On Brennan's conjecture in conformal mapping

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Doctoral Thesis, 1999

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Preface

The aim of this work was to prove Brennan's conjecture. Since this turned out to be a too difficult problem for me, we have to be content with the partial results in this thesis, some of which are interesting in their own right. A number of persons have helped me along the way. I thank

- my advisor professor Lennart Carleson for his knowledgeable support and encouragement.
- KTH and its Department of Mathematics (and hence all taxpayers in Sweden) for giving me a good salary during the years 1994–1999.
- professor Nikolai Makarov for inviting me to Caltech in fall 1997.
- KTH and Caltech for financial support for the stay at Caltech.
- my collegues at KTH for many interesting discussions.
- the computer programs Maple and Mathematica[1] for doing many boring computations without complaining. Without them, I would not have obtained theorems 5.2 and 6.2.
- everyone else who has contributed to this thesis with an amount that has positive real part.

Stockholm, October 1999

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[1] and LaTeX (typesetter's comment)

Abstract

Let f be a one-to-one analytic function in the unit disc with $f'(0) = 1$. Brennan's conjecture states that for every $\varepsilon > 0$

$$\int_0^{2\pi} |f'(re^{i\theta})|^{-2} \, \mathrm{d}\theta = O\left((1-r)^{-1-\varepsilon}\right) \qquad (*)$$

We do some work on the following reformulations, which we prove are equivalent.

$$\int_0^{2\pi} |f'(re^{i\theta})|^{-2} \, \mathrm{d}\theta \le \frac{C}{1-r}, \textit{where } C \textit{ is an absolute constant.} \qquad (1)$$

We propose the stronger conjecture that (for fixed r) the integral is maximized when f is the Koebe function $z(1+z)^{-2}$. To support this, we show that the Koebe function is a local maximum in the sense that analytic variations of the omitted arc decrease the integral.

If $p \ge 2$, the MacLaurin coefficients of $(f')^{-p}$ grow like $O(n^{p-1})$. (2)

We show that if $n \le 2p+1$, then the nth coefficient is maximized when f is the Koebe function. The proof is similar to de Branges' proof of the Bieberbach conjecture. As a consequence we get sharp estimates for certain higher-order Schwarzian derivatives of f. These are used to show that (*) holds with $\varepsilon = 0.547$. Earlier it was known that one can take $\varepsilon = 0.601$.

The Carleson-Makarov conjecture about β-numbers: $\sum \beta_j^2 \le 1$. (3)

We show the existence of extremal domains Ω for the sum $\sum_1^n \beta_j^p$, and use the second variation to prove that the boundary of Ω consists of trajectories of a quadratic differential which has no multiple zeros on the boundary of Ω. We also prove some estimates of extremal length, that give geometric criteria for a point to have positive β-number. This is related to the angular derivative problem.

Key words and phrases: geometric function theory, conformal mappings, univalent functions, beta numbers, extremal length estimates, the angular derivative problem, harmonic measure, integral means spectrum of the derivative, Brennan's conjecture, variational methods, second variation, the Koebe function, coefficient problems, Löwner's equation, de Branges' proof of the Bieberbach conjecture, higher-order Schwarzian derivatives.

Mathematics Subject Classification (MSC 2000): Primary 30Cxx.
Secondary 30C35, 30C50, 30C55, 30C70, 30C75, 30C85, 31A15.

Contents

CHAPTER 1

Introduction

1. Background

A differentiable map between two domains in Euclidean space $\mathbb{R}^n$ is called a *conformal mapping* if it preserves angles between intersecting curves. In dimension $n \geq 3$ the conformal mappings coincide with the Möbius transformations, that is transformations which are either a similarity transformation or an inversion

$$x \mapsto \frac{x-p}{|x-p|^2}$$

followed by a similarity transformation, see [**Nev60**] and [**Geh62**]. For this reason we restrict attention to the more interesting case of conformal maps in the plane $\mathbb{R}^2$, which we identify with the set of complex numbers $\mathbb{C}$. As usual here, we restrict the notion of conformal mapping to include only orientation-preserving maps. Then a conformal map is an analytic function of one complex variable with nonvanishing derivative. In this thesis we will in addition require that conformal maps are one-to-one.

In applications the following problem is common: Given a simply connected domain $\Omega \subset \mathbb{C}$, find a conformal mapping f of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto Ω. By the Riemann mapping theorem such an f exists (if $\Omega \neq \mathbb{C}$), but it is not easy to compute if the geometry of Ω is complicated. Therefore it is of interest to find general properties of conformal maps, as well as relations between properties of the Riemann map and the geometry of Ω. The study of conformal maps from this viewpoint is the main part of *geometric function theory*, and it was initiated in the beginning of the century with the works of Koebe.

Pointwise estimates. The most basic question about Riemann maps in general is: What can we say about the location of $f(z)$ (where z is fixed)? Of course, to say anything we need some normalization on f. As usual, let us require that $f(0) = 0$ and $f'(0) = 1$. The set of conformal maps of the unit disc with this normalization is denoted by S (from the German word "schlicht"). Koebe [**Koe07**] proved that there are upper and lower bounds on $|f(z)|$ that are independent of $f \in S$. Thus conformal maps have a certain rigidity which analytic functions in general do not have. Koebe's result implies that S is compact in the topology of locally uniform convergence. This implies that there is also bounds on the derivatives $f'(z)$, $f''(z)$, $\ldots$. To find the sharp bounds for these quantities it will suffice to consider $z = 0$, that is, to find the sharp bounds for the MacLaurin coefficients $a_n = f^{(n)}(0)/n!$. Using Gronwall's area theorem, Bieberbach [**Bie16**] proved that $|a_2| \leq 2$, with equality only for the *Koebe functions*

$$k_\lambda(z) = \frac{z}{(1+\lambda z)^2}, \qquad \text{where } |\lambda| = 1.$$

These map the unit disc onto the complement of a ray. Bieberbach's theorem leads to the distortion estimate

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{6}{1-|z|^2} \tag{1.1}$$

and to the sharp bounds of $|f'(z)|$ and $|f(z)|$:

$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3} \tag{1.2}$$

$$\frac{|z|}{(1+|z|)^2} \le |f(z)| \le \frac{|z|}{(1-|z|)^2}. \tag{1.3}$$

A more general version of (1.2) is *the distortion theorem*:

$$\left|\frac{f'(z)}{f'(\zeta)}\right| \le \frac{\left(|1-\overline{\zeta}z| + |z-\zeta|\right)^4}{(1-|\zeta|^2)(1-|z|^2)^3}, \qquad z,\zeta \in \mathbb{D}. \tag{1.4}$$

(1.3) contains the Koebe $\frac{1}{4}$ theorem:

$$f(\mathbb{D}) \supset \{w : |w| < \frac{1}{4}\}.$$

Again, the extremal functions in (1.2) and (1.3) are Koebe functions. Bieberbach expected that the Koebe functions are extremal also for $|a_n|$, that is, $|a_n| \le n$. This is the famous *Bieberbach conjecture*, which stimulated much research in conformal mapping theory. Let us mention a few results of this research that are relevant to this thesis. Löwner [**Löw23**] proved $|a_3| \le 3$ by representing conformal maps in terms of solutions to a certain differential equation. Schiffer developed a calculus of variations for conformal maps (see [**Ahl73**]), which Garabedian and Schiffer [**GaSc55**] used to prove $|a_4| \le 4$. Another version of Schiffer's method was given by Golusin, see [**Gol57**]. Milin conjectured about the MacLaurin coefficients c_n of $\log \frac{f(z)}{z}$ that the sum

$$\sum_{n=1}^{N} n(N+1-n)|c_n|^2$$

is maximized when $f \in S$ is a Koebe function. By using the relations between the a_n and the c_n one can show that Milin's conjecture implies Bieberbach's conjecture, see [**Dur83**, p. 155]. Finally, de Branges [**deB85**] proved Milin's conjecture by using Löwner's differential equation. De Branges' theorem $|a_n| \le n$ immediately gives the promised sharp bound for the derivatives of a function $f \in S$:

$$|f^{(n)}(z)| \le n! \frac{n+|z|}{(1-|z|)^{n+2}}.$$

Integral means. In the following we concentrate on how to estimate the quantity $|f'(z)|$, which is the local scaling factor of the conformal map $f \in S$. Inequalities (1.2) estimate the minimum and maximum of $|f'(z)|$ over a circle $|z| = r$. To get a measure of the overall size of $|f'|$ we can the use the integral means

$$\int_{|z|=r} |f'(z)|^t \, \mathrm{d}\theta \quad \text{or} \quad \iint_{\mathbb{D}} |f'(z)|^t \, \mathrm{d}A,$$

where $\mathrm{d}\theta$ is the angular measure $|dz|/r$ and $\mathrm{d}A$ is the area measure $\mathrm{d}x\mathrm{d}y$. If $t > 0$, these quantities measure how much the conformal map expands. If $t < 0$, they measure compression. The problem is to estimate

$$\int_{|z|=r} |f'(z)|^t \, \mathrm{d}\theta \tag{1.5}$$

as $r \to 1$. Since

$$\lim_{t\to+\infty} \left(\int_{|z|=r} |f'(z)|^t \, \mathrm{d}\theta \right)^{1/t} = \max_{|z|=r} |f'(z)|$$

we can expect that for large t, the Koebe function will maximize (1.5). Accordingly, Feng and MacGregor [**FeMG76**] proved that

$$\int_{|z|=r} |f'(z)|^t \,\mathrm{d}\theta = O\left(\left(\frac{1}{1-r}\right)^{3t-1}\right) \qquad \text{if } t > \frac{2}{5}. \tag{1.6}$$

Note that the Koebe functions k_λ have

$$\int_{|z|=r} |k_\lambda'(z)|^t \,\mathrm{d}\theta \asymp \left(\frac{1}{1-r}\right)^{\max\{3t-1,0,-t-1\}} \qquad \text{if } t \neq \frac{1}{3}, -1.$$

Moreover, it follows from de Branges' theorem and Parseval's formula that the Koebe functions maximize the integral (1.5) if $t = 2, 4, 6, \ldots$. Since

$$\lim_{t\to-\infty} \left(\int_{|z|=r} |f'(z)|^t \,\mathrm{d}\theta\right)^{1/t} = \min_{|z|=r} |f'(z)|$$

we can expect that the Koebe functions will be extremal for large negative t. This was confirmed by Carleson and Makarov [**CaMa94**] in the sense that

$$\int_{|z|=r} |f'(z)|^t \,\mathrm{d}\theta = O\left(\left(\frac{1}{1-r}\right)^{-t-1}\right) \qquad \text{if } t < t_0, \tag{1.7}$$

where t_0 is some (unspecified) absolute constant. The proof of this is based on extremal length estimates, and is much deeper than the proof of (1.6). When t has an intermediate value, one can expect that the function maximizing (1.5) should have $|f'(z)|^t$ fairly large on a fairly large set of z. For example, if $t < 0$ the extremal function (for a fixed r) should have several zeros of f' on the unit circle, and so the image domain should have several tips. To get a function with maximal growth of $\int_{|z|=r} |f'(z)|^t \,\mathrm{d}\theta$ as $r \to 1$, one should iterate this picture, and so get an image domain with a "fractal" boundary. These ideas were confirmed to some extent by a result of Makarov [**Mak86**], which was improved by Rohde [**Pom92**, Proposition 8.15]. Rohde shows that the function

$$f(z) = \int_0^z \exp\left(1.129 \sum_{\nu=1}^{\infty} \zeta^{15^\nu}\right) \mathrm{d}\zeta$$

belongs to S and satisfies

$$\int_{|z|=r} |f'(z)|^t \,\mathrm{d}\theta \geq C \left(\frac{1}{1-r}\right)^{0.117t^2} \qquad \text{for small } t.$$

The occurence of the lacunary series implies that the boundary of the image domain has a fractal structure.

To collect the information about the growth of integral means, Makarov [**Mak98**] introduced the *universal integral means spectrum* $B(t)$, which is defined as the infimum of all β such that

$$\int_{|z|=r} |f'(z)|^t \,\mathrm{d}\theta = O\left((1-r)^{-\beta}\right) \qquad \text{for all } f \in S.$$

It follows from Hölder's inequality that $B(t)$ is a convex function. The results stated above show that the graph of B looks something like the following figure. Notice the "phase transition" points t_- and $t_+ \leq 2/5$. If $t \leq t_-$ or $t \geq t_+$, then the Koebe function gives (almost) maximal growth of (1.5), but if $t_- < t < t_+$, maximal growth is faster than for the Koebe function, and is probably achieved by a function with "fractal" image domain. Carleson and Makarov [**CaMa94**] constructed such a fractal domain which shows that $t_- \leq -2$. *Brennan's conjecture*

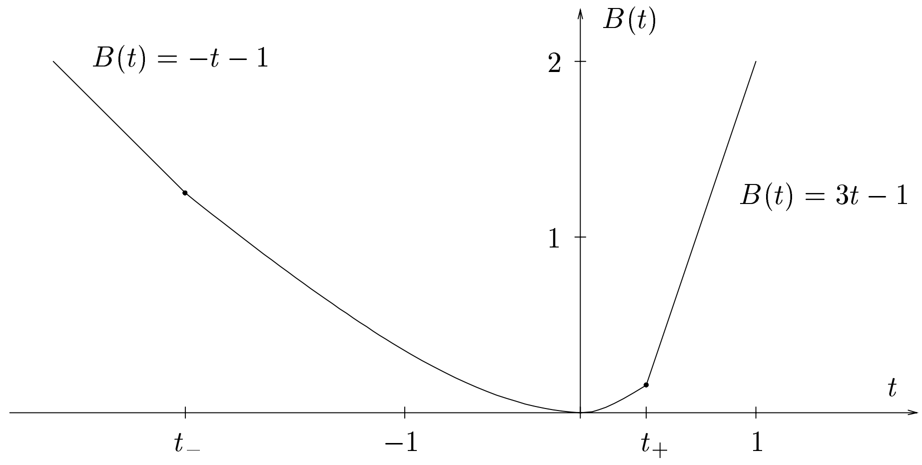

FIGURE 1. The universal integral means spectrum

states that $t_- = -2$. The known upper bound of $B(t)$ follows from the mentioned results and Pommerenke's estimates [**Pom92**, Theorem 8.5]

$$B(t) \le t - \frac{1}{2} + \left(4t^2 - t + \frac{1}{4}\right)^{1/2} \le 3t^2 + 7|t|^3, \tag{1.8}$$

$$B(-1) \le 0.601 \tag{1.9}$$

In particular,

$$B(t) \le |t| - 0.399 \qquad \text{for } t \le -1. \tag{1.10}$$

This has been improved by the author, see [**Ber98**] and Theorem 7.5.

Bounded functions. If the conformal map f is bounded, we can improve the estimates for positive t. Let $B_b(t)$ be the universal integral means spectrum for bounded conformal maps, with an analogous definition as $B(t)$. (In [**Mak98**] B_b is denoted by B, and B is denoted by $\breve{B}$.) It is easy to see that $B_b(t) = t - 1$ for $t \ge 2$. Jones and Makarov [**JoMa95**] proved that $B_b(t)$ has a phase transition point for $t = 2$. More precisely,

$$t - 1 + C_1(2-t)^2 \le B_b(t) \le t - 1 + C_2(2-t)^2 \qquad \text{for } 0 < t < 2, \tag{1.11}$$

where C_1 and C_2 are positive constants. For negative t, the spectra $B_b(t)$ and $B(t)$ agree, since [**Mak98**]

$$B(t) = \max\{B_b(t), 3t - 1\} \qquad \text{for } t \in \mathbb{R}.$$

The value of $B_b(1)$ has some special interest, since $\int_{|z|=r} |f'|\,|dz|$ is the length of the image of the circle $|z| = r$ under the conformal map f. Also, as was shown in [**CaJo92**], the slowest possible decay of the coefficients of bounded conformal maps of $\mathbb{D}$ is $O(n^{B_b(1)-1})$ (up to an ε in the exponent). The known upper bound $B_b(1) < 0.4884$ [**GrPo97**] is only slighty smaller than the trivial bound $\frac{1}{2}$. Carleson and Jones [**CaJo92**] considered Ω_c, the domain of attraction of ∞ for iterations of $z^2 + c$. Computer experiments for the Riemann map of Ω_c (when this is simply connected) indicated that $B_b(1) > 0.23$. This led Carleson and Jones to conjecture that $B_b(1) = \frac{1}{4}$. Kraetzer [**Kra96**] continued with computer experiments to calculate $\int_{|z|=r} |f'(z)|^t \, d\theta$ for f mapping $\mathbb{D}$ onto Ω_c. The results led Kraetzer to conjecture that

$$B_b(t) = \frac{t^2}{4} \quad \text{for } -2 \le t \le 2$$

Kraetzer's conjecture implies Brennan's conjecture $B(-2) = 1$, the Carleson-Jones conjecture as well as (1.11). An even more general conjecture has been proposed by Binder [**Bin97**]:

$$\int_{|z|=r} |f'(z)^t| \,\mathrm{d}\theta = O\left(\left(\frac{1}{1-r}\right)^{|t|^2/4+\varepsilon}\right) \qquad \text{for all } \varepsilon > 0,$$

where t is complex with $|t| \le 2$ and f is a bounded conformal map of $\mathbb{D}$. Notice that since $|f'(z)^t| = |f'(z)|^{\operatorname{Re} t} \exp(-\operatorname{Im} t \arg f'(z))$, this measures both the expansion/compression as well as the rotation.

We remark that the corresponding problem of estimation of integral means of $f \in S$ was solved completely by Baernstein [**Bae74**]:

$$\int_{|z|=r} |f(z)|^t \,\mathrm{d}\theta \le \int_{|z|=r} |k_\lambda(z)|^t \,\mathrm{d}\theta \qquad \text{for } t \in \mathbb{R}.$$

The correct order of growth had earlier been obtained by Prawitz [**Pra27**]. Leung [**Leu79**] used Baernstein's theory to prove

$$\int_{|z|=r} |f'(z)|^t \,\mathrm{d}\theta \le \int_{|z|=r} |k'_\lambda(z)|^t \,\mathrm{d}\theta \qquad \text{for } t \in \mathbb{R}, \tag{1.12}$$

when $f \in S$ maps onto a close-to-convex domain, that is, a domain whose complement is a union of disjoint rays (open or closed).

Brennan's conjecture. We now examine the source of inspiration of this thesis, Brennan's conjecture, more closely. Brennan [**Bre78**] originally formulated it as an estimate for conformal maps φ mapping some domain Ω onto the unit disc:

$$\iint_\Omega |\varphi'|^q \,\mathrm{d}A < \infty \tag{1.13}$$

for $\frac{4}{3} < q < 4$. Brennan's motivation for studying the L^q norm of φ' was a problem in approximation theory. By a change of variables (1.13) can be written in terms of $f = \varphi^{-1}$:

$$\iint_{\mathbb{D}} |f'|^{2-q} \,\mathrm{d}A < \infty. \tag{1.14}$$

When f is a Koebe function, and $q \le \frac{4}{3}$ or $q \ge 4$, this integral diverges. It follows from Feng's and MacGregor's result (1.6) and the distortion estimate (1.2) that the integral is finite if $\frac{4}{3} < q < 3$. Using a harmonic measure argument of Carleson, Brennan increased the upper bound to $3+\varepsilon$. It follows from Pommerenke's estimate (1.10) that (1.14) holds for $\frac{4}{3} < q < 3.399$. Dahlberg and Lewis [**Bre78**] proved that (1.13) holds for $\frac{4}{3} < q < 4$ if Ω is close-to-convex. (This also follows from (1.12).) It is easy to see that Brennan's conjecture in the form (1.14) is equivalent to

$$\int_{|z|=r} |f'(z)|^{-2} \,\mathrm{d}\theta = O\left(\left(\frac{1}{1-r}\right)^{1+\varepsilon}\right) \qquad \text{for all } \varepsilon > 0, f \in S, \tag{1.15}$$

in other words $B(-2) = 1$, or $t_- = -2$ as was stated above. We can heuristically get a "$r = 1$ version" of this estimate if we assume that f is analytic also on the unit circle. Each zero ζ of f' on the unit circle gives a contribution of the order $|f''(\zeta)|^{-2}(1-r)^{-1}$ to the integral in (1.15). Thus the "$r = 1$ version" of (1.15) should be something like

$$\sum_\zeta |f''(\zeta)|^{-2} \le \text{constant},$$

where the sum is taken over all zeros of f' on the unit circle. An estimate of this type occurs in [**CaMa94**]. The authors define a so-called β-number $\beta_\Omega(a,b)$ whenever Ω is a simply connected domain and a, b are distinct point in the closure of Ω. For the case of a function $f \in S$ with image domain Ω and ζ a zero of f' on the unit circle we have

$$\beta_\Omega(f(\zeta), 0) = \frac{2|f(\zeta)|^2}{|f''(\zeta)|}. \tag{1.16}$$

Carleson and Makarov showed that Brennan's conjecture is related to the estimate

$$\sum_\zeta \beta_\Omega(f(\zeta), 0)^2 \leq \text{absolute constant}, \tag{1.17}$$

where the sum again is taken over all zeros of f' on the unit circle. Note that the corresponding points $f(\zeta)$ are "tip points" of the boundary of Ω. If both ζ and ζ_0 are zeros of f' on the unit circle, we have

$$\beta_\Omega(f(\zeta), f(\zeta_0)) = \frac{4|f(\zeta) - f(\zeta_0)|^2}{|\zeta - \zeta_0|^4 |f''(\zeta) f''(\zeta_0)|}. \tag{1.18}$$

Carleson and Makarov proved that both Brennan's conjecture and (1.17) follows from

$$\sum_\zeta \beta_\Omega(f(\zeta), f(\zeta_0))^2 \leq 1, \tag{1.19}$$

where the sum is taken over all zeros $\zeta \neq \zeta_0$ of f' on the unit circle. The advantage with this formulation is that we know the extremal case: The Koebe function $z(1+z)^{-2}$. (The sum in (1.19) should actually be taken over tip points $f(\zeta)$. For the Koebe function these are $\frac{1}{4}$ and ∞. A definition more general than (1.18) is used when ∞ is a tip point.) More generally, Carleson and Makarov showed that $B(-p) = p - 1$ if

$$\sum_{j=1}^{n} \beta_\Omega(a_j, b)^p \leq 1, \tag{1.20}$$

where $a_1, \ldots, a_n, b$ are the tip points of the boundary of Ω. The case $n = 2$, $p = 2$ of this inequality was proved in [**CaMa94**] by computing the extremal domains of the sum in (1.20) using Schiffer's method of boundary variation.

In [**BaVoZd98**] the authors consider the conformal map from the unit disc to Ω_c, the domain of attraction of ∞ for iterations of $z^2 + c$ (when this is simply connected). They prove that (1.14) holds for $2 < q < 4$, thus establishing Brennan's conjecture in this special case. A similar (easier) argument works for iterations of $z^d + c$, where $d > 2$. In [**HuSt98**] the authors consider an analogue of Brennan's conjecture for K-quasiconformal maps φ mapping a domain Ω onto the unit disc. They prove that

$$\iint_\Omega |\nabla\varphi|^p \, \mathrm{d}A < \infty$$

holds for all such φ if and only if

$$\frac{4K}{2K+1} < p < \frac{2Kq_0}{(K-1)q_0 + 2},$$

where q_0 is the largest number such that (1.13) holds for $\frac{4}{3} < q < q_0$.

Boundary distortion. The universal integral means spectrum $B(t)$ for negative t gives a bound on how much a conformal map f from $\mathbb{D}$ to Ω may compress subsets of $\mathbb{D}$. This has a counterpart for the corresponding map of the unit circle to the boundary of Ω. (This map can be defined as a radial limit for nearly all points on the unit circle.) Namely, we have the following inequality between the Hausdorff dimensions of a Borel set A on the unit circle and its image $f(A)$, see [**Mak87**].

$$\dim f(A) \geq \frac{p \dim A}{B(-p) + p + 1 - \dim A} \quad \text{for } p > 0.$$

Together with Pommerenke's estimate $B(-p) < 3p^2$ for small $p > 0$, this implies Makarov's famous dimension theorem [**Mak85**]:

$$\dim A = 1 \quad \Longrightarrow \quad \dim f(A) \geq 1.$$

Another way to measure the compression of the boundary map is to count the number of disjoint discs D of a certain radius that have $|f^{-1}(D \cap \partial\Omega)| \geq h$. Here, $|\ |$ denotes angular measure on the unit circle, and $\partial\Omega$ is the boundary of Ω. Recall that the measure $\omega(E) = |f^{-1}(E \cap \partial\Omega)|/2\pi$ is called harmonic measure of Ω (with base point $f(0)$). Recall also Beurling's estimate $\omega(D) \leq C\sqrt{\rho}$ when D is a disc of radius ρ. Thus a disc D can be considered to have large harmonic measure if $\omega(D) \geq \rho^{\frac{1}{2}+\delta}$, where $\delta > 0$ is small. Let $N(\Omega, \rho, \delta)$ be the maximum number of disjoint such discs. Carleson and Makarov [**CaMa94**] proved that $B(-p) = p - 1$ follows from the estimate

$$N(\Omega, \rho, \delta) \leq C\rho^{-2p\delta}.$$

They also proved this estimate for some (unspecified) p. See [**Mak87**] and [**Mak98**] for relations between different ways to measure boundary distortion of conformal maps (equivalently, ways to characterize the distribution of harmonic measure).

2. Summary of the thesis

Chapter 2: β-numbers. As a preparation for Chapter 3 and 4 we start with a chapter about β-numbers. We define β-numbers in terms of extremal length, almost as in [**CaMa94**]. In [**CaMa94**] was given a formula for $\beta_\Omega(a, b)$ in terms of a conformal map f from the unit disc to Ω in the case when f is sufficiently regular at the boundary, cf. (1.16). We generalize this formula to the general case as follows. Let $b \in \Omega \setminus \{\infty\}$. We show that the boundary map gives a one-to-one correspondance between

(i) Points $a \in \partial\Omega \setminus \{\infty\}$ such that $\beta_\Omega(a, b) > 0$.
(ii) Points ζ on the unit circle with $\operatorname{anglim}_{z\to\zeta} \left|\frac{z-\zeta}{f'(z)}\right| > 0$.

(anglim denotes limit within Stolz angles.) For such points

$$\beta_\Omega(a, b) = \frac{2|a - b|^2}{|1 - \overline{z_b}\zeta|^4} \frac{1 - |z_b|^2}{|f'(z_b)|} \operatorname*{anglim}_{z\to\zeta} \left|\frac{z - \zeta}{f'(z)}\right|,$$

where $f(z_b) = b$, see Theorem 2.1. We also show that the angular limit above exists for all ζ on the unit circle, see Proposition 2.21. We give a similar formula in the case when both a and b are on the boundary of Ω, see Theorems 2.22 and 2.23.

We consider the problem of characterizing the geometry of the boundary $\partial\Omega$ around a point a with $\beta_\Omega(a, b) > 0$ (where $b \in \Omega$). We will see that $\partial\Omega$ has to be rather thin and straight, but it can spiral around a if does so sufficiently slowly. To be more precise, let $a = 0$ and let α_k be the minimal angle (with apex 0) that contains that part of the complement of Ω that is in the annulus $e^{-k} \leq |z| \leq e^{-k+1}$. Then

$$\sum_1^\infty \alpha_k < \infty \implies \beta_\Omega(0, b) > 0 \implies \sum_1^\infty \frac{\alpha_k^2}{\log(9/\alpha_k)} < \infty,$$

see Corollary 2.36 and Corollary 2.25. If the boundary of Ω is a curve $\phi = \phi(r)$ in polar coordinates, where $\phi(r)$ is monotone, then this can be improved:

$$\sum_1^\infty \alpha_k^2 \log\frac{9}{\alpha_k} < \infty \implies \beta_\Omega(0, b) > 0 \implies \sum_1^\infty \alpha_k^2 < \infty,$$

see Corollary 2.38. A version of the first implication also holds for certain non-monotone $\phi(r)$ not having too large and quick oscillations, see Theorem 2.39.

Proving these results boils down to estimating extremal distance in strip domains. A similar problem arises when one seeks geometric conditions on $\Omega = f(\mathbb{D})$ for the angular derivative

$$f'(\zeta) = \operatorname*{anglim}_{z\to\zeta} f'(z)$$

to exist, see [**RoWa77**]. Thus our extremal distance estimates also have some importance for the angular derivative problem, see Section 2.6.

Chapter 3: Brennan's conjecture. We prove that a number of reformulations of Brennan's conjecture are equivalent. Among other things, we prove that $B(-p) = p - 1$ is equivalent to the following statements:

(i) For any simply connected domain Ω and distinct points $a_1, \dots, a_m, b \in \partial\Omega$,

$$\sum_1^m \beta_\Omega(a_j, b)^p \le 1.$$

(ii) For any $\varepsilon > 0$ there exists a $C(\varepsilon)$ such that for any simply connected domain $\Omega \ni \infty$ with $\operatorname{diam} \partial\Omega = 1$

$$N(\Omega, \rho, \delta) \le C(\varepsilon)\rho^{-2p\delta-\varepsilon},$$

where $N(\Omega, \rho, \delta)$ is the maximum number of disjoint discs of radius ρ with harmonic measure $\ge \rho^{\frac{1}{2}+\delta}$ (with respect to ∞).

The proofs of most of these equivalences were sketched in [**CaMa94**]. Our proof is in part an elaboration of the ideas in [**CaMa94**]. What is new (and easy to prove) is that another equivalent formulation is: There exists a constant $C = C(p)$ such that for all $f \in S$

$$\int_{|z|=r} |f'(z)|^{-p}\,\mathrm{d}\theta \le \frac{C}{(1-r)^{p-1}}, \qquad 0 \le r < 1. \tag{1.21}$$

Chapter 4: Extremals for $\sum \beta^p$. If Brennan's conjecture is false, there exists a simply connected domain Ω and boundary points $a_1, \dots, a_m, b$ such that

$$\sum_{j=1}^m \beta_\Omega(a_j, b)^2 > 1, \tag{1.22}$$

where m is minimal, see statement (i) above. In an effort to prove Brennan's conjecture we consider configurations $(\Omega, a_1, \dots, a_m, b)$ which maximize this sum, and then try to derive a contradiction by using the first variation equation and the second variation inequality. First of all we prove that maximizing configurations exist, Theorem 4.1. By using Schiffer's method of interior variation, we prove that the complement of an extremal domain Ω consists of trajectories of the quadratic differential

$$\sum_{j=1}^m \frac{\beta_\Omega(a_j, \infty)^2}{(a_j - w)^2}\mathrm{d}w^2$$

(assuming $b = \infty$). Moreover, a conformal map f of the upper half plane onto Ω normalized by $f(\infty) = \infty$ satisfies the differential equation

$$f'(z)^2 \sum_{j=1}^m \frac{\beta_\Omega(a_j, \infty)^2}{(a_j - f(z))^2} = \sum_{j=1}^m \frac{4\beta_\Omega(a_j, \infty)^2}{(x_j - z)^2},$$

where $f(x_j) = a_j$, see Theorem 4.2. We also use the second variation to derive a number of inequalities that hold for the extremal configuration. Most of these results have been obtained independently by O'Neill [**O'N99**]. We use the second variational inequalities to rule out multiple zeros of the above-mentioned quadratic differential on the complement of Ω. This means that the complement of Ω is a tree consisting of arcs such at each junction point, exactly three arcs meet.

Unfortunately, the author has not been able to use the information contained in the mentioned equations and inequalities to derive a contradiction. For generality, we actually do all this with the exponent $p > 0$ instead of 2.

Chapter 5: Integral means. By (1.21) Brennan's conjecture can be stated: There exists a constant C such that for all $f \in S$

$$\int_{|z|=r} |f'(z)|^{-2} \, d\theta \le C \int_{|z|=r} |k'(z)|^{-2} \, d\theta, \qquad 0 \le r < 1, \tag{1.23}$$

where k is the Koebe function $k(z) = z(1+z)^{-2}$. We conjecture that we can take $C = 1$, which is not far-fetched in view of all instances where the Koebe function is extremal. To support this, we prove a result which almost shows that the Koebe function is a local maximum of the the functional

$$f \mapsto \int_{|z|=r} |f'(z)|^{-2} \, d\theta$$

on S (r is fixed). More precisely, we show that if distort the Koebe slit $[\frac{1}{4}, +\infty]$ with a conformal mapping $w \mapsto w + \varepsilon v_1(w)$, the corresponding Riemann map f_ε will satisfy

$$\int_{|z|=r} \left| \frac{f_\varepsilon'(0)}{f_\varepsilon'(z)} \right|^2 \, d\theta \le \int_{|z|=r} |k'(z)|^{-2} \, d\theta$$

if ε is small (depending on v_1 and r). (Actually, we prove this only when $0.91 < r < 1$.) Since the algebraic computations in the proof of this result are so laborious, we use the computer program Mathematica, see the Appendix.

