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A Least Squares Method for Identification of Feedback Cascade Systems

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Abstract: The problem of identification of systems in dynamic networks is considered. Although the prediction error method (PEM) can be applied to the overall system, the non-standard model structure requires solving a non-convex optimization problem. Alternative methods have been proposed, such as instrumental variables and indirect PEM. In this paper, we first consider acyclic cascade systems, and argue that these methods have different ranges of applicability. Then, for a network with feedback connection, we propose an approach to deal with the fact that indirect PEM yields a non-convex problem in that case. A numerical simulation may indicate that this approach is competitive with other existing methods.

Keywords: System identification, Least-squares algorithm, Feedforward networks, Feedback loops, Closed-loop identification

1. INTRODUCTION

Due to the rising complexity of systems encountered in engineering problems, identification of systems that are embedded in a dynamic network has become an increasingly relevant problem. Thus, several contributions have recently been provided in this area, e.g., Dankers et al. (2014), Everitt et al. (2014), Van den Hof et al. (2013), Dankers et al. (2013), Everitt et al. (2013), Van den Hof et al. (2012), Hågg et al. (2011), Wahlberg and Sandberg (2008), Wahlberg et al. (2008).

A common particular case of such networks is the identification of acyclic cascade structures, e.g., the system in Fig. 1. It contains one external input, \( u(t) \), and two outputs, \( y_1(t) \) and \( y_2(t) \), with measurement noises \( e_1(t) \) and \( e_2(t) \), respectively, which, for the purpose of this paper, are Gaussian, white, and uncorrelated to each other, with variances \( \lambda_1 \) and \( \lambda_2 \). A general discussion on identification and variance analysis of this type of cascade systems is taken in Wahlberg et al. (2008).

The goal of system identification is to estimate the transfer functions \( G_1(q) \) and \( G_2(q) \), where \( q \) is the forward-shift operator. Mathematically, the system in Fig. 1 can be described by

\[
\begin{align}
y_1(t) &= G_1(q)u(t) + e_1(t) \\
y_2(t) &= G_2(q)G_1(q)u(t) + e_2(t).
\end{align}
\]

First, notice that the transfer function \( G_1(q) \) can be estimated with (1a) from the signals \( u(t) \) and \( y_1(t) \), using standard system identification techniques. Likewise, the product \( G_2(q)G_1(q) \) can be estimated in a similar fashion from (1b), using \( u(t) \) and \( y_2(t) \) as data. However, the input to the transfer function \( G_2(q) \), indicated in Fig. 1 as \( u_2(t) \), is not known. Therefore, \( G_2(q) \) cannot be estimated directly using a similar approach. A possible strategy to obtain \( G_2(q) \) from the previously obtained estimates of \( G_1(q) \) and \( G_2(q) \) is to use the relation \( G_2(q) = G_2(q)G_1(q)^{-1}(q) \). However, that does not allow imposing a particular structure on \( G_2(q) \). Furthermore, if \( G_1(q) \) and \( G_2(q) \) are estimated in the previously presented way, information that could be useful for the estimation is neglected. For example, using also \( y_2(t) \) when estimating \( G_1(q) \) can improve the variance of the estimates (see Everitt et al. (2013)).

Another possibility, which solves the problem of imposing structure, is to estimate \( G_2(q) \) using \( y_2(t) \) and an estimate of \( u_2(t) \) as data. However, the presence of input noise makes this an errors-in-variables (EIV) problem (Söderström (2007)). When applied to this type of problem, standard system identification methods typically yield parameter estimates that are not consistent. Instrumental variable (IV) methods (Söderström and Stoica (2002)) can be used to solve this problem, since some choices of instruments provide consistent estimates (see, e.g., Söderström and Mahata (2002) and Thil et al. (2008)). A generalized IV approach for EIV identification in dynamic networks has been proposed in Dankers et al.
output, corrupted by noise \( \tilde{y} \) assumed to be known. We introduce the assumption that (1999)). The essential idea of PEM is to minimize a cost for which indirect PEM does not avoid non-convexity, we consider a feedback cascade structure in Section 5, for which indirect PEM does not avoid non-convexity, and propose an intermediate step using the method in Galrinho et al. (2014). A numerical simulation is presented in Section 6, followed by a discussion in Section 7.

