From Solving Problems to Problem Solving – Primary School Teachers Developing Their Mathematics Teaching through Collaborative Professional Development

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Abstract

In many countries, problem solving in mathematics plays a substantial role in the syllabus, promoting the basic aim that pupils develop a deeper understanding of mathematics. However, in many countries there are few classrooms in which mathematics teaching through problem solving has truly been implemented. This article reports findings from a study of Swedish primary school teachers who, within a collaborative professional-development initiative, worked to improve their mathematics teaching, focusing on problem solving. Several studies have reported on the mathematical knowledge primary school teachers have, do not have, and ought to have. This article, however, does not evaluate the knowledge possessed by the teachers but instead focuses on how the professional collaborative learning initiative influenced the teachers' awareness of problem solving as content. The collaborative professional-development initiative had a cyclic design in which changes in the teachers' awareness became visible in their jointly produced lesson plans and pre-tests. Changes in these documents show how the role of problem solving in the teachers' mathematics classroom shifted in accordance with the historical development of problem solving as content in school: problem solving as a context, as a skill, and as an art.

Keywords: problem solving, mathematics, learning study, professional development, teacher

Introduction

In many countries, problem solving in mathematics plays a substantial role in the syllabus, promoting the basic aim that pupils develop a deeper understanding of mathematics (Lesh & Zawojewski, 2007). However, across several countries there are few classrooms in which mathematics teaching through problem solving has seen significant implementation (Lester & Lambdin, 2007). The purpose of this article is to answer the need for research on teachers' learning indicated by other researchers(Carlgren, 2012; De Simone, 2014; Lo & Marton, 2012; Runesson & Gustafsson, 2012). In the Swedish primary school curriculum (Swedish National Agency for Education, 2011), problem solving in mathematics is both content for pupils to learn and an ability for pupils to develop. This article reports findings from a study of primary school teachers who, within a collaborative professional-development initiative, worked to improve their mathematics teaching, focusing in particular on problem solving. Like most primary school teachers around the world (Tatto, Lerman, & Novotná, 2009), these teachers have been educated as generalists and work in classrooms teaching several subjects, of which mathematics is one.

Collaborative professional development is a new experience for many teachers but has been strongly promoted by the Swedish National Agency for Education. One kind of collective professional-development initiative in particular, learning study, has been subsidized in recent years and has become common in Sweden. Learning study involves teachers and researchers working together to plan a research lesson. The lesson is taught by the teachers in one or several cycles and is observed, evaluated, and modified by the team before the next cycle is taught. This article presents findings from one such learning study. The teachers involved wanted to develop their mathematics teaching, specifically in the area of problem solving, and this article considers the pre-tests and the lesson plans that the teachers produced because they reflect changes in the teachers' awareness of problem solving. These changes were shown to have many similarities with the historical development of problem solving. This article discusses how the design of the professional collaborative-learning initiative made these changes possible.

First, teachers' professional development in general and learning study in particular will be presented. Second, the role that problem solving has played within mathematics education over time will be described. After that, the study and the empirical material, as well as the findings, are presented. Finally, these findings are discussed in relation to the potential of learning study as a model for professional-development initiatives.

Teachers' professional development

In order to successfully teach mathematics, teachers not only need to know mathematics itself but must also have knowledge on how to teach mathematics (Hill, Sleep, Lewis, & Ball, 2007; Ponte & Chapman, 2008). Studies of mathematics teachers and their teaching have often focused on the knowledge teachers have, the knowledge teachers do not have, the knowledge teachers use, or the knowledge they need in order to teach mathematics (Hill et al., 2007). This article, in contrast, does not evaluate the knowledge possessed by the teachers involved in the study but instead focuses on how their participation in the professional-development initiative influenced their awareness of teaching problem solving.