In Section 5.3 we give a some relations between integral means and coefficients for functions $(f')^p$, where $f \in S$. Denote the MacLaurin coeffients of a function g by $c_n(g)$. Let $p \in \mathbb{R}$ and $\beta > 0$. We prove that the following statements are equivalent:

(i) There exists a constant C_1 such that

$$\int_{|z|=r} |f'(z)|^p \, d\theta \le \frac{C_1}{(1-r)^\beta} \quad \text{for } 0 < r < 1 \text{ and all } f \in S.$$

(ii) There exists a constant C_2 such that

$$|c_n((f')^p)| \le C_2 n^\beta \quad \text{for } n \ge 1 \text{ and all } f \in S.$$

The implication (i) $\Longrightarrow$ (ii) follows easily from Cauchy's integral formula. The reversed implication was proved in [**CaJo92**] for $p = 1$, and our proof is an easy generalization of that special case. It follows that Brennan's conjecture can be stated

$$|c_n((f')^p)| \le C_p |c_n((k')^p)| \quad \text{for } n \ge 0,\ f \in S,\ p \le -2. \tag{1.24}$$

This does not hold for $-2 < p < 0$, since then $B(p) > \max\{-p-1, 0\}$.

Chapter 6: De Branges' method. We show how the ideas used in de Branges' proof of Milin's conjecture can be used to solve other extremal problems for conformal mappings. This method has been described at a higher level of sophistication in the papers [**VaNi91, VaNi92**] in an operator-theoretic setting. See [**HeWe96**] for an exposition in the language of systems theory. We analyse how to attack certain problems of the following type: Let $f \in S$ and form a function $G(f)$ analytic in $\mathbb{D}$ by

$$G(f)(z) = \Phi(z, f(z), f'(z), f''(z), \ldots, f^{(\nu)}(z)),$$

where Φ is some fixed analytic function. Let $H_1, H_2, \ldots$ be real constants. Show that the quantity

$$\sum_{n=0}^{\infty} H_n |c_n(G(f))|^2$$

has maximum when f is the Koebe function $k(z) = z(1+z)^{-2}$.

We use the method to prove the following results:
(a) If $p < -\frac{1}{8}$ and r is small, then

$$\int_{|z|=r} |f'(z)|^p \, d\theta \le \int_{|z|=r} |k'(z)|^p \, d\theta \qquad \text{for all } f \in S.$$

This follows from Theorem 6.1 and Lemma 6.6. Thus (1.23) holds with $C = 1$ if r is small.
(b) If $p < 0$ and $n \le 2|p| + 1$, then (1.24) holds with $C_p = 1$, see Theorem 6.2.

Chapter 7: Generalized Schwarzian derivatives. The differential operator

$$S_n f = (f')^{\frac{n-1}{2}} D^n (f')^{-\frac{n-1}{2}}$$

satisfies a "chain rule"

$$S_n(f \circ \tau) = ((S_n f) \circ \tau)\,(\tau')^n \quad \text{if } \tau \text{ is a Möbius transformation.} \tag{1.25}$$

In conjunction with (b) above this gives the sharp estimate

$$|S_n f(z)| \le \frac{K_n}{(1-|z|^2)^n}, \qquad |z| < 1, \tag{1.26}$$

for conformal maps f of the unit disc, where $K_n = (n-1)(n+1)(n+3)\ldots(3n-3)$, see Theorem 7.1. These results are well-known in the case $n = 2$, since S_2 is the Schwarzian derivative times a constant factor. Another set of operators that satisfy (1.25) is Peschl's generalized Schwarzian derivatives:

$$P_n f = (f')^n (D^{n-2} S(f^{-1})) \circ f,$$

where S is the Schwarzian derivative. Klouth and Wirths [**Klo89**] proved that P_n also satisfies an estimate of the type (1.26). In Theorem 7.3 we prove that $S_n f$ is a polynomial in $P_2 f, \ldots, P_n f$ with positive coefficients. Thus our estimate (1.26) follows from the estimate of Klouth and Wirths. We use (1.26) to derive an estimate for

$$\int_{|z|=r} |f'(z)|^{-n+1} \, d\theta, \quad f \in S,$$

see Theorem 7.5. The proof of this estimate is a generalization of Pommerenke's proof of the special case $n = 2$, see (1.9). In particular we get $B(-2) < 1.547$ and $B(-3) < 2.530$. We have thus improved the estimate (1.10). As a consequence, we get that (1.13) holds for $\frac{4}{3} < q < 3.421$.

The content of sections 6.4, 6.5, 7.1, 7.3 have earlier been published in [**Ber98**]. We reprint a modified version of this material with the permission of Institut Mittag-Leffler, which has the copyright.

3. Notation

$\mathbb{R}$ and $\mathbb{C}$ are the sets of real and complex numbers. $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is the upper half plane. $D(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$ is the disc with centre a and radius r. We define $D(\infty, r) = \{z \in \hat{\mathbb{C}} : |z| > r^{-1}\}$. $\mathbb{D} = D(0,1)$ is the unit disc, and $\mathbb{D}^* = D(\infty, 1)$ is the exterior of the unit circle. $C(a,r) = \{z \in \mathbb{C} : |z-a| = r\}$ is the circle with centre a and radius r. We define $C(\infty, r) = C(0, r^{-1})$. $\mathbb{T} = C(0,1)$ is the unit circle. $[a,b]$ is the closed line segment between the points $a, b \in \mathbb{C}$. (a,b) is the corresponding open segment. For a set $\Omega \subset \hat{\mathbb{C}}$, $\Omega^c = \hat{\mathbb{C}} \setminus \Omega$ is its complement, $\partial\Omega$ is its boundary, and $\overline{\Omega} = \Omega \cup \partial\Omega$ is its closure. $\operatorname{diam}\Omega = \sup\{|z-w| : z, w \in \Omega\}$ is the diameter of Ω. $\operatorname{dist}(z, \Omega) = \inf\{|z-w| : w \in \Omega\}$ is the distance from z to Ω.

Absolute signs $|\ |$ denote the length of a curve or a vector in $\mathbb{R}^2$. $\mathrm{d}\theta = |\mathrm{d}z|/r$ is the angular measure on a circle $|z| = r$. $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ is the area measure. $\omega(E, \Omega, z)$ is the harmonic measure of the set E with respect to the domain Ω and the base point z. If $x \in \mathbb{R}$ we let $x_+ = \max\{x, 0\}$. The statement $x \asymp y$ means that there exists positive constants A and B such that $Ax \le y \le Bx$. It should be clear from the context if A and B are absolute constants or if they depend on some parameters.

The set of all analytic (=holomorphic) functions in (a neighbourhood of) the set Ω is denoted by $A(\Omega)$. The angular limit of $f \in A(\mathbb{D})$ as $z \to \zeta \in \mathbb{T}$ is denoted by $\operatorname{anglim}_{z\to\zeta} f(z)$ or simply $f(\zeta)$. The nth MacLaurin coefficient of a function f is denoted by $c_n(f)$. We will use the term conformal mapping in a slightly more general way than above. Namely, a conformal mapping is an injective (one-to-one) meromorphic function in a domain in $\hat{\mathbb{C}}$. A univalent function is the same as a conformal mapping. S is the set of all univalent functions $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 0$ and $f'(0) = 1$. Σ is the set of all univalent functions $f : \mathbb{D}^* \to \hat{\mathbb{C}}$ with $f(\infty) = \infty$ and $f'(\infty) = 1$. Among the Koebe functions $z(1+\lambda z)^{-2}$, $|\lambda| = 1$ we single out $k(z) = z(1+z)^{-2}$ as *the* Koebe function. This is convenient for us since we will deal with $(k')^{-1}$, which has positive MacLaurin coefficients.

CHAPTER 2

β-numbers

1. Overview

Let Ω be a simply connected domain whose boundary has a tip point a. That is, in a neighbourhood of a the boundary is a smooth arc ending at a. The β-number of a (with respect to $b \in \Omega$) measures the density of harmonic measure (with respect to b) at a. Namely, in the special case $b = \infty$,

$$\beta_\Omega(a, \infty) = \frac{\operatorname{cap} \partial\Omega}{4} \lim_{|\gamma| \to 0} \frac{\omega(\gamma, \Omega, \infty)^2}{|\gamma|},$$

where γ denotes an arc of $\partial\Omega$ ending at a, with length $|\gamma|$. $\operatorname{cap} \partial\Omega$ is the logarithmic capacity of the boundary of Ω. In terms of a conformal mapping f from the exterior of the unit disc $\mathbb{D}^* = \{z \in \hat{\mathbb{C}} : |z| > 1\}$ onto Ω with $f(\infty) = \infty$, this can be written

$$\beta_\Omega(a, \infty) = \frac{2|f'(\infty)|}{|f''(\zeta_a)|}, \tag{2.1}$$

where $\zeta_a = f^{-1}(a)$ is on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. There is also a version of β-numbers when both a and b are tip points of $\partial\Omega$:

$$\beta_\Omega(a, b) = \frac{4|a - b|^2}{|\zeta_a - \zeta_b|^4 |f''(\zeta_a) f''(\zeta_b)|}. \tag{2.2}$$

For the purpose of discussing Brennan's conjecture, these formulas are sufficient. However, we consider the problem of extending the definition to arbitrary simply connected domains Ω and arbitrary points a, b in the closure $\overline{\Omega}$ of Ω. Such a definition was given in [**CaMa94**] in terms of extremal length. Using a variant of this definition we prove the following generalization of (2.1).

Theorem 2.1. *Let $f : \mathbb{D} \to \Omega$ be a conformal bijection. For every $\zeta \in \mathbb{T}$, the angular limit*

$$\Gamma_f(\zeta) := \operatorname*{anglim}_{z \to \zeta} \left| \frac{z - \zeta}{f'(z)} \right| \text{ exists (finite).} \tag{2.3}$$

Let $z_b \in \mathbb{D}$ and $b = f(z_b) \neq \infty$. If $\Gamma_f(\zeta) > 0$ then

$$a = \operatorname*{anglim}_{z \to \zeta} f(z) \text{ exists (finite)}$$

and

$$\beta_\Omega(a, b) = \frac{2|a - b|^2}{|1 - \overline{z_b}\zeta|^4} \frac{1 - |z_b|^2}{|f'(z_b)|} \Gamma_f(\zeta).$$

Moreover, every $a \in \partial\Omega \setminus \{\infty\}$ with $\beta_\Omega(a, b) > 0$ is gotten in this way from a unique $\zeta \in \mathbb{T}$.

An analogous generalization of (2.2) holds, see Theorem 2.22.

We also consider the problem of characterizing the geometry of $\partial\Omega$ around a point a with $\beta_\Omega(a, b) > 0$. It turns out that $\partial\Omega$ has to be thin and rather straight, but it can spiral around a if it does so sufficiently slowly. More precisely, let $a = 0$ and let

α_k be the minimal angle (with apex 0) that contains $\Omega^c \cap \{z : e^{-k} \le |z| \le e^{-k+1}\}$. ($\Omega^c = \hat{\mathbb{C}} \setminus \Omega$ is the complement of Ω.) Then, for $b \in \Omega$ we have the implications

$$\sum_1^\infty \alpha_k < \infty \implies \beta_\Omega(0,b) > 0 \implies \sum_1^\infty \frac{\alpha_k^2}{\log(9/\alpha_k)} < \infty.$$

If $\partial\Omega$ is a curve $\phi = \phi(r)$ in polar coordinates, where $\phi(r)$ is monotone, then this can be improved:

$$\sum_1^\infty \alpha_k^2 \log \frac{9}{\alpha_k} < \infty \implies \beta_\Omega(0,b) > 0 \implies \sum_1^\infty \alpha_k^2 < \infty.$$

A version of the first implication also holds for certain non-monotone $\phi(r)$ which do not have too large and quick oscillations, see Theorem 2.39. Suppose next that on each interval $e^{-k} \le r \le e^{-k+1}$ the function $\phi(r)$ is monotone and satisfies $|\phi'(r)| \asymp |\phi'(e^{-k})|$. That is, there are positive constants such that $A|\phi'(e^{-k})| \le |\phi'(r)| \le B|\phi'(e^{-k})|$. Then

$$\beta_\Omega(0,b) > 0 \iff \sum_1^\infty \alpha_k^2 < \infty.$$

Proving these results boils down to estimating extremal distance in strip domains. The same problem arises when one seeks geometric conditions on $\Omega = f(\mathbb{D})$ for the angular derivative

$$f'(\zeta) = \operatorname*{anglim}_{z\to\zeta} f'(z)$$

to exist, see [**RoWa77**]. The kind of strip domains arising in the angular derivative problem are however more "flabby", which probably makes that problem harder. Some of our extremal distance estimates give new insights for the angular derivative problem. For example, consider the strip domain in Figure 2 made up of $n+1$ unit squares.

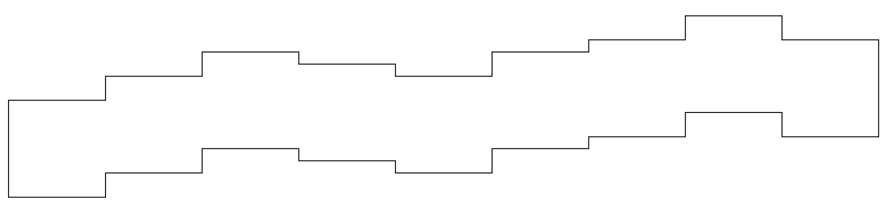

FIGURE 2. A strip domain

Let $\mu_1, \dots, \mu_n$ be the sizes of the "steps" (which are assumed to be small). Let d be the extremal distance between the two vertical sides (of length 1). We prove

$$d - n \asymp \sum_1^n \mu_k^2 \log \frac{1}{\mu_k}. \tag{2.4}$$

(Actually, we will do it for a slightly different domain.) A related statement was proved in [**War71**] and [**Eke71**] using the Poisson integral. We have thus solved Problem 2 of [**RoWa77**]: To prove (2.4) with extremal length methods.

2. Definition of β-numbers

The purpose of this section is to define β-numbers in terms of reduced extremal distance. First recall the definition of extremal length:

Definition 2.2. Let Γ be a family of rectifiable curves in a Borel set $\Omega \subset \hat{\mathbb{C}}$. By a *metric* we mean a non-negative Borel measureable function ρ in Ω. The *ρ-length* of Γ and the *ρ-area* of Ω are

$$L_\rho(\Gamma) = \inf_{\gamma\in\Gamma} \int_\gamma \rho(z)\,|\mathrm{d}z| \quad\text{and}\quad A_\rho(\Omega) = \iint_\Omega \rho^2\,\mathrm{d}A,$$

where $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ is the usual area measure. The *extremal length* of the family Γ is

$$\lambda(\Gamma) = \sup_\rho \frac{L_\rho(\Gamma)^2}{A_\rho(\Omega)},$$

where the supremum is taken over all metrics with $0 < A_\rho(\Omega) < +\infty$. Note that $\lambda(\Gamma)$ does not depend on Ω.

We will use the following properties of extremal length. See [**Ahl73**] for proofs.

Conformal invariance: If $f : \Omega \to \hat{\mathbb{C}}$ is conformal, then

$$\lambda(f(\Gamma)) = \lambda(\Gamma)$$

for every curve family Γ in Ω. Here, $f(\Gamma)$ is the family of curves $f(\gamma)$ where $\gamma \in \Gamma$.

The comparison principle: Let Γ and Γ' be two curve families such that every $\gamma \in \Gamma$ contains a $\gamma' \in \Gamma'$. Then

$$\lambda(\Gamma) \geq \lambda(\Gamma').$$

The serial rule: Let Γ_1 and Γ_2 be curve families in the disjoint domains Ω_1 and Ω_2. Let Γ be a curve family such that every $\gamma \in \Gamma$ contains both a $\gamma_1 \in \Gamma_1$ and a $\gamma_2 \in \Gamma_2$. Then

$$\lambda(\Gamma) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2).$$

Definition 2.3. Let Ω be a domain, and E_1 and E_2 subsets of $\hat{\mathbb{C}}$. The *extremal distance* between E_1 and E_2 in Ω is

$$d_\Omega(E_1, E_2) = \lambda(\Gamma),$$

where Γ is the family of curves in Ω connecting E_1 and E_2.

Example 2.4. If Ω is a sector of angle θ of the annulus $r < |z-a| < R$, then

$$d_\Omega(C(a,r), C(a,R)) = \frac{1}{\theta}\log\frac{R}{r},$$

where $C(a,r)$ is the circle with centre a and radius r. In particular,

$$d_{\hat{\mathbb{C}}}(C(a,r), C(a,R)) = \frac{1}{2\pi}\log\frac{R}{r} \qquad \text{if } r < R.$$

Extremal distance is a conformally invariant way of measuring the distance between sets E_1 and E_2 within Ω. When E_1 and E_2 reduce to points, the extremal distance becomes infinite. As a substitute for extremal distance for two points $a, b \in \Omega$ we can use the finite limit

$$\lim_{r,s\to 0} d_\Omega(C(a,r), C(b,s)) + \frac{1}{2\pi}\log r + \frac{1}{2\pi}\log s.$$

Since this quantity can be negative, we prefer to use instead the *reduced extremal distance* (cf. [**Ahl73**, Section 4-14])

$$\delta_\Omega(a,b) = \lim_{r,s\to 0} d_\Omega(C(a,r), C(b,s)) - d_{\hat{\mathbb{C}}}(C(a,r), C(b,s)) \geq 0, \tag{2.5}$$

which has the bonus of being invariant under Möbius transformations, see Proposition 2.12 below. Note that we define $C(\infty, r)$ to be the circle $C(0, r^{-1})$. Consider now the case when $a, b \in \partial\Omega$. If $\partial\Omega$ is piecewise smooth, then (2.5) is finite only when $\partial\Omega$ has inward cusps at a and b. There are however some exceptions like $\Omega = \mathbb{C}\backslash[0, +\infty)$, where (2.5) gives a finite $\delta_\Omega(1, \infty)$. This is due to the fact that the point 1 corresponds to two prime ends of Ω. To remedy this we use the following definition for general a and b.

Definition 2.5. Let Ω be a simply connected domain and let C_1 and C_2 be disjoint Jordan curves (or arcs). Define

$$\tilde{d}_\Omega(C_1, C_2) = \inf_{I_1, I_2} d_\Omega(I_1, I_2),$$

where the infimum is taken over all component arcs I_1, I_2 of the sets $C_1 \cap \Omega$ and $C_2 \cap \Omega$ respectively.
The *excess of extremal distance* in Ω between C_1 and C_2 is

$$X_\Omega(C_1, C_2) = \tilde{d}_\Omega(C_1, C_2) - d_{\hat{\mathbb{C}}}(C_1, C_2).$$

Let $a, b \in \overline{\Omega}$ be distinct. The *reduced extremal distance* between a and b in Ω is

$$\delta_\Omega(a, b) = \lim_{r,s \to 0} X_\Omega(C(a, r), C(b, s)).$$

The existence of the limit (which can be $+\infty$) follows from Lemma 2.8 and 2.9 below.

Lemma 2.6. $X_\Omega(C_1, C_2) \geq 0$.

Proof. This follows immediately from the comparison principle. □

Lemma 2.7. *If C separates C_1 from C_2 in Ω, then*

$$\tilde{d}_\Omega(C_1, C_2) \geq \tilde{d}_\Omega(C_1, C) + \tilde{d}_\Omega(C, C_2).$$

Proof. Let I_j be components of $C_j \cap \Omega$. There exists a component I of $C \cap \Omega$ that separates I_1 and I_2 in Ω, see [**Pom92**, Proposition 2.13]. By the serial rule

$$d_\Omega(I_1, I_2) \geq d_\Omega(I_1, I) + d_\Omega(I, I_2),$$

and the lemma follows. □

Lemma 2.8. $\tilde{d}_\Omega(C(a, r), C(b, s)) + \frac{1}{2\pi} \log rs$ *is a decreasing function of r (and of s) for small r and s.*

Proof. Let $r_1 < r_2$. By Lemma 2.7

$$\tilde{d}_\Omega(C(a, r_1), C(b, s)) \geq \tilde{d}_\Omega(C(a, r_1), C(a, r_2)) + \tilde{d}_\Omega(C(a, r_2), C(b, s)).$$

By Lemma 2.6 and Example 2.4

$$\tilde{d}_\Omega(C(a, r_1), C(a, r_2)) \geq \frac{1}{2\pi} \log \frac{r_2}{r_1}.$$

Thus

$$\tilde{d}_\Omega(C(a, r_1), C(b, s)) + \frac{1}{2\pi} \log r_1 \geq \tilde{d}_\Omega(C(a, r_2), C(b, s)) + \frac{1}{2\pi} \log r_2.$$
□

Lemma 2.9. *If $a \in \mathbb{C}$ then*

$$d_{\hat{\mathbb{C}}}(C(a, r), C(\infty, s)) = \frac{1}{2\pi} \log \frac{1}{rs} + O(s) \qquad \text{as } s \to 0. \tag{2.6}$$

If $a, b \in \mathbb{C}$ are distinct, then

$$d_{\hat{\mathbb{C}}}(C(a, r), C(b, s)) = \frac{1}{2\pi} \log \frac{|a - b|^2}{rs} + O(r + s) \qquad \text{as } r, s \to 0. \tag{2.7}$$

Proof. a) Let $f(z) = z - a$. The circle $f(C(\infty, s)) = C(-a, 1/s)$ separates two circles $C(0, 1/s_1)$ and $C(0, 1/s_2)$, where $s_j = s + O(s^2)$ and $s_1 < s_2$. By the comparison principle

$$d_{\hat{\mathbb{C}}}(C(0,r), C(0,1/s_2)) \leq d_{\hat{\mathbb{C}}}(f(C(a,r)), f(C(\infty,s))) \leq d_{\hat{\mathbb{C}}}(C(0,r), C(0,1/s_1)).$$

(2.6) now follows from the conformal invariance of extremal distance and Example 2.4.
b) Let $f(z) = 1/(z-b)$. Since f is conformal, the circle $f(C(a,r))$ lies between two circles $C(f(a), r_j))$ with $r_j = |f'(a)|r + O(r^2)$ and $r_1 < r_2$. Similarly, the circle $f(C(b,s))$ lies between two circles $C(\infty, s_j)$ with $s_j = s + O(s^2)$ and $s_1 < s_2$. By comparison

$$d_{\hat{\mathbb{C}}}(C(f(a), r_2), C(\infty, s_2)) \leq d_{\hat{\mathbb{C}}}(f(C(a,r)), f(C(b,s))) \leq d_{\hat{\mathbb{C}}}(C(f(a), r_1), C(\infty, s_1)).$$

Now (2.7) follows from (2.6) and the conformal invariance of $d_{\hat{\mathbb{C}}}$. □

We can now define β-numbers:

Definition 2.10. Let Ω be a simply connected domain and a, b distinct points in $\overline{\Omega}$. Define

$$\beta_\Omega(a,b) = e^{-\delta_\Omega(a,b)}.$$

Proposition 2.11. $\delta_\Omega(a,b) \geq 0$ *and* $0 \leq \beta_\Omega(a,b) \leq 1$.

Proof. This follows immediately from Lemma 2.6. □

Proposition 2.12. *β is a Möbius invariant, that is,*

$$\beta_{\sigma(\Omega)}(\sigma(a), \sigma(b)) = \beta_\Omega(a,b)$$

when σ is a Möbius transformation.

Proof. Let $r, s > 0$ be small. Then $\sigma(C(a,r))$ is contained in a disc $D(\sigma(a), r')$, where $r' = |\sigma'(a)|r + O(r^2)$. (In case $a = \infty$ or $\sigma(a) = \infty$ we have to interpret σ' in a special way.) Similarly, there is an $s' = |\sigma'(b)|s + O(s^2)$ so that $\sigma(C(b,s)) \subset D(\sigma(b), s')$. From conformal invariance and Lemma 2.7 it follows that

$$\begin{aligned} \tilde{d}_\Omega(C(a,r), C(b,s)) &= \tilde{d}_{\sigma(\Omega)}(\sigma(C(a,r)), \sigma(C(b,s))) \\ &\geq \tilde{d}_{\sigma(\Omega)}(C(\sigma(a), r'), C(\sigma(b), s')). \end{aligned} \tag{2.8}$$

In a similar way we get

$$d_{\hat{\mathbb{C}}}(C(a,r), C(b,s)) \leq d_{\hat{\mathbb{C}}}(C(\sigma(a), r''), C(\sigma(b), s'')), \tag{2.9}$$

where $r'' = |\sigma'(a)|r + O(r^2)$ and $s'' = |\sigma'(b)|s + O(s^2)$. By Lemma 2.9 the right-hand side of (2.9) can be written

$$d_{\hat{\mathbb{C}}}(C(\sigma(a), r'), C(\sigma(b), s')) + O(r+s).$$

Thus subtraction of (2.8) and (2.9) and letting $r, s \to 0$ gives

$$\delta_\Omega(a,b) \geq \delta_{\sigma(\Omega)}(\sigma(a), \sigma(b)).$$

Considering σ^{-1} we get the reverse inequality. Thus δ and β are Möbius invariants. □

3. Analytic formulas for β-numbers

In this section we express β-numbers in terms of the Riemann map. We start with some preliminary results for the case $b \in \Omega$.

Proposition 2.13. *Let Ω be a simply connected domain. Let $b \in \Omega, a \in \partial\Omega$. If $\rho_0 > 0$ is sufficiently small, the following are equivalent:*

$$\beta_\Omega(a,b) > 0 \tag{2.10}$$

$$X_\Omega(C(a,\rho), C(a,\rho_0)) \text{ is bounded for } \rho < \rho_0 \tag{2.11}$$

$$X_\Omega(C(a,\rho), C(a,r)) \to 0 \quad \text{as } \rho < r \to 0. \tag{2.12}$$

Proof. (2.10) $\Longrightarrow$ (2.11): By Lemma 2.7

$$\tilde{d}_\Omega(C(a,\rho), C(a,\rho_0)) + \tilde{d}_\Omega(C(a,\rho_0), C(b,s)) \le \tilde{d}_\Omega(C(a,\rho), C(b,s)).$$

By Lemma 2.9 and Example 2.4

$$d_{\hat{\mathbb{C}}}(C(a,\rho), C(a,\rho_0)) + d_{\hat{\mathbb{C}}}(C(a,\rho_0), C(b,s)) = d_{\hat{\mathbb{C}}}(C(a,\rho), C(b,s)) + O(1).$$

Subtraction of these gives

$$X_\Omega(C(a,\rho), C(a,\rho_0)) + X_\Omega(C(a,\rho_0), C(b,s)) \le X_\Omega(C(a,\rho), C(b,s)) - O(1).$$

If $\delta_\Omega(a,b) < +\infty$, the right-hand side is bounded as $\rho, s \to 0$, and (2.11) follows.
(2.11) $\Longrightarrow$ (2.12): As above, Lemma 2.7 and Example 2.4 gives

$$X_\Omega(C(a,\rho), C(a,r)) + X_\Omega(C(a,r), C(a,\rho_0)) \le X_\Omega(C(a,\rho), C(a,\rho_0)) \tag{2.13}$$

if $\rho < r < \rho_0$. Thus $X_\Omega(C(a,\rho), C(a,\rho_0))$ is a decreasing function of ρ, and by (2.11) it therefore has a finite limit as $\rho \to 0$. (2.12) now follows from (2.13).
(2.12) $\Longrightarrow$ (2.10): Since both conditions are invariant under Möbius transformations, it suffices to consider finite a and b. Then the implication follows from Proposition 2.17 below. □

To prove the implication (2.12) $\Longrightarrow$ (2.10) we need two lemmas and a definition.

Lemma 2.14. *If $1 < R \le 2$ and $I = \{e^{i\theta} : 0 \le \theta \le 1\}$ then*

$$d_{\mathbb{C}}(I, C(0,R)) > \frac{1}{5}\log R.$$

Proof. Let G be the rectangle $(-A, A) \times (-A, 1 + A)$, where $A = \log R$. By comparison and taking the logarithm we have

$$d_{\mathbb{C}}(I, C(0,R)) \ge d_{\mathbb{C}}(\log I, \partial G).$$

For the metric $\rho = 1$ and the family Γ of curves connecting $\log I$ and ∂G we have $L_\rho(\Gamma) = A$ and $A_\rho(G) = 2A(1 + 2A) < 5A$. Thus

$$d_{\mathbb{C}}(\log I, \partial G) = \lambda(\Gamma) \ge \frac{L_\rho(\Gamma)^2}{A_\rho(G)} > \frac{A}{5}.$$

□

Definition 2.15. Let $\{Q_\rho\}$ be a family of curves. We say that Q_ρ is approximately an arc of the circle $C(a, r(\rho))$ if

$$Q_\rho \subset \{z : r(\rho)(1 - \varepsilon(\rho)) < |z| < r(\rho)(1 + \varepsilon(\rho))\}, \text{ where } \varepsilon(\rho) \to 0 \text{ as } \rho \to 0.$$

We abbreviate this as

$$Q_\rho \approx C(a, r(\rho)).$$

Lemma 2.16. *Let Ω be a simply connected domain. Assume that $a \in \partial\Omega \setminus \{\infty\}$ satisfies* (2.12). *Define I_ρ to be a longest arc of $C(a,\rho) \cap \Omega$. Then*

$$|I_\rho|/2\pi\rho \to 1 \text{ as } \rho \to 0 \tag{2.14}$$

and

$$\tilde{d}_\Omega(C(a,\rho), C(a,r)) = d_\Omega(I_\rho, I_r) \text{ when } \rho < r \text{ are small.} \tag{2.15}$$

Let $f : \mathbb{D} \to \Omega$ be a conformal bijection, and define

$$Q_\rho = f^{-1}(I_\rho).$$

Then there is a point $\zeta_a \in \mathbb{T}$ and a number $\gamma_a > 0$ such that

$$a = \operatorname*{anglim}_{z\to\zeta_a} f(z) \quad \text{and} \quad Q_\rho \approx C(\zeta_a, \gamma_a\sqrt{\rho}). \tag{2.16}$$

Moreover

$$\operatorname*{anglim}_{z\to\zeta_a} \frac{|f(z)-a|}{|z-\zeta_a|^2} = \frac{1}{\gamma_a^2} \quad \text{and} \quad \Gamma_f(\zeta_a) := \operatorname*{anglim}_{z\to\zeta_a} \left|\frac{z-\zeta_a}{f'(z)}\right| = \frac{\gamma_a^2}{2}. \tag{2.17}$$

Proof. By Theorem 2.24 in the next section it follows from (2.12) that Ω contains a sector

$$S_\rho = \{z : \rho \le |z| \le 2\rho,\ \varphi(\rho) - \pi \le \arg z \le \varphi(\rho) + \pi - \alpha(\rho)\}$$

for every small ρ, where

$$\lim_{\rho\to 0} \alpha(\rho) = 0.$$

Thus

$$2\pi - \alpha(\rho) \le \frac{|I_\rho|}{\rho} \le 2\pi,$$

and (2.14) follows. To prove (2.15) we consider two cases.
Case 1: $\rho < r/2$. Let I'_ρ be a component of $C(a,\rho) \cap \Omega$ different from I_ρ, and let J_r be any component of $C(a,r) \cap \Omega$. Since $\operatorname{diam} I'_\rho \le \rho\alpha(\rho)$, the comparison principle gives (with $z' \in I'_\rho$)

$$d_\Omega(I'_\rho, J_r) \ge d_{\hat{\mathbb{C}}}(C(z', \rho\alpha(\rho)), C(z', r-\rho)) \ge \frac{1}{2\pi}\log\frac{r}{2\rho\alpha(\rho)}. \tag{2.18}$$

Together with (2.12) this shows that

$$d_\Omega(I'_\rho, J_r) \ge \tilde{d}_\Omega(C(a,\rho), C(a,r)) + 1 \qquad \text{if } r \text{ is small.}$$

A similar argument shows that

$$d_\Omega(J_\rho, I'_r) \ge \tilde{d}_\Omega(C(a,\rho), C(a,r)) + 1 \qquad \text{if } r \text{ is small.}$$

Thus (2.15) holds.
Case 2: $\rho < r \le 2\rho$. Let I'_ρ be a component of $C(a,\rho) \cap \Omega$ different from I_ρ, and let J_r be any component of $C(a,r) \cap \Omega$. By Lemma 2.14

$$d_\Omega(I'_\rho, J_r) \ge \frac{1}{5}\log\frac{r}{\rho},$$

if ρ is small, and similarly we get

$$d_\Omega(J_\rho, I'_r) \ge \frac{1}{5}\log\frac{r}{\rho}.$$

By comparison with the sector S_ρ we have

$$d_\Omega(I_\rho, I_r) \le \frac{1}{2\pi - \alpha(\rho)}\log\frac{r}{\rho}.$$

These inequalities show that (2.15) holds.