2. ERRORS-IN-VARIABLES METHODS

Consider the SISO system \( G(q) \), and assume that data is generated according to
\[
\begin{align*}
\{ y_o(t) &= G(q)u(t) \\
y(t) &= y_o(t) + \tilde{y}(t),
\end{align*}
\]
where \( y_o(t) \) is the true system output, \( y(t) \) is the measured output, corrupted by noise \( \tilde{y}(t) \), and the input \( u(t) \) is assumed to be known. We introduce the assumption that \( G(q) \) is FIR, and parametrize it accordingly as
\[
G(q, \theta) = \theta_1q^{-1} + \theta_2q^{-2} + \cdots + \theta_nq^{-n},
\]
and that \( \tilde{y}(t) \) is Gaussian white noise.

The prediction error method (PEM) serves as benchmark in the field, since it is well known to provide asymptotically efficient estimates if the model orders are correct (Ljung (1999)). The essential idea of PEM is to minimize a cost function of the prediction errors. In this setting, PEM consists on minimizing the cost function
\[
V(\theta) = \frac{1}{N} \sum_{t=1}^{N} (y(t) - G(q, \theta)u(t))^2,
\]
if a quadratic cost is used, and where \( N \) is the number of samples available. Then, the minimizer of (3) is an asymptotically efficient estimate of \( \theta \), if the model orders are correct. In general, PEM requires solving a non-convex optimization problem. However, for this particular model structure, the minimizer of (3) can be obtained by solving a least squares (LS) problem. Defining the regression vector as
\[
\varphi^\top(t) := [u(t-1) \ u(t-2) \ \cdots \ u(t-n)],
\]
it is possible to write
\[
y(t) = \varphi^\top(t)\theta + \tilde{y}(t),
\]
where
\[
\theta = [\theta_1 \ \theta_2 \ \cdots \ \theta_n]^\top.
\]

Further, if we define
\[
y = [y(1) \ y(2) \ \cdots \ y(N)]^\top, \quad \Phi^\top = [\varphi(1) \ \varphi(2) \ \cdots \ \varphi(N)]^\top,
\]
and \( \tilde{y} \) analogously to \( y \), we can write
\[
y = \Phi^\top \theta + \tilde{y}.
\]

An estimate of \( \theta \), which corresponds to the minimizer of (3), can be obtained by LS, computing
\[
\hat{\theta} = (\Phi\Phi^\top)^{-1}\Phi y.
\]

We consider now the case when the true input is not known, but it can be measured, and is corrupted by measurement noise. In this case, the data is generated according to
\[
\begin{align*}
y_o(t) &= G(q)u_o(t) \\
y(t) &= y_o(t) + \tilde{y}(t),
\end{align*}
\]
where \( u_o(t) \) is the true input, \( \tilde{u}(t) \) the input measurement noise, and \( u(t) \) the measured input. This setting corresponds to an errors-in-variables (EIV) problem (Söderström (2007)). In this scenario, we have that
\[
\begin{align*}
y(t) &= \varphi^\top(t)\theta + v(t, \theta) \\
v(t, \theta) &= \tilde{y}(t) - \varphi^\top(t)\theta,
\end{align*}
\]
where, if \( \varphi_o \) is defined analogously to \( \varphi \), but containing true input values \( u_o \), then
\[
\tilde{v}(t) = \varphi(t) - \varphi_o(t).
\]
Because \( v(t, \theta) \) is not white, if the parameter vector \( \theta \) is estimated according to (4), the obtained estimate is not consistent.