The issue of teachers' professional development is not new, and several concepts have been tried with varying results. For example, increased planning time for teachers has been shown not to result in improvements (Evans, 2012), while pupils of teachers who participate in ongoing professional development have shown improved performance in mathematics (Simpson & Linder, 2014). It is important that the effect of professional development on teachers is sustainable (Elliott & Yu, 2008; Soine & Lumpe, 2014), and several studies show that professional development does not occur when one person tells someone else what and how something is to be done (Males, Otten, & Herbel-Eisenmann, 2010). Instead, professional development is a process of teaching and learning that requires action and reflection over time. Given that knowledge emerges and is used in relation to context (Boaler, 2000), professional development ought to be conducted in the context of teaching. Collective professional-development initiatives can advantageously be carried out in collaboration between researchers and practicing teachers, but it is important that the teachers develop control of their own work, as well as autonomy (Males et al., 2010). One example of such collaboration between researchers and teachers was examined by Franke and Kazemi (2001). Having researched teacher development for several years, focusing on knowledge as a personal property of individual teachers, they started to develop collaboration with and between teachers. About their change of direction, they write:

> … we now consider teachers' classrooms, the work in professional development meetings, and teachers' informal interactions with colleagues and staff as sites for their learning and practice (Franke & Kazemi, 2001, p. 71).

Often, such interactions are seen as peripheral to understanding teaching and teachers, but Franke and Kazemi (2001) found them to be a significant aspect of teacher development. They stress that collective professional development should not be separated from the teachers' classrooms but instead ought to mirror the interactions and identities there. Similar collaborations between researchers and teachers with positive outcomes have, for example, been carried out by Graven (2004), Goos and Bennison (2008), and Gellert (2008), as well as by Cuddapah and Clayton (2011).

When diverse groups of teachers with different competences come together, they can create communities with deep new insights into teaching and learning. However, in many schools the teaching culture does not include critical or reflective discussions of current teaching practice (Putnam & Borko, 2000). Further, teachers who are told to collaborate seldom know what they are supposed to do or how the collaboration could improve their teaching (Stigler & Hiebert, 1999). According to Lord (1994), collegiality is more than sharing ideas; it implies confronting both one's own teaching practice and the teaching practices of one's colleagues.

Learning study

This investigation follows several teachers who participated in a collaborative professional-development initiative called a learning study. Learning study can be seen as a hybrid between lesson study and design research (Marton & Pang, 2004). As in a lesson study, teachers develop and improve their teaching through collaborative work but are supervised and supported by a researcher. As in design research, a cyclic structure allows the analysis and improvement of teaching. The structure applied in a learning study can be divided into several phases; that applied in the present study starts by choosing a focus for one lesson to be taught. This focus is named the object of learning (A). In the present study, the object of learning was problem solving. It is important to note that the object of learning was chosen by the teachers, not by the researcher or supervisor; ownership is an important tenant of a learning study (Holmqvist, 2011; Marton & Tsui, 2004). After choosing the object of learning, the teachers start to gather information regarding their object and then develop some kind of pre-test (B) to explore what their pupils already know or are able to do and what they do not know or cannot do. An analysis is conducted of the pre-test (C), and a first lesson plan is developed (D). The lesson is conducted by one of the teachers (E), and this is followed by a post-test (F). A new analysis (G) is performed in which differences between pre- and post-test results are related to the given lesson. Questions about the object of learning are posed: In what way was the object of learning made learnable? In what way did pupils experience the object of learning? A distinction between the intended (as planned beforehand), enacted (as performed in class by the teacher), and lived (as experienced by the pupils) object of learning is suitable, as it allows the teachers to analyze differences. After that, the lesson plan is revised (D) according to the results of the analysis of the pre- and post-tests in relation to the conducted lesson. The new lesson is taught to a new group of pupils: the cycle of pre-test, lesson, post-test is repeated. This process (Figure 1) continues until a satisfactory result is reached, and for each new group the lesson is adapted according to the results of the analysis.

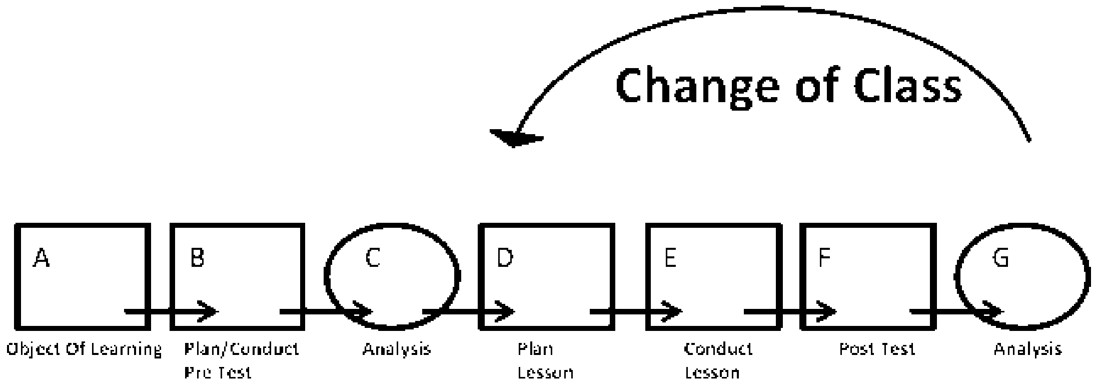

Figure 1. Learning study cycle (van Bommel, 2012, p. 50; 2014, p. 104)