The crosscuts I_ρ form a null-chain in Ω, see [**Pom92**, p. 29]. Or rather: For every strictly decreasing sequence ρ_n, the sequence I_{ρ_n} is a null-chain defining a prime end P. This follows from the sector property above. Thus, by the prime end theorem [**Pom92**, Theorem 2.15] the crosscuts Q_ρ form a null-chain in $\mathbb{D}$, corresponding to a point $\zeta_a \in \mathbb{T}$. a is obviously the unique principal point of the prime end P. By [**Pom92**, Corollary 2.17] this means that

$$a = \operatorname*{anglim}_{z \to \zeta_a} f(z).$$

By (2.15) we can write (2.12) as

$$d_\Omega(I_{\rho_1}, I_{\rho_2}) - \frac{1}{2\pi} \log \frac{\rho_2}{\rho_1} \to 0 \qquad \text{as } \rho_1 < \rho_2 \to 0. \tag{2.19}$$

Hence, the extremal distances $d_\Omega(I_\rho, I_r)$ are approximately additive:

$$d_\Omega(I_\rho, I_R) \le d_\Omega(I_\rho, I_r) + d_\Omega(I_r, I_R) + \varepsilon(R) \qquad \text{for } \rho < r < R,$$

where $\varepsilon(R) \to 0$ as $R \to 0$. Using a module theorem of Teichmüller (Corollary 2.34 on page 35) one can show that this implies

$$Q_\rho \approx C(\zeta_a, r(\rho))$$

for some function $r(\rho) > 0$, see [**Pom92**, Proposition 11.8]. By comparison this gives

$$d_\mathbb{D}(Q_{\rho_1}, Q_{\rho_2}) - \frac{1}{\pi} \log \frac{r(\rho_2)}{r(\rho_1)} \to 0 \qquad \text{as } \rho_1 < \rho_2 \to 0.$$

Together with (2.19) and conformal invariance this gives

$$\log \frac{r(\rho_2)}{\sqrt{\rho_2}} - \log \frac{r(\rho_1)}{\sqrt{\rho_1}} \to 0 \qquad \text{as } \rho_1 < \rho_2 \to 0,$$

so that

$$\gamma_a = \lim_{\rho \to 0} \frac{r(\rho)}{\sqrt{\rho}} \in (0, +\infty) \text{ exists,}$$

and (2.16) follows.

Finally, we prove (2.17). We may assume $a = 0$ and $\zeta_a = 1$. Consider the mappings

$$F_\rho(z) = \frac{e^{-i\varphi(\rho)}}{\rho} f(1 + \gamma_a \sqrt{\rho} z).$$

By the sector property, F_ρ^{-1} is defined in the sector

$$\{w : 1 \le |w| \le 2,\ -\pi \le \arg w \le \pi - \alpha(\rho)\}.$$

Let

$$\tilde{I}_\rho(t) = \frac{e^{-i\varphi(\rho)}}{\rho} I_{t\rho} \subset C(0, t)$$

and

$$\tilde{Q}_\rho(t) = F_\rho^{-1}(\tilde{I}_\rho(t)) = \frac{1}{\gamma_a \sqrt{\rho}} (Q_{t\rho} - 1).$$

By (2.16), $\tilde{Q}_\rho(t) \approx C(0, \sqrt{t})$. Thus

$$|F_\rho^{-1}(w)| \to \sqrt{|w|} \qquad \text{as } \rho \to 0$$

for w in the sector $S = \{w : 1 < |w| < 2,\ -\pi < \arg w < \pi\}$. Choose $\lambda(\rho) \in \mathbb{T}$ so that $\arg(\lambda(\rho) F_\rho^{-1}(3/2)) = 0$. Then

$$\log \lambda(\rho) F_\rho^{-1}(w) \to \frac{1}{2} \log w \text{ as } \rho \to 0 \tag{2.20}$$

locally uniformly in S (since this holds for the real part and for the point $3/2$). If for some sequence $\rho_n \to 0$ we have $\lambda(\rho_n) \to \lambda_0 \ne -1$, then (2.20) yields

$$F_{\rho_n}^{-1}(w) \to \lambda_0^{-1} \sqrt{w} \qquad \text{as } n \to \infty$$

locally uniformly in S. This contradicts the fact that for every $w \in S$ we have

$$F_{\rho_n}^{-1}(w) \in \frac{1}{\gamma_a \sqrt{\rho_n}} (\mathbb{D} - 1) \subset \{z : \operatorname{Re} z < 0\} \qquad \text{for large } n.$$

Thus $\lim_{\rho\to 0}\lambda(\rho) = -1$ and

$$F_\rho^{-1}(w) \to -\sqrt{w} \text{ locally uniformly in } S.$$

It follows that $F_\rho(z) \to z^2$ locally uniformly in the sector $\{z : 1 < |z| < 2, \operatorname{Re} z < 0\}$. Letting $s = 1 + \gamma_a\sqrt{\rho}z \in \mathbb{D}$ we get

$$\frac{|f(s)|}{|s-1|^2} = \frac{\rho|F_\rho(z)|}{\gamma_a^2\rho|z|^2} \to \frac{1}{\gamma_a^2}$$

as $s \to 1$ in a Stolz angle. Also,

$$\frac{|f'(s)|}{|s-1|} = \frac{|F_\rho'(z)|\sqrt{\rho}/\gamma_a}{\gamma_a\sqrt{\rho}|z|} \to \frac{2}{\gamma_a^2}$$

as $s \to 1$ in a Stolz angle. □

We can now compute β-numbers in terms of the Riemann map.

Proposition 2.17. *Let $f : \mathbb{D} \to \Omega$ be a conformal bijection, and let $f(z_b) = b \neq \infty$. Assume that $a \in \partial\Omega \setminus \{\infty\}$ satisfies (2.12). Let $\zeta_a \in \mathbb{T}$ and $\Gamma_f(\zeta_a) > 0$ be as in the preceding lemma. Then*

$$\beta_\Omega(a,b) = \frac{2|a-b|^2}{|1-\overline{z_b}\zeta_a|^4}\frac{1-|z_b|^2}{|f'(z_b)|}\Gamma_f(\zeta_a). \tag{2.21}$$

Proof. Using the notation of the preceding lemma, we first calculate

$$d_\Omega(I_\rho, C(b,\rho)) = d_{\mathbb{D}}(Q_\rho, C_\rho), \text{ where } C_\rho = f^{-1}(C(b,\rho)).$$

Since f is conformal and by (2.16) we have

$$C_\rho \approx C(z_b, \frac{\rho}{|f'(z_b)|}) \quad \text{and} \quad Q_\rho \approx C(\zeta_a, \gamma_a\sqrt{\rho}).$$

We now do some conformal mappings. First map $\varphi : \mathbb{D} \to \mathbb{D}$ using

$$\varphi(z) = \lambda\frac{z - z_b}{1-\overline{z_b}z},$$

where $\lambda \in \mathbb{T}$ is chosen so that $\varphi(\zeta_a) = 1$. By the conformality of φ,

$$C_\rho' = \varphi(C_\rho) \approx C(0, |\varphi'(z_b)|\frac{\rho}{|f'(z_b)|}) \text{ and } Q_\rho' = \varphi(Q_\rho) \approx C(1, |\varphi'(\zeta_a)|\gamma_a\sqrt{\rho}).$$

Next map $\psi : \mathbb{D} \to \mathbb{C} \setminus (-\infty, 0]$ using

$$\psi(z) = \left(\frac{z-1}{z+1}\right)^2.$$

We get

$$C_\rho'' = \psi(C_\rho') \approx C(1, 4|\varphi'(z_b)|\rho|f'(z_b)|^{-1}) \text{ and } Q_\rho'' = \psi(Q_\rho') \approx C(0, \frac{1}{4}|\varphi'(\zeta_a)|^2\gamma_a^2\rho).$$

Note that C_ρ'' is a Jordan curve, while Q_ρ'' is a crosscut of $\mathbb{C} \setminus (-\infty, 0]$. We can now compare $d_{\mathbb{C}\setminus(-\infty,0]}(C_\rho'', Q_\rho'')$ with extremal distances

$$d_{\mathbb{C}\setminus(-\infty,0]}(C(0,r_0), C(1,r_1)) = d_{\mathbb{C}}(C(0,r_0), C(1,r_1)) = -\frac{1}{2\pi}\log(r_0 r_1) + O(r_0 + r_1),$$

(The first equality follows from the symmetry in the real axis, while the second follows from Lemma 2.9.) The comparison principle yields

$$d_{\mathbb{C}\setminus(-\infty,0]}(C_\rho'', Q_\rho'') = -\frac{1}{2\pi}\log\left(|\varphi'(\zeta_a)|^2|\varphi'(z_b)||f'(z_b)|^{-1}\gamma_a^2\rho^2\right) + o(1) \text{ as } \rho \to 0.$$

By conformal invariance, this equals $d_\Omega(I_\rho, C(b,\rho))$. Using an estimate similar to (2.18) it is now easy to see that

$$\tilde{d}_\Omega(C(a,\rho), C(b,\rho)) = d_\Omega(I_\rho, C(b,\rho)).$$

Subtracting

$$d_{\hat{\mathbb{C}}}(C(a,\rho),C(b,\rho)) = \frac{1}{2\pi}\log\frac{|a-b|^2}{\rho^2}+o(1)$$

(see Lemma 2.9) we get

$$X_\Omega(C(a,\rho),C(b,\rho)) = -\frac{1}{2\pi}\log\left(|\varphi'(\zeta_a)|^2|\varphi'(z_b)||f'(z_b)|^{-1}\gamma_a^2|a-b|^2\right)+o(1)$$

and thus

$$\beta_\Omega(a,b) = |\varphi'(\zeta_a)|^2|\varphi'(z_b)||f'(z_b)|^{-1}\gamma_a^2|a-b|^2.$$

Calculating φ' and inserting (2.17) we get (2.21). □

In particular Lemma 2.16 states:

Corollary 2.18. *Let $f:\mathbb{D}\to\Omega$ be a conformal bijection. If $a\in\partial\Omega\setminus\{\infty\}$ satisfies* (2.12) *then*

$$\lim_{r\to1}\frac{1-r}{|f'(r\zeta_a)|}>0.$$

We now prove a converse of this.

Proposition 2.19. *If $f:\mathbb{D}\to\Omega$ is a conformal bijection, and $\zeta\in\mathbb{T}$ is a point with*

$$\lim_{r\to1}\frac{1-r}{|f'(r\zeta)|}>0,$$

then

$$a = \operatorname*{anglim}_{z\to\zeta} f(z) \text{ exists (finite)},$$

a satisfies (2.12) *and $\zeta_a=\zeta$. (ζ_a is as in Lemma 2.16.)*

Proof. Since there is a finite B so that $|f'(r\zeta)|\le B(1-r)$ we easily get that $a=\lim_{r\to1}f(r\zeta)$ exists (finite). As is well-known then also $a=\operatorname{anglim}_{z\to\zeta}f(z)$. By integration

$$|f(r\zeta)-a|\le\frac{B}{2}(1-r)^2. \tag{2.22}$$

Let P be the prime end of Ω that corresponds to ζ. Since $a=\operatorname{anglim}_{z\to\zeta}f(z)$, the point a is the unique principal point of P, see [**Pom92**, Corollary 2.17]. Thus P can be represented by a null-chain $J_n\subset C(a,\rho_n)$. Then $W_n=f^{-1}(J_n)$ is a null-chain in $\mathbb{D}$ corresponding to ζ. We can find $r_n\in(0,1)$ so that $r_n\zeta\in W_n$. By (2.22),

$$\rho_n\le\frac{B}{2}(1-r_n)^2. \tag{2.23}$$

Map $\mathbb{D}$ onto the strip $S=\{s:|\operatorname{Im}s|<\pi/2\}$ by

$$s=\varphi(z)=\log\frac{\zeta+z}{\zeta-z}.$$

Then $W_n'=\varphi(W_n)$ form a null-chain in S corresponding to $+\infty$. Now W_n' contains the point

$$s_n=\varphi(r_n\zeta)=\log\frac{1+r_n}{1-r_n}.$$

Thus, by Ahlfors' distortion theorem [**Ahl73**, p. 76-78]

$$d_S(W_1',W_n')\le\frac{1}{\pi}(s_n-s_1+\log 32).$$

Together with (2.23) this gives

$$d_\Omega(J_1,J_n)=d_S(W_1',W_n')\le\frac{1}{2\pi}\log\frac{1}{\rho_n}+C.$$

Thus $X_\Omega(C(a,\rho_1),C(a,\rho_n))$ is bounded, and (2.12) follows from Proposition 2.13.

Let $\alpha(\rho), I_\rho, Q_\rho, \zeta_a$ be as in Lemma 2.16. If I'_{ρ_n} is a component of $C(a,\rho_n)\cap\Omega$ different from I_{ρ_n}, then by (2.18)

$$d_\Omega(I'_{\rho_n}, J_1) \geq \frac{1}{2\pi}\log\frac{1}{2\rho_n\alpha(\rho_n)}.$$

Thus $J_n = I_{\rho_n}$ for large n. Hence the null-chains W_n and Q_n are also equal, and therefore the corresponding points ζ and ζ_a are equal. □

It follows that we have a 1-1 correspondance between

(i) Points $a \in \partial\Omega\setminus\{\infty\}$ satisfying $\beta_\Omega(a,b) > 0$ (where $b\in\Omega$).
(ii) Points $\zeta\in\mathbb{T}$ with $\lim_{r\to1}\frac{1-r}{|f'(r\zeta)|} > 0$.

In this context it is comforting to know:

Lemma 2.20. *For every conformal mapping f in $\mathbb{D}$ and every $\zeta\in\mathbb{T}$ the radial limit*

$$\lim_{r\to1}\frac{1-r}{|f'(r\zeta)|} \text{ exists (finite).}$$

Proof. We may assume that $\zeta = 1$. Suppose first that $\infty\notin f(\mathbb{D})$. By the distortion theorem (1.4)

$$|f'(r)| \geq |f'(s)|\frac{(1-s^2)^3(1-r^2)}{(1-rs+r-s)^4} \qquad \text{for } 0<s<r<1.$$

Thus

$$\liminf_{r\to1}\frac{|f'(r)|}{1-r} \geq |f'(s)|\frac{(1+s)^3}{8(1-s)} \qquad \text{for } 0<s<1$$

and

$$\liminf_{r\to1}\frac{|f'(r)|}{1-r} \geq \limsup_{s\to1}\frac{|f'(s)|}{1-s}.$$

Hence $\lim_{r\to1}|f'(r)|/(1-r)$ exists in $(0,+\infty]$.

In case $\infty\in f(\mathbb{D})$, let $R>0$ be small and let $\varphi:\mathbb{D}\cap D(1,R)\to\mathbb{D}$ be a conformal bijection with $\varphi(1)=1$. Then the preceding case applies to $g = f\circ\varphi^{-1}$. By the distortion theorem

$$\left|\frac{g'(\varphi(r))}{g'(1+\varphi'(1)(r-1))}\right| \to 1 \text{ as } r\to1.$$

Thus

$$\lim_{r\to1}\frac{|f'(r)|}{1-r} = \lim_{r\to1}\frac{|g'(1+\varphi'(1)(r-1))|}{1-r}|\varphi'(r)| = \lim_{s\to1}\frac{|g'(s)|}{1-s}|\varphi'(1)|^2 > 0.$$ □

We can even prove that an angular limit exists:

Proposition 2.21. *For every conformal mapping f in $\mathbb{D}$ and every $\zeta\in\mathbb{T}$ the angular limit*

$$\Gamma_f(\zeta) := \operatorname*{anglim}_{z\to\zeta}\frac{|\zeta-z|}{|f'(z)|} \text{ exists (finite).}$$

Proof. By the preceding lemma we have two cases:
Case 1:

$$\lim_{r\to1}\frac{1-r}{|f'(r\zeta)|} > 0.$$

Proposition 2.19 says that $a = \operatorname{anglim}_{z\to\zeta} f(z)$ satisfies (2.12) and $\zeta_a = \zeta$. Lemma 2.16 now shows that the angular limit exists.
Case 2:

$$\lim_{r\to1}\frac{1-r}{|f'(r\zeta)|} = 0.$$

The distortion theorem (1.4) shows that

$$|f'(z)| \asymp |f'(|z|\zeta)|$$

for z in a Stolz angle Δ at ζ. Since $|z-\zeta| \le C_1(1-|z|)$ for $z \in \Delta$ we get

$$\left|\frac{z-\zeta}{f'(z)}\right| \le C_2 \frac{1-|z|}{|f'(|z|\zeta)|} \to 0 \qquad \text{as } z \to \zeta,\, z \in \Delta.$$

□

Collecting what we have obtained this far, we see that Theorem 2.1 follows from Propositions 2.21, 2.19, 2.17 and 2.13. Note also that formula (2.1) can be generalized to

$$\beta_\Omega(a,\infty) = 2|f'(\infty)|\Gamma_f(\zeta_a). \tag{2.24}$$

Here, $f : \mathbb{D}^* \to \Omega$ is a conformal bijection with $f(\infty) = \infty$ and $a = f(\zeta_a) \in \partial\Omega$. The proof is similar to that of Proposition 2.17. We now deal with the case of β-numbers where both points are on the boundary.

Theorem 2.22. *With the notation of Theorem 2.1, suppose that $\zeta_1, \zeta_2 \in \mathbb{T}$ satisfy $\Gamma_f(\zeta_j) > 0$. Then*

$$\beta_\Omega(f(\zeta_1), f(\zeta_2)) = \frac{4|f(\zeta_1)-f(\zeta_2)|^2}{|\zeta_1-\zeta_2|^4}\Gamma_f(\zeta_1)\Gamma_f(\zeta_2). \tag{2.25}$$

Conversely, if $a_j \in \partial\Omega \setminus \{\infty\}$ satisfy $\beta_\Omega(a_1,a_2) > 0$, then $a_j = f(\zeta_j)$ for some uniquely determined $\zeta_j \in \mathbb{T}$ with $\Gamma_f(\zeta_j) > 0$.

Proof. The proof is similar to that of Proposition 2.17. By Proposition 2.19, $a_j = f(\zeta_j)$ satisfy (2.12). Thus, by Lemma 2.16,

$$Q_\rho(a_j) = f^{-1}(I_\rho(a_j)) \approx C(\zeta_j, \gamma_{a_j}\sqrt{\rho}).$$

Map $\mathbb{D}$ onto the upper half plane $\mathbb{H}$ using

$$\varphi(z) = \frac{z-\zeta_1}{z-\zeta_2}.$$

Then

$$Q'_\rho(a_j) = \varphi(Q_\rho(a_j)) \approx C(\varphi(\zeta_j), \frac{\gamma_{a_j}\sqrt{\rho}}{|\zeta_1-\zeta_2|}).$$

By comparison with sectors of angle π,

$$d_{\mathbb{H}}(Q'_\rho(a_1), Q'_\rho(a_2)) = \frac{1}{\pi}\log\frac{|\zeta_1-\zeta_2|^2}{\gamma_{a_1}\gamma_{a_2}\rho} + o(1) \qquad \text{as } \rho \to 0.$$

By conformal invariance this is also equal to $d_\Omega(I_\rho(a_1), I_\rho(a_2))$. Using an estimate similar to (2.18) it is easy to see that

$$\tilde{d}_\Omega(C(a_1,\rho), C(a_2,\rho)) = d_\Omega(I_\rho(a_1), I_\rho(a_2)) \qquad \text{for small } \rho.$$

Subtracting

$$d_{\hat{\mathbb{C}}}(C(a_1,\rho), C(a_2,\rho)) = \frac{1}{2\pi}\log\frac{|a_1-a_2|^2}{\rho^2} + o(1)$$

(see Lemma 2.9) we get

$$X_\Omega(C(a_1,\rho), C(a_2,\rho)) = \frac{1}{\pi}\log\frac{|\zeta_1-\zeta_2|^2}{\gamma_{a_1}\gamma_{a_2}|a_1-a_2|} + o(1) \qquad \text{as } \rho \to 0.$$

Thus

$$\beta_\Omega(a_1,a_2) = \frac{\gamma_{a_1}^2\gamma_{a_2}^2|a_1-a_2|^2}{|\zeta_1-\zeta_2|^4},$$

which proves (2.25)

For the converse, suppose that $\beta_\Omega(a_1,a_2) > 0$, where $a_j \in \partial\Omega \setminus \{\infty\}$. The proof of (2.10) $\Longrightarrow$ (2.11) in Proposition 2.13 applies almost unchanged. We get that a_j satisfy (2.12). By Lemma 2.16, $a_j = f(\zeta_j)$ for some $\zeta_j \in \mathbb{T}$ with $\Gamma_f(\zeta_j) > 0$. Proposition 2.19 shows that ζ_j is uniquely determined by a_j. □

The formula (2.25) is valid also for a conformal map f from the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ onto Ω. Namely, if $x_1, x_2 \in \mathbb{R}$ have $\Gamma_f(x_j) > 0$ then

$$\beta_\Omega(f(x_1), f(x_2)) = \frac{4|f(x_1) - f(x_2)|^2}{(x_1 - x_2)^4}\Gamma_f(x_1)\Gamma_f(x_2). \tag{2.26}$$

For the limit case $x_2 = f(x_2) = \infty$ we get a neat formula:

Theorem 2.23. *Let $f : \mathbb{H} \to \mathbb{C}$ be an univalent function. For every $x \in \mathbb{R}$ the angular limit*

$$\Gamma_f(x) := \operatorname*{anglim}_{z \to x} \left|\frac{z - x}{f'(z)}\right| \qquad \text{exists (finite).} \tag{2.27}$$

Also

$$\Gamma_f(\infty) := \operatorname*{anglim}_{z \to \infty} \left|\frac{f'(z)}{z}\right| \qquad \text{exists (finite).} \tag{2.28}$$

If $\Gamma_f(x) > 0$ and $\Gamma_f(\infty) > 0$ then

$$a = \operatorname{anglim}_{z \to x} f(z) \qquad \text{exists (finite),} \tag{2.29}$$

$$\infty = \operatorname{anglim}_{z \to \infty} f(z) \tag{2.30}$$

and

$$\beta_{f(\mathbb{H})}(a, \infty) = \Gamma_f(x)\Gamma_f(\infty). \tag{2.31}$$

Conversely, if $\Omega \subset \mathbb{C}$ is a simply connected domain with boundary points a and ∞ satisfying $\beta_\Omega(a, \infty) > 0$, then there is a conformal bijection $f : \mathbb{H} \to \Omega$ with $\Gamma_f(0) > 0$, $\Gamma_f(\infty) > 0$ and $a = \operatorname{anglim}_{z \to 0} f(z)$.

Proof. (2.27) and (2.29) follow from Theorem 2.1 by a simple transformation. A transformation of the distortion theorem (1.4) shows that

$$\left|\frac{f'(z)}{f'(s)}\right| \le \frac{(|z - s| + |z - \overline{s}|)^4}{16 \operatorname{Im} s (\operatorname{Im} z)^3} \qquad \text{for } z, s \in \mathbb{H}. \tag{2.32}$$

It follows from this that

$$C = \lim_{y \to +\infty} \frac{|f'(iy)|}{y} \qquad \text{exists (finite),}$$

cf. Proposition 2.20. To prove (2.28), (2.30) and (2.31) we may suppose that $0 \notin f(\mathbb{H})$, $a = 1$ and $x = 0$. We consider two cases.
Case 1: $C > 0$. By the Koebe 1/4 theorem

$$f(\mathbb{H}) \supset D\left(f(iy), \frac{1}{4}\frac{Cy}{2}y\right) \qquad \text{for large } y > 0.$$

This implies that $|f(iy)| \ge Cy^2/8$ for large $y > 0$. Define $g : \mathbb{D} \to \mathbb{C}$ by $g(z) = 1/f(\varphi(z))$, where $\varphi(z) = i(1 + z)/(1 - z)$. We get for $0 < r < 1$ that

$$\frac{1 - r}{|g'(r)|} = \frac{1 - r}{|f'(\varphi(r))|}\frac{|f(\varphi(r))|^2}{|\varphi'(r)|} \ge \left(\frac{C}{8}\right)^2 \frac{(1 - r)|\varphi(r)|^4}{|f'(\varphi(r))||\varphi'(r)|} \to \frac{C}{16}$$

as $r \to 1$. By Proposition 2.19 and Lemma 2.16 this gives $\Gamma_g(1) > 0$,

$$\operatorname*{anglim}_{z \to 1} g(z) = 0 \tag{2.33}$$

and

$$\operatorname*{anglim}_{z \to 1} \frac{|g(z)|}{|z - 1|^2} = \frac{1}{2\Gamma_g(1)}.$$

Letting $\psi(z) = \varphi^{-1}(z) = (z - i)/(z + i)$ we get

$$\left|\frac{f'(z)}{z}\right| = \left|\frac{g'(\psi(z))}{\psi(z) - 1}\right|\left|\frac{(\psi(z) - 1)^2}{g(\psi(z))}\right|^2\left|\frac{\psi'(z)}{(\psi(z) - 1)^3 z}\right| \to \frac{1}{\Gamma_g(1)}(2\Gamma_g(1))^2\frac{1}{4} = \Gamma_g(1) \tag{2.34}$$

as $z \to \infty$ in a Stolz angle, so that $\Gamma_f(\infty) = \Gamma_g(1)$. (2.30) follows from (2.33). To prove (2.31), note that

$$\left|\frac{z+1}{g'(z)}\right| = \left|\frac{\varphi(z)}{f'(\varphi(z))}\right| \left|\frac{f(\varphi(z))^2(z+1)}{\varphi(z)\varphi'(z)}\right| \to \Gamma_f(0)\frac{1}{|\varphi'(-1)|^2} \tag{2.35}$$

as $z \to -1$ in a Stolz angle, so that $\Gamma_g(-1) = 4\Gamma_f(0)$. Theorem 2.22 now gives

$$\beta_{f(\mathbb{H})}(1,\infty) = \beta_{g(\mathbb{D})}(1,0) = \frac{4}{2^4}\Gamma_g(-1)\Gamma_g(1) = \Gamma_f(0)\Gamma_f(\infty).$$

Case 2: $C = 0$. By (2.32) we get

$$\frac{|f'(z)|}{|z|} \le C_2 \frac{|f'(i\,\mathrm{Im}\, z)|}{\mathrm{Im}\, z}$$

for z in a Stolz angle Δ at ∞. It follows that

$$\operatorname*{anglim}_{z\to\infty} \frac{|f'(z)|}{|z|} = 0.$$

To prove the converse, suppose that $\Omega \subset \mathbb{C}$ is a simply connected domain with $a, \infty \in \partial\Omega$ satisfying $\beta_\Omega(a,\infty) > 0$. We may assume that $a = 1$. Let $g : \mathbb{D} \to \Omega^{-1}$ be a conformal bijection. By the Möbius invariance

$$\beta_{\Omega^{-1}}(1,0) = \beta_\Omega(1,\infty) > 0.$$

By Theorem 2.22 there exists $\zeta_j \in \mathbb{T}$ with $\Gamma_g(\zeta_j) > 0$ and $g(\zeta_0) = 0, g(\zeta_1) = 1$. By composing with a Möbius transformation we may arrange so that $\zeta_0 = 1, \zeta_1 = -1$. Letting $f(z) = 1/g(\psi(z))$, where ψ is as above, we see that (2.34) and (2.35) hold. □

4. Geometric criteria for $\beta > 0$: General domains.

We study the following problem: Given a simply connected domain Ω and points $a \in \partial\Omega$, $b \in \Omega$, find necessary or sufficient geometric conditions on $\partial\Omega$ for $\beta_\Omega(a,b)$ to be positive. By Proposition 2.13, our task is to estimate

$$X_\Omega(C(a,\rho), C(a,\rho_0)) = \tilde{d}_\Omega(C(a,\rho), C(a,\rho_0)) - \frac{1}{2\pi}\log\frac{\rho_0}{\rho} \quad \text{as } \rho \to 0. \tag{2.36}$$

It is enough to consider $\rho = e^{-An}$, where $n > 0$ is integer and $A > 0$ is a constant. We may as well assume that $\rho_0 = 1$ and $a = 0$. We will estimate (2.36) in terms of the angles $\alpha_1, \alpha_2, \ldots \alpha_n$ which are defined as follows: α_k is the minimal angle (with apex 0) that contains $\Omega^c \cap \{z : e^{-k} \le |z| \le e^{-k+1}\}$. This notation will be kept for the rest of this chapter.

Theorem 2.24. *There is a constant C (depending on A) such that*

$$X_\Omega(C(0,e^{-An}),\mathbb{T}) = \tilde{d}_\Omega(C(0,e^{-An}),\mathbb{T}) - \frac{An}{2\pi} \ge C\sum_1^n \frac{\alpha_k^2}{\log(9/\alpha_k)}.$$

Corollary 2.25. *If $\beta_\Omega(0,b) > 0$ then*

$$\sum_1^\infty \frac{\alpha_k^2}{\log(9/\alpha_k)} < \infty.$$

To understand Theorem 2.24 it is instructive to consider the special kind of domains $\Omega = \mathbb{C} \setminus J$, where J is a Jordan arc. Assume that J can be described by an equation $\phi = \phi(r)$ in polar coordinates, where $\phi(r)$ is increasing. Consider the domain

$$G = -\log\Omega = \{x + iy : \varphi(x) < y < \varphi(x) + 2\pi\},$$

where $\varphi(x) = \phi(e^x)$, see Figure 3 below. The case $n = 1$ of Theorem 2.24 states

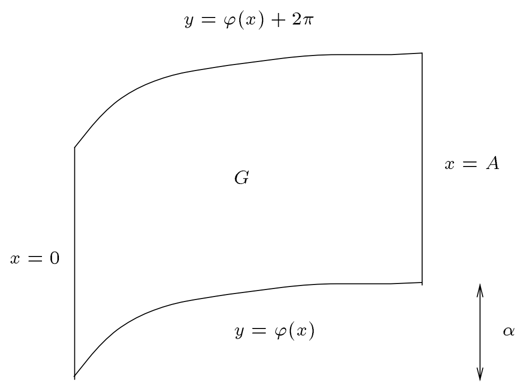

FIGURE 3

that

$$d_G(\{z : \operatorname{Re} z = 0\}, \{z : \operatorname{Re} z = A\}) - \frac{A}{2\pi} \geq C \frac{\alpha^2}{\log(9/\alpha)}, \tag{2.37}$$

where $\alpha = \min\{\varphi(A) - \varphi(0), 2\pi\}$. The following example shows that this estimate is best possible (up to the constant).

Example 2.26. Let $G = \{x + iy : 0 < x < A,\ \varphi(x) < y < \varphi(x) + 2\pi\}$, where

$$\varphi(x) = \frac{\alpha}{\log(1/\alpha + 1)} \log \frac{x + \alpha}{\alpha} \qquad (\alpha > 0 \text{ is small}).$$

Then

$$d_G(\{z : \operatorname{Re} z = 0\}, \{z : \operatorname{Re} z = A\}) \leq \frac{A}{2\pi} + C \frac{\alpha^2}{\log(1/\alpha)}.$$

Theorem 2.24 has been in the air for some time. The easier estimate

$$X_\Omega(C(0, e^{-A})) \geq C\alpha_1^3$$

appears in [**CaMa94**, p. 38], see also (2.52) below. Theorem 2.24 implies a module estimate of Teichmüller, see Corollary 2.34 below. To prove Theorem 2.24 and Example 2.26, we will use the connection between extremal distance and Dirichlet integrals.

Definition 2.27. The Dirichlet integral of a function $u : G \to \mathbb{R}$ is

$$D_G(u) = \iint_G |\nabla u|^2 \, \mathrm{d}A.$$

Lemma 2.28. *Let G be a Jordan domain. Let I_0, I_1 by two disjoint arcs on $\partial\Omega$, and let J_0, J_1 be the complementary arcs. Then*

$$d_G(I_0, I_1) = \frac{1}{d_G(J_0, J_1)} = \min_u D_G(u),$$

where the minimum is taken over all continuous $u : \overline{G} \to \mathbb{R}$ in the Sobolev space $W_1^2(G)$ with $u = 0$ on J_0 and $u = 1$ on J_1.
The minimizing function is $u = \operatorname{Re} f$, where f is a conformal mapping of G onto a rectangle $(0, 1) \times (0, \lambda)$ with $f(J_j) = \{j\} \times [0, \lambda]$.