Instrumental variable (IV) methods are appropriate to deal with EIV problems. The basic idea of IV methods is to choose a vector of instruments \( z(t) \) that is uncorrelated with the error \( v(t, \theta) \), while being highly correlated with \( \varphi(t) \). Then, for such an instrument vector, computing
\[
\hat{\theta} = (Z\Phi^\top)^{-1}Zy,
\]
where
\[
Z = [z(1) \ z(2) \ \cdots \ z(N)],
\]
yields a consistent estimate of \( \theta \) under certain excitation conditions.

There is no unique way to define \( z(t) \). One approach, proposed in Söderström and Mahata (2002), is to choose
\[
z^\top(t) = [u(t-1 - d_u) \ \cdots \ u(t - d_u - n_{z_u})],
\]
where \( d_u \geq 1 \). Another possibility, proposed in Thil et al. (2008), is to also include past outputs in the instrument vector, according to
\[
z^\top(t) = [-y(t-1 - d_y) \ \cdots \ -y(t - d_y - n_{z_y}) \ u(t-1 - d_u) \ \cdots \ u(t - d_u - n_{z_u})],
\]
where \( d_u \) is at least the order of the filter (it must be a moving average (MA) filter) applied to the noise. For the considered FIR case, \( d_u \geq 0 \).
Returning to the cascade example of Fig. 1, it is possible to observe that such methods are appropriate for the estimation of \( G_2(q) \). Using the notation introduced in this Section, \( u_0(t) \) plays the role of \( u_2(t) \), the unknown input to the second transfer function, and \( y_1(t) \) corresponds to \( u(t) \), the measurable input. Furthermore, the fact that the transfer functions are part of a network creates new possibilities for choices of instruments, as discussed in Dankers et al. (2014). In this case, a natural instrument candidate when estimating \( G_2(q) \) is the external input \( u(t) \).

However, note that although \( G_1(q) \) can be obtained from data \( \{u(t), y_1(t)\} \) using any standard system identification method, discarding the output \( y_2(t) \) in the estimation of \( G_1(q) \) increases the variance of the estimated parameters (Everitt et al. (2013)). The contribution Gunes et al. (2014) addresses a similar issue for the two-stage method. Moreover, although \( G_2(q) \) can be estimated consistently from \( \{y_1(t), y_2(t)\} \) using the IV approach, it was shown in Hjalmarsson et al. (2011) that the EIV setting, even for asymptotically efficient estimator, does not reach the asymptotic properties of PEM.

3. INDIRECT PEM

In the cascade case, due to the product \( G_2(q)G_1(q) \), structures that are linear in the model parameters, such as FIR or ARX, cannot be directly applied, and PEM requires solving a non-convex optimization problem. Indirect PEM (Söderström et al. (1991)) offers, in some cases, a workaround for the non-convexity of PEM.

The general idea is as follows. Consider two nested model structures \( M_1 \) and \( M_2 \), such that \( M_1 \subset M_2 \). Moreover, suppose that \( M_1 \) is parametrized by the parameter vector \( \theta \), while \( M_2 \) is parametrized by \( \alpha \). In a first step, \( \alpha \) is estimated with PEM, by minimizing

\[
V(\alpha) = \sum_{t=1}^{N} \epsilon^2(t, \alpha),
\]

where \( \epsilon(t, \alpha) \) are the prediction errors associated with the model structure \( M_2 \). In some situations, \( M_2 \) can be chosen such that minimizing (6) becomes an easy problem, even if applying PEM to the model structure \( M_1 \) is difficult. For examples, see Söderström et al. (1991).

Then, in a second step, an estimate of \( \theta \) is obtained by minimizing

\[
V(\theta) = \frac{1}{N} \frac{1}{2} [\hat{\alpha} - \alpha(\theta)]^\top P_{\alpha}^{-1} [\hat{\alpha} - \alpha(\theta)],
\]

where \( \hat{\alpha} \) is the estimate of \( \alpha \) obtained by minimizing (6), and \( P_{\alpha} \) is a consistent estimate of the covariance of \( \hat{\alpha} \), obtained from minimizing (6).

