0 \frac{\partial}{\partial x} n_1, \\
\frac{\partial}{\partial t} n_1 + \frac{\partial}{\partial x} (n_0 v_1) = 0,
\]  

(5.14)

where index “1” indicates the perturbed quantities. The background plasma is at rest, so that the convective terms proportional to \( n_1 \) in the momentum equation and the term \( n_1 v_0 \) in the continuity equation disappear. With harmonic time variation \( \frac{\partial}{\partial t} \to i \omega \) and the total current \( j_T = i \omega_0 E_1 - e n_0 v_1 \) equations (5.14) can be transformed into a single second order, ordinary differential equation with inhomogeneous coefficient functions

\[
0 = \gamma \frac{k_0}{m} \frac{\partial^2}{\partial x^2} E_1 + \frac{e E_0(x)}{m} \frac{\partial}{\partial x} E_1 + (\omega^2 - \omega_p^2(x)) E_1 + \frac{i \omega}{\varepsilon_0} j_T,
\]

BC: \( E_1(0) = E_1(l) = \frac{1}{i \omega \varepsilon_0} j_T. \)  

(5.15)

The boundary conditions BC are fixed by the total current, which is conserved. The inhomogeneity enters the differential equation through the zeroth order field \( E_0(x) \) and the local plasma frequency \( \omega_p(x) = \sqrt{e^2 n_0(x)/\varepsilon_0 m} \). The total current is a pure displacement current at the boundaries, when the zeroth order equations are solved kinetically with nonemitting electrodes, because then \( n_0(0) = n_0(l) = 0 \).

The boundary value problem for \( E_1(x) \) is solved with an FEM method for piecewise linear functions. Expressing the FEM-solution \( E_{\text{FEM}}(x) = \sum_k E_k \Lambda_k(x) \) this the usual hat-function basis \( \{ \Lambda_k \} \), the FEM formulation of eq. (5.15) leads to the linear system

\[
0 = (A_{ik} + B_{ik} + C_{ik}) E_k + \frac{1}{i \omega \varepsilon_0} j_T - E_{BC}^k,
\]

BC: \( E_{BC} = (\delta_{i,0} + \delta_{i,N}) \frac{1}{i \omega \varepsilon_0} j_T. \)  

(5.16)

The matrices, which are again derived by multiplying eq. (5.15) with a basis function \( \Lambda_i(x) \) and integrating over space, are

\[
A_{ik} = \begin{cases} 
\frac{1}{h} (\delta_{i,k+1} - 2 \delta_{i,k} + \delta_{i,k-1}), & i \in \{1, \ldots, N - 1\}, \\
\delta_{il}, & i \in \{0, N\}, 
\end{cases}
\]
Figure 5.2. Solution of the high frequency fluid model. a) Reactance $X(f) = \text{Imag}(1/Z(f))$. $X(f) = 0 \Omega$ defines the eigenmodes of the diode. The 16 lowest are shown. b) - d) Electric field profiles $E_1(x)$ of four eigenmodes with a frequency $f_m$ below the maximum plasma frequency $f_{pe,max}$ and e), f) two successive eigenmodes with frequencies above $f_{pe,max}$. Vertical lines mark the resonance point $f_m = f_{pe}(x)$. For modes with $f_m < f_{pe,max}$ the field concentrates in the low density regions near the electrodes.
\[ B_{kk} = \begin{cases} 
\frac{e}{\gamma \kappa_{\text{B}} T} \left( -\delta_{i,k+1} [E_{0i-1} + 2E_{0i}] + \delta_{ik} [E_{0i-1} - E_{0i+1}] \right), & i \in \{1, \ldots, N - 1\}, \\
0, & i \in \{0, N\},
\end{cases} \]

\[ C_{kk} = \begin{cases} 
\frac{m}{\gamma \kappa_{\text{B}} T} \frac{\hbar}{12} \left( \delta_{i,k+1} \left[ 2\omega^2 - \omega_{p_{i-1}}^2 - \omega_{p_i}^2 \right] + \delta_{ik} \left[ 8\omega^2 - \omega_{p_{i-1}}^2 - 6\omega_{p_i}^2 \right] + \delta_{i,k-1} \left[ 2\omega^2 - \omega_{p_{i-1}}^2 - \omega_{p_{i+1}}^2 \right] \right), & i \in \{1, \ldots, N - 1\}, \\
0, & i \in \{0, N\},
\end{cases} \]

First and last line implement again the boundary conditions. For simplicity eq. (5.16) is solved on the same equidistant as the zeroth order problem. The grid size is chosen to \( h \leq 0.2 \lambda_D \), where \( \lambda_D = \sqrt{\kappa_{\text{B}} T e \epsilon_0 / (e^2 N_{\text{im}}) \) is the minimum Debye length in the diode. A description of the numerics under the use of a non-equidistant mesh and error estimates for \( E_1 \) can be found in [26].

The eigenmode structure resulting from this model, when applied to the experiment in manuscript 2, are summed up in Figure 5.2. Figure 5.2(a) shows the diode reactance \( X(f) = \text{Im}(Z(f)) \) for \( j_\text{F} = 1 \text{ Am}^{-2} \) and \( A = 1 \text{ m}^2 \). For frequencies up to 140 MHz, the short circuited diode shows 16 eigenmodes. Nine have of them have frequencies below the maximum plasma frequency, which is indicated by the dashed line. Panels (b) to (g) show \( \text{Im}(E_1(m(x))) \) for some eigenfrequencies. For the eigenmodes with \( f_m < f_{\text{pe, max}} \), the electric field is concentrated in the regions \( f_{\text{pe}}(x) < f_m \). Towards the resonance points \( (f_m(x) = f_m) \) the local wavelength increases. In the regions where \( f_{\text{pe}}(x) > f_m \) the wave field approaches the particular solution

\[ E_{1,p}(x) = \frac{i \omega}{\epsilon_0 (\omega^2 - \omega_{p_0}^2)} j_\text{F}. \]

The particular solution \( E_{1,p}(x) \) and the resonance points are included in the Figures. The number of local maxima and minima of successive eigenmodes differs by one (see (d) and (e)). For eigenmodes with \( f_m > f_{\text{pe, max}} \) this corresponds to an additional half wavelength within the diode (see (f) and (g)).
Chapter 6

Particle-in-cell Simulations

A powerful tool for investigations of periodic and bounded plasma systems are particle-in-cell (PIC) simulations. The basic idea of particle-in-cell simulations is to directly integrate Newton’s equations of motion for $10^4$–$10^6$ superparticles, moving in the electromagnetic field, self-consistently determined from the Maxwell equations. Each superparticle itself represents $10^3$–$10^5$ real particles, depending on the application. As particle-in-cell simulations use only basic physical principles, they capture all possible nonlinear aspects of the plasma behavior. The use of particles to simulate physical systems is common in for example astrophysics and solid state physics [49, 50].