Proof. By conformal invariance of d and D it suffices to consider $G = (0, 1) \times (0, \lambda)$ and $J_j = \{j\} \times [0, \lambda]$. Then

$$d_G(I_0, I_1) = \lambda \qquad \text{and} \qquad d_G(J_0, J_1) = \frac{1}{\lambda}.$$

If $u = j$ on J_j, then

$$1 = (u(1,y) - u(0,y))^2 \le \left(\int_0^1 |\nabla u(x,y)| \,\mathrm{d}x \right)^2 \le \int_0^1 |\nabla u(x,y)|^2 \,\mathrm{d}x$$

for almost every y. Integration yields $\lambda \le D_G(u)$, with equality if $u(z) = \operatorname{Re} z$. □

Proof of Example 2.26. Let $u : \overline{G} \to \mathbb{R}$ be any continuous function which is C^1 in G and satisfies $u(0,\cdot) = 0$ and $u(A,\cdot) = 1$. By Lemma 2.28 we only have to prove that

$$D_G(u) \ge \frac{2\pi}{A} - C_1 \frac{\alpha^2}{\log(1/\alpha)}.$$

Writing $u(x,y) = x/A + h(x,y)$, we have

$$D_G(u) = \frac{2\pi}{A} + \frac{2}{A} \int_{\partial G} h \,\mathrm{d}y + D_G(h).$$

Hence it suffices to prove

$$\left| \int_{\partial G} h \,\mathrm{d}y \right| \le C_2 \frac{\alpha}{\sqrt{\log(1/\alpha)}} \sqrt{D_G(h)}. \tag{2.38}$$

Define $H : [0,A] \times [0,2\pi] \to \mathbb{R}$ by

$$H(x,y) = h(x, \varphi(x) + y).$$

Since $\varphi'(x) < 1/2$ we get $D(H) \asymp D_G(h)$, so that (2.38) can be written

$$\left| \int_0^{2\pi} (H(x,0) - H(x,2\pi)) \frac{\mathrm{d}x}{x+\alpha} \right| \le C_3 \sqrt{\log \frac{1}{\alpha} \cdot D(H)} \tag{2.39}$$

Here we have used also that $H(0,\cdot) = H(A,\cdot) = 0$. Let L_ξ be the line segment from $(\xi, 0)$ to $(0, 2\pi\xi/A)$. Then

$$|H(\xi,0)| = \left| H(\xi,0) - H\left(0, \frac{2\pi\xi}{A}\right) \right| \le \int_{L_\xi} |\nabla H| \,\mathrm{d}s \le \left(C_4 \xi \int_{L_\xi} |\nabla H|^2 \,\mathrm{d}s \right)^{1/2},$$

so that by Schwarz' inequality

$$\int_0^1 |H(\xi,0)| \frac{\mathrm{d}\xi}{\xi+\alpha} \le \left(C_4 \int_0^1 \int_{L_\xi} |\nabla H|^2 \,\mathrm{d}s \,\mathrm{d}\xi \int_0^1 \frac{\xi}{(\xi+\alpha)^2} \,\mathrm{d}\xi \right)^{1/2}$$
$$\le C_5 \left(D(H) \log \frac{1}{\alpha} \right)^{1/2}.$$

Together with the similar estimate

$$\int_0^1 |H(\xi,2\pi)| \frac{\mathrm{d}\xi}{\xi+\alpha} \le C_5 \left(D(H) \log \frac{1}{\alpha} \right)^{1/2}$$

this proves (2.39). □

To prove Theorem 2.24, we first prove the special case (2.37). Actually, we will need the following more general case:

Lemma 2.29. *Let G be a domain of the type*

$$G = \{x + iy : \varphi(x) < y < \psi(x),\ 0 < x < A\}$$

where φ and ψ are increasing functions. Assume that $\operatorname{area} G = 2\pi A$, $\psi(A) - \varphi(0) \le 8\pi$ *and* $\psi(0) - \varphi(A) = 2\pi - \alpha$, *where* $\alpha < \min\{\frac{1}{16A}, \frac{A}{16}\}$. *Let* $I_x = \{x + iy : \varphi(x) < y < \psi(x)\}$. *Then*

$$d_G(I_0, I_A) - \frac{A}{2\pi} \ge C \frac{\alpha^2}{\log(9/\alpha)}.$$

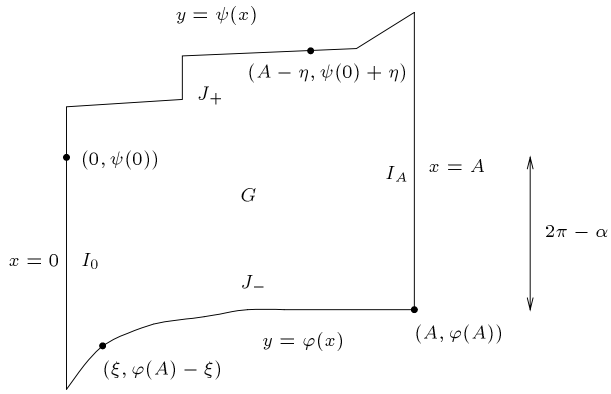

FIGURE 4

Proof. Let J_+ and J_- denote the arcs of ∂G that are complementary to I_0 and I_A, chosen so that J_+ is the upper part (corresponding to ψ), see Figure 4. Let $\xi \geq 0$ and $\eta \geq 0$ be the unique numbers satisfying $(\xi, \varphi(A) - \xi) \in J_-$ and $(A - \eta, \psi(0) + \eta) \in J_+$. Using that the height of G is at most 8π we get

$$2\pi A = \operatorname{area} G \leq (\xi + 2\pi - \alpha + \eta)A + 8\pi\xi + 8\pi\eta.$$

Hence $\xi \geq c\alpha$ or $\eta \geq c\alpha$, where $c = A/(2A + 16\pi)$. It suffices to consider the case $\xi \geq c\alpha$. By comparing areas we have

$$2\pi A = \operatorname{area} G \geq \xi^2 + \eta^2 + (2\pi - \alpha)A,$$

and it follows that $\xi \leq \sqrt{A\alpha} < \min\{1/4, A/4\}$.

Below we construct a continuous function $h : \overline{G} \to \mathbb{R}$ satisfying

$$h = 0 \text{ on } I_0 \cup J_+ \cup I_A, \tag{2.40}$$

$$\int_{J_-} h \, \mathrm{d}y \geq \frac{\xi}{4}, \tag{2.41}$$

$$D_G(h) \leq C \log \frac{1}{\xi}. \tag{2.42}$$

Using this, let

$$u(x, y) = \frac{x}{A} - \varepsilon h(x, y), \qquad \text{where } \varepsilon = \frac{\xi}{4AC\log(1/\xi)}.$$

Then

$$\begin{aligned} D_G(u) &= \frac{\operatorname{area} G}{A^2} - \frac{2\varepsilon}{A} \iint_{\partial G} \frac{\partial h}{\partial x} \, \mathrm{d}A + \varepsilon^2 D_G(h) \\ &= \frac{2\pi}{A} - \frac{2\varepsilon}{A} \int_{J_-} h \, \mathrm{d}y + \varepsilon^2 D_G(h) \\ &\leq \frac{2\pi}{A} - \frac{\varepsilon\xi}{2A} + C\varepsilon^2 \log\frac{1}{\xi} = \frac{2\pi}{A} - \frac{\xi^2}{16A^2C\log(1/\xi)}. \end{aligned}$$

Since $u = 0$ on I_0 and $u = 1$ on I_A, Lemma 2.28 gives

$$d_G(I_0, I_A) \geq \frac{1}{D_G(u)} \geq \frac{A}{2\pi} + C_2 \frac{\xi^2}{\log(1/\xi)},$$

which proves the lemma.

To construct the function h, we consider two cases:
Case 1: $\varphi(A - \xi) \geq \varphi(A) - \xi/2$. We may assume that $\varphi(A - \xi) = 0$. Let $R_1, \ldots, R_7$ be the closed polygonal regions in Figure 5.

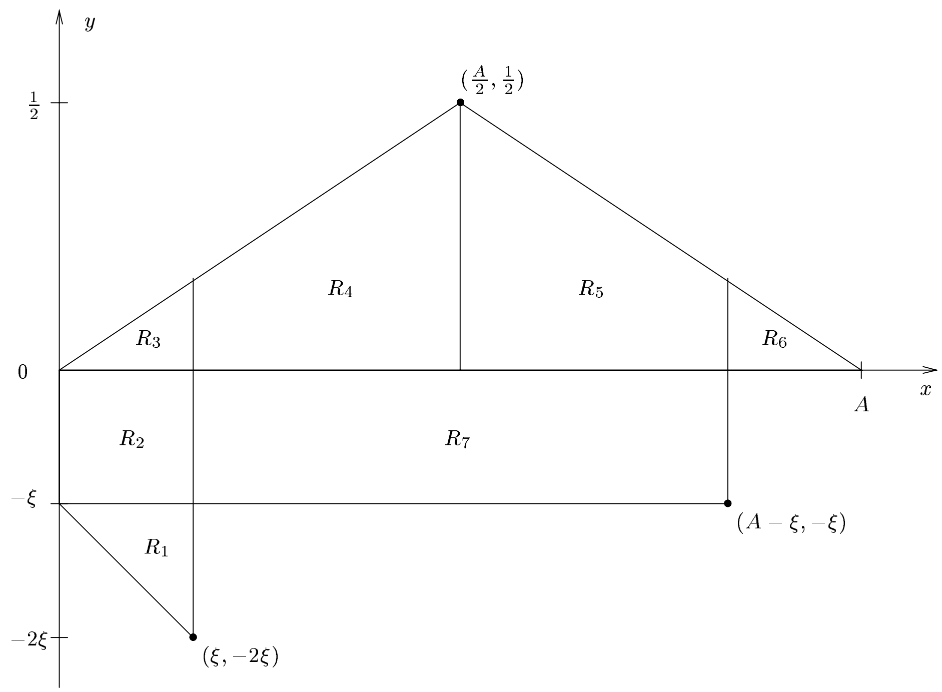

FIGURE 5. The domains $R_1, \ldots, R_7$

We define

$$h(x,y) = \begin{cases} (x+y)/\xi + 1 & \text{in } R_1 \\ x/\xi & \text{in } R_2 \\ (x-Ay)/\xi & \text{in } R_3 \\ 1 - Ay/x & \text{in } R_4 \\ 1 - Ay/(1-x) & \text{in } R_5 \\ (1-x-Ay)/\xi & \text{in } R_6 \\ 1 & \text{in } R_7 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that (2.40) and (2.42) are satisfied. (2.41) follows from

$$\int_{J_-} h \, \mathrm{d}y \geq \int_{J_- \cap R_7} \mathrm{d}y \geq \varphi(A-\xi) - \varphi(\xi) = \varphi(A-\xi) - (\varphi(A) - \xi) \geq \frac{\xi}{2}.$$

Case 2: $\varphi(A-\xi) < \varphi(A) - \xi/2$. Let $z_1 \in J_-$ be a point with $\operatorname{Im} z_1 = \varphi(A-\xi) + \xi/4$. Let R_1, R_2 and R_3 be the polygonal regions in Figure 6.

Define

$$h(x,y) = \begin{cases} \frac{4}{\xi}\left(x - \left(A - \frac{5\xi}{4}\right)\right) & \text{in } R_1 \\ 1 + \frac{4}{\xi}(\operatorname{Im} z_1 - y) & \text{in } R_2 \\ 1 & \text{in } R_3 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that (2.40)–(2.42) are satisfied. □

We will use rearrangement inequalities to reduce the proof of Theorem 2.24 to Lemma 2.29. First a one-dimensional version.

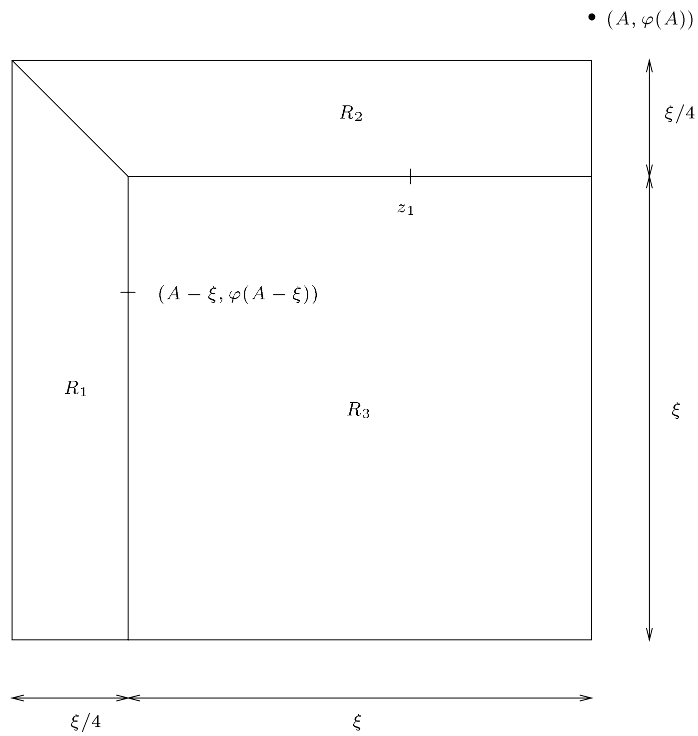

FIGURE 6. The domains R_1, R_2, R_3

Definition 2.30. Let $u : [a, b] \to [0, 1]$ be a measurable function. Then the function

$$m(y) = |\{x : u(x) \le y\}|$$

is increasing and right-continuous. ($|\ |$ denotes Lebesgue measure on $\mathbb{R}$.) The "inverse" of m defined by

$$u^*(x) = \sup\{y : m(y) \le x\}, \qquad a \le x \le b$$

is called the increasing (right-continuous) rearrangement of u.

Lemma 2.31. *Rearrangement has the following properties:*

a) *u^* and u are equimeasurable, that is*

$$\int_a^b \varphi(u^*)\, \mathrm{d}x = \int_a^b \varphi(u)\, \mathrm{d}x$$

for any Borel measurable function $\varphi : [0, 1] \to [0, 1]$.

b) $\int_a^b u_1 u_2\, \mathrm{d}x \le \int_a^b u_1^* u_2^*\, \mathrm{d}x$.

c) $\int_a^b (u_1 - u_2)^2\, \mathrm{d}x \ge \int_a^b (u_1^* - u_2^*)^2\, \mathrm{d}x$.

d) $|u_1 - u_2| \le c \implies |u_1^* - u_2^*| \le c \qquad$ (*c constant*).

e) *If u is Lipschitz, then u^* is also Lipschitz with the same (or smaller) constant, and*

$$\int_a^b u'(x)^2\, \mathrm{d}x \ge \int_a^b (u^*)'(x)^2\, \mathrm{d}x.$$

Proof. a) See [**HaLiPó59**, Section 10.12].
b) See [**HaLiPó59**, Theorem 378].
c) Follows from a) and b).
d) Follows from the implication $u \le v \implies u^* \le v^*$.
e) Assume that u is Lipschitz with constant k, and $a \le x_1 < x_2 \le b$. Choose

$\xi_1, \xi_2 \in [a,b]$ such that $u(\xi_j) = u^*(x_j)$ and $u(I) = (u^*(x_1), u^*(x_2))$, where I is the open interval between ξ_1 and ξ_2. We have

$$\begin{aligned} |\xi_2 - \xi_1| = |I| &\le |\{x : u^*(x_1) < u(x) < u^*(x_2)\}| \\ &= |\{x : u^*(x_1) < u^*(x) < u^*(x_2)\}| \le x_2 - x_1. \end{aligned} \tag{2.43}$$

Thus we see that u^* is Lipschitz:

$$u^*(x_2) - u^*(x_1) = u(\xi_2) - u(\xi_1) \le k|\xi_2 - \xi_1| \le k(x_2 - x_1).$$

By Schwarz' inequality

$$(u^*(x_2) - u^*(x_1))^2 = (u(\xi_2) - u(\xi_1))^2 \le \left(\int_I |u'|\,\mathrm{d}x\right)^2 \le |I| \int_I |u'|^2\,\mathrm{d}x,$$

so that by (2.43)

$$\frac{(u^*(x_2) - u^*(x_1))^2}{x_2 - x_1} \le \int_I |u'|^2\,\mathrm{d}x.$$

Hence, for a partition $a = x_0 < x_1 < \ldots < x_n = b$ we get

$$\sum_{k=1}^{n} \left(\frac{u^*(x_k) - u^*(x_{k-1})}{x_k - x_{k-1}}\right)^2 (x_k - x_{k-1}) \le \int_a^b |u'|^2\,\mathrm{d}x.$$

Now e) follows from the bounded convergence theorem. □

The following proposition is a consequence of an inequality for the Dirichlet integral under Steiner symmetrization, see [**PóSz51**, p. 185-186]. For the corresponding result for circular symmetrization, see [**Hay94**, Section 4.7.1].

Proposition 2.32. *Let $u : [a,b] \times [c,d] \to [0,1]$ be Lipschitz. Define the increasing rearrangement of u in the x-variable by*

$$u^*(x,y) = (u(\cdot,y))^*(x).$$

Then u^* is Lipschitz and

$$\iint |\nabla u|^2\,\mathrm{d}A \ge \iint |\nabla u^*|^2\,\mathrm{d}A. \tag{2.44}$$

Proof. Since $|u(\cdot,y) - u(\cdot,y')| \le k|y - y'|$, Lemma 2.31d) gives

$$|u^*(x',y) - u^*(x',y')| \le k|y - y'|$$

By Lemma 2.31e)

$$|u^*(x,y) - u^*(x',y)| \le k|x - x'|.$$

These add to

$$|u^*(x,y) - u^*(x',y')| \le k(|x - x'| + |y - y'|),$$

and it follows that u^* is Lipschitz.
For a partition $c = y_0 < y_1 < \ldots < y_n = d$, Lemma 2.31c) gives

$$\begin{aligned} &\int_a^b \sum_{k=1}^{n} \left(\frac{u(x,y_k) - u(x,y_{k-1})}{y_k - y_{k-1}}\right)^2 (y_k - y_{k-1})\,\mathrm{d}x \\ &\qquad \ge \int_a^b \sum_{k=1}^{n} \left(\frac{u^*(x,y_k) - u^*(x,y_{k-1})}{y_k - y_{k-1}}\right)^2 (y_k - y_{k-1})\,\mathrm{d}x. \end{aligned}$$

By the bounded convergence theorem

$$\int_a^b \int_c^d \left(\frac{\partial u}{\partial y}\right)^2 \mathrm{d}y\,\mathrm{d}x \ge \int_a^b \int_c^d \left(\frac{\partial u^*}{\partial y}\right)^2 \mathrm{d}y\,\mathrm{d}x.$$

Together with Lemma 2.31e) this gives (2.44). □

Before the proof of Theorem 2.24, just one more lemma.

Lemma 2.33. *If $J \subset \mathbb{T}$ is an arc and $A > 0$, then*

$$d_{\mathbb{C}}(J, C(0, e^A)) \geq \frac{A}{2\pi} + C(2\pi - |J|)^2$$

for some constant C depending on A.

Proof. Let $J = \{e^{i\theta} : \varepsilon \leq \theta \leq 2\pi - \varepsilon\}$. Let $\Omega = \mathbb{H} \cap D(0, e^A) \setminus J$. By symmetry

$$d_{\mathbb{C}}(J, C(0, e^A)) = \frac{1}{2} d_\Omega(J, C(0, e^A)). \tag{2.45}$$

(To see this, let $\varphi : \Omega \to \{z : 1 < |z| < R, \operatorname{Im} z > 0\}$ be a conformal bijection with $\varphi((-e^A, -1)) = (-R, -1)$ and $\varphi((-1, e^A)) = (1, R)$. By reflection φ extends to a conformal mapping of $D(0, e^A) \setminus J$ onto $\{z : 1 < |z| < R\}$. Now (2.45) follows from Example 2.4.)

Consider $G = \log \Omega = \{x + iy : 0 < y < \pi,\ x < A\} \setminus [i\varepsilon, i\pi]$. We have to prove that

$$d_G([i\varepsilon, i\pi], [A, A + i\pi]) \geq \frac{A}{\pi} + C_2 \varepsilon^2.$$

Define $c = \min\{A/2\pi, 1\}$, $h(z) = (c\varepsilon - |z|)_+/2A$ and $u(x, y) = (x/A)_+ + h(x, y)$. Then

$$\begin{aligned} D_G(u) &= D_G((x/A)_+) + \frac{2}{A} \iint_{(0,A)\times(0,\pi)} \frac{\partial h}{\partial x}\, \mathrm{d}A + D_G(h) \\ &= \frac{\pi}{A} - \frac{c^2\varepsilon^2}{2A^2} + \frac{\pi c^2 \varepsilon^2}{8A^2}. \end{aligned}$$

Since $u = 0$ on $[i\varepsilon, i\pi]$ and $u = 1$ on $[A, A + i\pi]$, Lemma 2.28 gives

$$d_G([i\varepsilon, i\pi], [A, A + i\pi]) \geq \frac{1}{D_G(u)} \geq \frac{A}{\pi} + C_2 \varepsilon^2.$$

□

Proof of Theorem 2.24. By Lemma 2.7 we only have to consider the case $n = 1$. Let γ_0 and γ_A be components of $\mathbb{T} \cap \Omega$ and $C(0, e^{-A}) \cap \Omega$, respectively. We have to prove that

$$d_{\Omega'}(\gamma_0, \gamma_A) - \frac{A}{2\pi} \geq C \frac{\alpha_1^2}{\log(9/\alpha_1)},$$

where Ω' is the component of $\Omega \setminus \gamma_0 \setminus \gamma_A$ that has both γ_0 and γ_A on its boundary. By the comparison principle, we may assume that γ_0 and γ_A are not separated in Ω' by any other components of $\mathbb{T} \cap \Omega$ or $C(0, e^{-A}) \cap \Omega$. Let $f : \Omega' \to (0, \lambda) \times (0, 1)$ be a conformal bijection such that $f(\gamma_0) = \{0\} \times (0, 1)$ and $f(\gamma_A) = \{\lambda\} \times (0, 1)$. Consider the Jordan domain

$$\Omega'' = f((0, \lambda) \times (\varepsilon, 1 - \varepsilon)) \subset \Omega',$$

which is bounded by the analytic curves

$$\sigma''_- = f((0, \lambda) \times \{\varepsilon\}) \quad \text{and} \quad \sigma''_+ = f((0, \lambda) \times \{1 - \varepsilon\})$$

and the circular arcs

$$\gamma''_0 = f(\{0\} \times (\varepsilon, 1 - \varepsilon)) \quad \text{and} \quad \gamma''_A = f(\{\lambda\} \times (\varepsilon, 1 - \varepsilon)).$$

Since

$$d_{\Omega''}(\gamma''_0, \gamma''_A) = \frac{\lambda}{1 - 2\varepsilon} \quad \text{and} \quad d_{\Omega'}(\gamma_0, \gamma_A) = \lambda,$$

we need only prove that

$$d_{\Omega''}(\gamma''_0, \gamma''_A) - \frac{A}{2\pi} \geq C \frac{\alpha_1^2}{\log(9/\alpha_1)}. \tag{2.46}$$

We divide the proof into three cases:

Case 1: $2\pi - e^A |\gamma''_A| \geq \alpha_1/4$. By Lemma 2.33

$$d_{\Omega''}(\gamma''_0, \gamma''_A) \geq d_{\mathbb{C}}(\mathbb{T}, \gamma''_A) \geq \frac{A}{2\pi} + C \left(\frac{\alpha_1}{4}\right)^2.$$

Case 2: $2\pi - |\gamma_0''| \geq \alpha_1/4$.
This case is reduced to Case 1 using the transformation $w = e^{-A}/z$.

Case 3: $2\pi - e^A|\gamma_A''| < \alpha_1/4$ and $2\pi - |\gamma_0''| < \alpha_1/4$.
Consider $G = -\log \Omega''$, $I_0 = -\log \gamma_0''$ $I_A = -\log \gamma_A''$ and $J_\pm = -\log \sigma_\pm''$. Assume that J_+ is located above J_-, as in Figure 7.

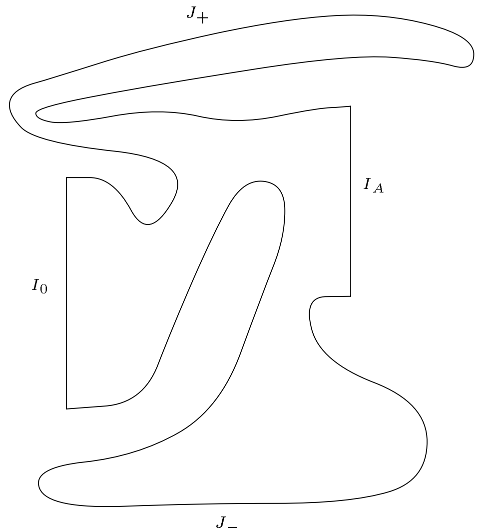

FIGURE 7. The domain G

Let $u : \overline{G} \to [0,1]$ be the minimizing function given by Lemma 2.28. u is harmonic in a neighbourhood of $\overline{G}$, $u = 0$ on J_+, $u = 1$ on J_-, and

$$d_{\Omega''}(\gamma_0'', \gamma_A'') = d_G(I_0, I_A) = D_G(u) \geq D_V(u),$$

where V is the component of $(G \cup I_0 \cup I_A) \cap ([0,A] \times \mathbb{R})$ that has contains I_0 and I_A.

Let $R = [0,A] \times [a,b]$ be a rectangle containing V. The set $R \setminus V$ has two components $K_+ \supset J_+$ and $K_- \supset J_-$. Extend $u|_V$ to R by letting $u = 0$ on K_+ and $u = 1$ on K_-. Now $u : R \to [0,1]$ is a Lipschitz function. Let $u^* : R \to [0,1]$ be the increasing rearrangement of u in the x-variable. By Proposition 2.32

$$D_V(u) = D_R(u) \geq D_R(u^*).$$

Let now u^{**} be the *decreasing* rearrangement of u^* in the *y-variable.* If follows from Proposition 2.32 that

$$D_R(u^*) \geq D_R(u^{**}).$$

The set $V^{**} = \{z \in R : 0 < u^{**}(z) < 1\}$ can be written

$$V^{**} = \{(x,y) \in R : a + L_1(x) < y < b - L_0(x)\},$$

where $L_j(x) = |\{y : u^*(x,y) = j\}|$. Since $u^*(x,y)$ is increasing in x it follows that L_1 is increasing and L_0 is decreasing.

Let $I_0^{**} = V^{**} \cap (\{0\} \times \mathbb{R})$ and $I_A^{**} = V^{**} \cap (\{A\} \times \mathbb{R})$. By Lemma 2.28

$$D_R(u^{**}) = D_{V^{**}}(u^{**}) \geq d_{V^{**}}(I_0^{**}, I_A^{**}).$$

Consider the set

$$W = \{(x,y) : \varphi(x) < y < \psi(x),\, 0 \leq x \leq A\} \supset V^{**},$$

where

$$\varphi(x) = a + L_1(x) \;\text{ and }\; \psi(x) = b - L_0(x) + 2(2\pi A - \operatorname{area} V^{**})x/A.$$

Since area V^{**} = area $V \leq 2\pi A$ the functions φ and ψ are increasing, and area $W = 2\pi A$. By comparison

$$d_{V^{**}}(I_0^{**}, I_A^{**}) \geq d_W(Q_0, Q_A),$$

where $Q_0 = W \cap (\{0\} \times \mathbb{R})$ and $Q_A = W \cap (\{A\} \times \mathbb{R})$.

Let $B = (\psi(0) - \varphi(A))_+$ and $\alpha = 2\pi - B$. We claim that

$$\alpha \geq \alpha_1/2. \tag{2.47}$$

and

$$d_W(Q_0, Q_A) \geq \frac{A}{2\pi} + C\frac{\alpha^2}{\log(9/\alpha)} \tag{2.48}$$

Together with the inequalities above, these prove (2.46).

Proof of (2.47): Consider

$$y_- = \max\{y : u(x,y) = 1 \text{ for some } x \in [0, A]\}$$

and

$$y_+ = \min\{y : u(x,y) = 0 \text{ for some } x \in [0, A]\}.$$

If $y_- < y_+$ then $[0, A] \times (y_-, y_+) \subset V \subset G$. Thus, by the definition of α_1,

$$y_+ - y_- \leq 2\pi - \alpha_1.$$

To prove (2.47) it thus suffices to prove

$$\psi(0) \leq y_+ + \frac{\alpha_1}{4} \tag{2.49}$$

and

$$\varphi(A) \geq y_- - \frac{\alpha_1}{4} \tag{2.50}$$

We prove (2.49). The proof of (2.50) is similar. Let $I_0^* = \{(0,y) : 0 < u^*(0,y) < 1\}$. It easy to see that

$$\psi(0) = y_+ + |\{(0,y) \in I_0^* : y > y_+\}|. \tag{2.51}$$

Further, there is a curve $\gamma \subset \{z : u(z) = 0\}$ that connects some point (x, y_+) with a point $(0, y')$ or with a point $(1, y')$. Assume the first case (the other case is dealt with similarly). Since $u^*(0,y) = \min\{u(x,y) : 0 \leq x \leq A\}$ it follows that the segment $[(0, y_+), (0, y')]$ is disjoint from I_0^*. Since $u(0, y') = 0$, the segment I_0 is located in $y < y'$. Thus

$$\{(0,y) \in I_0^* : y > y_+\} = \{(0,y) \in I_0^* : y > y'\} \text{ is disjoint from } I_0.$$

Since $u^*(0,y) = \min\{u(x,y) : 0 \leq x \leq A\}$ we have $u > 0$ on I_0^*. Since $u(0,y) < 1$ for $y > \inf\{y : (0,y) \in I_0\}$ we get

$$\{(0,y) \in I_0^* : y > y_+\} = \{(0,y) \in I_0^* : y > y'\} \subset V.$$

By the second of the inequalities defining Case 3, we have

$$|\{y : (0,y) \in V \setminus I_0\}| \leq 2\pi - |I_0| < \alpha_1/4.$$

Hence (2.49) follows from (2.51).

Proof of (2.48): We first prove that

$$d_W(Q_0, Q_A) \geq \frac{A}{2\pi} + C\alpha^3. \tag{2.52}$$

Define a metric ρ in W by

$$\rho(x,y)^2 = \begin{cases} 1 + \frac{\alpha^2}{16A^2} & \text{if } \varphi(A) - \frac{\alpha}{4} < y < \varphi(A) + B + \frac{\alpha}{4} \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$A_\rho(W) \leq 2\pi A + \frac{\alpha^2}{16A^2}\left(B + \frac{\alpha}{2}\right)A = \frac{2\pi}{A}\left(A^2 + \frac{\alpha^2}{16}\right) - \frac{\alpha^3}{32A},$$

and some calculus shows that

$$L_\rho(\Gamma) \geq \sqrt{A^2 + \frac{\alpha^2}{16}}$$

for the family Γ of curves in W connecting Q_0 and Q_A. Thus

$$\lambda(\Gamma) \geq \frac{L_\rho(\Gamma)^2}{A_\rho(W)} \geq \frac{A}{2\pi} + C\alpha^3.$$

When proving (2.48) we can thus assume that $\alpha < \min\{\frac{1}{16A}, \frac{A}{16}\}$. By (2.47) we get $\alpha_1 < 2\pi$, which means that Ω contains some sector of $\{z : e^{-A} < |z| < 1\}$. Hence the height of V is at most 4π, that is, we can take $b - a \leq 4\pi$. It follows that the height of W is at most 8π, that is, $\psi(A) - \varphi(0) \leq 8\pi$. Now (2.48) follows from Lemma 2.29. □

As a consequence of Theorem 2.24 we get the following module theorem due to Teichmüller [**Tei82**, p. 233-244], see also [**Pom92**, Proposition 9.5].

Corollary 2.34. *Let J be a Jordan curve separating $\mathbb{T}$ and $C(0,R)$, where $R > 1$. If $\varepsilon < 1/2$ and*

$$d_{\mathbb{C}}(\mathbb{T}, J) + d_{\mathbb{C}}(J, C(0,R)) > \frac{1}{2\pi}\log R - \varepsilon,$$

then

$$\max_{z\in J}|z| / \min_{z\in J}|z| < 1 + C\sqrt{\varepsilon \log\frac{1}{\varepsilon}}, \tag{2.53}$$

where the constant C may depend on R.

The proof in [**Pom92**] shows that C can be taken independent of R.

Proof. By the serial rule, conformal invariance and the assumption we get

$$\begin{aligned} d_{\mathbb{C}}(J, RJ) &\geq d_{\mathbb{C}}(J, C(0,R)) + d_{\mathbb{C}}(C(0,R), RJ) \\ &= d_{\mathbb{C}}(J, C(0,R)) + d_{\mathbb{C}}(\mathbb{T}, J) > \frac{1}{2\pi}\log R - \varepsilon. \end{aligned} \tag{2.54}$$

Consider the domain G bounded by $\log J$, $\log(RJ)$ and line segments $I_0 \subset \{z : \operatorname{Im} z = 0\}$, $I_1 \subset \{z : \operatorname{Im} z = 2\pi\}$. By Lemma 2.28 and comparison

$$d_G(I_0, I_1)^{-1} = d_G(\log J, \log(RJ)) \geq d_{\mathbb{C}}(J, RJ),$$

which together with (2.54) gives

$$d_G(I_0, I_1) \leq \frac{2\pi}{\log R} + C_1\varepsilon.$$

After a scaling, Theorem 2.24 yields

$$d_G(I_0, I_1) \geq \frac{2\pi}{\log R} + C_2\frac{\alpha^2}{\log(9/\alpha)},$$

where

$$\alpha = \min\{2\pi, \frac{2\pi}{\log R}(\max \operatorname{Re}\log J - \min \operatorname{Re}\log J)\}.$$

It follows that

$$\alpha \leq C_3\sqrt{\varepsilon\log\frac{1}{\varepsilon}},$$

which is (2.53). □

We also would like to mention the following rather trivial *upper* estimate of the excess of extremal distance.