Note that (7) is still non-quadratic; however, it is rather easy to handle. Asymptotically, a single step of the Gauss-Newton algorithm, initialized with a consistent estimate of \( \theta \), provides an efficient estimate, i.e., it has the same asymptotic properties as if PEM had been applied to the model of interest directly. In Söderström et al. (1991), more details are given on the algorithm.

Indirect PEM can be applied to the cascade system in Fig. 1 as follows. First, since the transfer functions considered are FIR, \( G_1(q) \) can be estimated from \( \{u(t), y_1(t)\} \), and \( G_2(q) \) from \( \{u(t), y_2(t)\} \), by solving LS problems as in (4). Since the parameters estimated in this step were, in the exposition above, designated by \( \alpha \), we shall write that estimates \( G_1(q, \hat{\alpha}) \) and \( G_2(q, \hat{\alpha}) \) have been obtained in this step.

In the second step, we solve

\[
G_1(q, \theta) = G_1(q, \hat{\alpha})
\]

(8a)

\[
G_2(q, \theta)G_1(q, \theta) = G_2(q, \hat{\alpha})
\]

(8b)

for \( \theta \) in a weighted least squares (WLS) sense, where the weighting is a consistent estimate of the inverse covariance of \( \alpha \), obtained from the LS estimator by

\[
\hat{P}_{\alpha} = \begin{bmatrix}
\hat{\alpha} (\Phi_1 \Phi_1^\top) & 0 \\
0 & \frac{1}{\hat{\alpha}} (\Phi_2 \Phi_2^\top)
\end{bmatrix},
\]

and \( \Phi_1 \) and \( \Phi_2 \) are the regressor matrices for the estimation of \( G_1(q) \) and \( G_2(q) \), respectively. To be able to obtain an asymptotically efficient estimate in one step of the Gauss-Newton algorithm, consistent estimates of \( G_1(q, \theta) \) and \( G_2(q, \theta) \) are required. An estimate of \( G_1(q, \theta) \) is readily available from (8a), while a consistent estimate of \( G_2(q, \theta) \) can be obtained by applying LS to (8b), replacing \( G_1(q, \hat{\alpha}) \) by \( G_1(q, \hat{\alpha}) \). So, if the estimates obtained in this way, \( G_1(q, \hat{\theta}) \) and \( G_2(q, \hat{\theta}) \), are used as initializations to the Gauss-Newton algorithm, the next step yields asymptotically efficient estimates, \( \hat{\theta} \).

Thus, the asymptotic efficiency achieved by applying indirect PEM is an advantage in comparison to the EIV approach. On the other hand, for other structures, such as output-error, IV techniques can still be applied to obtain consistent estimates of \( G_1(q) \) and \( G_2(q) \), while indirect PEM suffers from non-convexity. There are, however, certain cascade structures for which the standard IV approach is not applicable, but to which indirect PEM still applies. An example is discussed in the next section.

4. ANOTHER CASCADE STRUCTURE

Consider the cascade structure in Fig. 2, which can be written as

\[
\begin{bmatrix}
y_1(t) \\
y_2(t)
\end{bmatrix} = \begin{bmatrix} 0 & G_2(q) \\
G_3(q)G_1(q) & G_3(q)G_2(q)
\end{bmatrix} \begin{bmatrix} u_1(t) \\
u_2(t)
\end{bmatrix} + \begin{bmatrix} e_1(t) \\
e_2(t)
\end{bmatrix},
\]

where we maintain the FIR model assumption.

Fig. 2. Cascade system with three transfer functions.