6.1 Description of the Simulation Code xpdp1

The code used for the simulations presented here and in Paper 2 is the public particle-in-cell code “xpdp1” which has been developed by the “Plasma Physics and Simulation Group” in Berkeley [51]. The code simulates the electrostatic behavior of a one dimensional, plane plasma diode of length $l$ with an outer LRC-circuit. The code includes a Monte-Carlo model to treat the effects of impact ionization, secondary emission, elastic and charge exchange collisions. The basic principles of particle-in-cell simulations, excluding the Monte-Carlo part, will be describes below.

The code of xpdp1 runs in an infinite loop consisting of

1. calculating the forces acting on the superparticles from the electrostatic potential and advancing the particle positions
2. calculating the electrostatic potential from the charge density and the outer circuit current

Each loop the simulation time $t$ is increased by a discrete step $\Delta t$. 43
Chapter 6. Particle-in-cell Simulations

Figure 6.1. Illustrations for xdp1 a) Simulated system (after [52]). The Poisson equation is solved on the equidistant grid $x_i^1$. The right boundary condition $\phi_{N_G} = 0$ is chosen to define a reference potential. A general outer circuit interacts with the plasma through the left boundary condition $E_0 = \sigma_L / \epsilon_0$. In the case of a short cut outer circuit $L = R = C = 0$, the boundary condition at $x = 0$ reduces to $\phi_0 = \phi_{N_G} - V_0$. b) Leap-frog algorithm for integration of the continuous particle positions $x_k$ and velocities $v_k$ (after [49]).

The forces $F_{k}^{n}$ acting on particles $k$ at time $t^n = n \Delta t$ are calculated by solving Poisson’s equation on the equidistant grid $x_i^1 = (i-1)\Delta x$, $i \in \{1, \ldots, N_G\}$, with a finite difference method. The charge density $\rho_i^n$, which needs to be known at the grid points, is defined with the help of a spatial weighting function $W$

$$\rho_i^n = \frac{1}{\Delta x} \sum_{k=1}^{N_s} q_k W \left( \frac{x_k^n - x_i^1}{\Delta x} \right) + \rho_0. \quad (6.1)$$

where $q_k$ are the particle charges and $\rho_0$ is a neutralizing background charge. xdp1 uses

$$W(x) = \begin{cases} 1 - |x|, & \text{for } |x| < 1, \\ 0, & \text{else}, \end{cases} \quad (6.2)$$

which distributes $q_k$ to the two closest grid points, so that the total charge is conserved. The potential is then calculated by solving the system

$$\frac{\phi_i^n - 2\phi_{i+1}^n + \phi_{i+2}^n}{(\Delta x)^2} = -\frac{\rho_i^n}{\epsilon_0}. \quad (6.3)$$

The boundary conditions are the arbitrary choice of the reference potential $\phi_{N_G}^n = 0$ at the right electrode and the electric field at the left boundary $E_0 = (\phi_1^n - \phi_2^n)/\Delta x = \sigma_L / \epsilon_0$. Here, $\sigma_L$ is the surface charge density on the left electrode. From the solution $\phi_i^n$ the electric field at the grid points is calculated with the symmetric finite difference

$$E_i^n = \frac{\phi_{i-1}^n - \phi_{i+1}^n}{2\Delta x}. \quad (6.4)$$
6.1. Description of the Simulation Code xpdpl

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>grid size</td>
<td>$\Delta x$ $0.4 \times 10^{-3}$ m</td>
</tr>
<tr>
<td>time step</td>
<td>$\Delta t$ $2.5 \times 10^{-11}$ s</td>
</tr>
<tr>
<td>simulation time</td>
<td>$1.5 \mu$s</td>
</tr>
<tr>
<td>superparticles</td>
<td>$N_S$ $\approx 110 000$</td>
</tr>
<tr>
<td>saving time step</td>
<td>$\Delta t_s$ $1.25$ ns</td>
</tr>
<tr>
<td>Nyquist frequency</td>
<td>$f_{Ny}$ $400$ MHz</td>
</tr>
</tbody>
</table>

Table 6.1. Simulation parameters.

and then extrapolated to the particle positions by

$$F^m_k = q_k \sum_{i=0}^{N_k} W \left( \frac{x^n_k - x^G_i}{\Delta x} \right) F^n_k$$

(6.5)

The particle positions and velocities are then advanced according to Newton’s equations of motion

$$m_k \frac{dv_k}{dt} = q_k (E + v_k \times B) = F_k$$

(6.6)

$$\frac{dx_k}{dt} = v_k.$$ Here, $m_k$ is the mass of particle $k$. To solve these equations for discretized time the “leap-frog” algorithm is used (see Fig. 6.1(b)). The equations have the form

$$m_k \frac{v_{k+1}^{n+1/2} - v_{k-1}^{n-1/2}}{\Delta t} = F^n_k,$$

(6.7)

$$\frac{x_{k+1}^{n+1} - x_k^n}{\Delta t} = v_{k+1}^{n+1/2}.$$ (6.8)

The leap-frog algorithm discretizes the particle positions at times $t^n = n \Delta t$, but the velocities at times $(n - 1/2) \Delta t$. With known forces $F^n_k$ and velocity $v_{k}^{n-1/2} = v_k((n-1/2)\Delta t)$, first the new particle velocities $v_{k}^{n+1/2}$ and then their new positions $x_{k}^{n+1}$ are calculated. The symmetry of the force balance around time $t^n$ ensures numerical stability of the iteration for $\omega \Delta t < 2$, where $\omega$ is the highest oscillation frequency of the plasma. In the simulations presented in Paper 2, $\Delta t$ is chosen so that the fastest particle moved approximately $\Delta x/2$ per time step. For the given plasma densities this results in $\omega_p \Delta t \approx 0.01$. At the start of the simulation the code calculates the velocities $v_{k}^{-1/2}$, by moving the particle half a time step backwards, before advancing them a full time step forward.