Theorem 2.35. *If $\alpha_k < \varepsilon\,(k = 1, \dots, n)$, where ε is small (depending on A), then*

$$X_\Omega(C(0, e^{-An}), \mathbb{T}) = \tilde{d}_\Omega(C(0, e^{-An}), \mathbb{T}) - \frac{An}{2\pi} \leq CA \sum_1^n \alpha_k, \tag{2.55}$$

where C is an absolute constant.

Corollary 2.36. *If $\sum_1^\infty \alpha_k < \infty$ then $\beta_\Omega(0, b) > 0$.*

Proof of Theorem 2.35. Let $U = -\log \Omega$, and let I_0 and I_n be components of $U \cap \{z : \operatorname{Re} z = 0\}$ and $U \cap \{z : \operatorname{Re} z = An\}$, respectively. Let G be the component of U that has both I_0 and I_1 on its boundary. As in the proof of Theorem 2.24, we may assume that G is a Jordan domain. Let J_+ and J_- be the complementary arcs of I_0 and I_n. Define a metric ρ in G by

$$\rho(x, y) = \rho_k = \frac{1}{2\pi - \alpha_{k-1} - \alpha_k - \alpha_{k+1}} \qquad \text{for } k - 1 \leq x < k$$

for $k = 1, 2, \dots, n$ and $\rho = 0$ otherwise. (We let $\alpha_{-1} = \alpha_{n+1} = 0$.) Let Γ be the family of curves in G connecting J_+ and J_-. If ε is sufficiently small (depending on A), then $L_\rho(\Gamma) \geq 1$, so that

$$d_G(J_+, J_-) = \lambda(\Gamma) \geq \frac{L_\rho(\Gamma)^2}{A_\rho(G)} \geq \frac{1}{A_\rho(G)}$$

and by Lemma 2.28

$$d_G(I_0, I_n) = \frac{1}{d_G(J_+, J_-)} \leq A_\rho(G) \leq 2\pi A \sum_1^n \rho_k^2 \leq \frac{An}{2\pi} + CA \sum_1^n \alpha_k.$$

This proves (2.55). □

This estimate is also in a sense best possible. Namely, consider the domain $-\log \Omega = U = \{(x, y) : \varphi(x) < y < \varphi(x) + 2\pi\}$, where $\varphi(x) = \frac{\alpha}{2} \sin Nx$. As $N \to +\infty$ we have

$$d_U(\{z : \operatorname{Re} z = 0\}, \{z : \operatorname{Re} z = An\}) \to \frac{An}{2\pi - \alpha} \geq \frac{An}{2\pi} + CAn\alpha.$$

5. Geometric criteria for $\beta > 0$: Slit domains

We now turn attention to the special kind of domains $\Omega = \hat{\mathbb{C}} \setminus J$, where J is a Jordan arc with one endpoint at 0. We assume that J can be described by an equation $\phi = \phi(r)$ in polar polar coordinates. We allow $\phi(r)$ to have jump discontinuities — in that case we include the corresponding circular arc in J. (That is, the arc $\{re^{i\theta} : \phi(r-) \leq \theta \leq \phi(r+)\}$ if $\phi(r-) < \phi(r+)$, and the arc $\{re^{i\theta} : \phi(r-) \geq \theta \geq \phi(r+)\}$ if $\phi(r-) > \phi(r+)$.) In case $\phi(r)$ is increasing, we can improve Theorems 2.24 and 2.35:

Theorem 2.37. *With Ω as above and $\phi(r)$ increasing we have*

$$c \sum_2^{n-1} \alpha_k^2 \leq X_\Omega(C(0, e^{-An}), \mathbb{T}) \leq C \sum_1^n \alpha_k^2 \log \frac{9}{\alpha_k},$$

where for the second inequality we assume that α_k are smaller than some small constant ε_0 (depending on A). The constants c and C may depend on A.

Corollary 2.38. *For such Ω and for $b \in \Omega$ we have the following implications.*

$$\sum_1^\infty \alpha_k^2 \log \frac{9}{\alpha_k} < \infty \implies \beta_\Omega(0, b) > 0 \implies \sum_1^\infty \alpha_k^2 < \infty.$$

Proof of Theorem 2.37. Consider the set $G = (-\log\Omega) \cap \{(x,y) : 0 \le x \le An\}$. We have

$$G = \{(x,y) : \varphi(x) < y < \psi(x),\ 0 \le x \le An\},$$

where φ and ψ are increasing functions with $\psi = \varphi + 2\pi$ except for at most countably many discontinuity points. We have to prove

$$c\sum_{2}^{n-1} \alpha_k^2 \le d_G(I_0, I_n) - \frac{An}{2\pi} \le C\sum_{1}^{n} \alpha_k^2 \log\frac{9}{\alpha_k}, \tag{2.56}$$

where $I_j = G \cap \{(Aj, y) : y \in \mathbb{R}\}$. Note that $\alpha_k = \min\{2\pi, \varphi(Ak) - \varphi(A(k-1)-)\}$. To prove the lower estimate, we first prove the case $n = 3$:

$$d_G(I_0, I_3) - \frac{3A}{2\pi} \ge c\alpha_2^2. \tag{2.57}$$

By Theorem 2.24, we need only consider the case $\alpha_1 + \alpha_2 + \alpha_3 < 1$. Let J_+ and J_- be the arcs that together with I_0 and I_3 form the boundary of G. Choose J_+ to be the upper arc (corresponding to ψ). We construct a continuous function $h : \overline{G} \to \mathbb{R}$ such that

$$h = 0 \text{ on } I_0 \cup J_+ \cup I_3, \qquad \int_{J_-} h\,\mathrm{d}y \ge \alpha_2 \quad \text{and} \quad D_G(h) \le C_1,$$

where the constant C_1 only depends on A. We can assume $\varphi(2A) = 0$ and define

$$h(x,y) = \begin{cases} x/A & \text{for } 0 \le x \le A,\ y \le 1 - x/A \\ 1 - y & \text{for } 0 \le y \le 1,\ A - Ay \le x \le 2A + Ay \\ 3 - x/A & \text{for } 2A \le x \le 3A,\ 0 \le y \le x/A - 2 \\ 1 & \text{for } A \le x \le 2A,\ y \le 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that h has the properties above. Define $u = x/3A - \varepsilon h$, where $\varepsilon = \alpha_2/2AC_1$. As in the proof of Lemma 2.29 we get

$$D_G(u) \le \frac{2\pi}{3A} - \frac{\alpha_2^2}{12A^2C_1},$$

and then (2.57) follows from Lemma 2.28.

By the serial rule (2.57) implies

$$d_G(I_0, I_n) - \frac{An}{2\pi} \ge c(\alpha_2^2 + \alpha_5^2 + \alpha_8^2 + \ldots + \alpha_{2+3k}^2),$$

where $2 + 3k \le n - 1$. Similarly

$$d_G(I_0, I_n) - \frac{An}{2\pi} \ge c(\alpha_3^2 + \alpha_6^2 + \ldots + \alpha_{3k'}^2)$$

and

$$d_G(I_0, I_n) - \frac{An}{2\pi} \ge c(\alpha_4^2 + \alpha_7^2 + \ldots + \alpha_{1+3k''}^2).$$

Taking the mean of these gives the lower estimate in (2.56).

We prove the upper estimate in (2.56) in a more general situation, see the next theorem (take $a = 2\pi/A$). □

We prove a version of the upper estimate in Theorem 2.37 in a more general situation. We consider graphs $y = \varphi(x)$ which, when thought of as a road, does not have too large and narrow holes or bumps.

Theorem 2.39. *Let $a > 0$ and let n be a positive integer. Let J be a Jordan arc which is the graph of a function $\varphi : [0,n] \to \mathbb{R}$ of bounded variation. We allow φ to have jump discontinuities of size $< a$ — in that case we include the corresponding vertical segment in J. Let G be the set in $[0,n] \times \mathbb{R}$ between J and $K = J + ai$.*

Let $\Delta < 1$. Assume that for every $z \in J \cup K$ there is a sector of angle Δ of the disc $D(z, a/2)$ that is contained in the domain

$$G' = G \cup \Big((-\infty, 0) \times (\varphi(0), \varphi(0)+a)\Big) \cup \Big((n, +\infty) \times (\varphi(n), \varphi(n)+a)\Big).$$

Let μ_k be the total variation of φ on $[k-1, k]$. Assume that $\mu_k < \varepsilon_0$ for $k = 1, \ldots, n$, where ε_0 is small (depending on a and Δ). Let $I_j = G \cap \{z : \operatorname{Re} z = j\}$. Then

$$d_G(I_0, I_n) - \frac{n}{a} \leq C \sum_1^n \mu_k^2 \log \frac{1}{\mu_k},$$

where the constant C only depends on a and Δ.

The proof of Theorem 2.39 uses the technique of Example 2.26. The key inequality (2.58) below is proved by using the estimate

$$|h(a_\xi) - h(b_\xi)| \leq \int_{[a_\xi, b_\xi]} |\nabla h| \, \mathrm{d}s.$$

One then integrates over ξ and uses Schwarz' inequality.

Lemma 2.40. *Under the assumptions of the theorem, let $h : \overline{G} \to \mathbb{R}$ be a continuous function which is C^1 on G and satisfies $h = 0$ on $I_0 \cup I_n$. Let J_k be the subarc of J with endpoints $k - 1 + i\varphi(k-1)$ and $k + i\varphi(k)$. Let $K_k = J_k + ia$ and $G_k = G \cap \{z : k-2 \leq z \leq k+1\}$. Then*

$$\left| \int_{J_k} h \, \mathrm{d}y - \int_{K_k} h \, \mathrm{d}y \right| \leq C\mu_k \sqrt{\log \frac{1}{\mu_k} \cdot D_{G_k}(h)}, \qquad k = 1, \ldots, n. \tag{2.58}$$

Proof of Lemma. Let $q = \min\{\frac{a}{2} \sin(\Delta/2), 1\}$. For every $z \in J$ either

(R) the segment $(z, z + q + ia/2)$ is contained in G', or
(L) the segment $(z, z - q + ia/2)$ is contained in G'.

Fix k. Let J_R be the set of $z \in J_k$ that satisfy (R). Let $y_0 = \max\{\operatorname{Im} z : z \in J_k\}$. Let L_0 be the horizontal line segment $[k - 2 + y_0 i, k + 1 + y_0 i]$. Define a projection $p_R : J_R \to L_0$ by letting $p_R(z)$ be the intersection point of the segments $(z, z + q + ai)$ and L_0. For each $z \in J_R$, the segment $(z, p_R(z))$ is contained in G'. Define a (signed) measure ν_R on L_0 by

$$\nu_R(E) = \int_{p_R^{-1}(E)} \mathrm{d}y.$$

It is easy to see that $\mathrm{d}\nu_R = \delta_R(z)\, \mathrm{d}z$, where $|\delta_R(z)| \leq C_1 = C_1(a, q)$. Extend $h(x, y)$ to be 0 for $x < 0$ and $x > n$. Now

$$\begin{aligned}
\left| \int_{J_R} h \, \mathrm{d}y - \int_{L_0} h \, \mathrm{d}\nu_R \right| &= \left| \int_{p_R(J_R)} (h(p_R^{-1}(z)) - h(z)) \, \mathrm{d}\nu_R(z) \right| \\
&\leq \int_{p_R(J_R)} \int_{(z, p_R^{-1}(z))} |\nabla h| \, \mathrm{d}s \, |\delta_R(z)| \, \mathrm{d}z \\
&\leq \left(\int_{p_R(J_R)} \left(\int_{(z, p_R^{-1}(z))} |\nabla h| \, \mathrm{d}s \right)^2 \mathrm{d}z \right)^{1/2} \left(\int_{L_0} |\delta_R(z)|^2 \, \mathrm{d}z \right)^{1/2} \\
&\leq \left(\int_{p_R(J_R)} \int_{(z, p_R^{-1}(z))} |\nabla h|^2 \, \mathrm{d}s \, |p_R^{-1}(z) - z| \, \mathrm{d}z \right)^{1/2} \left(C_1 \int_{L_0} |\delta_R(z)| \, \mathrm{d}z \right)^{1/2} \\
&\leq C_2 \mu_k \sqrt{D_{G_k}(h)}.
\end{aligned}$$

In the last step we used that

$$|p_R^{-1}(z) - z| \le C_3\mu_k \text{ and } \int_{L_0} |\delta_R(z)|\,\mathrm{d}z = |\nu_R|(L_0) = \int_{J_R} |\mathrm{d}y| \le \mu_k.$$

Let $J_L = J \setminus J_R$. Defining p_L, ν_L and δ_L similarly we have

$$\left|\int_{J_L} h\,\mathrm{d}y - \int_{L_0} h\,\mathrm{d}\nu_L\right| \le C_2\mu_k\sqrt{D_{G_k}(h)},$$

and thus

$$\left|\int_{J_k} h\,\mathrm{d}y - \int_{L_0} h\,\mathrm{d}\nu\right| \le 2C_2\mu_k\sqrt{D_{G_k}(h)}, \tag{2.59}$$

where $\nu = \nu_R + \nu_L$. The signed measure ν has density $\delta = \delta_R + \delta_L$ satisfying $|\delta| \le 2C_1$.

Let $y_2 = \min\{\operatorname{Im} z : z \in K_k\}$ and $y_1 = (y_0 + y_2)/2$. Let L_1 be the horizontal line segment $[k - 1 + y_1 i, k + y_1 i]$. We now claim that

$$\left|\int_{L_0} h\,\mathrm{d}\nu - \nu(L_0)\int_{L_1} h\,\mathrm{d}z\right| \le C_4\mu_k\sqrt{\log\frac{1}{\mu_k}\cdot D_{G_k}(h)}. \tag{2.60}$$

Together with (2.59) this gives

$$\left|\int_{J_k} h\,\mathrm{d}y - (\varphi(k) - \varphi(k-1))\int_{L_1} h\,\mathrm{d}z\right| \le C_5\mu_k\sqrt{\log\frac{1}{\mu_k}\cdot D_{G_k}(h)}. \tag{2.61}$$

(since $\nu(L_0) = \int_{J_k}\,\mathrm{d}y = \varphi(k) - \varphi(k-1)$). The same argument with K in place of J gives

$$\left|\int_{K_k} h\,\mathrm{d}y - (\varphi(k) - \varphi(k-1))\int_{L_1} h\,\mathrm{d}z\right| \le C_5\mu_k\sqrt{\log\frac{1}{\mu_k}\cdot D_{G_k}(h)}. \tag{2.62}$$

(2.61) and (2.62) prove the lemma.

To prove (2.60) it suffices to prove

$$\left|\int_{L_0} h\,\mathrm{d}\nu_+ - \nu_+(L_0)\int_{L_1} h\,\mathrm{d}z\right| \le C_6\mu_k\sqrt{\log\frac{1}{\mu_k}\cdot D_{G_k}(h)}. \tag{2.63}$$

and the corresponding inequality for the negative part ν_-. Define a mapping $F : L_0 \to L_1$ by

$$F(x + iy_0) = f(x) + iy_1, \text{ where } f(x) = k - 1 + \frac{\nu_+([k - 2 + iy_0, x + iy_0])}{\nu_+(L_0)}.$$

We have

$$\begin{aligned}
\left|\int_{L_0} h\,\mathrm{d}\nu_+ - \nu_+(L_0)\int_{L_1} h\,\mathrm{d}z\right| &= \left|\int_{L_0}(h(z) - h(F(z)))\delta_+(z)\,\mathrm{d}z\right| \\
&\le \int_{L_0}\int_{(z,F(z))} |\nabla h|\,\mathrm{d}s\,\delta_+(z)\,\mathrm{d}z \le C_7\int_{L_0}\int_{(z,F(z))} |\nabla h|\,\mathrm{d}y\,\delta_+(z)\,\mathrm{d}z \\
&\le C_7\left(\int_{L_0}\left(\int_{(z,F(z))} |\nabla h|\,\mathrm{d}y\right)^2 \delta_+(z)\,\mathrm{d}z\right)^{1/2}\left(\int_{L_0}\delta_+(z)\,\mathrm{d}z\right)^{1/2}.
\end{aligned} \tag{2.64}$$

Write $z = \xi + iy_0$. We can parametrize the segment $(z, F(z))$ as

$$\begin{cases} x = \xi + t(f(\xi) - \xi) \\ y = y_0 + t(y_1 - y_0) \end{cases} \qquad 0 < t < 1.$$

By Schwarz' inequality

$$\left(\int_{(z,F(z))} |\nabla h|\,\mathrm{d}y\right)^2 \le (y_1 - y_0)^2\int_0^1 |\nabla h|^2(1 + tf'(\xi) - t)\,\mathrm{d}t\int_0^1 \frac{\mathrm{d}t}{1 + tf'(\xi) - t}.$$

The last factor can be estimated as follows.

$$\int_0^1 \frac{\mathrm{d}t}{1 + tf'(\xi) - t} = \frac{\log f'(\xi)}{f'(\xi) - 1} \le 2\frac{\nu_+(L_0)}{\delta_+(z)} \log \frac{2C_1}{\nu_+(L_0)} \le C_8 \frac{\mu_k}{\delta_+(z)} \log \frac{1}{\mu_k}.$$

(To see the first inequality, let $\kappa = f'(\xi) = \delta_+(\xi)/\nu_+(L_0)$. In case $\kappa < 2$ one uses

$$\frac{\log \kappa}{\kappa - 1} < \frac{2}{\kappa} \qquad \text{and} \qquad \nu_+(L_0) \le \mu_k < \varepsilon_0 < \frac{2C_1}{e}.$$

In case $\kappa \ge 2$ one uses $1/(\kappa - 1) \le 2/\kappa$ and $\delta_+(z) \le 2C_1$.) Hence

$$\begin{aligned} &\int_{L_0} \left(\int_{(z,F(z))} |\nabla h| \,\mathrm{d}y \right)^2 \delta_+(z) \,\mathrm{d}z \\ &\qquad \le (y_1 - y_0)^2 C_8 \mu_k \log \frac{1}{\mu_k} \cdot \int_{L_0} \int_0^1 |\nabla h|^2 (1 + tf'(\xi) - t) \,\mathrm{d}t \,\mathrm{d}\xi \\ &\qquad\qquad \le C_9 \mu_k \log \frac{1}{\mu_k} \cdot D_{G_k}(h), \end{aligned}$$

where the last inequality follows from the change of variables formula. Substituting this in (2.64) and using

$$\int_{L_0} \delta_+(z) \,\mathrm{d}z = \nu_+(L_0) \le |\nu|(L_0) \le |\nu_R|(L_0) + |\nu_L|(L_0) = \int_{J_k} |\mathrm{d}y| = \mu_k$$

we get (2.63). The inequality (2.63) with ν_- in place of ν_+ is proved in the same way. Hence (2.60) holds, and the lemma is proved. □

Proof of Thereom 2.39. Let $u : \overline{G} \to [0,1]$ be the minimizing function given by Lemma 2.28 that satisfies

$$u|_{I_0} = 0, \quad u|_{I_n} = 1 \quad \text{and} \quad d_G(I_0, I_n) = \frac{1}{D_G(u)}.$$

Write $u = x/n + h$. Summation of (2.58) and using Cauchy-Schwarz' inequality gives

$$\left| \int_J h \,\mathrm{d}y - \int_K h \,\mathrm{d}y \right| \le C \left(\sum_1^n \mu_k^2 \log \frac{1}{\mu_k} \right)^{1/2} (3D_G(h))^{1/2}.$$

Thus

$$\begin{aligned} D_G(u) &= \frac{\operatorname{area} G}{n^2} + \frac{2}{n} \iint_G \frac{\partial h}{\partial x} \,\mathrm{d}A + D_G(h) \\ &= \frac{a}{n} + \frac{2}{n} \left(\int_J h \,\mathrm{d}y - \int_K h \,\mathrm{d}y \right) + D_G(h) \\ &\ge \frac{a}{n} - \frac{2}{n} C \left(3 \sum_1^n \mu_k^2 \log \frac{1}{\mu_k} \right)^{1/2} D_G(h)^{1/2} + D_G(h) \\ &\ge \frac{a}{n} - \frac{3C^2}{n^2} \sum_1^n \mu_k^2 \log \frac{1}{\mu_k}. \end{aligned}$$

Choosing ε_0 sufficiently small we get

$$d_G(I_0, I_n) = \frac{1}{D_G(u)} \le \frac{n}{a} + \frac{6C^2}{a^2} \sum_1^n \mu_k^2 \log \frac{1}{\mu_k}$$

□

Another class of domains $\Omega = \hat{\mathbb{C}} \setminus J$ is obtained when J is piecewise C^1. Consider the corresponding strip domain

$$G = -\log \Omega = \{(x, y) : \varphi(x) < y < \psi(x)\},$$

where $\psi(x) = \varphi(x) + 2\pi$ is piecewise C^1. For strip domains G with $\theta(x) = \psi(x) - \varphi(x)$ not necessarily equal to 2π, one has the following estimate, see [**Beu89**, p. 379–380] or [**Pom92**, Proposition 11.15].

$$d_G(\{x_1\} \times \mathbb{R}, \{x_2\} \times \mathbb{R}) \leq \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\theta(x)} + \frac{1}{2} \int_{x_1}^{x_2} \frac{\varphi'(x)^2 + \psi'(x)^2}{\theta(x)} \,\mathrm{d}x. \tag{2.65}$$

(This follows from Lemma 2.28 with $u(x, y) = (y - \varphi(x))/\theta(x)$.) This gives immediately the upper estimate in the following theorem.

Theorem 2.41. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be piecewise C^1. Assume that on each interval $[k-1, k]$ (k integer), φ is monotone and*

$$\frac{1}{C} \left|\varphi'\left(k - \frac{1}{2}\right)\right| \leq |\varphi'(x)| \leq C \left|\varphi'\left(k - \frac{1}{2}\right)\right| \qquad \text{for } k - 1 < x < k.$$

Let $G = \{(x, y) : \varphi(x) < y < \varphi(x) + a\}$. Then

$$d_G(\{0\} \times \mathbb{R}, \{n\} \times \mathbb{R}) - \frac{n}{a} \asymp \sum_1^n |\varphi(k) - \varphi(k-1)|^2. \tag{2.66}$$

Corollary 2.42. *Let J be the curve $\phi = A\varphi(\frac{1}{A} \log r)$ in polar coordinates, where φ is as in the theorem, and $A = 2\pi/a$. Let $b \in \Omega = \hat{\mathbb{C}} \setminus J$. Then*

$$\beta_\Omega(0, b) > 0 \iff \sum_1^\infty \alpha_k^2 < \infty.$$

Proof of Theorem 2.41. It follows from Theorem 2.37 that

$$d_G(\{z : \operatorname{Re} z = 0\}, \{z : \operatorname{Re} z = 1\}) - \frac{1}{a} \geq C_1 \left|\varphi\left(\frac{2}{3}\right) - \varphi\left(\frac{1}{3}\right)\right|^2 \tag{2.67}$$

if $|\varphi(\frac{2}{3}) - \varphi(\frac{1}{3})| \leq 2(a+1)$, say. To see that (2.67) holds also when $|\varphi(\frac{2}{3}) - \varphi(\frac{1}{3})| > 2(a+1)$, consider the metric $\rho = 1$ and the family Γ of curves in G connecting $\{z : \operatorname{Re} z = 0\}$ and $\{z : \operatorname{Re} z = 1\}$. Then

$$L_\rho(\Gamma) \geq \left|\varphi\left(\frac{2}{3}\right) - \varphi\left(\frac{1}{3}\right)\right| - a > 1 + \frac{1}{2}\left|\varphi\left(\frac{2}{3}\right) - \varphi\left(\frac{1}{3}\right)\right|$$

and

$$A_\rho(G \cap \{z : 0 < \operatorname{Re} z < 1\}) = a.$$

Hence

$$\lambda(\Gamma) \geq \frac{L_\rho(\Gamma)^2}{A_\rho(G \cap \{z : 0 < \operatorname{Re} z < 1\})} \geq \frac{1}{a} + C_2 \left|\varphi\left(\frac{2}{3}\right) - \varphi\left(\frac{1}{3}\right)\right|^2.$$

From the assumptions it follows that

$$\left|\varphi\left(\frac{2}{3}\right) - \varphi\left(\frac{1}{3}\right)\right| \asymp |\varphi(1) - \varphi(0)|.$$

Thus (2.67) and the serial rule gives the lower estimate in (2.66). ☐

6. The angular derivative problem

We have seen that giving geometric criteria for $\beta_\Omega(0, b) > 0$ boils down to estimating when

$$d_G(\{z : \operatorname{Re} z = x_1\}, \{z : \operatorname{Re} z = x_2\}) - \frac{x_2 - x_1}{2\pi} \to 0 \quad \text{as } x_2 > x_1 \to +\infty. \tag{2.68}$$

Here we suppose for simplicity that $G = -\log \Omega$ is a strip domain

$$G = \{x + iy : \varphi(x) < y < \psi(x)\}$$

with $\theta(x) = \psi(x) - \varphi(x) \leq 2\pi$.

A similar problem appears in the angular derivative problem. Let $f : \mathbb{D} \to \Omega$ be a conformal bijection with $f(1) = 0 \in \partial\Omega$. It is known [**Pom92**, Theorem 11.9] that the angular derivative $f'(1) = \operatorname{anglim}_{z\to 1} f'(z)$ exists and is $\neq 0, \infty$) if and and only if

$$d_G(\{z : \operatorname{Re} z = x_1\}, \{z : \operatorname{Re} z = x_2\}) - \frac{x_2 - x_1}{\pi} \to 0 \quad \text{as } x_2 > x_1 \to +\infty \tag{2.69}$$

and

$$\psi_\infty = \liminf_{x\to+\infty} \psi(x) \text{ and } \varphi_\infty = \limsup_{x\to+\infty} \varphi(x) \text{ are finite, and } \psi_\infty - \varphi_\infty = \pi. \tag{2.70}$$

Characterizing (2.68) should be easier than characterizing (2.69). Namely, we have seen that the quantity in (2.68) is non-negative, while the quantity in (2.69) can be negative. On the other hand, the condition (2.70) means that the known geometric criteria in the angular derivative problem are of little interest for the β-number problem, at least when rotation of spirals is considered. These criteria deal mostly with the cases $G \subset S$ or $G \supset S$, where $S = (0, +\infty) \times (0, \pi)$.

Our estimates apply to the angular derivative problem for domains with $\theta(x) \leq \pi$ (Theorems 2.24 and 2.35) or $\theta(x) = \pi$ (Theorems 2.39 and 2.41). This might be interesting, since these types of domains are generally not of the types $G \subset S$ or $G \supset S$. The following $\theta(x) = \pi$ type domains have however been dealt with before.

Example 2.43. Consider the union of the rectangles $[k - \frac{1}{2}, k + \frac{1}{2}] \times [y_k, y_k + \pi]$ (k integer), where the "jumps" $\mu_k = |y_k - y_{k-1}|$ are small. Let G be the interior domain. In [**War71**] and [**Eke71**] it was proved that if $\lim_{k\to+\infty} y_k = 0$ then the following equivalence holds.

$$(2.69) \iff \sum_1^\infty \mu_k^2 \log \frac{1}{\mu_k} < \infty.$$

(Actually they consider a more general class of domains.) The proof is very complicated and is based on estimates for the conformal map of a strip onto G using the Poisson integral. We now give a simpler proof. We prove that (if $\mu_k < \varepsilon_0$)

$$d_G(\{z : \operatorname{Re} z = 0\}, \{z : \operatorname{Re} z = n\}) - \frac{n}{\pi} \asymp \sum_1^n \mu_k^2 \log \frac{1}{\mu_k}.$$

The upper estimate follows immediately from Theorem 2.39. The lower estimate follows from a modification of the proof of Lemma 2.29. By the serial rule it suffices to consider $n = 1$. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be the continuous function

$$h(x, y) = h(z) = \begin{cases} 1 & \text{for } r \leq \mu_1/2 \\ (1 - 2r)/(1 - \mu_1) & \text{for } \mu_1/2 \leq r \leq 1/2 \\ 0 & \text{for } r \geq 1/2, \end{cases}$$

where $r = |z - (\frac{1}{2} + \frac{y_0+y_1}{2} i)|$. It is easily checked that

$$h(0, \cdot) = h(1, \cdot) = 0, \qquad \int_{\partial G_1} h \, \mathrm{d}y = y_1 - y_0 \qquad \text{and} \qquad D_{G_1}(h) \leq \frac{2\pi}{\log(1/\mu_1)},$$

where $G_1 = G \cap \{z : 0 < \operatorname{Re} z < 1\}$. As in the proof of Lemma 2.29, with $u = x - \varepsilon h$ this yields

$$d_G(\{z : \operatorname{Re} z = 0\}, \{z : \operatorname{Re} z = 1\}) \geq \frac{1}{D_{G_1}(u)} \geq \frac{1}{\pi} + c\mu_1^2 \log \frac{1}{\mu_1}.$$

CHAPTER 3

Brennan's conjecture

1. Equivalent formulations

Let $\Omega \subset \mathbb{C}$ be a simply connected domain, and let $\varphi : \Omega \to \mathbb{D}$ be a conformal bijection. In [**Bre78**] J. E. Brennan posed the following problem.

Brennan's problem: For which $q \in \mathbb{R}$ is

$$\iint_{\Omega} |\varphi'|^q \, \mathrm{d}A < \infty \qquad \text{for every } \varphi? \tag{3.1}$$

In terms of the inverse map $\varphi^{-1} = f : \mathbb{D} \to \Omega$ we may write this

$$\iint_{\mathbb{D}} |f'|^t \, \mathrm{d}A < \infty \qquad \text{for every } f, \tag{3.2}$$

where $t = 2 - q$. This is also related to integral means over circles $|z| = r$.

Definition 3.1. Let $f : \mathbb{D} \to \mathbb{C}$ be univalent and $t \in \mathbb{R}$. Define $\beta_f(t)$ to be the infimum of all β such that

$$\int_0^{2\pi} |f'(re^{i\theta})|^t \, \mathrm{d}\theta = O\left(\left(\frac{1}{1-r}\right)^{\beta}\right) \qquad \text{as } r \to 1.$$

β_f is called the *integral means spectrum* of f'. The *universal integral means spectrum* is

$$B(t) = \sup\{\beta_f(t) : f : \mathbb{D} \to \mathbb{C} \text{ is univalent }\}.$$

Example 3.2. For the Koebe function $k(z) = z(1+z)^{-2}$ we have

$$\beta_k(t) = \max\{-t-1, 0, 3t-1\}.$$

Thus (3.2) implies $-2 < t < 2/3$.

Proposition 3.3. $\beta_f : \mathbb{R} \to [0, +\infty)$ *is convex, and*

$$\beta_f(t+s) \le \beta_f(t) + 3s \quad \textit{if } s > 0, \tag{3.3}$$

$$\beta_f(t-s) \le \beta_f(t) + s \quad \textit{if } s > 0, \tag{3.4}$$

$$B(t) = 3t - 1 \qquad \textit{if } t \ge 2/5. \tag{3.5}$$

Proof. This follows from Hölder's inequality and the distortion theorem. See [**Pom92**, Proposition 8.3 and Theorem 8.4]. □

Proposition 3.4.

$$\beta_f(t) < 1 \implies \iint_{\mathbb{D}} |f'|^t \, \mathrm{d}A < \infty \implies \beta_f(t) \le 1.$$

Proof. The first implication is obvious. Since $|f'|^t$ is subharmonic, the integral $\int_0^{2\pi} |f'(re^{i\theta})|^t \, \mathrm{d}\theta$ is an increasing function of r. Hence

$$\iint_{\mathbb{D}} |f'|^t \, \mathrm{d}A \ge (1-r) r \int_0^{2\pi} |f'(re^{i\theta})|^t \, \mathrm{d}\theta,$$

and the second implication follows. □

It follows from Example 3.2 and Proposition 3.3 that the equation $B(t) = 1$ has two roots $t = 2/3$ and $t_{\mathrm{Br}} \in [-2, -1]$. Proposition 3.4 shows that (3.2) is equivalent to either

$$t_{\mathrm{Br}} < t < 2/3 \quad \text{or} \quad t_{\mathrm{Br}} \le t < 2/3.$$

Brennan proved that $t_{\mathrm{Br}} < -1$, and he conjectured that $t_{\mathrm{Br}} = -2$.