This article focuses on stages A, B, C, and D, reflecting on changes in the formulation of the object of learning (A) during these stages. The documents analyzed are two version of pre-tests (B) and two versions of lesson plans (D). It is important to realize that the teachers' analysis in the process (C, G) and their planning of the test (B) and lesson (D) are guided by a theoretical tool: variation theory (Runesson, 2006). Variation theory is used both as a planning tool for the teachers and a tool for analyzing the data gathered (van Bommel & Liljekvist, 2008). This is similar to most learning studies conducted. Variation theory enables teachers to focus on the content (Holmqvist, Gustavsson, & Wernberg, 2008; Pang, Linder, & Fraser, 2006; Pang & Marton, 2005).

> The primary contribution of variation theory to learning study is that it brings the focus of the study sharply on the object of learning and provides a theoretical grounding to understand some of the necessary conditions for learning (Lo & Marton, 2012, p. 9).

According to variation theory, learning is always the learning of *something*, a phenomenon. When a (qualitative) change in viewing or understanding this phenomenon occurs, learning is said to take place (Marton & Pang, 2004). Such change in understanding results in a more powerful way of seeing the phenomenon (Marton & Tsui, 2004). The phenomenon within a classroom situation is called the *object of learning*. In a learning study, teachers choose an object of learning, often inspired by pupils' difficulties understanding it or by teachers' difficulties teaching it. A change in one's understanding of the object of learning requires insight into some aspects of that object. Such aspects are called *critical features* and should be experienced in order for a change in understanding to take place. During a learning study, critical features of the object of learning are looked for. When critical features are dealt with, a change in understanding the object of learning can be established. Dealing with the critical features means that pupils have to be able to *discern* the critical features *simultaneously*. The use of *patterns of variation* supports such simultaneous discernment. In all cases the distinction between the intended, enacted, and lived object of learning makes it possible to examine the results from each perspective. Differences between the three manifestations of the object of learning can explain why learning did or did not take place. What parts of the object of learning were discerned? Which critical features could be discerned?

As stated earlier, this article concerns phases A, B, C, and D, which include only the intended object of learning. The object of learning as enacted and the lived constituted part of the learning study but is not part of this article. The following sections provide more background on the mathematical area related to the object of learning chosen by the teachers—problem solving.

Problem solving

Over time, the role of problem solving in mathematics education has changed (Lester, 1994; Schoenfeld, 1985). There seems to be a consensus that pupils should be educated to become competent problem solvers (Schoenfeld, 1992). However, a competent problem solver will be defined differently depending on the role that problem solving plays in mathematics. Stanic and Kilpatrick (1989) have pointed out three main themes concerning the role of problem solving. The first theme they describe comprises problem solving seen as a *context*, when it is used to reach and facilitate other valuable ends. In such cases, problem solving can be used to justify the teaching of mathematics, to motivate students and get them interested in mathematics, to stimulate further this gained interest; problem solving can also be used as recreation and even as a vehicle "through which a new concept or skill might be learned" (Stanic & Kilpatrick, 1989, p. 14). Problem solving as a practice is also included in this theme to describe situations in which problem solving is used to reinforce and practice previously taught concepts and skills.

In the second theme, problem solving is seen as a *skill*. Here, problem solving itself is the goal, part of the curriculum, added as one of a number of skills to be taught. The difference between problem solving as a context, in which also skills are central, and problem solving as a skill is that in the latter, the skill itself is problem solving, whereas, in the former, problem solving is used to address other skills. When problem solving was seen as part of the curriculum, different types of problems came in focus. Routine and non-routine problems were defined in response to the changing role from context to skill. The skill was well defined, and students had to be able to correctly solve the problems they were given. Non-routine problems required more then a correct answer of the pupils and resembled the approach to problem solving as a skill.