In the case of a general LRC-circuit, the plasma interacts with the outer circuit through the surface charges $\sigma_L$. $\sigma_L$ obeys the time discretized charge conservation

$$\sigma_L^n = \sigma_L^{n-1} + \frac{Q_{con}^n + Q^n - Q^{n-1}}{A}.$$ (6.9)
Here, $Q_{\text{conv}}^n$ is the charge transported to the surface by particle convection, and $Q$ is the charge of the circuit capacitor. Each time step $Q_{\text{conv}}^n$ is determined by the charge in the first grid cell, weighted to the left electrode. Before solving the Poisson equation, the contribution of the outer circuit to the surface charge $(Q^n - Q^{n-1})/A$ is determined from a finite difference representation of Kirchhoff’s voltage law

$$L \ddot{Q} + R \dot{Q} + \frac{Q}{C} = \phi_{N_G} - \phi_1 - V_0(t),$$

which typically includes the capacitor charge for a few steps backwards in time. A detailed summary of the treatment of the outer circuit can be found in reference [52].

In the case of a short circuit the boundary condition at $x = 0$ is simply

$$\phi_1 = \phi_{N_G} - V_0(t).$$

### 6.2 Application to the Experimental Parameters of Paper 2

Particle-in-cell simulations were used to interpret the high frequency measurements in Paper 2. The system length in the $l = 0.1\text{m}$ case only five times the wavelength of the observed wave. In the simulations presented in Paper 2 the beam is accelerated in a cathode sheath within the plasma volume. Here, simulations with a pre-accelerated beam are presented. This is a better representation of the actual experimental setup. The plasma parameters for the simulation are those in Table 5.1 and the simulation parameters are the same as in Paper 2 (Table 6.1). During the simulation, which runs with approximately 110 000 superparticles for $1.5\mu\text{s}$, the electric field and potential are saved each $1.25\mu\text{s}$. This leads to 1201 samples of the electric field in each grid point. The Nyquist frequency $f_{Ny}$ equals $400\text{MHz}$ and corresponds to approximately $3.1 f_{pe,max}$. The observations and conclusions are in principle the same as in Paper 2.

Figure 6.2 shows the phase space at the end of the simulation ($t = 1.5\mu\text{s}$), a gray scale plot of the high frequency electric field $E_{\text{HF}} = E(x,t) - \langle E(x,t) \rangle$, in the interval from $t = 1.4\mu\text{s}$ to $t = 1.5\mu\text{s}$, and the total electrostatic energy $W_{\text{ES}}(t) = \int_0^L 0.5\epsilon_0 E^2_t(x,t)dx$. The initial phase space of the simulation is loaded with a distribution function, resembling the DC solution, described in Chapter 5. Even though the initial distribution fulfills Vlasov and Poisson equation, transient oscillations still appear. The electrostatic energy $W_{\text{ES}}$ integrated over the interval $[0, 0.02]\text{m}$ shows that the simulation reached a stationary state after approximately $0.25\mu\text{s}$. For $t > 0.25\mu\text{s}$, the hf electric field $E_{\text{HF}}(x,t)$ can be divided into three regions showing different behavior (Fig. 6.2(b)) In the region $0 < x < 0.03\text{m}$ standing waves can be seen, while the region $0.03\text{m} < x < 0.08\text{m}$ is dominated by propagating waves. In the region $x > 0.08\text{m}$ the beam has merged with the background component and the wave amplitude is strongly damped.

Figure 6.3 shows a contour and grey scale plot of the power spectral density $P(f,x)$ [53]. For the estimation of $P(f,x)$ the electric field in the time interval $0.25\mu\text{s} \leq t \leq 1.50\mu\text{s}$ is used. The simulation yields for this interval 1000
samples of $E(x,t)$ in each grid point, which leads to a frequency resolution of $\Delta f = 1/\Delta t_s/1000 = 0.80$ MHz. In the regions close to the left electrode, through which the pre-accelerated electron beam enters the diode, standing wave fields with one to four local field maxima can be found. Their center frequencies $f$ are 99.2, 108.0, 116.0 and 120.8 MHz for 1, 2, 3 and 4 local maxima, respectively. The power in the middle region is also concentrated to these frequencies. The presence of standing waves in the regions to the left of the first resonance point and the dominance of propagating waves in the region $f < f_{pe}(x)$ is again shown in Figure 6.4. The left Panels show the spectral components of $P(f,x)$ at the mode frequencies. The resonance points are indicated by vertical lines. The right charts show the phase of the electric field, relative to the outer circuit current $I$. As expected for standing waves the phase is roughly constant in between field nodes and differs by $\pi$ between successive maxima. The linear change of the phase with position in the regions $f < f_{pe}(x)$, indicates again propagating waves.