An important difference when comparing to the structure in Fig. 1, besides the presence of one more transfer function block and one more input, is that not all block inputs and outputs are measured. In particular, the output of \( G_1(q) \), as well as the input of \( G_3(q) \), are unknown. Therefore, the IV approach delineated in Section 2 is not directly applicable for estimating \( G_1(q) \) and \( G_3(q) \).
On the other hand, indirect PEM is still applicable. By using the parametrization
\[
G_2(q) = G_2(q, \alpha) \\
G_3(q)G_1(q) = G_{31}(q, \alpha) \\
G_3(q)G_2(q) = G_{32}(q, \alpha)
\]
we can use (9) to estimate \(G_2(q, \alpha)\), \(G_{31}(q, \alpha)\), and \(G_{32}(q, \alpha)\). Then, the second step of indirect PEM concerns solving
\[
G_2(q, \theta) = G_2(q, \hat{\alpha}) \quad (10a) \\
G_3(q)G_1(q, \theta) = G_{31}(q, \hat{\alpha}) \quad (10b) \\
G_3(q)G_2(q, \theta) = G_{32}(q, \hat{\alpha}) \quad (10c)
\]
in the discussed WLS sense. Since (10a) can be used to obtain a consistent estimate of \(G_2(q, \theta)\), \(G_2(q, \hat{\theta})\), \(10c\) can then be used to obtain a consistent estimate of \(G_3(q, \theta)\), by replacing \(G_2(q, \theta)\) with \(G_2(q, \hat{\theta})\). With a consistent estimate \(G_3(q, \hat{\theta})\), we can use (10b) to obtain a consistent estimate of \(G_1(q, \theta)\). With a consistent estimate \(\hat{\theta}\) of the complete vector \(\theta\), one step of the Gauss-Newton algorithm provides an asymptotically efficient estimate of \(\theta\).

5. CASCADE WITH FEEDBACK

In this section, we consider a cascade system with two blocks connected in a feedback loop, represented in Fig. 3. Mathematically, we can write it as
\[
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \frac{1}{1 - G_1(q)G_2(q)} \begin{bmatrix} G_1(q) & G_1(q)G_2(q) \\ G_1(q)G_2(q) & G_2(q) \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} + \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}.
\]
(11)

![Fig. 3. Block diagram of a feedback cascade system.](image)

Notice that this case is different from the typical feedback setting, with the measured output included in the loop. Instead, it can be seen as a natural extension of the previous cascade systems, where the systems are connected physically. A well-known historical example of a system physically connected in feedback is a centrifugal governor.

Application of an IV method, in the EIV setting, to estimate \(G_1(q)\) and \(G_2(q)\) is straightforward, as both outputs are measured, and these are also the inputs to the other transfer function, summed with a known external reference. Note, however, that the same limitation of Section 4 would be encountered if, for example, output \(y_2(t)\) could not be measured.

Concerning indirect PEM, it is difficult to find a model structure of finite order that is easy to estimate, and that contains the true model. The purpose of this section is to propose an intermediate step, based on the method presented in Galinho et al. (2014), which reduces the problem to a setting that allows the application of the indirect PEM algorithm as in the previous sections.

To do so, we use the following procedure. In a first step, we estimate high-order FIR models of each closed-loop SISO transfer function in (11). In a second step, we reduce these estimates to estimates of \(G_1(q)\), \(G_2(q)\), and the product \(G_1(q)G_2(q)\), with the desired orders, by LS. Then, we re-estimate by WLS, where we use the estimate obtained in step 2 to construct the weighting matrix. Finally, the indirect PEM algorithm can be applied.

In detail, the first step consists on approximating (11) by the indirect PEM algorithm as in the previous sections.

In detail, the first step consists on approximating (11) by
\[
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} G_{11}^{CL}(q, g_{11}) & G_{12}^{CL}(q, g_{12}) \\ G_{21}^{CL}(q, g_{21}) & G_{22}^{CL}(q, g_{22}) \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} + \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix},
\]
(12)
where
\[
G_{ij}^{CL}(q, g_{ij}) = \sum_{k=1}^{m} g_{ij}^{(k)} q^{-k},
\]
for \(i, j = 1, 2\). If the order \(m\) is chosen large enough, we have that
\[
\frac{1}{1 - G_1(q)G_2(q)} \begin{bmatrix} G_1(q) & G_1(q)G_2(q) \\ G_1(q)G_2(q) & G_2(q) \end{bmatrix} \approx \begin{bmatrix} G_{11}^{CL}(q, g_{11}) & G_{12}^{CL}(q, g_{12}) \\ G_{21}^{CL}(q, g_{21}) & G_{22}^{CL}(q, g_{22}) \end{bmatrix}.
\]
(13)