Stanic and Kilpatrick refer to Polya when describing the third view. In the middle of the last century, Pólya (1957) introduced the view of problem solving as an *art*, as the heart of mathematics. Although Mason et al. (1982) described the stages of problem solving in a way that differed from Pólya's, all address problem solving as an art. When problem solving is seen in this way, the idea of merely being able to solve the problem is no longer central. Instead, the process of being able to solve a problem becomes focal; this modern heuristic renders mathematics a hands-on activity.

These three views of problem solving—as a *context*, as a *skill*, and as an *art*—describe the different roles problem solving can have within mathematics education. The teachers in this investigation worked with problem solving in their learning study. Moreover, their choice of problems and the way they described and implemented the object of learning into lesson plans seemed to relate to their views of problem solving's role.

Data

In this section some background is given around the participants, further the way the data was collected will be described.

Participants

Four primary school teachers participated in the learning study. Three of them had conducted two learning studies together previously, and the fourth was new to this group but had participated in one past learning study. The study was conducted during spring semester 2012 in Swedish grade 5 (students age 11). The teachers had worked together as a team for some years. One of the authors had supervised all the previous learning studies this group had done and served as supervisor for this learning study as well. The teachers themselves decided to continue with a supervisor at this stage. The supervision focused on the learning study, and only afterwards was the aim addressed in this article - to describe the change in the participating teachers' awareness concerning the object of learning - examined and met by analyzing the collected data, as described in the following section.

Data collection and analysis

The data used in this study consist of the suggested problems in the pre-tests developed during the learning study, as well as the suggested lesson plans. At different stages the object of learning was reformulated and specified in e-mail correspondence between the group of teachers and the supervisor; these e-mail messages are also treated as data. Both the pre-test and the lesson plan were revised during the first phases of the learning study (phases B, C, and D in figure 1), and the differences in problems, in the formulation of the object of learning, and in the suggested lesson plans form the units of analysis here. After the learning study was completed, the authors of this article decided to describe the learning study and went through the documents; the journey of the participating teachers thus became visible. Because the teachers' experience had not been in focus during the learning study itself, the data had not yet been analyzed from this perspective. Re-reading the planning documents and e-mail correspondence made apparent a correlation between the different roles problem solving has played over time and the role it had played in the learning study. This role also seemed to reflect a different awareness amongst the participating teachers concerning the teaching of problem solving. Different versions of the various documents were set side by side, and the way the object of learning had been treated was described in relation to the role that problem solving was assigned in those documents. In such analysis, the object of learning itself (problem solving) could explain the differences in awareness about the object of learning.

Findings

The analysis showed three cases in which changes were visible. First, there were changes in the different versions of the pre-test. Furthermore, changes were visible in the different versions of the lesson plan. A third change was visible in the teachers' way of expressing their object of learning. In this section these three cases will be described.

Changes in pre-test

One of the problems to be solved by pupils in the first draft of the pre-test was the following:

> In a square field, with sides 60 meters long, we have goats, sheep, and chickens. In total, we can count 128 legs. How many goats, sheep, and chickens are there?
>
> (Problem 1, version 1)

One of the teachers, eager to start the learning study, tested this problem in one of his classes. The outcome was clear: 15% of the pupils could solve the problem. As for the remaining 85%, there were no indications as to what their difficulties had been; all that was tested here was whether students could answer a given question. This started a discussion in the team, expressed in the e-mail messages: In what way could the pre-test provide information about the critical aspects of problem solving? The problem was transformed into the following:

> In a square field, with sides 80 meters long, we have goats, sheep, and chickens. In total, we can count 128 legs. How many goats, sheep, and chickens are there?
>
> a) Underline in red information that is important.
> b) Underline in blue information that is not important.
>
> (Problem 1, version 2)

Although the problems are similar, there is a major difference between the two. In the first draft, pupils were asked to solve the problem. The answers they gave communicated information about whether pupils were able to work (start) on the problem and, if they could start, whether they could perform the correct calculation. As many pupils did not explain how they had calculated, the teachers did not obtain information about the strategy pupils had used. But in the final draft, the teachers focused more on the aspect of data processing as formulated in their object of learning. They now gained information about the pupils' ability to detect and evaluate information.