Comparing the field structure left of the first resonance point with the eigenmodes of the fluid model (see Figure 5.2) it is obvious that, the standing waves correspond to the fluid eigenmodes of the background electrons expected for this plasma density profile. even though the frequencies are not precisely the same, they correspond to each other. The fact that the electric field and the outer circuit current $I$ are correlated to each other is shown in Figure 6.5. While Panels (a) to (c) show the power spectrum of the electric field at $x = 0.02$ m and $0.08$ m and the power spectrum of $I$, Panels (d) to (f) show the cross power spectrum $C(f)$ of the two fields and the current [53]. The power spectrum of $I$ in (c) shows distinct peaks at 17.2, 99.2, 116.4 and 120.3 MHz. This proves that the waves, observed in the diode, have a nonzero total current. The cross power spectra in Panels (e) and (f) show that the peaks in $I$ do not coincidentally appear at the same frequencies as the dominating electric field components. The cross power spectrum between $E_1$ at $x = 0.01$ and 0.08 m shows that there is some physical process, connecting the phases of standing and propagating waves. It is natural to assume that this process is modulation of the beam, as it passes the region close to the left electrode.
Figure 6.2. Simulation results: (a) phase space at the end of the simulation $t = 1.5 \mu s$. Background [·-·] and beam electrons [·×·]. (b) Greyscale plot of the high frequency electric field $E_{HF}(x, t) = E(x, t) - \langle E(x, t) \rangle$. In the region close to the left electrode field nodes, indicating standing waves, can be seen, while the center region is clearly dominated by propagating waves. (c) Electrostatic waves energy $W_{ES}(t)$ integrated over the region close to the left electrode $0 \text{ m} \leq x \leq 0.02 \text{ m}$ [lower curve] and the total length [upper curve]. The simulation reaches a stationary state at $t \approx 0.25 \mu s$. 
Figure 6.3. Power spectral density $P(f, x)$ of the stationary electric field $E_4$. The power concentrates in narrow bands. Left of the first resonance point $f = f_{pe}(x)$ (two of them are indicated by vertical lines), local maxima of the field strength indicate the presence of standing waves. The full line is the plasma frequency of the background component at the start of the simulation.
Figure 6.4. Profiles of the Fourier components of the power of $E$ at the dominating frequency bands, scaled to field strength (left) and their phase, relative to the circuit current $I$ (right). The amplitudes of the frequency bands above 99 MHz show with increasing frequency one to four local maxima in the low density region in close agreement with the fluid model.
Figure 6.5. Power spectra of $E$ in the standing wave region (a), in the center region (b) and the circuit current $I$ (c). Cross power spectrum of the electric field (d) and of the circuit current and the fields (e) and (f).
Chapter 7

High Frequency Oscillations during the Transient Current Pulse

During the transient current pulse in the initial phase of the double layer formation investigated in Paper 4, high frequency fluctuations of the electric field are known to appear. They have already been observed by M. Bohm and S. Torvén in 1987 [54]. The spatial and temporal extension of some frequency components was investigated, but no attempt was made to measure a dispersion relation. The measurements presented below were intended to test whether it would be possible to determine an experimental dispersion relation.

7.1 Experimental Setup and Results

The triple plasma device at the Alfvén Laboratory is described in detail in Paper 4. Figure 7.1 shows a typical time series measured with a high frequency probe at \( z = 0.1 \) m and average spectra along the symmetry axis of the plasma column \((r = 0 \) mm\). The neutral gas pressure is \( p = 0.4 \) mTorr, the source discharge currents \( I_1 = 1.0 \) A and \( I_2 = 2.0 \) A. The magnetic field gives a cyclotron frequency of 200 MHz. A voltage drop of \( U_a = 200 \) V is applied at \( t = 0 \) \( \mu \)s. The time series in Panel (a) shows that the fluctuations have decayed after approximately 2.5 \( \mu \)s to the noise level. This is the case at all positions on the low potential side of the forming double layer. The power spectral density of the probe signal shows that the fluctuation level increases for positions closer to the high potential aperture. While there is a clear upper limit at \( f \approx 160 \) MHz in the spectrum close to the low potential side, the other spectra extend to higher frequencies. As the spectra for \( z > 0.1 \) m did not show a limiting frequency, I restricted the measurement of the dispersion relation to \( z = 0.1 \) m.
Chapter 7. High Frequency Oscillations during the Transient Current Pulse

Figure 7.1. a) Typical time series measured with a high frequency probe at \( z = 0.1 \text{m} \). The probe signals in the interval \( 0.5 \mu \text{s} < t < 1 \mu \text{s} \) are used to measure the dispersion relation of the fluctuations. b) Average spectra over 100 time series as in a) on the symmetry axis \( (r = 0 \text{mm}) \) for different axial positions \( z \). Near the low potential aperture the power spectrum is limited at approximately 100 MHz. The fluctuation level increases and the limit disappears with increasing axial position \( z \).
7.1. Experimental Setup and Results

Figure 7.2. Distributions $C$ according to eq. (7.5) measured for $I_B = 8$ A at $z = 0.1$ m and probe separation $\Delta z = 13, 23$, and 33 mm for Panels a), b) and c), respectively. Crosses are the mean phase difference $\Theta$. The error bars indicate the variance $\sqrt{\sigma^2}$ of the distribution in the direction of the phase difference.
Figure 7.3. Absolute part of the transfer functions $A_1(f)$ and $A_2(f)$ of the reference probe, Panel a) and the movable probe b), respectively. c) Spectral density at $z = 0.1$ m for different magnetic field strengths.
7.1. Experimental Setup and Results

Figure 7.4. Dispersion relations $k(f)$ for $I_B = 8$, 9, and 10 A. $\Delta k$ is the error of $k$ following from the width $\sqrt{\sigma^2}$ of the power spectra. The smooth line in the region $k < 0$ are the dispersion relations of the azimuthally symmetric waves of a partially filled, magnetized waveguide. The lowest order plasma branch ('□') fits well the measured dispersion relation in their lower part.
The high frequency fluctuations are measured with two hf probes as described in Chapter 2. Their signals are analyzed with a statistical method described by Beall, Kim and Powers [55]. The statistical method is based on the idea to use the phase difference \( \Theta \) of the signals of two probes to build up a histogram of the power distribution as a function of frequency \( f \) and phase difference \( \vartheta \). For each frequency a mean phase difference \( \bar{\Theta}(f) \) is then determined and can be related to the wave vector \( k \), if the transfer functions of the probes are known.