Since (12) is a multi-input multi-output (MIMO) FIR model, \(G_{ij}^{CL}(q, g_{ij})\) can be estimated by LS. This is done by solving, analogously to what was done from (2) to (4),
\[
\phi_{\hat{y}} := \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \left( \Phi \Phi^\top \right)^{-1} \Phi y,
\]
\[
\Phi^\top = \begin{bmatrix} \Phi_{r_1} \Phi_{r_2} \end{bmatrix},
\]
where \(\Phi_{r_1}\) and \(\Phi_{r_2}\) are the regression matrices for \(r_1(t)\) and \(r_2(t)\), respectively. The estimated parameters are, then, distributed as
\[
\hat{y}_i \sim \mathcal{N} \left( \bar{y}_i, \Phi_{\hat{y}_i} \right),
\]
where the covariance matrix is given by
\[
\Phi_{\hat{y}_i} = \lambda_i \left( \Phi \Phi^\top \right)^{-1}.
\]

In a second step, we use the approximation (13) to write
\[
Z_{ii}(q, \alpha, g_{ii}) := G_{ii}^{CL}(q, g_{ii}) (1 - G_{12}(q, \alpha_{12}))
\]
\[
- G_i(q, \alpha_i) = 0
\]
\[
Z_{ij}(q, \alpha, g_{ij}) := G_{ij}^{CL}(q, g_{ij}) (1 - G_{12}(q, \alpha_{12}))
\]
\[
- G_{i2}(q, \alpha_{12}) = 0
\]
(14)

(14) only, \(i \neq j\), where we parametrize \(G_1(q)\) and \(G_2(q)\) as
\[
G_1(q, \alpha_i) = \alpha_i^1 q^{-1} + \cdots + \alpha_i^{n_i} q^{-n_i},
\]
and, to avoid the product \(G_1(q)G_2(q)\),
\[
G_{12}(q, \alpha_{12}) = \alpha_{12}^2 q^{-2} + \cdots + \alpha_{12}^{n_{12}+n_{12}-1} q^{-(n_{12}+n_{22}-1)}.
\]