Brennan's conjecture: (3.1) holds for $4/3 < q < 4$.

By Proposition 3.3 and Example 3.2 an equivalent formulation is

$$B(t) = -t - 1 \quad \text{for } t \le -2.$$

In [**CaMa94**] Carleson and Makarov indicated that the condition $B(t) = -t - 1$ can be reformulated as an estimate of β-numbers, or as an estimate of the number of discs of large harmonic measure. In the following theorem, we prove that these formulations are equivalent.

Definition 3.5. Σ is the set of all conformal maps $g : \mathbb{D}^* \to \hat{\mathbb{C}}$ with the normalization $g(\infty) = \infty$, $g'(\infty) = 1$.

Definition 3.6. Let $\Omega \ni \infty$ be a simply connected domain. Let $\rho, h > 0$. Define $N_\Omega(\rho, h)$ to be the maximal number of disjoint discs Δ_j of radius ρ such that

$$\omega(\Delta_j \cap \partial\Omega, \Omega, \infty) \ge h.$$

Theorem 3.7. *Let $p > 0$. The following are equivalent:*

a) $B(-p) = p - 1$.

b) *For any polygonal tree Γ with leaves $a_1, \dots, a_m, \infty$,*

$$\sum_1^m \beta_{\mathbb{C}\setminus\Gamma}(a_j, \infty)^p \le 1.$$

c) *For any simply connected domain Ω and distinct points $a_1, \dots, a_m, b \in \partial\Omega$,*

$$\sum_1^m \beta_\Omega(a_j, b)^p \le 1.$$

d) *There exists a constant C such that for any simply connected domain Ω and distinct points $a_1, \dots, a_m \in \partial\Omega$, $b \in \Omega$ we have*

$$\sum_1^m \beta_\Omega(a_j, b)^p \le C.$$

e) *There exists a constant C such that for all $g \in \Sigma$*

$$\int_0^{2\pi} |g'(re^{i\theta})|^{-p} \, \mathrm{d}\theta \le \frac{C}{(r-1)^{p-1}}, \qquad r > 1.$$

f) *There exists a constant C such that for all $f \in S$*

$$\int_0^{2\pi} |f'(re^{i\theta})|^{-p} \, \mathrm{d}\theta \le \frac{C}{(1-r)^{p-1}}, \qquad 0 \le r < 1.$$

g) *For any $\varepsilon > 0$ there exists a $C(\varepsilon)$ such that for any simply connected domain $\Omega \ni \infty$ with $\operatorname{diam} \partial\Omega = 1$*

$$N_\Omega(\rho, h) \le C(\varepsilon) \frac{\rho^{p-\varepsilon}}{h^{2p}}, \qquad \rho, h > 0.$$

The heart of the theorem lies in the implication a) $\Longrightarrow$ b). This is proved with an iterative construction. Starting with a tree Γ with

$$\sum_1^m \beta_{\mathbb{C}\setminus\Gamma}(a_j, \infty)^p = K > 1$$

one may build "dandelions" Γ_n, which are trees with

$$\sum_1^{m^n} \beta_{\mathbb{C}\setminus\Gamma_n}(a'_j,\infty)^p \approx K^n$$

In the limit $n \to \infty$ one gets a domain $\mathbb{C}\setminus\Gamma_\infty$ for which the Riemann map $f:\mathbb{D}\to\mathbb{C}\setminus\Gamma_\infty$ satisfies

$$\int_0^{2\pi} |f'(re^{i\theta})|^{-p}\,\mathrm{d}\theta \geq C\left(\frac{1}{1-r}\right)^{p-1+\delta}$$

for some $\delta > 0$.

Brennan's conjecture states that for every univalent $f:\mathbb{D}\to\mathbb{C}$ and every $\varepsilon > 0$ there is a $C(f,\varepsilon)$ such that

$$\int_0^{2\pi} |f'(re^{i\theta})|^{-2}\,\mathrm{d}\theta \leq C(f,\varepsilon)\left(\frac{1}{1-r}\right)^{1+\varepsilon}.$$

The equivalence a) $\iff$ f) shows that an equivalent formulation is

$$\int_0^{2\pi} |f'(re^{i\theta})|^{-2}\,\mathrm{d}\theta \leq \frac{C}{1-r} \qquad \text{for every } f\in S,$$

where C is an absolute constant.

The main theorem of [**CaMa94**] is that g) holds with $\varepsilon = 0$ for some $p = p_0 > 0$. Thus $B(-p) = p-1$ for $p \geq p_0$. Moreover they showed that for the domain $\Omega = \mathbb{C}\setminus[-1,+\infty)\setminus[-i\varepsilon, i\varepsilon]$ one has for $p < 2$

$$\beta_\Omega(-1,\infty)^p + \beta_\Omega(i\varepsilon,\infty)^p + \beta_\Omega(-i\varepsilon,\infty)^p > 1 \tag{3.6}$$

if ε is small. By the implication a) $\Longrightarrow$ b) this gives $B(-p) > p-1$ for $p < 2$. In conclusion, there exists a $t_{\mathrm{CM}} \in (-\infty,-2]$ such that

$$B(t) = -t-1 \text{ for } t \leq t_{\mathrm{CM}},$$
$$B(t) > -t-1 \text{ for } t > t_{\mathrm{CM}}.$$

Brennan's conjecture states that $t_{\mathrm{CM}} = -2$.

The rest of the chapter is devoted to the proof of Theorem 3.7. We prove the implications a) $\Longrightarrow$ b) $\Longrightarrow$ c) $\Longrightarrow$ d) $\Longrightarrow$ e) $\Longrightarrow$ f) $\Longrightarrow$ a) and e) $\Longrightarrow$ g) $\Longrightarrow$ a).

2. The dandelion construction

In this section we prove the implication a) $\Longrightarrow$ b) of Theorem 3.7. The idea is taken from [**CaMa94**, Section 4.3]. Let Γ_1 be a polygonal tree with leaves $a_1,\dots,a_m,\infty$. Assume that $[-\infty,0]\subset\Gamma_1$.

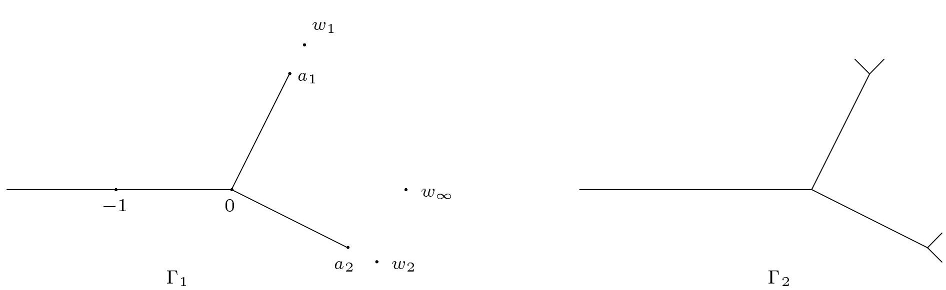

FIGURE 8

By induction we construct polygonal trees Γ_n $(n = 2,3,\dots)$ with leaves ∞ and a_X, where $X = X_1\dots X_n$ ranges over all sequences of length n with $X_j \in \{1,2,\dots,m\}$.

Γ_{n+1} is constructed from Γ_n as follows. Fix a small $q > 0$. For each leaf a_X, let s_X be the line segment of Γ_n with length q^n that ends at a_X. Construct Γ_{n+1} by glueing to each s_X a scaled down and rotated copy of $\Gamma_1' = \Gamma_1 \setminus [-\infty, -1)$ so that $[-1, 0]$ corresponds to s_X. The leaves of Γ_{n+1} are coded in the natural way, namely, the leaves of the tree glued on at a_X are denoted $a_{X1}, \ldots, a_{Xm}$. We choose q so small that no intersections occur in this process. The set

$$\Gamma_\infty = \overline{\cup_{n=1}^{\infty} \Gamma_n}$$

is called the *dandelion* constructed from Γ_1 with scale q. The implication a) $\Longrightarrow$ b) follows from the following lemma.

Lemma 3.8. *Let $g_\infty : \mathbb{D} \to \Omega_\infty = \mathbb{C} \setminus \Gamma_\infty$ be a conformal bijection. Let $\Omega = \mathbb{C} \setminus \Gamma_1$, $\beta_j = \beta_\Omega(a_j, \infty)$ and $p > 0$. Assume that*

$$\sum_{j=1}^{m} \beta_j^p > 1.$$

Then, if q is sufficiently small, we have

$$\int_0^{2\pi} |g_\infty'(re^{i\theta})|^{-p} \, \mathrm{d}\theta \geq C \left(\frac{1}{1-r} \right)^{p-1+\varepsilon}, \qquad 0 \leq r < 1, \tag{3.7}$$

where $\varepsilon = \varepsilon(q) > 0$ and $C = C(g_\infty) > 0$.

Proof. Fix $b > 0$ such that $b \in \Omega$. We first study the conformal bijection $f_\infty : \mathbb{H} \to \Omega_\infty$ with $f_\infty(i) = b$, $f_\infty(\infty) = \infty$. Let $w_\infty = 1/\sqrt{q} \in \Omega$. For each leaf a_X, let $w_X \in \Omega_\infty$ be the point corresponding to w_∞ in the glueing operation. That is, the segment (a_X, w_X) is parallel to s_X and $|w_X - a_X| = |s_X| w_\infty$. Let $z_X = f_\infty^{-1}(w_X) \in \mathbb{H}$ and $z_\infty = f_\infty^{-1}(w_\infty) \in \mathbb{H}$. The proof rests on the following estimates:

$$\left| \frac{f_\infty'(z_{Xj})}{f_\infty'(z_X)} \right| = \sqrt{\frac{q}{\beta_j}} (1 + o(1)) \tag{3.8}$$

$$\frac{\operatorname{Im} z_{Xj}}{\operatorname{Im} z_X} = \sqrt{q \beta_j} (1 + o(1)), \tag{3.9}$$

where $o(1) \to 0$ as $q \to 0$ uniformly in X and j. Moreover

$$|\operatorname{Re} z_X - \operatorname{Re} z_Y| \geq C q^{1/4} \operatorname{Im} z_X \tag{3.10}$$

if $X \neq Y$ have the same length. Before proving these estimates, let us see how the proof is finished. Define

$$s = \sum_1^m \beta_j^p \qquad \text{and} \qquad \alpha_k = \beta_k^p / s.$$

Given n, select integers $n_k \approx \alpha_k n$ so that $\sum_1^m n_k = n$. Let S_n be the set of all sequences $X = X_1 \ldots X_n$ that contain n_k symbols k $(k = 1, 2, \ldots, m)$. The number of such X is

$$\# S_n = \frac{n!}{n_1! \ldots n_m!} \asymp \frac{\sqrt{n}(n/e)^n}{\prod_1^m \sqrt{n_k}(n_k/e)^{n_k}} = \left(\frac{1 + o(1)}{\prod_1^m \alpha_k^{\alpha_k}} \right)^n,$$

where $o(1) \to 0$ as $n \to \infty$. By (3.8) and (3.9) we have for $X \in S_n$

$$|f_\infty'(z_X)| = |f_\infty'(z_\infty)| \left(\frac{q(1 + o(1))}{\prod_1^m \beta_k^{\alpha_k}} \right)^{n/2} \tag{3.11}$$

and

$$\operatorname{Im} z_X = \operatorname{Im} z_\infty \left(q(1 + o(1)) \prod_1^m \beta_k^{\alpha_k} \right)^{n/2}, \tag{3.12}$$

where $o(1) \to 0$ as $q \to 0$ and $n \to \infty$ (uniformly in X).

Let

$$y_n = \operatorname{Im} z_\infty \left(q \prod_1^m \beta_k^{\alpha_k} \right)^{n/2}.$$

By (3.12) and the distortion theorem we have for $X \in S_n$

$$|f'_\infty(\operatorname{Re} z_X + \xi + iy_n)| = |f'_\infty(z_X)|(1+o(1))^n \tag{3.13}$$

for real ξ with $|\xi| \leq \operatorname{Im} z_X$. There is a bounded interval I containing all $\operatorname{Re} z_X$. By (3.10) and (3.13) we get

$$\int_I |f'_\infty(x+iy_n)|^{-p}\, \mathrm{d}x \geq \sum_{X \in S_n} |f'_\infty(z_X)|^{-p}(1+o(1))^n C q^{1/4} \operatorname{Im} z_X.$$

By (3.11) and (3.12) this is greater than

$$\begin{aligned}
&\# S_n |f_\infty(z_\infty)|^{-p} \left(\frac{\prod_1^m \beta_k^{\alpha_k}}{q(1+o(1))} \right)^{np/2} C q^{1/4} y_n \\
&= C_2(q) \left(\prod_1^m \frac{\beta_k^{\alpha_k p}}{\alpha_k^{\alpha_k}} (1+o(1)) \right)^n \frac{y_n}{(q \prod_1^m \beta_k^{\alpha_k})^{np/2}} = C_3(q)(s(1+o(1)))^n y_n^{1-p}.
\end{aligned}$$

Fix q and n_0 so that $s(1+o(1)) > s_0 > 1$ for $n > n_0$. Let $\varepsilon > 0$ solve

$$\left(q \prod_1^m \beta_k^{\alpha_k} \right)^{\varepsilon/2} = 1/s_0.$$

We have

$$\int_I |f'_\infty(x+iy_n)|^{-p}\, \mathrm{d}x \geq C_3(q) s_0^n y_n^{1-p} = C_4(q) y_n^{1-p-\varepsilon}, \qquad n > n_0.$$

Now consider any conformal bijection $g_\infty : \mathbb{D} \to \Omega_\infty$. Using the distortion theorem it is straightforward to see that

$$\int_{|z|=1-y_n} |g'_\infty(z)|^{-p}\, \mathrm{d}\theta \geq C_5 \int_I |f'_\infty(x+iy_n)|^{-p}\, \mathrm{d}x,$$

and (3.7) follows.

It remains to prove the estimates (3.8)–(3.10). To do this, introduce scaled variants of f_∞:

$$f_X(z) = \frac{e^{i\theta_X}}{q^n}(f_\infty(\operatorname{Re} \zeta_X + z \operatorname{Im} \zeta_X) - a_X) \qquad \text{for } X = X_1 \ldots X_n.$$

$\theta_X \in \mathbb{R}$ is chosen so that the domain $\Omega_X = f_X(\mathbb{H})$ has $\Gamma'_1 = \Gamma_1 \setminus [-\infty, -1)$ on its boundary. $\zeta_X \in \mathbb{H}$ is chosen so that $f_X(i) = b$. The domains Ω_X and Ω differ only within the discs $D(\infty, C_1 q)$ and $D(a_j, C_1 q)$, $j = 1, 2, \ldots, m$. We need a lemma which estimates how much $f_X^{-1}(w)$ and $f^{-1}(w)$ differ for w not close to these discs.

Lemma 3.9. *There are constants C, A_1, A_2, A_3 depending on the positive integer N such that the following holds. Let $\Omega_1 \supset \Omega_0$ be simply connected domains such that*

$$\Omega_1 \cap \partial\Omega_0 = Q = \cup_1^N Q_k,$$

where Q_k are crosscuts of Ω_1. Let $w_0 \in \Omega_0$ and assume for the harmonic measure that

$$\omega(Q, \Omega_0, w_0) = \varepsilon \leq 1/C.$$

Let Ω_2 be another simply connected domain such that $\Omega_2 \supset \Omega_0$ and $\Omega_2 \cap \partial\Omega_0 = Q$. Let $g_j : \Omega_j \to \mathbb{D}$ be conformal bijections with $g_j(w_0) = 0$. Let $1 \geq \delta \geq C\varepsilon$ and

$$F = \{w \in \Omega_0 : \operatorname{dist}(g_1(w), g_1(Q)) \geq \delta\}.$$

We consider different normalizations:
a) Suppose that $g_j'(w_0) > 0$. *Then*

$$\left|\frac{g_2(w)}{g_1(w)} - 1\right| \le A_1\varepsilon^2/\delta \qquad \textit{for } w \in F,$$
$$\left|\frac{g_2'(w)}{g_1'(w)} - 1\right| \le A_1\varepsilon^2/\delta^2 \qquad \textit{for } w \in F.$$

b) Suppose for $j = 1, 2$ *that* P_j *is a prime end of* Ω_j *that is separated from* w_0 *in* Ω_j *by the crosscut* Q_k. *Assume that* $g_j(P_j) = 1$. *Then*

$$\left|\frac{g_2(w)}{g_1(w)} - 1\right| \le A_2\varepsilon \qquad \textit{for } w \in F,$$
$$\left|\frac{g_2'(w)}{g_1'(w)} - 1\right| \le A_2\varepsilon/\delta \qquad \textit{for } w \in F.$$

c) Let P_j *be as in b). Let* $h_j : \Omega_j \to \mathbb{H}$ *be conformal bijections with* $h_j(w_0) = i$ *and* $h_j(P_j) = \infty$. *Then*

$$\left|\frac{h_2'(w)}{h_1'(w)} - 1\right| \le A_3\varepsilon/\delta \qquad \textit{for } w \in F.$$

We use part c) of the lemma with $\Omega_1 = \Omega$, $\Omega_2 = \Omega_X$, $h_1 = f^{-1}$, $h_2 = f_X^{-1}$, $\overline{Q} = C(\infty, C_1 q) \cup \cup_1^m C(a_j, C_1 q)$, $\varepsilon \asymp \sqrt{q}$ and $\delta = C_3 q^{1/3}$. We get

$$(f_X^{-1})'(w) = (f^{-1})'(w)(1 + O(q^{1/6})) \tag{3.14}$$

for w outside the discs $D(\infty, q^{2/3})$, $D(a_j, q^{2/3})$, $j = 1, \ldots, m$. In particular

$$\frac{(f_X^{-1})'(w_\infty)}{(f_X^{-1})'(w_j)} = \frac{(f^{-1})'(w_\infty)}{(f^{-1})'(w_j)}(1 + O(q^{1/6})).$$

The left-hand side here is

$$\frac{f_X'(f_X^{-1}(w_j))}{f_X'(f_X^{-1}(w_\infty))} = \frac{f_\infty'(z_{Xj})}{f_\infty'(z_X)}.$$

Since f is analytic at $x_j = f^{-1}(a_j)$ and meromorphic at ∞ we get by Theorem 2.23

$$\left|\frac{(f^{-1})'(w_\infty)}{(f^{-1})'(w_j)}\right| = \sqrt{q\left|\frac{f''(x_j)}{f''(\infty)}\right|}(1 + O(q^{1/4})) = \sqrt{\frac{q}{\beta_j}}(1 + O(q^{1/4})),$$

and (3.8) follows.

To prove (3.9), integrate (3.14) around one of the semicircles $\sigma \subset C(a_j, \sqrt{q})$ with endpoints w_j and $\tilde{w}_j \in \Gamma_1$. We get

$$f_X^{-1}(w_j) - f_X^{-1}(\tilde{w}_j) = f^{-1}(w_j) - f^{-1}(\tilde{w}_j) + |f^{-1}(\sigma)|O(q^{1/6}).$$

Since $f_X^{-1}(\tilde{w}_j)$ and $f^{-1}(\tilde{w}_j)$ are real, and $|f^{-1}(\sigma)| \asymp \operatorname{Im} f^{-1}(w_j)$ this gives

$$\operatorname{Im} f_X^{-1}(w_j) = \operatorname{Im} f^{-1}(w_j)(1 + O(q^{1/6})).$$

Similarly we get

$$\operatorname{Im} f_X^{-1}(w_\infty) = \operatorname{Im} f^{-1}(w_\infty)(1 + O(q^{1/6})).$$

Since

$$f_X^{-1}(w_j) = \frac{z_{Xj} - \operatorname{Re}\zeta_X}{\operatorname{Im}\zeta_X} \qquad \text{and} \qquad f_X^{-1}(w_\infty) = \frac{z_X - \operatorname{Re}\zeta_X}{\operatorname{Im}\zeta_X}$$

we get

$$\frac{\operatorname{Im} z_{Xj}}{\operatorname{Im} z_X} = \frac{\operatorname{Im} f^{-1}(w_j)}{\operatorname{Im} f^{-1}(w_\infty)}(1 + O(q^{1/6}))$$

Since f is analytic at x_j and meromorphic at ∞ we get by Theorem 2.23

$$\frac{\operatorname{Im} f^{-1}(w_j)}{\operatorname{Im} f^{-1}(w_\infty)} = \sqrt{q\left|\frac{f''(\infty)}{f''(x_j)}\right|}(1+O(q^{1/4})) = \sqrt{q\beta_j}(1+O(q^{1/4})),$$

and (3.9) follows.

Finally, integrating (3.14) from any point $w'_j \in C(a_j, \sqrt{q})$ to any point $w'_k \in C(a_k, \sqrt{q})$ we get

$$f_X^{-1}(w'_j) - f_X^{-1}(w'_k) = f^{-1}(w'_j) - f^{-1}(w'_k) + O(q^{1/6}),$$

so that

$$\operatorname{Re}(f_X^{-1}(w'_j) - f_X^{-1}(w'_k)) \geq C \qquad \text{if } j \neq k \text{ and } q \text{ is small.}$$

It follows that

$$\operatorname{Re}(f_X^{-1}(w_{jY}) - f_X^{-1}(w_{kZ})) \geq C \qquad \text{if } j \neq k.$$

Now

$$f_X^{-1}(w_{jY}) = \frac{z_{XjY} - \operatorname{Re}\zeta_X}{\operatorname{Im}\zeta_X}$$

so we have

$$\begin{aligned}\operatorname{Re}(z_{XjY} - z_{XjZ}) \geq C \operatorname{Im}\zeta_X = C\frac{\operatorname{Im} z_X}{\operatorname{Im} f_X^{-1}(w_\infty)} \\ \asymp \frac{\operatorname{Im} z_X}{\operatorname{Im} f^{-1}(w_\infty)} \asymp q^{1/4}\operatorname{Im} z_X \geq q^{1/4}\operatorname{Im} z_{XjY}\end{aligned}$$

and (3.10) follows. □

Proof of Lemma 3.9. a) $\varphi_j = g_j \circ g_0^{-1}$ maps $\mathbb{D}$ conformally onto the Jordan domain $g_j(\Omega_j) \subset \mathbb{D}$, and $\varphi_j(0) = 0$, $\varphi'_j(0) > 0$. The set $\tilde{Q} = g_0(Q)$ consists of the arcs $\tilde{Q}_k = g_0(Q_k) \subset \mathbb{T}$. Let

$$\tilde{F} = g_0(F) = \{z \in \mathbb{D} : \operatorname{dist}(\varphi_1(z), \varphi_1(\tilde{Q})) \geq \delta\}.$$

It suffices to prove

$$\left|\frac{\varphi_j(z)}{z} - 1\right| \leq C_1\frac{\varepsilon^2}{\delta} \qquad \text{for } z \in \tilde{F} \tag{3.15}$$

and

$$\left|\varphi'_j(z) - 1\right| \leq C_2\frac{\varepsilon^2}{\delta^2} \qquad \text{for } z \in \tilde{F}. \tag{3.16}$$

We have

$$\frac{|\tilde{Q}|}{2\pi} = \omega(\tilde{Q}, \mathbb{D}, 0) = \omega(Q, \Omega_0, w_0) = \varepsilon, \tag{3.17}$$

$$|\varphi_j| = 1 \text{ on } \mathbb{T} \setminus \tilde{Q} \tag{3.18}$$

and with an absolute constant $C_3 > 0$,

$$|\varphi_j| \geq 1 - C_3\varepsilon \text{ on } \tilde{Q}. \tag{3.19}$$

To see (3.19), let I be the circular projection of $\varphi_j(\tilde{Q})$:

$$I = \{|z| : z \in \varphi_j(\tilde{Q})\} = [r, 1).$$

By Beurling's projection theorem [**Ahl73**, Theorem 3-6]

$$\omega(I, \mathbb{D} \setminus I, 0) \leq \omega(\varphi_j(\tilde{Q}), \mathbb{D} \setminus \varphi_j(\tilde{Q}), 0) = \omega(\tilde{Q}, \mathbb{D}, 0) = \varepsilon$$

and hence $r \geq 1 - C_3\varepsilon$ and (3.19) follows. The Poisson-Schwarz formula

$$\log\frac{\varphi_j(z)}{z} = \int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log|\varphi_j(e^{i\theta})|\,\frac{\mathrm{d}\theta}{2\pi} \tag{3.20}$$

together with (3.17)–(3.19) gives

$$\left|\log \frac{\varphi_j(z)}{z}\right| \le \frac{4C_3\varepsilon^2}{\operatorname{dist}(z,\tilde{Q})} \tag{3.21}$$

if ε is small.

We now show that

$$z \in \tilde{F} \implies \operatorname{dist}(z,\tilde{Q}) \ge \frac{\delta}{C} \tag{3.22}$$

if our constant C is chosen large enough. Let L be the union of the arcs of $\mathbb{T} \setminus \tilde{Q}$ that have length $> 2\delta/C$. Let σ be a component arc of $\mathbb{T} \setminus L$. Thus

$$|\sigma| \le |\tilde{Q}| + (N-1)\frac{2\delta}{C} \le 2\pi\varepsilon + \frac{(2N-2)\delta}{C} \le \frac{2\pi + 2N - 2}{C}\delta.$$

Let G be the domain

$$G = \{z \in \mathbb{D} : \operatorname{dist}(z, \tilde{Q} \cap \sigma) < \frac{\delta}{C}\}.$$

The arc $\gamma = \partial G \cap \mathbb{D}$ has

$$\operatorname{diam}\gamma \le |\sigma| + 2\frac{\delta}{C} \le \frac{2\pi + 2N}{C}\delta. \tag{3.23}$$

For $z \in \gamma$ we have $\operatorname{dist}(z,\tilde{Q}) = \delta/C$, so (3.21) gives

$$|\varphi_j(z) - z| \le \frac{8C_3C\varepsilon^2}{\delta} \le \frac{8C_3}{C}\delta. \tag{3.24}$$

Thus $\operatorname{diam}\varphi_j(\gamma) \le (2\pi+2N+16C_3)\delta/C$, so that $\operatorname{diam}\varphi_j(G) < \delta$ is C is sufficiently large. Hence

$$z \in G \implies \operatorname{dist}(\varphi_1(z), \varphi_1(\tilde{Q})) < \delta,$$

which proves (3.22).

The estimate (3.15) follows from (3.21) and (3.22). By (3.20),

$$\frac{\varphi_j'(z)}{\varphi_j(z)} - \frac{1}{z} = \int_0^{2\pi} \frac{2}{(e^{i\theta} - z)^2} \log|\varphi_j(e^{i\theta})| \frac{d\theta}{2\pi}$$

so that (3.17)–(3.19) and (3.22) gives for $z \in \tilde{F}$

$$\left|\varphi_j'(z) - \frac{\varphi_j(z)}{z}\right| \le \frac{2}{\operatorname{dist}(z,\tilde{Q})^2} 2C_3\varepsilon\frac{|\tilde{Q}|}{2\pi} \le \frac{4C_3C^2\varepsilon^2}{\delta^2}.$$

Together with (3.15) this proves (3.16).

b) Keep the normalization in a). We need only prove that

$$|g_2(P_2) - g_1(P_1)| \le C_4\varepsilon. \tag{3.25}$$

Let L, σ, G and γ be as before, with $\sigma \supset \tilde{Q}_k$. Let $z', z'' \in \mathbb{T}$ be the endpoints of γ. By (3.23) and (3.24) with $\delta = C\varepsilon$ we have

$$|z' - z''|,\ |\varphi_j(z') - z'|,\ |\varphi_j(z'') - z''| \le C_5\varepsilon.$$

It follows that

$$|g_j(P_j) - z'| \le 2C_5\varepsilon,$$

which proves (3.25).

c) With $g_j(P_j) = 1$ and $\psi(z) = i(1+z)/(1-z)$ we have $h_j = \psi \circ g_j$, so that

$$\frac{h_2'(w)}{h_1'(w)} = \left(\frac{1 - g_1(w)}{1 - g_2(w)}\right)^2 \frac{g_2'(w)}{g_1'(w)}. \tag{3.26}$$

For $w \in F$, $g_1(Q)$ separates $g_1(w)$ and $g_1(P_1) = 1$ in $\mathbb{D}$, so

$$|g_1(w) - 1| \ge \operatorname{dist}(g_1(w), g_1(Q)) \ge \delta.$$

Together with b) this gives

$$\left|\frac{1-g_2(w)}{1-g_1(w)}-1\right|\le\frac{A_2\varepsilon}{\delta}.$$

By b)

$$\left|\frac{g_2'(w)}{g_1'(w)}-1\right|\le\frac{A_2\varepsilon}{\delta}$$

so (3.26) gives

$$\left|\frac{h_2'(w)}{h_1'(w)}-1\right|\le\frac{A_3\varepsilon}{\delta}.$$

□

3. Polygonal approximation

We prove the implication b) $\Longrightarrow$ c) of Theorem 3.7. Let $\Omega\ni\infty$ be a simply connected domain, and $a_j\in\partial\Omega$ with $\beta_\Omega(a_j,a_0)>0$ for $j=1,2,\dots,m$. Let $\varepsilon>0$. We will approximate $\partial\Omega$ with a polygonal tree Γ with leaves a_j'' such that

$$\beta_{\mathbb{C}\setminus\Gamma}(a_j'',a_0'')\ge\beta_\Omega(a_j,a_0)-\varepsilon\qquad\text{for } j=1,\dots,m. \tag{3.27}$$

Using a Möbius transformation we may arrange so that $a_0''=\infty$. This will prove the implication b) $\Longrightarrow$ c).

The first problem is that $\partial\Omega$ may look like a spiral around a_j. To get rid of this we consider the modified domain $\Omega_1=f(\mathbb{H}\setminus[0,i\delta])$, where $f:\mathbb{H}\to\Omega$ is a conformal bijection such that $f(0)=a_1$ and $\Gamma_f(0)=\operatorname{anglim}_{z\to0}|z/f'(z)|>0$. The point a_1 is replaced by $\tilde a_1=f(i\delta)$, and $\partial\Omega_1$ is an analytic arc in a neighbourhood of $\tilde a_1$. We will prove that

$$|\beta_{\Omega_1}(\tilde a_1,a_0)-\beta_\Omega(a_1,a_0)|<\varepsilon \tag{3.28}$$

and

$$|\beta_{\Omega_1}(a_j,a_0)-\beta_\Omega(a_j,a_0)|<\varepsilon\qquad(j\ne1) \tag{3.29}$$

if δ is sufficiently small. We then do the same modification for the other points $a_2,a_3,\dots,a_m,a_0$. We end up with a domain $\Omega'\ni\infty$ and points $a_0',\dots,a_m'\in\partial\Omega'$ such that

$$|\beta_{\Omega'}(a_j',a_0')-\beta_\Omega(a_j,a_0)|<(m+1)\varepsilon,\qquad j=1,\dots,m.$$

$\partial\Omega'$ is an analytic arc in a neighbourhood of each point a_j'.

To get closer to a polygonal domain, we consider the modified domain $\Omega_1'=\Omega'\setminus[a_1',\tilde a_1']$, where the segment $[a_1',\tilde a_1']$ is chosen so that $\partial\Omega_1'$ has a tangent at a_1'. We will prove that

$$|\beta_{\Omega_1'}(\tilde a_1',a_0')-\beta_{\Omega'}(a_1',a_0')|<\varepsilon \tag{3.30}$$

and

$$|\beta_{\Omega_1'}(a_j',a_0')-\beta_{\Omega'}(a_j',a_0')|<\varepsilon\qquad(j\ne1) \tag{3.31}$$

if $[a_1',\tilde a_1']$ is sufficiently short. We then add line segments at the other points a_j' and end up with a domain Ω'' and points $a_0'',\dots,a_m''\in\partial\Omega''$ such that

$$|\beta_{\Omega''}(a_j'',a_0'')-\beta_{\Omega'}(a_j',a_0')|<(m+1)\varepsilon,\qquad j=1,\dots,m.$$

$\partial\Omega''$ is a line segment in a neighbourhood of each a_j''.

We now approximate Ω'' with a domain $\Omega_r''\subset\Omega''$ such that

$$\partial\Omega_r''=J\cup\cup_{j=0}^m[b_j,a_j'']$$

where $[b_j,a_j'']$ are parts of the line segments $[a_j',a_j'']$ and J is an analytic Jordan curve. Let $g:\mathbb{D}\to\Omega'$ and $h:\mathbb{D}\to\Omega''$ be conformal bijections with $g(0)=h(0)$, $g'(0)>0$ and $h'(0)>0$. Then $\psi=g^{-1}\circ h$ maps $\mathbb{D}$ onto $G=\mathbb{D}\setminus\gamma_0\setminus\dots\setminus\gamma_m$, where γ_j are analytic arcs ending orthogonally on $\mathbb{T}$. We let $G_r=G\cap D(0,r)$ and

$\Omega_r'' = g(G_r)$. Note that $J = g(C(0,r))$ is an analytic Jordan curve. We will prove that

$$|\beta_{\Omega_r''}(a_j'', a_0'') - \beta_{\Omega''}(a_j'', a_0'')| < \varepsilon, \qquad j = 1, \dots, m \tag{3.32}$$

if r is sufficiently close to 1.