The probes 1 and 2 at the positions \( z \) and \( z + \Delta z \) give the time series \( y_1(t) \) and \( y_2(t) \) with the Fourier transforms
\[
Y_1(\omega) = \int_0^T y_1(t) \exp(i\omega t) dt \\
Y_2(\omega) = \int_0^T y_2(t) \exp(i\omega t) dt,
\]
the sample spectral densities
\[
P_1(\omega) = Y_1^*(\omega)Y_1(\omega), \\
P_2(\omega) = Y_2^*(\omega)Y_2(\omega),
\]
and the sample cross spectral density
\[
H(\omega) = Y_1^*(\omega)Y_2(\omega) \\
= |H(\omega)| \exp(i\Theta(\omega)).
\]
Here, $T$ is the length of the time series and $\Theta(\omega)$ the sample phase difference. $\Theta(\omega)$ is defined to lie in the interval $[-\pi, \pi]$. The spectrum $S(\omega, \vartheta)$ is now built up by summation of the average power of $M$ samples $j$ according to

\[
S(\omega, \vartheta) = \sum_{j=1}^{M} \sum_{l=0}^{N-1} \left( \frac{1}{2} (P_{1}^{(j)}(\omega) + P_{2}^{(j)}(\omega)) \right) W\left( \frac{N}{2\pi} (\Theta^{(j)} + \pi) - l \right). \tag{7.1}
\]

The summation over $l$ is a summation over the phase difference $\vartheta = 2\pi t / N - \pi$. The weight function

\[
W(x) = \begin{cases} 
1, & \text{for } 0 \leq x < 1 \\
0, & \text{else}
\end{cases}
\tag{7.2}
\]

is used only to describe that the average power is added to the “box” corresponding to the phase difference interval $\vartheta \in 2\pi[l, l + 1]/N - \pi$. Its width $2\pi/N$ is freely choseable. If the probes have the transfer functions $A_{1}(\omega) = \alpha_{1}(\omega) \exp(i\vartheta_{1}(\omega))$ and $A_{2}(\omega) = \alpha_{2}(\omega) \exp(i\vartheta_{2}(\omega))$, a wave proportional to $\exp(i(kz - \omega t))$ will give rise to a phase difference

\[
\Theta(\omega) = k(\omega) \Delta z - \vartheta_{1}(\omega) + \vartheta_{2}(\omega). \tag{7.3}
\]

An attempt to calibrate the probes showed that the transfer functions vary with the cable positioning in the experimental setup, so that $\vartheta_{1}(\omega)$ and $\vartheta_{2}(\omega)$ could not be determined with good reproducibility. In order to eliminate the need of an accurate knowledge of $\vartheta_{1}$ and $\vartheta_{2}$, the statistical method is used only to determine a mean phase difference $\bar{\Theta}(\omega)$ as a function of the probe separation $\Delta z$. The measured wave vector $k$ becomes then

\[
k(\omega) = \Delta \bar{\Theta}(\omega) / \Delta(\Delta z). \tag{7.4}
\]

The statistical method described in [55] assumes that the process causing the investigated fluctuations is stationary. This implies that the true power frequency is independent of time. In the experimental situation this is not the case. The spread of the power over phase and frequency is a mixture of nonlinear effects and simple averaging over variations of the plasma parameters, which might change from sample to sample.

Dispersion relations have been measured at $z = 0.1 \text{ m}$ for an applied voltage $U_{a}$ of 100 V and magnetic coil currents $I_{B} = 5$ to 10 A corresponding to field strengths from $B = 4.3$ to 14.3 mT. The spectra $S$ for 6 probe separations ($\Delta z = 3$–53 mm) have been determined from 200 samples. Only the interval between 0.5 and 1.0 $\mu$s is used in the analysis. The sampling frequency of the time series is 1 GHz. For a minimum probe separation of 1 cm, only wave vectors in the interval $[\pm 314, 314]$ m$^{-1}$ can be measured.

Figure 7.2 shows three examples of the measured, normalized spectrum

\[
C(\omega, \vartheta) = S(\omega, \vartheta) / \int_{-\pi}^{\pi} S(\omega, \vartheta) d\vartheta \tag{7.5}
\]
They are measured for $I_B = 8 \, \text{A}$ and probe separations 13, 23 and 33 mm for Panels (a) to (c), respectively. The mean phase difference $\bar{\Theta}(\omega)$ is determined so that the variance

$$\sigma^2(\omega) \equiv \int_{\bar{\Theta}(\omega) - \pi}^{\bar{\Theta}(\omega) + \pi} \left( \Theta(\omega) - \bar{\Theta} \right)^2 C(f, \vartheta) \, d\vartheta$$

is minimal. $\bar{\Theta}$ is plotted as crosses in Figure 7.2. For some frequencies the root of the variance is indicated as an error bar. During the integration in eq. (7.6), $\Theta$ is folded back into the interval $[-\pi, \pi]$. As the noise level is not subtracted when calculating $\sigma^2$, it assumes a maximal value $\pi^2/3$ if $C(f, \vartheta)$ is equally distributed over the phase difference, e.g., if the probes detect only noise.

Before going to the dispersion relations, we consider the power spectra in Figure 7.3. Panel (c) shows the power spectra as a function of the magnetic field. The power spectra are obtained by integrating the $S$ over $\vartheta$. One can conclude that the fluctuation level increases with increasing magnetic field. The increasing upper limit in the spectra reflects the increase of the plasma density with increasing magnetic field strength. Unfortunately it is not certain that the maxima in the spectra correspond to maxima in the wave field, because all lie close to resonances in the transfer functions of the probes. They are displayed in Panels (a) and (b). Dividing the spectra with $|A_1(\omega)|$ produces artificial maxima at the minima of $|A_1(\omega)|$ so that the uncertainty is not removed.

Figure 7.4 shows the statistical dispersion relations $k(f)$ obtained from the $\bar{\Theta}(\omega)$ measurements for $I_B = 8$ to $I_B = 10 \, \text{A}$. $k(\omega)$ is taken from a linear least square fit of the variation of $\bar{\Theta}(\omega)$ with $\Delta \omega$. At the right hand side of the Panels an error estimate

$$\Delta k = 2 \sqrt{\sigma^2} / d,$$

is displayed. Here, $d = 0.05 \, \text{m}$ is the maximal probe separation. The error $\Delta k$ is also plotted as a gray shaded region around the wave vector $k$. $\Delta k$ has a maximum of 72.6 m$^{-1}$ when the power is equally distributed over all frequencies. For frequencies below the observed limit frequency shown in Fig. 7.4 $\Delta k$ lies below the maximum, indicating the presence of waves.