The pre-test problems addressed different roles of problem solving. Whereas the first version of the pre-test treated problem solving as a skill, the final version of the pre-test treated problem solving as an art. Simultaneously, the formulation of the object of learning changed, as mentioned earlier. The first formulation of the object of learning was ambiguous and could be interpreted within all three themes described: problem solving as a context, as a skill, and as an art. Although the teachers did not express changed perspectives, their reformulation of the object of learning led to a clearer formulation that corresponded to only one of the themes: problem solving as an art. Problem solving as an art is sometimes reduced to problem solving as a skill when too much focus is placed on the steps (Stanic & Kilpatrick, 1989).

The teachers hypothesized that pupils' major difficulty was that they did not know how to start on a problem; the problem was not clear to them. To address this anticipated critical feature, one teacher suggested asking pupils to formulate the problem in their own words; however, the team members did not consider an understanding of the problem and the ability to reformulate the problem to be the same thing. Two of the problems in the test's final draft addressed this critical feature. The problem above was used twice in the final test, and the final version was as follows:

> In a square field, with sides 80 meters long, we have goats, sheep, and chickens. In total, we can count 128 legs. How big is the field?

Which one of the following answers is reasonable?

a) 48 chickens, 23 sheep, 22 goats
b) 320 m
c) 308 m^2
d) 6,400 m^2

(Additional questions to problem 1, version 2)

Asking what the answer would look like gave teachers insight into pupils' understanding of the problem (Pólya, 1957). This type of questioning was adapted in a third problem as well:

Oscar has 21 meters of cloth. He has to cut the cloth into pieces 3 meters long. How many times does he have to cut?

(Problem 2, version 1)

In this version, pupils are merely asked to solve the problem. As a result, their answers varied not only in numerical value but also in the units they chose. Not being able to choose the right unit was, according to the teachers, linked to not having understood the problem. Because the teachers wanted to know whether the pupils had understood the problem, it was revised:

Oscar has 21 meters of cloth. He has to cut the cloth into pieces 3 meters long. How many times does he have to cut?
The answer to this question is 6, but what is the unit?

a) pieces
b) times
c) meter
d) no unit

Explain your answer.

(Problem 2, version 2)

A focus on the unit implies a shift from problem solving as a skill to problem solving as an art.

Changes in the lesson plan

The first rough draft of the lesson plan consisted of three parts. The lesson would start with a formal introduction to problem solving. At that stage, the steps *understand the problem*, *carry out a plan*, and *reflect on the plan* were put forward. Second, a hand-out would be distributed that highlighted important factors to consider while engaging in problem solving. These were mostly strategies (i.e., *draw a picture*, *make a table*, *find a pattern*), but also other aspects like *do not forget the unit* and *is your answer reasonable?* Third, a set of problems was put together for the pupils to solve using the information given in the formal introduction and the hand-out.

This first draft of the lesson plan showed diversity across the themes. The formal introduction and the hand-out seemed to relate to problem solving as an art. The problems given, however, were presented in such a way that problem solving as a skill would be central at that stage of the lesson. Stanic and Kilpatrick write that the problem for teachers who see problem solving as an art "is how to develop such artistic ability" (Stanic & Kilpatrick, 1989, p. 17) in pupils.

The same shift in perspectives described earlier when comparing the two drafts of the pre-tests was observable when comparing the two versions of the lesson plan. In the final version of the lesson plan, problem solving was dealt with as follows: one problem was stated at the beginning of the lesson. Based on that problem, problem-solving strategies were discussed, and the importance of understanding the problem was considered. This even included a discussion of the outcome—what the answer would look like. In the final version of the lesson plan, the teachers decided to keep the context and situation invariant and to vary the questions within that context. The focus was clearly on the *process* of problem solving, on problem solving as an art. The lesson plan did not specifically aim at solving the problems worked with. During the lesson, the problem would be treated by asking questions like these: What strategy will be useful? What strategy will not be useful? What information is important? What information is not important? What unit will the answer have?

The intended object of learning described in this lesson plan seemed to correspond well with the described (intended) object of learning of the learning study itself. Moreover, the object of learning was addressed in the same way in the pre-test and in the lesson plan. Through such coherence, the pre-test could really inform teachers about how to adjust the lesson plan further and provided an opportunity to analyze pre- and post-test results in relation to the given lesson.