Then, if \(g_{ij}\) is replaced by its estimate, \(\hat{g}_{ij}\), (14) can be solved by LS. Write, first, (14) in vector form, as
\[
Q(\hat{\gamma}) \alpha = \hat{\gamma},
\]
(15)
where
\[
\alpha = \begin{bmatrix} \alpha_1 \alpha_2 \alpha_{12} \end{bmatrix}^\top,
\]
\[
g = \begin{bmatrix} g_{11}^\top \ g_{12}^\top \ g_{21}^\top \ g_{22}^\top \end{bmatrix}^\top,
\]
and obtain estimates \(\hat{\alpha}_1\), \(\hat{\alpha}_2\), and \(\hat{\alpha}_{12}\) by computing the LS estimate of \(\alpha\) from (15), i.e.,
\[
\hat{\alpha} = (Q(\hat{\gamma})^\top Q(\hat{\gamma}))^{-1} Q(\hat{\gamma})^\top \hat{\gamma}.
\]
In a third step, we re-solve (14), now using WLS, where the weighting is obtained as follows. Let
\[ \hat{g}_{ij} = g_{ij} + \tilde{g}_{ij}. \]
Then, we can write, in transfer function form,
\[ G_{ij}^{\text{CL}}(q, \hat{g}_{ij}) = G_{ij}^{\text{CL}}(q, g_{ij}) + G_{ij}^{\text{CL}}(q, \tilde{g}_{ij}). \]  
Replacing (16) in (14) yields
\[ Z_{ij}(q, \alpha, \hat{g}_{ij}) = (1 - G_{12}(q, \alpha_{12})) G_{ij}^{\text{CL}}(q, \hat{g}_{ij}). \]  
Rewriting the equations defined in (17) in vector form, we have
\[ z(\alpha, \hat{g}) := \begin{bmatrix} z_{11}(\alpha, \hat{g}_{11}) \\ z_{12}(\alpha, \hat{g}_{12}) \\ z_{21}(\alpha, \hat{g}_{21}) \\ z_{22}(\alpha, \hat{g}_{22}) \end{bmatrix} = T(\alpha_{12}) \begin{bmatrix} \hat{g}_{11} \\ \hat{g}_{12} \\ \hat{g}_{21} \\ \hat{g}_{22} \end{bmatrix} =: T(\alpha_{12})\bar{g}. \]
Then, the optimal weighting for the WLS is the inverse of the covariance of $z(\alpha, \hat{g})$, i.e.,
\[ W(\alpha_{12}) = (T(\alpha_{12}) P_{g} T(\alpha_{12})^\top)^{-1}, \]
where $P_{g}$ is the covariance of $\bar{g}$, given by
\[ P_{g} = \begin{bmatrix} P_{g_{1}} & 0 \\ 0 & P_{g_{2}} \end{bmatrix}. \]
Since the true $\alpha_{12}$ is not available, we use, instead, the estimate obtained in step 2,
\[ W(\hat{\alpha}_{12}) = (T(\hat{\alpha}_{12}) P_{g} T(\hat{\alpha}_{12})^\top)^{-1}, \]
and compute
\[ \hat{\alpha} = (Q(\hat{g})^\top W(\hat{\alpha}_{12}) Q(\hat{g}))^{-1} Q(\hat{g})^\top W(\hat{\alpha}_{12})\hat{g}, \]
which is an improved estimate of $\alpha$. Asymptotically, we have that these estimates have covariance
\[ \text{cov}(\hat{\alpha}) = (Q(\hat{g})^\top W(\hat{\alpha}_{12}) Q(\hat{g}))^{-1}. \]
With these estimates, we can construct transfer functions $G_{1}(q, \hat{\alpha}_{1}), G_{2}(q, \hat{\alpha}_{2})$, and $G_{12}(q, \hat{\alpha}_{12})$, and solve
\[ G_{1}(q, \theta) = G_{1}(q, \hat{\alpha}_{1}) \]
\[ G_{2}(q, \theta) = G_{2}(q, \hat{\alpha}_{2}) \]
\[ G_{12}(q, \theta) = G_{12}(q, \hat{\alpha}_{12}) \]
with the algorithm previously mentioned for indirect PEM, using (18) as covariance estimate for $\hat{\alpha}$.

We note that, like indirect PEM for the acyclic cascade systems, this method does not require all the transfer function outputs to be measured. For example, if $\hat{y}_{2}(t)$ were not measured, the method would be applicable without major changes. In particular, (19b) would be absent in (19).

6. NUMERICAL SIMULATION

In this section, we perform a numerical simulation to evaluate the method previously described for identification of the system in Fig. 3. One hundred Monte Carlo runs are performed, for sample sizes $N = \{100, 300, 600, 1000, 3000, 6000\}$. The systems are given by
\[ \begin{cases} G_{1}(q) = \theta_{1}^{(1)} q^{-1} + \theta_{2}^{(1)} q^{-2} \\ G_{2}(q) = \theta_{1}^{(2)} q^{-1} + \theta_{2}^{(2)} q^{-2} \end{cases} \]
The closed-loop, whose poles are given by
\[ 1 - G_{1}(q) G_{2}(q) = 0, \]
is, thus, a fourth order system. The poles are placed randomly inside the unit circle, under the structure imposed by (20). The same colored input as in Söderström and Mahata (2002) and Thil et al. (2008) is used, i.e.,
\[ r_{i}(t) = \frac{1}{1 - 0.2 q^{-1} + 0.5 q^{-2}} w_{r_{i}}(t), \]
for $i = 1, 2$, where $w_{r_{i}}(t)$ is Gaussian white noise with unitary variance, and uncorrelated for each $i$. The sensor noises $e_{1}(t)$ and $e_{2}(t)$ are also Gaussian and white, and uncorrelated to each other. The variances are changed at each run in order to achieve a signal-to-noise ratio of 5dB at the outputs.