The measured dispersion relations are compared with the electrostatic wave-modes of a partially filled cylindrical plasma waveguide with the radial dimensions of the device [56]. The plasma density profile used for the calculation is compared with the measured radial density profile (‘×’) in Figure 7.5. The middle chamber of the triple plasma device is considered as a cylindrical waveguide with radius $a = 0.15 \, \text{m}$ which contains a homogeneous, cold plasma column with radius $b = 0.015 \, \text{m}$ and zeros drift velocity. The wavenodes of such a waveguide are given by the dispersion relation

$$0 = pbK_p \frac{J_m'(pb)}{J_m(pb)} + mK_x - \frac{K_m(ka)J_m'(kb) - J_m(ka)K_m'(kb)}{K_m(ka)J_m(kb) - J_m(ka)K_m(kb)}$$

(7.8)
\[ p^2 = -k^2 \frac{K_{||}}{K_{\perp}} \]  \hspace{1cm} (7.9)

\[ K_{||} = 1 - \frac{\omega_p^2}{\omega^2}, \quad K_{\perp} = 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2}, \quad K_\chi = \frac{\omega_0 \omega_p^2}{\omega^2 - \omega_0^2}. \]  \hspace{1cm} (7.10)

K_{||}, K_{\perp} and K_\chi are elements of the dielectric tensor, J_m is the Bessel function of the first kind, K_m and I_m are the modified Bessel functions of the first and the second kind. The prime denotes derivative with respect to the argument and m is the azimuthal mode number.

As plasma frequency the maximum frequency with \( \Delta k \) less then 60 m\(^{-1}\) is chosen. The externally applied magnetic field gives a cyclotron frequency of \( f_c = 40\) MHz \( \times I_B/\)A. The theoretical dispersion relations displayed in Figure 7.4 for negative k are the cyclotron wave (‘\( \sigma \)’), and the plasma waves with one (‘\( \Delta \)’), two (‘\( \Delta' \)’) and three (‘\( \sigma' \)’) zeros in the axial electric field \( E_z(r) \). All shown branches are azimuthally symmetric (\( m = 0 \)).

### 7.2 Discussion

The measured dispersion relations agree well with the plasma branch up to a “cut-off” frequency. Above this “cut-off” frequency, which increases from approximately 180 MHz for \( I_B = 8\) A to 220 MHz for \( I_B = 10\) A, the measured dispersion relation shows wave vectors between approximately zero and -70 m\(^{-1}\).

The good agreement between the measured dispersion relation below the “cut-off” frequency and the model dispersion relations indicates that the plasma column at \( z = 0.1\) m is at rest, in spite of the applied potential drop of 100 V. This is, however, in agreement with the fact that the position of the potential minimum is in this region. If the plasma electrons would propagate towards the high potential side with a velocity \( v_0 \) corresponding to 25 V, the Doppler shift \( kv_0 \) would result in a decrease of the oscillation frequency of the displayed modes. The Doppler shifted plasma modes (‘\( \sigma' \)’ with one zero in \( E_z(r) \) are plotted as dotted lines in Fig. 7.4(c).

The cause of the measured dispersion curves above the “cut-off” frequency is uncertain. In the cold plasma model used above there is no wave mode in this region. Electromagnetic modes can be excluded, because the lowest cutoff frequency of the E-modes is approximately 850 MHz. The fact that the theoretical wave vector of the electrostatic plasma wave pass the Nyquist wave vector 314 m\(^{-1}\) suggests that aliasing may be important in this frequency interval.

The fact that the observed waves propagate against the flow of the electrons excludes that they are excited locally by a beam-plasma like instability. It is more likely that the waves observed at \( z = 0.1\) m are the products of nonlinear fluctuations and parametric instability during the saturation of the beam-plasma and the Buneman instability, which may take place in the region closer to the high potential side. The maximal growth rate of the Buneman instability, when cold electron are
streaming with a drift velocity \( v_0 \) relative to the ions without a DC electric field, is \([57]\)

\[
\omega_1 = \frac{1}{2} \left( \frac{m_e}{2m_i} \right)^{1/3} \left[ 1 + \left( \frac{m_e}{2m_i} \right)^{1/3} \right] \omega_p
\]

occuring at the frequency

\[
\omega_r = \frac{\sqrt{3}}{2} \left( \frac{m_e}{2m_i} \right)^{1/3} \left[ 1 - \left( \frac{m_e}{2m_i} \right)^{1/3} \right] \omega_p.
\]

The wavevector is determined by the resonance condition \( kv_0 = \omega_p \). Using the limiting frequency 238 MHz in the dispersion for \( I_p = 8.0 \) as plasma frequency, the maximum growth rate of the Buneman instability in an the Argon plasma \((m_i = 40 \times 1836 \times m_e)\) becomes \( \omega_1 = 1.44 \times 10^{-7} \text{ rad s}^{-1} \). The oscillation frequency is \( f_r = 3.8 \text{ MHz} \). Ishihara, Hirose and Langdon found in their investigation of the nonlinear development of the Buneman instability that its saturation due to electron trapping occurs at \( t\omega_i \approx 7 \) \([58]\). This time corresponds to \( t \approx 0.48 \mu s \) and is accordingly of the same order as the time after which the current maximum is reached. Direct observation of the Buneman frequency is difficult in the experiment due to the rapid growth and saturation of this instability. Some evidence of the Buneman frequency can be discerned in the fluctuations measured by emissive probes, but the results are not reproducible.