It is interesting that these teachers had experienced similar difficulties during previous learning studies. The idea of discernment was grasped but was difficult to implement systematically. During both studies, they struggled with the idea of having only one context or situation while varying the specific questions within that context. In their first study, a set of eight problems was suggested at first; this was subsequently reduced to four and finally to just three problems. During their second study, four problems addressed different aspects of the object of learning, and towards the end of the study they became convinced that one of the problems was rich enough to address all these aspects. During this study, they recognized the comments given and were able to adjust their approach so as to formulate a rich invariant context and a way within that context to address the critical aspects. It would then be possible for the pupils to discern the object of learning, as the variation appeared in the questions accompanying a single problem (problem solving as an art) and not in different problems (problem solving as a skill).

Problem solving as the object of learning

The object of learning was first described by the teachers thus:

> Problem solving
> (First formulation of object of learning - emailcorrespondence)

Their motivation was partly that problem solving plays a central part in the curriculum (Swedish National Agency for Education, 2011). As the problem-solving competency is stated specifically in the curriculum, the teachers' view on it, expressed through their formulation of the object of learning, could be categorized as treating problem solving as a skill. Furthermore, the participating teachers related in their e-mail messages that pupils often encountered difficulty choosing a strategy but also had difficulty starting work on a problem. During the planning phase of the learning study, the formulation of the object of learning was modified. It became more precise and now addressed the underlying obstacles first described as difficulty getting started. After some sessions, the object of learning was formulated as follows:

> Problem solving: understanding the problem, being able to decode given data and non-data in a problem, being able to choose between strategies.
> (Final formulation of the object of learning – email correspondence)

In light of such refinement concerning the object of learning, the view of problem solving seemed to change focus from problem solving as a skill to problem solving as an art. The last section returns to the object of learning and considers some formulated critical features, but for now the reformulation of the object of learning remains central as it relates to changes in the pre-test and the lesson plan.

The table below summarizes the focus on the themes of problem solving throughout the learning study. As stated, a clear shift is visible away from problem solving as a skill towards problem solving as an art. Possible reasons for this shift are discussed the next section.

Table 1. Problem-solving themes in three sets of data.

	First version	Final version
Pre-test question 1	Problem solving as a skill	Problem solving as an art
Pre-test question 2	Problem solving as a skill	Problem solving as an art
Lesson plan	Problem solving as a skill	Problem solving as an art
Formulation of the object of learning	Problem solving as a skill Problem solving as a context Problem solving as an art	Problem solving as an art

Conclusions and discussion

This learning study informs teachers about how teaching problem solving can be enacted (Runesson & Gustafsson, 2012). Earlier, the formulation of the object of learning was modified from simply "problem solving" (email correspondence at start of learning study) to a more detailed description: "Problem solving: understanding the problem, being able to decode given data and non-data in a problem, being able to choose between strategies." (email correspondence after second version of pre-test). Some of the critical features discerned were incorporated into this description. In order to understand the phenomenon problem solving, one must be able to distinguish between data and non-data. One must be able to reformulate the question, to formulate different strategies, and to compare the outcome of various strategies.

The enactment has not been described in this article. Instead, the first phases of the learning study, immediately before the implementation in class, have been in focus. These initial phases reflect the aim of this article: to exemplify the change in the awareness of participating teachers concerning the object of learning. The teachers' awareness changed, and instances of such changes were visible in the variations between the produced and modified materials during the learning study.

This investigation also indicates that the professional collaborative-learning initiative made these changes in the teachers' awareness possible. Collaborative learning established interaction amongst these four teachers, and elaboration has been pointed out as a key feature by Franke and Kazemi (2001). Participating in collaborative professional development also leads to mutual language and terminology amongst participants. As the specific focus within a learning study is the object of learning, such language is created around that object of learning. Furthermore, the importance of the autonomy of the teachers has been pointed out by several researchers (Holmqvist, 2011; Males et al., 2010; Marton & Tsui, 2004), and the process described in this article shows that it was the teachers who made the changes at their own pace. Several trials initiated by the teachers were conducted before they decided upon final versions.

Interaction, mutual language, and ownership all contributed to changes in the teachers' awareness. Such change in awareness can be described as *learning*, given that a change in understanding the phenomenon (Marton & Tsui, 2004) occurred. As Carlgren (2012) suggests, a learning study can be seen as clinical research informing one about the object of learning. Such a development might have been the case for these teachers. Perhaps they had in mind problem solving as an art from the beginning of the study but were unable to articulate the idea clearly. Their literature review might have provided them with new terminology that helped them express precisely what they meant. Or their perspectives *did* change; perhaps the literature search resulted in a change in their own understanding.

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