Finally, let T be a polygonal tree in the interior domain of J with leaves b_j. Let Γ be the tree $T \cup \cup_{j=0}^m [b_j, a_j'']$. Since $\mathbb{C} \setminus \Gamma \supset \Omega_r''$ the comparison principle gives

$$\beta_{\mathbb{C}\setminus\Gamma}(a_j'', a_0'') \geq \beta_{\Omega_r''}(a_j'', a_0'') \geq \beta_\Omega(a_j, a_0) - (2m+3)\varepsilon \qquad \text{for } j = 1, 2, \dots, m$$

and (3.27) is proved.

To see that we may take $a_0'' = \infty$, let σ be a Möbius transformation with $\sigma(a_0'') = \infty$. Had we instead of the line segments $[a_j', a_j'']$ taken appropriate circular arcs C_j we would get that $\sigma(\Omega_r'')$ has boundary $\sigma(J) \cup \cup_{j=0}^m [\sigma(b_j), \sigma(a_j'')]$. Let $\tilde{T}$ be a tree inside $\sigma(J)$ as before. Now $\tilde{\Gamma} = \tilde{T} \cup \cup_{j=0}^m [\sigma(b_j), \sigma(a_j'')]$ is a polygonal tree with root at ∞, and

$$\beta_{\mathbb{C}\setminus\tilde{\Gamma}}(\sigma(a_j''), \infty) \geq \beta_{\sigma(\Omega_r'')}(\sigma(a_j''), \infty) = \beta_{\Omega_r''}(a_j'', a_0'') \geq \beta_\Omega(a_j, a_0) - (2m+3)\varepsilon.$$

It remains to prove the estimates (3.28)–(3.32). The estimates (3.28)–(3.31) follow from the following lemma.

Lemma 3.10. *Let a_0, a_1, a_2 be distinct points on $\partial\Omega$ such that $\beta_\Omega(a_1, a_0) > 0$ and $\beta_\Omega(a_2, a_0) > 0$. Let $f : \mathbb{H} \to \Omega$ be a conformal bijection, and let $x_j \in \mathbb{R}$ be the point satisfying $f(x_j) = a_j$ and $\Gamma_f(x_j) = \operatorname{anglim}_{z\to x_j} |(z - x_j)/f'(z)| > 0$ (see Theorem 2.22). Let $\gamma : [0, t_0] \to \overline{\mathbb{H}}$ be an analytic arc with $\gamma(0) = x_1$ and $\gamma'(0) = i$. Let $\Omega_\varepsilon = f(\mathbb{H} \setminus \gamma([0, \varepsilon]))$ and $\tilde{a}_1 = f(\gamma(\varepsilon))$. Then*

$$\beta_{\Omega_\varepsilon}(\tilde{a}_1, a_0) \to \beta_\Omega(a_1, a_0) \quad \text{and} \quad \beta_{\Omega_\varepsilon}(a_2, a_0) \to \beta_\Omega(a_2, a_0) \quad \text{as } \varepsilon \to 0.$$

Proof. We may assume that $x_1 = 0$. Let $\varphi_\varepsilon : \mathbb{H} \to \mathbb{H} \setminus \gamma([0, \varepsilon])$ be the conformal bijection with $\varphi_\varepsilon(i) = i$ and $\varphi_\varepsilon(0) = \gamma(\varepsilon)$. We claim that

$$\varphi_\varepsilon^{-1}(x_0) \to x_0, \tag{3.33}$$

$$(\varphi_\varepsilon^{-1})'(x_0) \to 1, \tag{3.34}$$

$$|\varphi_\varepsilon''(0)|\varepsilon \to 1 \qquad \text{as } \varepsilon \to 0. \tag{3.35}$$

(3.33) and (3.34) follow from Lemma 3.9a) with $\Omega_1 = \mathbb{H}$, $\Omega_2 = \mathbb{H} \setminus \gamma([0, \varepsilon])$ and $Q = C(0, 2\varepsilon) \cap \mathbb{H}$. When proving (3.35), we may change the normalisation to $\varphi_\varepsilon(0) = \gamma(\varepsilon)$, $\varphi_\varepsilon(\infty) = \infty$, $\varphi_\varepsilon'(\infty) = 1$ (because of (3.33) and (3.34)). Consider the map $f_\varepsilon(z) = -\varphi_\varepsilon(z)^2$, which maps $\mathbb{H}$ onto $G_\varepsilon = \mathbb{C} \setminus (-\infty, 0] \setminus \Gamma_\varepsilon$, where Γ_ε is a curve

$$y = a_3 x^{3/2} + a_4 x^2 + a_5 x^{5/2} + \dots, \qquad 0 \leq x \leq x_\varepsilon = \varepsilon^2 + o(\varepsilon^2) \tag{3.36}$$

with endpoints 0 and $a_\varepsilon = f(\gamma(\varepsilon))$. By Theorem 2.23

$$\beta_{G_\varepsilon}(a_\varepsilon, \infty) = \left| \frac{f_\varepsilon''(\infty)}{f_\varepsilon''(0)} \right| = \frac{2}{2|\varphi_\varepsilon(0)\varphi_\varepsilon''(0)|}.$$

Thus it suffices to prove that $\beta_{G_\varepsilon}(a_\varepsilon, \infty) \to 1$ as $\varepsilon \to 0$. By the estimate (2.65)

$$d_{G_\varepsilon}(C(a_\varepsilon, \rho), C(a_\varepsilon, R)) \leq \frac{1}{2\pi} \log \frac{R}{\rho} + \frac{1}{2\pi} \int_\rho^R \phi'(r)^2 r \, \mathrm{d}r,$$

where $\phi = \phi(r)$ is the equation of the curve $(-\infty, 0] \cup \Gamma_\varepsilon$ in polar coordinates with centre a_ε. From (3.36) it follows that $|\phi'(r)| \leq C/\varepsilon$, while for the ray $(-\infty, 0]$ we have $|\phi'(r)| \leq C\varepsilon^2/r^2$. This gives

$$\delta_{G_\varepsilon}(a_\varepsilon, \infty) \leq \frac{1}{2\pi} \int_0^\infty \phi'(r)^2 r \, \mathrm{d}r = O(\varepsilon^2),$$

and $\lim_{\varepsilon\to 0} \beta_{G_\varepsilon}(a_\varepsilon, \infty) = 1$ follows.

Now let $g = f \circ \varphi_\varepsilon : \mathbb{H} \to \Omega_\varepsilon$. By (2.26) and (3.33)–(3.35)

$$\begin{aligned}\beta_{\Omega_\varepsilon}(\tilde a_1, a_0) &= \frac{4|\tilde a_1 - a_0|^2}{\varphi_\varepsilon^{-1}(x_0)^4} \operatorname*{anglim}_{z\to 0}\left|\frac{z}{g'(z)}\right| \operatorname*{anglim}_{z\to\varphi_\varepsilon^{-1}(x_0)}\left|\frac{z-\varphi_\varepsilon^{-1}(x_0)}{g'(z)}\right| \\ &= \frac{4|\tilde a_1 - a_0|^2}{\varphi_\varepsilon^{-1}(x_0)^4}\frac{1}{|f'(\gamma(\varepsilon))\varphi_\varepsilon''(0)|}\frac{1}{|\varphi_\varepsilon'(\varphi_\varepsilon^{-1}(x_0))|^2}\operatorname*{anglim}_{w\to x_0}\left|\frac{w-x_0}{f'(w)}\right| \\ &\to \frac{4|a_1-a_0|^2}{x_0^4}\lim_{\varepsilon\to 0}\left|\frac{\gamma(\varepsilon)}{f'(\gamma(\varepsilon))}\right|\operatorname*{anglim}_{w\to x_0}\left|\frac{w-x_0}{f'(w)}\right| = \beta_\Omega(a_1,a_0) \quad \text{as } \varepsilon\to 0.\end{aligned}$$

Similarly

$$\begin{aligned}&\beta_{\Omega_\varepsilon}(a_2, a_0) = \\ &\quad = \frac{4|a_2-a_0|^2}{|\varphi_\varepsilon^{-1}(x_2)-\varphi_\varepsilon^{-1}(x_0)|^4}\operatorname*{anglim}_{z\to\varphi_\varepsilon^{-1}(x_0)}\left|\frac{z-\varphi_\varepsilon^{-1}(x_0)}{g'(z)}\right|\operatorname*{anglim}_{z\to\varphi_\varepsilon^{-1}(x_2)}\left|\frac{z-\varphi_\varepsilon^{-1}(x_2)}{g'(z)}\right| \\ &\quad = \frac{4|a_2-a_0|^2}{|\varphi_\varepsilon^{-1}(x_2)-\varphi_\varepsilon^{-1}(x_0)|^4}\frac{1}{|\varphi_\varepsilon'(\varphi_\varepsilon^{-1}(x_0))|^2}\operatorname*{anglim}_{w\to x_0}\left|\frac{w-x_0}{f'(w)}\right| \cdot \\ &\quad\ \frac{1}{|\varphi_\varepsilon'(\varphi_\varepsilon^{-1}(x_2))|^2}\operatorname*{anglim}_{w\to x_2}\left|\frac{w-x_2}{f'(w)}\right| \to \beta_\Omega(a_2,a_0) \qquad \text{as } \varepsilon\to 0.\end{aligned}$$

□

To prove (3.32), let $\psi_r : \mathbb{D} \to G_r$ be the conformal bijection with $\psi_r(0) = 0$ and $\psi_r'(0) > 0$. Then $h_r = g \circ \psi_r$ maps $\mathbb{D}$ onto Ω_r''. Let $\zeta_j = h_r^{-1}(a_j'')$. By Theorem 2.22

$$\beta_{\Omega_r''}(a_j'', a_0'') = \frac{4|a_j''-a_0''|^2}{|\zeta_j-\zeta_0|^4}\frac{1}{|h''(\zeta_j)h''(\zeta_0)|},$$

where $h_r''(\zeta_j) = g'(g^{-1}(a_j''))\psi_r''(\zeta_j)$. It thus suffices to prove

$$\zeta_j \to h^{-1}(a_j'') \qquad \text{and} \qquad \psi_r''(\zeta_j) \to \psi''(h^{-1}(a_j'')) \qquad \text{as } r\to 1. \tag{3.37}$$

The sets $\mathbb{C}\backslash G_r$ are uniformly locally connected, and $G_r \to G$ in the sense of kernel convergence. By [**Pom92**, Corollary 2.4] this means that $\psi_r \to \psi$ uniformly on $\mathbb{D}$. Since ψ_r extends to neighbourhoods of ζ_j by reflection, we get $\psi^{-1}(\psi_r(z)) \to z$ and $\psi_r'' \to \psi''$ uniformly on these neighbourhoods. Hence also $\psi_r^{-1}(\psi(z)) \to z$ on these neighbourhoods, and (3.37) follows. This completes the proof of the implication b) $\Longrightarrow$ c) of Theorem 3.7.

4. Proofs of the implications c) $\Longrightarrow$ d) $\Longrightarrow$ e) $\Longrightarrow$ f) $\Longrightarrow$ a)

Proof of c) $\Longrightarrow$ d). This is taken from [**CaMa94**, Corollary 4]. Let Ω be a simply connected domain, and $a_1, \ldots, a_m \in \partial\Omega$, $b \in \Omega$. We may suppose that $b = \infty$ and $\beta_\Omega(a_j, \infty) > 0$. Let $f : \mathbb{D}^* \to \Omega$ be a conformal bijection with $f(\infty) = \infty$, and let $\zeta_j \in \mathbb{T}$ be the points such that $f(\zeta_j) = a_j$ and $\Gamma_f(\zeta_j) > 0$ (see Theorem 2.1). Let $\varphi : \mathbb{D}^* \to \mathbb{D}^* \setminus [1, 2]$ be the conformal bijection with $\varphi(\infty) = \infty$ and $\varphi(1) = 2$. $g = f \circ \varphi$ maps $\mathbb{D}^*$ onto a domain $\tilde\Omega$ with a new tip point $b = g(1) = f(2)$. We claim that

$$\beta_{\tilde\Omega}(a_j, b) \asymp \beta_\Omega(a_j, \infty) \qquad \text{if } \operatorname{Re}\zeta_j \le 0. \tag{3.38}$$

Together with the corresonding estimate for the points with $\operatorname{Re}\zeta_j > 0$, this proves the implication c) $\Longrightarrow$ d).

Let $\tilde{\zeta}_j = \varphi^{-1}(\zeta_j)$. By Theorem 2.22 and equation (2.24)

$$\begin{aligned}\beta_{\tilde{\Omega}}(a_j, b) &= \frac{4|a_j - b|^2}{|\tilde{\zeta}_j - 1|^4} \operatorname*{anglim}_{z\to\tilde{\zeta}_j} \left|\frac{z-\tilde{\zeta}_j}{g'(z)}\right| \frac{1}{|g''(1)|} \\ &= \frac{4|a_j - b|^2}{|\tilde{\zeta}_j - 1|^4} \operatorname*{anglim}_{z\to\tilde{\zeta}_j} \left|\frac{\varphi(z)-\zeta_j}{f'(\varphi(z))}\right| \frac{1}{|\varphi'(z)|} \left|\frac{z-\tilde{\zeta}_j}{\varphi(z)-\zeta_j}\right| \frac{1}{|f'(2)\varphi''(1)|} \\ &= \frac{4|a_j - b|^2}{|\tilde{\zeta}_j - 1|^4} \frac{\beta_\Omega(a_j,\infty)}{2|f'(\infty)|} \frac{1}{|\varphi'(\tilde{\zeta}_j)|^2 |f'(2)\varphi''(1)|}.\end{aligned}$$

Since $\operatorname{Re}\zeta_j \leq 0$ we have $|\tilde{\zeta}_j - 1| \asymp 1 \asymp |\varphi'(\tilde{\zeta}_j)|$. By the distortion theorem $|f'(2)| \asymp |f'(\infty)|$. Together with

$$|a_j - b| \asymp |f'(\infty)| \tag{3.39}$$

this proves (3.38). The estimate (3.39) follows from [**Pom92**, Corollary 1.3 and Theorem 1.4] and the Koebe 1/4 theorem. □

Proof of d) $\Longrightarrow$ e). Let $g \in \Sigma$. Let $N \geq 1$ be an integer and let $\zeta_k = e^{2\pi i k/N}$. Let

$$\varphi_N : \mathbb{D}^* \to \mathbb{D}^* \setminus \cup_{k=0}^{N-1}[\zeta_k, 2^{1/N}\zeta_k]$$

be the conformal bijection with $\varphi_N(\infty) = \infty$ and $\varphi_N'(\infty) > 0$. Consider the domain $\Omega_N = g(\varphi_N(\mathbb{D}^*))$, which has tip points $a_k = g(2^{1/N}\zeta_k)$. We claim that

$$|g'(2^{1/N}\zeta_k)|^{-1} \asymp N\beta_{\Omega_N}(a_k,\infty). \tag{3.40}$$

Together with the distortion theorem this gives

$$\int_0^{2\pi} |g'(2^{1/N}e^{i\theta})|^{-p}\,\mathrm{d}\theta \asymp \frac{1}{N}\sum_{k=0}^{N-1} |g'(2^{1/N}\zeta_k)|^{-p} \asymp N^{p-1}\sum_{k=0}^{N-1}\beta_{\Omega_N}(a_k,\infty)^p$$

so that e) follows from d).

By (2.24)

$$\beta_{\Omega_N}(a_k,\infty) = \frac{2|(g\circ\varphi_N)'(\infty)|}{|(g\circ\varphi_N)''(\zeta_k)|} = \frac{2|\varphi_N'(\infty)|}{|g'(2^{1/N}\zeta_k)\varphi_N''(\zeta_k)|}.$$

A calculation using $\varphi_N(z) = \varphi(z^N)^{1/N}$ gives

$$|\varphi_N'(\infty)| \asymp 1 \qquad \text{and} \qquad |\varphi_N''(\zeta_k)| \asymp N,$$

so (3.40) follows. □

Proof of e) $\Longrightarrow$ f). Let $f \in S$. Then $g(z) = f(z^{-1})^{-1} \in \Sigma$ and

$$|g'(z)| = \frac{|f'(z^{-1})|}{|f(z^{-1})z|^2} \leq \frac{|f'(z^{-1})|}{16}, \qquad z \in \mathbb{D}^*,$$

where the last inequality is the Koebe 1/4 theorem. Thus

$$\int_0^{2\pi} |f'(re^{i\theta})|^{-p}\,\mathrm{d}\theta \leq 16^{-p}\int_0^{2\pi}\left|g'\left(\frac{1}{r}e^{i\theta}\right)\right|^{-p}\mathrm{d}\theta, \qquad r<1 \tag{3.41}$$

so that f) follows from e). □

The implication f) $\Longrightarrow$ a) is trivial.

5. Concentration of harmonic measure

In this section we prove the implications e) $\Longrightarrow$ g) $\Longrightarrow$ a) of Theorem 3.7, which show that Brennan's conjecture can be formulated as an estimate of the number of disjoint discs of large harmonic measure. To prove e) $\Longrightarrow$ g) we use the following lemma.

Lemma 3.11. *There are absolute constants $\rho_0, C > 0$ such that the following holds. Assume that $f \in \Sigma$ and $\Omega = f(\mathbb{D}^*)$. If $\rho < \rho_0$ and*

$$\omega(D(w,\rho) \cap \partial\Omega, \Omega, \infty) \geq \rho$$

then there exists a disc $D(z,r) \subset f^{-1}(D(w, C\rho))$ such that

$$r = \frac{|z|-1}{2} \geq \frac{\omega(D(w,\rho) \cap \partial\Omega, \Omega, \infty)}{C \log(1/\rho)}.$$

Proof. By [**Mak85**, Lemma 2.3] there exists a crosscut σ of Ω such that $\sigma \subset D(w, 2\rho)$ and

$$\omega(\infty, \beta, \Omega) \geq \frac{\omega(D(w,\rho) \cap \partial\Omega, \Omega, \infty)}{4\pi \log(1/\rho)},$$

where β is the part of $\partial\Omega$ that σ separates from ∞. (In [**Mak85**] this is stated for Jordan domains, but the proof also works for simply connected domains if β is considered as a set of prime ends.) Let S be the hyperbolic geodesic in $\mathbb{D}^*$ that has the same endpoints as $f^{-1}(\sigma)$. With z as the midpoint of S we have

$$|z| - 1 \asymp \omega(\beta, \Omega, \infty).$$

With $r = (|z|-1)/2$ we have by the distortion theorem

$$\operatorname{diam} f(D(z,r)) \asymp \operatorname{dist}(f(z), \partial\Omega) \leq \operatorname{diam} f(S) \leq K \operatorname{diam} \sigma,$$

where the last inequality is the Gehring-Hayman theorem [**Pom92**, Theorem 4.20]. Thus

$$f(D(z,r)) \subset D(w, 2\rho + \operatorname{diam} f(S) + \operatorname{diam} f(D(z,r))) \subset D(w, C\rho). \qquad \square$$

Proof of e) $\Longrightarrow$ g). Assume that e) holds. Thus $p \geq 2$. Since g) holds trivially for $h < \rho$, we may assume $h \geq \rho$. If $N_\Omega(\rho, h) > 0$ then $h \leq C_0\sqrt{\rho}$ (this follows from Beurling's projection theorem). Thus we may also assume that $\rho < \rho_0$. Let $f \in \Sigma$ and $\Omega = f(\mathbb{D}^*)$. Assume that $D(w_j, \rho)$, $j = 1, \ldots, N$ are disjoint discs with

$$\omega(D(w_j,\rho) \cap \partial\Omega, \Omega, \infty) \geq h.$$

Let $D(z_j, r_j)$ be the corresponding discs given by Lemma 3.11, so that

$$r_j \geq \eta = \frac{h}{C \log(1/\rho)}.$$

By the distortion theorem

$$\iint_{D(z_j,r_j)} |f'|^{-p} \, dA \asymp \frac{r_j^2}{|f'(z_j)|^p} \asymp \frac{r_j^{2+p}}{\operatorname{dist}(f(z_j), \partial\Omega)^p} \geq \frac{\eta^{2+p}}{(C+1)^p \rho^p}.$$

Since each point is covered by a most C_1 sets $f(D(z_j, r_j))$ and since $r_j^2 = (|z_j| - 1)^2/4 \leq C_2 \operatorname{dist}(f(z_j), \partial\Omega) \leq C_2(C+1)\rho$ by the distortion theorem, we get

$$C_1 \iint_{1+\eta \leq |z| \leq 1 + C_3\sqrt{\rho}} |f'|^{-p} \, dA \geq \sum_1^N \iint_{D(z_j,r_j)} |f'|^{-p} \, dA \geq N C_4 \frac{\eta^{2+p}}{\rho^p}.$$

By e)

$$\iint_{1+\eta \leq |z| \leq 1 + C_3\sqrt{\rho}} |f'|^{-p} \, dA \leq \int_{1+\eta}^{1+C_3\sqrt{\rho}} \frac{C_5}{(r-1)^{p-1}} r \, dr,$$

and thus

$$N \le C_6 \frac{\rho^p}{\eta^{2p}} = C_6 \rho^p \left(\frac{C\log(1/\rho)}{h}\right)^{2p} \qquad \text{if } p > 2$$

and

$$N \le C_7 \frac{\rho^2}{\eta^4} \log \frac{\sqrt{\rho}}{\eta} = C_7 \rho^2 \left(\frac{C\log(1/\rho)}{h}\right)^4 \log\left(\frac{C\sqrt{\rho}\log(1/\rho)}{h}\right) \qquad \text{if } p = 2.$$

In both cases g) is satisfied. □

Proof of g) $\Longrightarrow$ a). This is taken from [**CaMa94**, Section 3.5]. Let $g \in \Sigma$ and $\Omega = g(\mathbb{D}^*)$. Let n be positive integer, let $r = 1 + 2/n$ and $\zeta_k = e^{2\pi i k/n}$. Consider the disjoint discs $D_k = D(r\zeta_k, 1/n)$, $k = 1, 2, \dots, n$. By the Koebe 1/4 theorem, $g(D_k)$ contains a disc B_k with centre $g(r\zeta_k)$ and radius $\rho_k = |g'(r\zeta_k)|/4n$. Similarly, $g^{-1}(B_k) \supset D(r\zeta_k, 1/16n)$. Hence, for the domain $\Omega_n = g(\{z : |z| > r\})$ we have

$$\omega(B_k \cap \partial\Omega_n, \Omega_n, \infty) \ge C/n, \qquad k = 1, 2, \dots, n.$$

Let $\varepsilon > 0$. By the distortion theorem

$$\int_0^{2\pi} |g'(re^{i\theta})|^{-p+2\varepsilon}\, \mathrm{d}\theta \asymp \frac{1}{n}\sum_1^n |g'(r\zeta_k)|^{-p+2\varepsilon} = \frac{1}{4^{p-2\varepsilon} n^{p+1-2\varepsilon}} \sum_1^n \rho_k^{-p+2\varepsilon}.$$

By g), the number of k:s with $2^{-\nu} < \rho_k \le 2^{-\nu+1}$ is at most

$$C_2(\varepsilon)\frac{(2^{-\nu+1})^{p-\varepsilon}}{(C/n)^{2p}}.$$

Thus

$$\sum_1^n \rho_k^{-p+2\varepsilon} \le C_3 + C_2(\varepsilon)\left(\frac{n}{C}\right)^{2p} \sum_{\nu=1}^{\infty} \left(2^{-\nu+1}\right)^{p-\varepsilon} 2^{\nu(p-2\varepsilon)} \le C_4(\varepsilon) n^{2p},$$

so that

$$\int_0^{2\pi} |g'(re^{i\theta})|^{-p+2\varepsilon}\, \mathrm{d}\theta \le C_5(\varepsilon) n^{p-1+2\varepsilon} = C_6(\varepsilon)\left(\frac{1}{r-1}\right)^{p-1+2\varepsilon}, \qquad r > 1.$$

By (3.41) this gives

$$B(-p+2\varepsilon) \le p - 1 + 2\varepsilon$$

and $B(-p) \le p - 1$ follows by continuity. □

CHAPTER 4

Extremals for $\sum \beta^p$

1. Overview

By Theorem 3.7, Brennan's conjecture can be stated: For any simply connected domain Ω and distinct points $a_1, \dots, a_m, b \in \partial\Omega$,

$$\sum_1^m \beta_\Omega(a_j, b)^2 \leq 1. \tag{4.1}$$

In an attempt to prove this formulation of Brennan's conjecture, we examine configurations $(\Omega, a_1, \dots, a_m, b)$ that maximize the sum in (4.1) (for fixed m). (We say that $(\Omega, a_1, \dots, a_m, b)$ is a configuration if Ω is a simply connected domain, and $a_1, \dots, a_m, b$ are distinct points on $\partial\Omega$.) We will consider the more general case with an exponent $p > 0$.

Theorem 4.1. *Assume that there exists a configuration with $\sum_1^m \beta_\Omega(a_j, b)^p > 1$. Let $m_0 \geq 2$ be the smallest m for which such a configuration exists. Then there is a configuration $(\Omega, a_1, \dots, a_{m_0}, b)$ that maximizes*

$$\sum_1^{m_0} \beta_\Omega(a_j, b)^p.$$

Remark. For $m > m_0$ it is conceiveable that no maximizing configuration exists. For example, let $(\hat\Omega, \hat a_1, \dots, \hat a_{m_0}, \infty)$ be a maximizing configuration. We will see that $\Gamma = \mathbb{C} \setminus \Omega$ is a tree of analytic arcs. Let Γ_2 be the first iterate in a dandelion construction similar to Section 3.2. One can prove that

$$\beta_{\mathbb{C}\setminus\Gamma_2}(a_{jk}, \infty) \to \beta_\Omega(\hat a_j, \infty)\beta_\Omega(\hat a_k, \infty) \qquad (1 \leq j, k \leq m_0)$$

as the scale $q \to 0$. It is conceivable that

$$\beta_{\mathbb{C}\setminus\Gamma_2}(a_{jk}, \infty) < \beta_\Omega(\hat a_j, \infty)\beta_\Omega(\hat a_k, \infty)$$

and

$$\sup \sum_1^{m_0^2} \beta_\Omega(a_j, \infty)^p = \lim_{q\to 0} \sum_{j,k=1}^{m_0} \beta_{\mathbb{C}\setminus\Gamma_2}(a_{jk}, \infty)^p.$$

To get information about a maximizing configuration we use Schiffer's method of interior variation [**Ahl73**, Chapter 7]. We will use the version given by Chang, Schiffer and Schober in [**ChScSc81**], which gives a simple derivation of both first and second variations. The first variation gives the following theorem.

Theorem 4.2. *Fix $m \geq 1$. Assume that $(\Omega, a_1, \dots, a_m, \infty)$ is a maximizing configuration for $\sum_1^m \beta_\Omega(a_j, \infty)^p$. Assume that $\beta_j = \beta_\Omega(a_j, \infty) > 0$ for $j = 1, \dots, m$. Then $\Gamma = \mathbb{C} \setminus \Omega$ is a tree of trajectories of the quadratic differential*

$$Q(w)\,\mathrm{d}w^2 = \sum_{j=1}^m \frac{\beta_j^p}{(a_j - w)^2}\,\mathrm{d}w^2.$$

Let $f : \mathbb{H} \to \Omega$ be a conformal bijection with $f(\infty) = \infty$. Let $x_j = f^{-1}(a_j)$. Then

$$Q(f(z))f'(z)^2 = P(z) = \sum_{j=1}^{m} \frac{4\beta_j^p}{(x_j - z)^2} \qquad \text{for } z \in \mathbb{H}. \tag{4.2}$$

Equation (4.2) should almost determine the configuration completely, as can be seen from the following heuristic argument. Given a_j, x_j and β_j, the equation (4.2) determines f up to an integration constant c (and a choice of branch). The condition that f should be univalent gives the constraints $f(\zeta_k) = z_k$, where $\zeta_1, \dots, \zeta_{m-1}$ are the zeros of Q in Ω, and $z_1, \dots, z_{m-1}$ are the zeros of P in $\mathbb{H}$. Moreover, the zeros $\zeta_m, \dots, \zeta_{2m-2}$ of Q on Γ should be mapped by f^{-1} to points on $\mathbb{R}$. Together with the constraints $|f''(\infty)|/|f''(x_j)| = \beta_j$ we have $4m-3$ real constraints on the $4m+2$ parameters $\operatorname{Re} a_j$, $\operatorname{Im} a_j$, x_j, β_j, $\operatorname{Re} c$, $\operatorname{Im} c$. This means that, modulo the freedom of rotating, scaling and translating Ω, as well as the freedom of translating x_j, there should only be a finite number of solutions $(f, a_1, \dots, a_m, x_1, \dots, x_m, \beta_1, \dots, \beta_m)$ to (4.2) (with f univalent).

This approach was used in [**CaMa94**] to prove (4.1) for $m = 2$. In this case, equation (4.2) is integrable in terms of elementary functions. It turns out that the above-mentioned constraints imply that $\beta_1 = \beta_2 < 1/\sqrt{2}$ (when $p = 2$). Hence the configuration is uniquely determined (up to Euclidean motions), and it is not a maximum. By Theorem 4.1 this means that (4.2) holds for $m = 2$. When $m \geq 3$, the difficulty of integrating (4.2) makes this method troublesome.

Another source of information comes from the second variation. The resulting inequality is quite complicated, see (4.26) and the special cases (4.28), (4.31) and (4.32). From this it is however possible to extract the following piece of information.

Theorem 4.3. *Under the assumptions of Theorem 4.2, the function Q has no multiple zeros on Γ.*

This means that at each branch point of the tree Γ, exactly three arcs join. Unfortunately, the author has not been able to rule out simple zeros on Γ.

2. Existence of extremals

We prove Theorem 4.1. We will use the following form of the distortion theorem, which follows from (1.4) by a Möbius transformation.

Lemma 4.4. *If $f : \mathbb{H} \to \mathbb{C}$ is univalent, then*

$$\left| \frac{f'(z)}{f'(s)} \right| \leq \frac{(|z-s| + |z-\overline{s}|)^4}{16 \operatorname{Im} s (\operatorname{Im} z)^3} \qquad \text{for } z, s \in \mathbb{H}.$$

Recall that by Theorem 2.23 the angular limits

$$\Gamma_f(x) = \operatorname*{anglim}_{z \to x} \left| \frac{z-x}{f'(z)} \right| \quad \text{and} \quad \Gamma_f(\infty) = \operatorname*{anglim}_{z \to \infty} \left| \frac{f'(z)}{z} \right|$$

exist (finite) for every univalent $f : \mathbb{H} \to \mathbb{C}$ and every $x \in \mathbb{R}$. By Lemma 4.4,

$$\Gamma_f(x) \leq \frac{|z-x|^4}{(\operatorname{Im} z)^3 |f'(z)|} \quad \text{and} \quad \Gamma_f(\infty) \leq \frac{|f'(s)|}{\operatorname{Im} s}. \tag{4.3}$$

To get an idea of the proof, let $f_k : \mathbb{H} \to \mathbb{C}$ be univalent functions converging locally uniformly to $f : \mathbb{H} \to \mathbb{C}$. The inequalities (4.3) enable us to get a relation between the β-numbers of $f_k(\mathbb{H})$ and $f(\mathbb{H})$. Namely, we get using Theorem 2.23,

$$\beta_{f_k(\mathbb{H})}(f(x_k), \infty) \leq \frac{|f_k'(s)|}{\operatorname{Im} s} \frac{|z - x_k|^4}{(\operatorname{Im} z)^3 |f_k'(z)|}.$$

Letting first $k \to \infty$ and then $s = iy \to \infty$, $z \to x = \lim_{k\to\infty} x_k$ we get

$$\limsup_{k\to\infty} \beta_{f_k(\mathbb{H})}(f(x_k), \infty) \le \Gamma_f(\infty)\Gamma_f(x) = \beta_{f(\mathbb{H})}(f(x), \infty).$$

Now we have m_0 points $a_{j,k} \in \partial f_k(\mathbb{H})$, and the corresponding $x_{j,k} = f_k^{-1}(a_{j,k}) \in \mathbb{R}$ may converge to the same x as $k \to \infty$. Hence a rescaling argument is called for.