The saturation of the Buneman instability is connected to a strong decrease of the electron drift velocity and an increase of the electron temperature \([58]\). Consequently, the Buneman instability may contribute to the decrease of the current between the sources as discussed in Paper 4. Ion density fluctuations produced by parametric decay may also contribute to the resistance. Here, electron waves produced by beam-plasma interaction may decay into ion waves and backward moving electron waves. The relative importance of radial ion losses and resistance due to waves is unknown so far. This question might be answered with help of two-dimensional particle-in-cell simulations.
Chapter 8

Summaries of the Papers

8.1 P1: Amplitude collapse of nonlinear double-layer oscillations

In Paper 1 the nonlinear low frequency oscillations ($\omega \ll \omega_{pi}$) arising from the negative differential resistance of the double layer within the triple plasma machine are investigated. When the double layer is coupled in series with a voltage source $U_{dc}$ and an LC-circuit, Kirchhoff’s voltage law for the voltage $U$ between the source anodes has the form of a harmonic oscillator with a Newtonian damping term $RCdU/dt$ and an inductive term $LdI/dt$. Here $I$ is the current passing the double layer and $R$ the resistance of the inductor. For low oscillation frequencies $f < 1/\tau_i$, where $\tau_i$ is the ion transit time from the high potential region to the current limiting potential minimum, the inductive term can be replaced by $L'(U)dU/dt$, where $I(U)$ is the static current voltage characteristic of the double layer. After a normalization the circuit equation finally gets the form of a generalized van der Pol equation

$$\ddot{V} + V = -\epsilon[g(V + U_{dc}) + D]\dot{V}.$$  (8.1)

Here $\epsilon$ is the nonlinear coupling term, $g$ the normalized differential conductance of the double layer and $D$ the damping coefficient of the outer circuit. The main difference to the classical van der Pol equation

$$\ddot{V} + V = -\epsilon[(V + U_{dc})^2 - 1]\dot{V}$$  (8.2)

is that here the nonlinear damping is asymmetric and goes to a constant, positive value and its second derivative is negative as $U_{dc}$ goes to $+\infty$. As the nonlinearity is small in the experiment, the finite oscillation amplitudes $a(U_{dc})$ could be calculated [39]. The oscillation $V(t)$ is assumed to be harmonic $V(t) = a(t)\cos(\omega t)$ and the slow variation of the amplitude $a$ is given by the average damping over one oscillation period (see eqs. (10) and (11) in P1). A development of $g(a\cos(\omega t) + U_{dc})$ around $U_{dc}$ shows that the amplitude $a = 0$ is unstable if and only if $g(U_{dc}) + D < 0$.  

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We investigated experimentally the oscillation amplitudes as a function of the dc voltage $U_{dc}$ for different capacitances $C$. With decreasing $C$, $D$ decreases so that the interval $[U_1, U_2]$ where $g(U_{dc}) + D < 0$ increases. Accordingly the interval with self-excited oscillation and their amplitudes increase with decreasing $C$. For $C < 4\mu F$ large jumps in the amplitude (amplitude collapse) occur in form of a hysteresis. They can be explained by the existence of a stationary, but unstable, oscillation amplitude separating two stable oscillation amplitudes.

Including the frequency dependence of the inductor's resistance $R$, we could show that the measured amplitude characteristics and the hysteresis agree well with the theory. If the inductance was absent in the outer circuit, the differential resistance would not appear in the circuit equation and consequently there would be no oscillations. The appearance of the hysteresis is a consequence of the special form of the current voltage characteristic.

8.2 P2: Localization of the High Frequency Field in a Beam-Plasma Diode due to Density Gradients

In Paper 2 the beam-plasma interaction in an inhomogeneous and unmagnetized plasma of finite length has been investigated. This investigation was motivated by the work of my colleagues on beam-plasma interaction in the inhomogeneous density region near a double layer and in the inhomogeneous density region in from of a hot, electron emitting cathode. In the double layer experiment a strongly localized, propagating wave was found. Its phase velocity and spatial growth rate agrees with the most unstable mode of the dispersion relations for homogeneous and infinite plasmas. However, the spatial damping rate exceeded the expected Landau damping rate. This was ascribed to be an effect of the density gradient. In the hot cathode experiments the electric field was found to be strongly localized to a small region in the density gradient. In the experiment as well as in the particle-in-cell simulations the wave field was found to form standing waves at the highest amplitudes, whereas propagating waves with lower amplitude existed in neighboring regions. This observation lead to the interpretation that the beam excites one or two of the electrostatic eigenmodes of the plasma [41].

It was accordingly of interest to see whether it is possible to excite single eigenmodes of an inhomogeneous system with an electron beam. The double plasma machine at the Kiel university is a device, where the length of the system, and by this the theoretical number of eigenmodes, can be controlled by moving the introduced extra anode.

The experiment showed that propagating space-charge-waves dominate the high frequency electric field in the center of the plasma. The high frequency field is measured with the techniques described in Section 2. A comparison with the kinetic dispersion relations for homogeneous plasmas showed that the measured growth and damping rates are in reasonable agreement with the most unstable modes. However, the experimental phase velocities were smaller than those predicted the-
The experiments are compared with particle-in-cell simulations. A frequency analysis of the simulated high frequency electric field shows a concentration of the electrostatic energy to a few dominant frequency bands. These frequencies are determined by Langmuir waves trapped between the beam injection grid and a resonance point, where \( f = f_{pe}(x) \). These waves excite slow-space-charge waves in the center region, where \( f < f_{pe}(x) \). We conclude that even moderate density gradients can have a significant influence on the high frequency behavior of beam-plasma experiments.

8.3 P3: Experimental Investigations and Simulations of Quasi-Steady Potential Drops in Plasmas

This Paper relates to theory and simulations by S. Torvén and M. Bohm on the initial phase of the formation of strong electric double layers in inhomogeneous plasmas [38,54,60]. In experiments in Q-machines, where a step voltage was applied to a homogeneous plasma column, the applied potential drop concentrated first in a cathode sheath and propagates then as a double layer into the plasma. However, preliminary experiments in a triple plasma device where the initial ion density had a sufficiently deep minimum, showed that the initial potential drop extends over the inhomogeneous region instead of being concentrated in a sheath [38].