The following methods are compared: PEM, as implemented in MATLAB2014b, starting at the true parameters (note that MATLAB does not provide an estimated point to initialize the algorithm, due to the non-standard model structure); the proposed method, referred to as WLSiPEM; the IV method with the external references as instruments (IVr); the IV method with instrument vector (5) (IVy). Concerning WLSiPEM, the order $m$ of the high-order FIR models is optimized over a grid of ten values, linearly spaced between $m = 20$ and $m = N/3$, and the one minimizing the prediction errors is chosen. A similar procedure is used to choose the length of the instrument vector for IV methods, by considering a grid of ten values linearly spaced between 4 and 22, and selecting the one minimizing the prediction errors. All the data available is used in the estimation.

The accuracy of the estimates is measured by the FIT, in percent, defined as
\[ \text{FIT} = 100 \left( 1 - \frac{||\theta - \hat{\theta}||_{2}}{||\theta - \hat{\theta}||_{2}} \right), \]
where $\theta$ is a vector containing the true parameters, $\hat{\theta}$ its mean, and $\hat{\theta}$ the estimated parameter.

The results are presented in Fig. 4, with the average FIT as function of sample size. In this simulation, the presented method performed better than both IV methods. On the other hand, other simulations showed the IV methods...
to be more robust in the presence of extremely colored external excitations. It is also observed that using other variables in the network to construct the instrument vector improves the estimate, in comparison to using delayed measured input and output values. Finally, we verified that, in this case, the estimates obtained by any of the methods were appropriate as initial estimates for PEM, since the algorithm converged to the same parameters as when it was started at the true values.

7. DISCUSSION

In this paper, we discussed two approaches for identification of network systems. One is based on the application of an IV method to an EIV setting, while the other uses indirect PEM. We point out that the applicability of each method depends on the particular problem setting and the assumptions on the network. The IV methods require that the input and output of all transfer functions are measured, while that is not a requirement for indirect PEM. Moreover, while the accuracy of IV methods depends on the choice of instruments and the correlation to the regression vector, indirect PEM guarantees asymptotic efficiency. On the other hand, standard IV methods admit rational transfer functions, while indirect PEM suffers from non-convexity in that case.

If there is feedback in the network, indirect PEM does not avoid a non-convex problem. To deal with that limitation, we propose a method that uses the approach in Galrinhoo et al. (2014), and transforms the problem into one where the indirect PEM algorithm can be applied. Analogously to the acyclic case, measuring all outputs is not a necessary condition to obtain all the transfer functions.

We perform a simulation study with two FIR transfer functions randomly generated and connected in feedback, where both outputs are measured, comparing PEM, the proposed method, and two IV methods with different instrument choices. The proposed approach performed better than the IV methods, but it did not prove as robust in the presence of extremely colored external excitations. This reinforces our discussion on the most appropriate approach depending on the settings and the network assumptions. To provide initial values for PEM, any of the techniques were successful.

The noise sources were considered white, but most methods discussed allow the noise to be colored. For the IV method in Thil et al. (2008), the output noise must be given by a MA filter of white noise, although the input noise is assumed white. In the network context, the more general case, with an autoregressive moving average (ARMA) filter, is addressed in Dankers et al. (2014). Concerning indirect PEM, we still have a linear regression problem if the noise is given by an autoregressive filter driven by white noise. Finally, the proposed method is still applicable in the ARMA filter case, for which a high-order ARX model is estimated in the first step of the algorithm.

Future work includes establishing conditions on the network structure for the methodology to be applicable, and extending the approach to the case where the blocks in the network are rational transfer functions.

REFERENCES