It was found that the electron injection is limited by a potential minimum on the low potential side of the potential drop. After a few ion plasma periods the potential profiles steepen and a double layer starts to form. The extended potential drops were explained by a one dimensional model considering steady electron motion through a prescribed, stationary ion density profile under quasi-neutral conditions. A comparison between the quasi-neutral potential profiles in this model and potential profiles from one-dimensional particle-in-cell simulations, which solve the Poisson equation, showed good agreement.

Paper 3 reviews these results and presents a preliminary comparison between the model and experiments. Experimentally it was found that the initially extended potential drops are connected with a transient current pulse. Its maximum is shown to increase with the applied voltage \( U_a \) approximately as \( U_a^{1/2} \). This is the dependency predicted by the quasi-neutrality model. The axial potential profile measured at 1\( \mu \)s is shown to disagree with the potential profiles expected from steady electron motion. It is argued that electron heating caused by the Buneman instability and beam-plasma interaction, in the region where the electrons from the low potential source meet the electrons from the high potential source, may be responsible for this deviation.
8.4 P4: Current Limitation and Inertial Resistance of an Inhomogeneous Plasma Diode

Paper 4 is an experimental investigation of the current limitation during the formation of strong electric double layers, when the low and high potential side of the developing double layer are separated by an ion density minimum. The length scale of the ion density gradient is of the order of a few 100 Debye lengths. A detailed comparison between the quasi-neutral model and experiments is presented. The current traces show an initial current pulse of 2 to 4 μs duration. Its maximum $I_{max}$, which is reached after 0.4 to 0.6 μs shows good agreement with the model values, when experimental initial plasma parameters are used. Particle-in-cell simulations show that the rise time of the current is determined by the electron inertia. The variation of $I_{max}$ with the axial magnetic field, confining the electrons, and the density in the plasma sources could both be tracked back to a variation of the minimum ion density. The potential minimum on the symmetry axis of the plasma column is extracted from time and space resolved emissive probe measurements averaged over 50 shots. The variation of its level and position is as predicted by the model. For comparison with the experiment the theoretical model presented in [38,60] is extended to asymmetric ion density profiles and a beam consisting of primary electrons is included in the electron distribution function.

For high magnetic field strengths ($B = 14.3 \text{ mT}$) the measured current maximum is proportional to $U_0^{0.55}$, which is close to the dependency following from the model. We conclude that for high magnetic field strengths the transient current maximum is well described by the quasi-neutrality model. The current is limited by the potential minimum, which assumes a value, so that the electrons injected from the low potential side pass the ion density minimum under quasi-neutral conditions.

The rapid decrease of the current after its maximum is partly caused by the radial redistribution of the initial ion density, as it reacts to the transient radial electric field. This picture is supported by reconstruction of the ion distribution function by numerical integration of ion trajectories under the influence of the measured electric field. As the ions propagate away from the symmetry axis, the density at the boundary of the plasma column is expected to increase. Such an increase is observed in the electron saturation current of a small cylindrical Langmuir probe. However, the measured high frequency fluctuations may decrease the mean flow velocity of the electrons and contribute to the increased resistance during the current decrease. An answer to the relative importance of both effects might be found with help of two-dimensional particle-in-cell simulations.
Chapter 9

Conclusions

Particular plasma processes are excited when a high voltage drop exists in unmagnetized plasma or along the magnetic field in magnetized plasma.

Electrons accelerated in the potential drop give beam-plasma interaction and excite high frequency electric fields which may be concentrated to regions with a width of only one to two wavelengths in the laboratory. The usual interpretation of such a localizations in unmagnetized plasma in based on Langmuir collapse. The work in Paper 2 offers an alternative explanation which is supported by experimental evidence and particle-in-cell simulations. In the pre-sheath in front of the cathode, that injects the beam, wave eigenmodes are excited. The electric field, which concentrates to the region where \( f \geq f_{pe} \) (oscillation frequency, \( f_{pe} \) plasma frequency), has a maximum at the reflection point (\( f = f_{pe} \)). This field excites a beam-plasma instability in the region where \( f \leq f_{pe} \). Growth and damping of this instability give another field maximum. In typical experiments these two maxima are closely spaced due to the rapid growth of the beam instability, and may be observed as a single “spike”.

A fundamental problem not yet solved is the electrical resistance of a collision-free plasma for currents along the magnetic field. In Paper 3 and Paper 4 the time evolution of the resistance is studied experimentally when a sudden voltage drop is applied to an inhomogeneous plasma column with an axial magnetic field. Besides currents, time resolved densities and axial potential profiles are measured. It is shown that inertial resistance dominates during the first few ion plasma periods (\( \omega_i^{-1} \)) when the current increases and assumes a maximum. For applied potential drops \( U_a \gg k_B T_e/e \), the resistance at the current maximum is found to be approximately proportional to \( U_a^{1/2} \) in agreement with the theory for inertial resistance which is modified to apply to the experiment. Then the current drops due to the density decrease caused by the transport in the radial electric field which is associated with the applied potential drop along the magnetic field. During this phase, resistance due to waves may also contribute. The conditions for a Buneman type of instability are fulfilled, but clear evidence of the Buneman frequency is not
obtained, probably due to the rapid growth of and saturation of the instability. Instead a broad spectrum extending up to the plasma frequency is observed near the current controlling potential minimum. The measured dispersion relations show that the waves propagate in the direction opposite to the accelerated electrons. It is proposed that these waves are excited by parametric instabilities. After the decrease, the current tends to saturate to a value independent of the applied voltage drop, that is, the plasma resistance is approximately proportional to $U_a$. At this time the axial potential profile has steepened, and the current is limited by the same mechanism as in the steady double layer that finally forms.

The strong double layer in the triple plasma device is intrinsically unstable due to its negative differential resistance. Non-linear oscillations may be excited at resonance frequencies of the current circuit. An investigation of the oscillation amplitudes is presented in Paper 1. As a peculiar property of these oscillations, it was discovered that a small change on the DC voltage drop over the double layer can cause sharp jumps in the oscillation amplitude. The oscillation amplitudes and their jumps, which appear in the form of a hysteresis, can be explained by a simple theoretical model if the particular form of the negative differential resistance is taken into account.
References


References

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Paper 4


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