Lemma 4.5. *Let $2 \le m \le m_0$, where m_0 is as in Theorem 4.1. Let $f_k : \mathbb{H} \to \mathbb{C}$ $(k = 1, 2, \dots)$ be univalent functions, converging locally uniformly to f as $k \to \infty$. Let $0 = x_{1,k} < x_{2,k} < \dots < x_{m,k} = 1$ be sequences converging to $0 = x_1 \le x_2 \le \dots \le x_m = 1$ as $k \to \infty$. Then there are sequences $z_{1,k}, z_{2,k}, \dots, z_{m,k} \in \mathbb{H}$ and a sequence K of positive integers such that*

$$\limsup_{\substack{k\to\infty\\ k\in K}} \sum_{j=1}^{m} \left(\frac{|z_{j,k} - x_{j,k}|^4}{(\operatorname{Im} z_{j,k})^3 |f_k'(z_{j,k})|} \right)^p \le \sum_{x\in\{x_1,\dots,x_m\}} \Gamma_f(x)^p.$$

Proof. We use induction over m. Fix m so that $2 \le m \le m_0$. Assume that the lemma is true for $m := 2, 3, \dots, m-1$. Consider a set of equal x_j: $x_{j_1} = x_{j_1+1} = \dots = x_{j_2}$. It suffices to show the existence of sequences $z_{j_1,k}, \dots, z_{j_2,k} \in \mathbb{H}$ and a sequence K of positive integers such that

$$\limsup_{\substack{k\to\infty\\ k\in K}} \sum_{j=j_1}^{j_2} \left(\frac{|z_{j,k} - x_{j,k}|^4}{(\operatorname{Im} z_{j,k})^3 |f_k'(z_{j,k})|} \right)^p \le \Gamma_f(x_{j_1})^p. \tag{4.4}$$

Case 1: $j_1 = j_2$. Take $z_{j_1,k} = x_{j_1,k} + i/k$ and let K be the set of positive integers. Fix $\varepsilon > 0$. By Lemma 4.4,

$$\frac{k^{-1}}{|f_k'(z_{j_1,k})|} \le \frac{\varepsilon}{|f_k'(x_{j_1,k} + i\varepsilon)|} \qquad \text{if } k^{-1} < \varepsilon.$$

It follows that the left-hand side of (4.4) is $\le (\varepsilon/|f'(x_{j_1} + i\varepsilon)|)^p$. Letting $\varepsilon \to 0$ we get (4.4).

Case 2: $j_1 < j_2$. (This case does not occur for $m = 2$.) Let $\delta_k = x_{j_2,k} - x_{j_1,k}$ and $\xi_{j,k} = (x_{j,k} - x_{j_1,k})/\delta_k$ for $j_1 \le j \le j_2$. Let g_k be the univalent functions

$$g_k(\zeta) = A_k f_k(x_{j_1,k} + \zeta\delta_k) + B_k,$$

where A_k and B_k are chosen so that $g_k(i) = 0$ and $g_k'(i) = 1$. By compactness there exists a sequence K' of positive integers, a univalent function $g : \mathbb{H} \to \mathbb{C}$ and numbers $0 = \xi_{j_1} \le \xi_{j_1+1} \le \dots \le \xi_{j_2} = 1$ such that $g_k \to g$ locally uniformly and $\xi_{j,k} \to \xi_j$ as $k \in K'$ tends to ∞. By the induction hypothesis, there exist sequences $\zeta_{j,k} \in \mathbb{H}$ $(j_1 \le j \le j_2)$ and a sequence $K \subset K'$ such that

$$\limsup_{\substack{k\to\infty\\ k\in K}} \sum_{j=j_1}^{j_2} \left(\frac{|\zeta_{j,k} - \xi_{j,k}|^4}{(\operatorname{Im} \zeta_{j,k})^3 |g_k'(\zeta_{j,k})|} \right)^p \le \sum_{\xi\in\{\xi_{j_1},\dots,\xi_{j_2}\}} \Gamma_g(\xi)^p. \tag{4.5}$$

Now let $z_{j,k} = x_{j_1,k} + \zeta_{j,k}\delta_k$ for $j_1 \le j \le j_2$. (4.5) implies

$$\limsup_{\substack{k\to\infty\\ k\in K}} \sum_{j=j_1}^{j_2} \left(\frac{|z_{j,k} - x_{j,k}|^4}{(\operatorname{Im} z_{j,k})^3 |f_k'(z_{j,k})|} \right)^p \le \left(\limsup_{k\to\infty} |A_k|\delta_k^2 \right)^p \sum_{\xi\in\{\xi_{j_1},\dots,\xi_{j_2}\}} \Gamma_g(\xi)^p. \tag{4.6}$$

It thus suffices to consider the case $\limsup_{k\to\infty} |A_k|\delta_k^2 > 0$. Let $y > 0$ and $\varepsilon > 0$. By Lemma 4.4

$$|A_k|\delta_k^2 \frac{y}{|g_k'(iy)|} = \frac{y\delta_k}{|f_k'(x_{j,k} + iy\delta_k)|} \le \frac{\varepsilon}{|f_k'(x_{j,k} + i\varepsilon)|} \qquad \text{if } y\delta_k < \varepsilon.$$

Letting $k \to \infty$ we get

$$\limsup_{k\to\infty} |A_k|\delta_k^2 \frac{y}{|g'(iy)|} \le \frac{\varepsilon}{|f'(x_{j_1} + i\varepsilon)|}.$$

Hence $\Gamma_g(\infty) > 0$ and we get

$$\limsup_{k\to\infty} |A_k|\delta_k^2 \le \Gamma_g(\infty)\Gamma_f(x_{j_1}).$$

Thus the right-hand side of (4.6) is

$$\le \Gamma_f(x_{j_1})^p \sum_{\xi} (\Gamma_g(\xi)\Gamma_g(\infty))^p = \Gamma_f(x_{j_1})^p \sum_{\xi} \beta_{g(\mathbb{H})}(g(\xi),\infty)^p,$$

where the sums are taken over those $\xi \in \{\xi_{j_1},\dots,\xi_{j_2}\}$ with $\Gamma_g(\xi) > 0$. Since $j_2 - j_1 < m \le m_0$, those sums are ≤ 1, and so (4.4) follows. ☐

Proof of Theorem 4.1. Let $S > 1$ be the supremum of $\sum_1^{m_0} \beta_\Omega(a_j,b)^p$. Let $(\Omega_k, a_{1,k},\dots,a_{m_0,k},\infty)$ be a sequence of configurations with

$$\lim_{k\to\infty} \sum_{j=1}^{m_0} \beta_{\Omega_k}(a_{j,k},\infty)^p = S.$$

We can arrange so that $\beta_\Omega(a_{j,k},\infty) > 0$. By Theorem 2.23 there are conformal bijections $f_k : \mathbb{H} \to \Omega_k$ and points $x_{1,k},\dots,x_{m_0,k} \in \mathbb{R}$ such that $f_k(x_{j,k}) = a_{j,k}$, $f_k(\infty) = \infty$ and

$$\beta_{\Omega_k}(a_{j,k},\infty) = \Gamma_{f_k}(x_{j,k})\Gamma_{f_k}(\infty). \tag{4.7}$$

By rescaling and renumbering we may assume that

$$0 = x_{1,k} < x_{2,k} < \dots < x_{m_0,k} = 1, \quad f_k(i) = 0, \quad f_k'(i) = 1.$$

By compactness we may, by taking a subsequence, assume that $f_k \to f$ locally uniformly and $x_{j,k} \to x_j$ as $k \to \infty$. Now (4.7) and (4.3) imply

$$\sum_1^{m_0} \beta_{\Omega_k}(a_{j,k},\infty)^p \le \left(\frac{|f_k'(s)|}{\operatorname{Im} s}\right)^p \sum_{j=1}^{m_0} \left(\frac{|z_{j,k} - x_{j,k}|^4}{(\operatorname{Im} z_{j,k})^3 |f_k(z_{j,k})|}\right)^p$$

for $s, z_{j,k} \in \mathbb{H}$. Applying Lemma 4.5 we get

$$S \le \left(\frac{|f'(s)|}{\operatorname{Im} s}\right)^p \sum_{x\in\{x_1,\dots,x_{m_0}\}} \Gamma_f(x)^p.$$

Thus $\Gamma_f(\infty) > 0$. By Theorem 2.23 we get

$$S \le \sum_x \beta_{f(\mathbb{H})}(f(x),\infty)^p,$$

where the sum is taken over those $x \in \{x_1,\dots,x_{m_0}\}$ for which $\Gamma_f(x) > 0$. By the definition of m_0, these x:s must be m_0 in number. Hence the configuration $(f(\mathbb{H}), f(x_1),\dots,f(x_{m_0}),\infty)$ is a maximizing configuration. ☐

3. Calculus of variations

Let $f : \mathbb{H} \to \Omega \subset \mathbb{C}$ be a conformal bijection. For $k = 1,2,\dots,n$, let $w_k \in \Omega$, $p_k, q_k, r_k \in \mathbb{C}$ and define

$$v_1(w) = \sum_{k=1}^n \frac{p_k}{w - w_k} \qquad \text{and} \qquad v_2(w) = \sum_{k=1}^n \frac{q_k}{(w-w_k)^2} + \frac{r_k}{w - w_k}. \tag{4.8}$$

For small $\varepsilon \in \mathbb{R}$ the mapping

$$v_\varepsilon(w) = w + \varepsilon v_1(w) + \varepsilon^2 v_2(w) \tag{4.9}$$

is univalent in a neighbourhood of $\Gamma = \hat{\mathbb{C}} \setminus \Omega$. Thus the domain $\Omega_\varepsilon = \hat{\mathbb{C}} \setminus v_\varepsilon(\Gamma)$ is simply connected. We say that Ω_ε is gotten by *interior variation* of Ω. We will calculate the conformal mappings $f_\varepsilon : \mathbb{H} \to \Omega_\varepsilon$ to second order in ε. The function $g_\varepsilon = f_\varepsilon^{-1} \circ v_\varepsilon \circ f$ is univalent in $N \cap \mathbb{H}$, where N is a neighbourhood of

$\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Since $g_\varepsilon(z) \to \hat{\mathbb{R}}$ as $z \to \hat{\mathbb{R}}$, we can extend g_ε by reflection to a neighbourhood of $\hat{\mathbb{R}}$. We normalize f_ε and g_ε so that

$$g_\varepsilon(z) = z + O(z^{-1}) \qquad \text{as } z \to \infty.$$

We can write

$$f_\varepsilon(z) = f(z) + \varepsilon f_1(z) + \varepsilon^2 f_2(z) + O(\varepsilon^3), \tag{4.10}$$

$$g_\varepsilon(z) = z + \varepsilon g_1(z) + \varepsilon^2 g_2(z) + O(\varepsilon^3), \tag{4.11}$$

where f_j are analytic in $\mathbb{H}$, g_j are analytic in a neighbourhood of $\hat{\mathbb{R}}$, and $O(\varepsilon^3)$ is uniform on compact subsets. This follows from the following lemma.

Lemma 4.6. *$f_\varepsilon(z)$ and $g_\varepsilon(z)$ can be extended to complex ε so that $f_\varepsilon(z)$ is analytic in $(\varepsilon, z) \in D(0,\varepsilon_0) \times \mathbb{H}$, and $g_\varepsilon(z) - z$ is analytic in $(\varepsilon, z) \in D(0,\varepsilon_0) \times N$, where N is a neighbourhood of $\hat{\mathbb{R}}$.*

Proof. Let $\varphi : \mathbb{D} \to \Omega$ be the conformal bijection $\varphi = f \circ \sigma^{-1}$, where $\sigma(z) = (z-i)/(z+i)$. It follows from the proof of Satz 1 in [**Gol57**, p. 96-102] that there exist functions $\varphi_\varepsilon(z)$ analytic in $(\varepsilon, z) \in D(0,\varepsilon_0) \times \mathbb{D}$ and $\psi_\varepsilon(z)$ analytic in $(\varepsilon, z) \in D(0,\varepsilon_0) \times \{z : r_0 < |z| < 1/r_0\}$, such that for real ε the following holds: $\varphi_\varepsilon : \mathbb{D} \to \Omega_\varepsilon$ is a conformal bijection, $\psi_\varepsilon : \{z : r_0 < |z| < 1/r_0\} \to \mathbb{C}$ is a univalent function with $\psi_\varepsilon(\mathbb{T}) = \mathbb{T}$, and

$$\varphi_\varepsilon \circ \psi_\varepsilon = v_\varepsilon \circ \varphi \qquad \text{on } \{z : r_0 < |z| < 1/r_0\}.$$

Using a rotation we may arrange so that $\psi_\varepsilon(1) = 1$ for real ε. Now let $\tilde{f}_\varepsilon(z) = \varphi_\varepsilon(\sigma(z))$ and $\tilde{g}_\varepsilon(z) = \sigma^{-1}(\psi_\varepsilon(\sigma(z)))$. For real ε we have that $\tilde{f}_\varepsilon : \mathbb{H} \to \Omega_\varepsilon$ is conformal bijection, $\tilde{g}_\varepsilon$ is univalent in a neighbourhood of $\hat{\mathbb{R}}$, $\tilde{g}_\varepsilon(\hat{\mathbb{R}}) = \hat{\mathbb{R}}$, $\tilde{g}_\varepsilon(\infty) = \infty$ and $\tilde{f}_\varepsilon \circ \tilde{g}_\varepsilon = v_\varepsilon \circ f$. It follows that for real ε,

$$f_\varepsilon(z) = \tilde{f}_\varepsilon\left(\tilde{g}_\varepsilon'(\infty)(z + A_\varepsilon)\right) \tag{4.12}$$

and

$$g_\varepsilon(z) = \tilde{g}_\varepsilon(z)/\tilde{g}_\varepsilon'(\infty) - A_\varepsilon, \tag{4.13}$$

where $A_\varepsilon = \lim_{z\to\infty} \tilde{g}_\varepsilon(z)/\tilde{g}_\varepsilon'(\infty) - z$. The right-hand sides of (4.12) and (4.13) have the required analyticity in ε and z. □

Inserting (4.9), (4.10) and (4.11) in the relation $f_\varepsilon(g_\varepsilon(z)) = v_\varepsilon(f(z))$ and comparing coefficients gives

$$f_1(z) + f'(z)g_1(z) = v_1(f(z)) \tag{4.14}$$

$$f_2(z) + f_1'(z)g_1(z) + \tfrac{1}{2}f''(z)g_1(z)^2 + f'(z)g_2(z) = v_2(f(z)). \tag{4.15}$$

By analytic continuation this holds for all $z \in \mathbb{H}$. To determine g_1, note that $v_1(f(z))$ is analytic in $\mathbb{H}$, except for simple poles at $z_k = f^{-1}(w_k)$ with residues $p_k/f'(z_k)$. By (4.14), g_1 is therefore analytic in $\mathbb{H}$ except for simple poles at z_k with residues $\alpha_k = p_k/f'(z_k)^2$. Since $g_1(\mathbb{R}) \subset \mathbb{R}$ and $g_1(\infty) = 0$ it follows that g_1 is rational:

$$g_1(z) = \sum_{k=1}^{n} \frac{\alpha_k}{z - z_k} + \frac{\overline{\alpha_k}}{z - \overline{z_k}}. \tag{4.16}$$

To calculate g_2, we use the trick in [**ChScSc81**] to choose q_k and r_k so that the principal parts of $f_1'(z)g_1(z) + \frac{1}{2}f''(z)g_1(z)^2$ and $v_2(f(z))$ coincide at the poles $z_1, \dots, z_n$. By (4.15) this implies that g_2 is analytic in $\mathbb{H}$, and since $g_2(\mathbb{R}) \subset \mathbb{R}$ and $g_2(\infty) = 0$ we get $g_2 = 0$. Let

$$\sigma_k(z) = g_1(z) - \frac{\alpha_k}{z - z_k} = \frac{\overline{\alpha_k}}{z - \overline{z_k}} + \sum_{l \neq k} \frac{\alpha_l}{z - z_l} + \frac{\overline{\alpha_l}}{z - \overline{z_l}}.$$

The principal part of $f_1'(z)g_1(z) + \frac{1}{2}f''(z)g_1(z)^2$ at z_k is

$$\frac{\alpha_k^2 f''(z_k)}{2(z-z_k)^2} + \frac{\alpha_k\sigma_k(z_k)f''(z_k) + \frac{1}{2}\alpha_k^2 f'''(z_k) + \alpha_k f_1'(z_k)}{z-z_k}.$$

The principal part of $v_2(f(z))$ at z_k is

$$\frac{q_k}{f'(z_k)^2(z-z_k)^2} + \left(\frac{r_k}{f'(z_k)} - \frac{q_k f''(z_k)}{f'(z_k)^3}\right)\frac{1}{z-z_k}.$$

Thus we choose

$$q_k = \frac{1}{2}\alpha_k^2 f''(z_k)f'(z_k)^2 \tag{4.17}$$

and

$$r_k = \frac{1}{2}\alpha_k^2\big(f''(z_k)^2 + f'''(z_k)f'(z_k)\big) + \alpha_k f'(z_k)\big(f_1'(z_k) + \sigma_k(z_k)f''(z_k)\big).$$

A calculation with (4.14) and (4.16) gives

$$\begin{aligned} f_1'(z_k) =& \frac{\alpha_k}{f'(z_k)}\left(\frac{1}{4}f''(z_k)^2 - \frac{2}{3}f'''(z_k)f'(z_k)\right) \\ &- \sigma_k(z_k)f''(z_k) - \sigma_k'(z_k)f'(z_k) - f'(z_k)\sum_{l\neq k}\frac{p_l}{(w_k-w_l)^2} \end{aligned}$$

so we get

$$\begin{aligned} r_k =& \alpha_k^2\left(\frac{3}{4}f''(z_k)^2 - \frac{1}{6}f'''(z_k)f'(z_k)\right) \\ &- \alpha_k f'(z_k)^2\left(\sigma_k'(z_k) + \sum_{l\neq k}\frac{p_l}{(w_k-w_l)^2}\right). \end{aligned} \tag{4.18}$$

The first and second variations f_1 and f_2 can now be computed from (4.14) and (4.15).

We now compute the first and second variations of the β-numbers. Assume that $a, \infty \in \partial\Omega$ have $\beta_\Omega(a,\infty) > 0$. By Theorem 2.23 we can choose f so that $\Gamma_f(\infty) > 0$, $f(\infty) = \infty$, and then we can find $x \in \mathbb{R}$ such that $\Gamma_f(x) > 0$ and $f(x) = a$. For the varied domain Ω_ε the corresponding points are $a_\varepsilon = v_\varepsilon(a)$ and $x_\varepsilon = g_\varepsilon(x)$. By Theorem 2.23

$$\beta_{\Omega_\varepsilon}(a_\varepsilon,\infty) = \Gamma_{f_\varepsilon}(x_\varepsilon)\Gamma_{f_\varepsilon}(\infty) \qquad \text{and} \qquad \beta_\Omega(a,\infty) = \Gamma_f(x)\Gamma_f(\infty).$$

Now

$$f_\varepsilon'(g_\varepsilon(z))g_\varepsilon'(z) = v_\varepsilon'(f(z))f'(z),$$

which implies

$$\begin{aligned} \Gamma_{f_\varepsilon}(x_\varepsilon) &= \operatorname*{anglim}_{z\to x}\left|\frac{g_\varepsilon(z)-x_\varepsilon}{f_\varepsilon'(g_\varepsilon(z))}\right| \\ &= \operatorname*{anglim}_{z\to x}\left|\frac{(g_\varepsilon(z)-x_\varepsilon)g_\varepsilon'(z)}{(z-x)v_\varepsilon'(f(z))}\right|\left|\frac{z-x}{f'(z)}\right| = \frac{|g_\varepsilon'(x)|^2}{|v_\varepsilon'(a)|}\Gamma_f(x) \end{aligned}$$

and

$$\Gamma_{f_\varepsilon}(\infty) = \operatorname*{anglim}_{z\to\infty}\left|\frac{f_\varepsilon'(g_\varepsilon(z))}{g_\varepsilon(z)}\right| = \operatorname*{anglim}_{z\to\infty}\left|\frac{v_\varepsilon'(f(z))z}{g_\varepsilon(z)g_\varepsilon'(z)}\right|\left|\frac{f'(z)}{z}\right| = \Gamma_f(\infty).$$

Thus

$$\beta_{\Omega_\varepsilon}(a_\varepsilon,\infty) = \beta_\Omega(a,\infty)\frac{|g_\varepsilon'(x)|^2}{|v_\varepsilon'(a)|}.$$

Substituting (4.9) and (4.11) gives

$$\beta_{\Omega_\varepsilon}(a_\varepsilon,\infty)^p = \beta_\Omega(a,\infty)^p\Bigg(1 + p\,\mathrm{Re}(2g_1'(x) - v_1'(a))\varepsilon + \frac{p}{2}\Big[|2g_1'(x) - v_1'(a)|^2$$
$$+ (p-2)\big(\,\mathrm{Re}\,(2g_1'(x) - v_1'(a))\,\big)^2 + 2\,\mathrm{Re}\,\big((g_1'(x) - v_1'(a))^2 - v_2'(a)\big)\Big]\varepsilon^2 + O(\varepsilon^3)\Bigg).$$

Thus, if $(\Omega, a_1, \ldots, a_m, \infty)$ is a maximizing configuration for $\sum_1^m \beta_\Omega(a_j, b)^p$, we must have

$$\sum_{j=1}^m \beta_j^p\,\mathrm{Re}(2g_1'(x_j) - v_1'(a_j)) = 0 \tag{4.19}$$

and

$$\sum_{j=1}^m \beta_j^p\Big[|2g_1'(x_j) - v_1'(a_j)|^2 + (p-2)\big(\,\mathrm{Re}\,(2g_1'(x_j) - v_1'(a_j))\,\big)^2$$
$$+ 2\,\mathrm{Re}\,\Big(\big(g_1'(x_j) - v_1'(a_j)\big)^2 - v_2'(a_j)\Big)\Big] \le 0, \tag{4.20}$$

where $\beta_j = \beta_\Omega(a_j, \infty)$. Equation (4.19) with $n = 1$ and $z_1 = z$ is

$$\sum_{j=1}^m \beta_j^p\left(\frac{-4}{(x_j - z)^2} + \frac{f'(z)^2}{(a_j - f(z))^2}\right) = 0 \qquad \text{for } z \in \mathbb{H}, \tag{4.21}$$

which is (4.2) of Theorem 4.2. It follows that the boundary of Ω consists of trajectories of the quadratic differential

$$Q(w)\,\mathrm{d}w^2 = \sum_{j=1}^m \frac{\beta_j^p}{(a_j - w)^2}\,\mathrm{d}w^2.$$

$\partial\Omega$ must be a tree, since otherwise we could enlarge the domain Ω by removing a part of the boundary, and so get all β_j increased. Thus $\partial\Omega = \hat{\mathbb{C}} \setminus \Omega$, and Theorem 4.2 is proved.

From the local structure of trajectories of a quadratic differential (see [**Pom75**, Section 8.2]) it follows that $f(z)$ continues analytically across $\mathbb{R}$, except for branch points where $Q(f(z)) = 0$. Substituting the Taylor expansion of f at x_k in (4.21) we get the following relations:

$$f'''(x_k) = 0 \tag{4.22}$$

$$\beta_k^p \frac{f^{(4)}(x_k)}{f''(x_k)} = 6\sum_{j\neq k} \frac{\beta_j^p}{(x_j - x_k)^2} \tag{4.23}$$

$$\beta_k^p \frac{f^{(5)}(x_k)}{f''(x_k)} = 40\sum_{j\neq k} \frac{\beta_j^p}{(x_j - x_k)^3} \tag{4.24}$$

$$\frac{2\beta_k^p}{45}\frac{f^{(6)}(x_k)}{f''(x_k)} = \frac{\beta_k^p}{36}\frac{f^{(4)}(x_k)^2}{f''(x_k)^2} + 12\sum_{j\neq k}\frac{\beta_j^p}{(x_j - x_k)^4} - f''(x_k)^2\sum_{j\neq k}\frac{\beta_j^p}{(a_j - a_k)^2} \tag{4.25}$$

4. The second variational inequality

We now try to extract information from (4.20). Noting that

$$\sum_{j=1}^m \beta_j^p v_2'(a_j) = \sum_{k=1}^n -q_k Q'(w_k) - r_k Q(w_k)$$

we can write (4.20) as

$$\sum_{j=1}^{m}\beta_j^p\Big[|2g_1'(x_j)-v_1'(a_j)|^2+(p-2)\big(\operatorname{Re}(2g_1'(x_j)-v_1'(a_j))\big)^2+ \\ 2\operatorname{Re}(g_1'(x_j)-v_1'(a_j))^2\Big]+2\operatorname{Re}\sum_{k=1}^{n}q_kQ'(w_k)+r_kQ(w_k)\le 0. \quad (4.26)$$

The special case $n=1$ is, with $\alpha=\alpha_1$, $z=z_1$, $f(z)=w=w_1$,

$$\begin{aligned}\sum_{j=1}^{m}\beta_j^p\Bigg[&\left|-\frac{2\alpha}{(x_j-z)^2}-\frac{2\overline{\alpha}}{(x_j-\overline{z})^2}+\frac{\alpha f'(z)^2}{(a_j-w)^2}\right|^2\\ &+(p-2)\left(\operatorname{Re}\left(-\frac{2\alpha}{(x_j-z)^2}-\frac{2\overline{\alpha}}{(x_j-\overline{z})^2}+\frac{\alpha f'(z)^2}{(a_j-w)^2}\right)\right)^2\\ &+2\operatorname{Re}\left(-\frac{\alpha}{(x_j-z)^2}-\frac{\overline{\alpha}}{(x_j-\overline{z})^2}+\frac{\alpha f'(z)^2}{(a_j-w)^2}\right)^2\Bigg]\\ +&\operatorname{Re}\alpha^2f''(z)^2f'(z)^2Q'(w)\\ +&2\operatorname{Re}\left(\alpha^2\left(\frac{3}{4}f''(z)^2-\frac{1}{6}f'''(z)f'(z)\right)+|\alpha|^2\frac{f'(z)^2}{(z-\overline{z})^2}\right)Q(w)\le 0.\end{aligned}$$

Expanding the squares and optimizing over α gives

$$\begin{aligned}&\Bigg|\sum_{j=1}^{m}\beta_j^p\left[\frac{p}{2}\left(\frac{4}{(x_j-z)^2}-\frac{f'(z)^2}{(a_j-w)^2}\right)^2-\frac{4}{(x_j-z)^4}+\frac{f'(z)^4}{(a_j-w)^4}\right]\\ &+f''(z)f'(z)^2Q'(w)+\left(\frac{3}{2}f''(z)^2-\frac{1}{3}f'''(x)f'(z)\right)Q(w)\Bigg|\\ &+\sum_{j=1}^{m}\beta_j^p\left[\frac{p}{2}\left|\frac{4}{(x_j-z)^2}-\frac{f'(z)^2}{(a_j-w)^2}\right|^2-\frac{4}{|x_j-z|^4}\right]\\ &-\frac{1}{2(\operatorname{Im}z)^2}\operatorname{Re}f'(z)^2Q(w)\ \le\ 0.\end{aligned} \quad (4.27)$$

Using the relation $Q(f(z))f'(z)^2=P(z)$ and its derivative we can write (4.27) as

$$\begin{aligned}&\Bigg|\sum_{j=1}^{m}\beta_j^p\Bigg[\frac{p}{2}\left(\frac{4}{(x_j-z)^2}-\frac{f'(z)^2}{(a_j-w)^2}\right)^2-\frac{4}{(x_j-z)^4}+\frac{f'(z)^4}{(a_j-w)^4}\\ &+\frac{8f''(z)}{f'(z)}\frac{1}{(x_j-z)^3}-\frac{2f''(z)^2}{f'(z)^2}\frac{1}{(x_j-z)^2}-\frac{4f'''(z)}{3f'(z)}\frac{1}{(x_j-z)^2}\Bigg]\Bigg|\\ &+\sum_{j=1}^{m}\beta_j^p\Bigg[\frac{p}{2}\left|\frac{4}{(x_j-z)^2}-\frac{f'(z)^2}{(a_j-w)^2}\right|^2-\frac{4}{|x_j-z|^4}\\ &-\frac{2}{(\operatorname{Im}z)^2}\operatorname{Re}\frac{1}{(x_j-z)^2}\Bigg]\le 0.\end{aligned} \quad (4.28)$$

Proof of Theorem 4.3. Assume that $w_0\in\Gamma$ is a zero of $Q(w)$ of order $d-2$. Thus

$$Q(w)=C(w-w_0)^{d-2}+O\left((w-w_0)^{d-1}\right)\qquad \text{as } w\to w_0.$$

It follows from (4.2) that

$$f(z)=w_0+A(z-x_0)^{2/d}+o((z-x_0)^{2/d}),$$

where $CA^d=d^2P(x_0)/4$. Let $x_0\in\mathbb{R}$ satisfy $f(x_0)=w_0$. Let $z=x_0+i\delta$, $\delta>0$ in (4.27), and let $\delta\to 0$. We get that (4.28) can be written

$$|\ldots|+\sum_{j=1}^{m}\beta_j^p\frac{p}{2}\left|\frac{2A}{d(a_j-w_0)}\right|^4\delta^{8/d-4}+o(\delta^{8/d-4})+O(\delta^{-2})\le 0.$$

If $d > 4$ this is a contradiction.

To rule out the case $d = 4$ we use the general inequality (4.26). Let $z_k = x_0 + ic_k\delta$, where $c_k > 0$ and $\delta > 0$. We fix α_k and c_k, but let $\delta \to 0$. We have $g_1'(x_j) = O(1)$ and

$$-v_1'(a_j) = \frac{A^2}{4i(a_j - w_0)^2\delta}\sum_{k=1}^{n}\frac{\alpha_k}{c_k} + o(\delta^{-1}).$$

Thus the first sum in (4.26) is

$$\begin{aligned}
&\sum_{j=1}^{m}\beta_j^p\left[\frac{p}{2}|v_1'(a_j)|^2 + \left(\frac{p}{2}+1\right)\operatorname{Re} v_1'(a_j)^2 + O(\delta^{-1})\right]\\
&= \frac{p}{32\delta^2}|A|^4\sum_{j=1}^{m}\frac{\beta_j^p}{|a_j - w_0|^4}\left|\sum_{k=1}^{n}\frac{\alpha_k}{c_k}\right|^2 \\
&- \frac{p+2}{32\delta^2}\operatorname{Re} A^4\sum_{j=1}^{m}\frac{\beta_j^p}{(a_j - w_0)^4}\left(\sum_{k=1}^{n}\frac{\alpha_k}{c_k}\right)^2 + o(\delta^{-2}).
\end{aligned} \tag{4.29}$$

Since

$$A^4\sum_{j=1}^{m}\frac{\beta_j^p}{(a_j - w_0)^4} = A^4\frac{Q''(w_0)}{6} = A^4\frac{C}{3} = \frac{4}{3}P(x_0)$$

we get that

$$(4.29) \geq \frac{P(x_0)}{\delta^2}\left(\frac{p}{24}\left|\sum_{k=1}^{n}\frac{\alpha_k}{c_k}\right|^2 - \frac{p+2}{24}\operatorname{Re}\left(\sum_{k=1}^{n}\frac{\alpha_k}{c_k}\right)^2\right) + o(\delta^{-2}).$$

We have

$$q_kQ'(w_k) = \frac{P(x_0)}{4}\frac{\alpha_k^2}{c_k^2\delta^2} + o(\delta^{-2})$$

and

$$r_kQ(w_k) = \frac{P(x_0)}{\delta^2}\left(-\frac{1}{16}\frac{\alpha_k^2}{c_k^2} - \frac{1}{4}\frac{|\alpha_k|^2}{c_k^2} + \frac{1}{4}\sum_{l\neq k}\frac{\alpha_k\alpha_l}{c_l(\sqrt{c_k} - \sqrt{c_l})^2}\right) + o(\delta^{-2}).$$

Putting all this into (4.26) we get from the leading term

$$\begin{aligned}
&\frac{p}{24}\left|\sum_1^n\frac{\alpha_k}{c_k}\right|^2 - \frac{p+2}{24}\operatorname{Re}\left(\sum_1^n\frac{\alpha_k}{c_k}\right)^2 + \frac{3}{8}\operatorname{Re}\sum_1^n\frac{\alpha_k^2}{c_k^2}\\
&-\frac{1}{2}\sum_1^n\frac{|\alpha_k|^2}{c_k^2} + \frac{1}{2}\operatorname{Re}\sum_{k=1}^{n}\sum_{l\neq k}\frac{\alpha_k\alpha_l}{c_l(\sqrt{c_k} - \sqrt{c_l})^2} \leq 0
\end{aligned}$$

Choosing $\alpha_k = c_k$ we get

$$-\frac{n^2}{12} - \frac{n}{8} + \frac{1}{4}\sum_{k=1}^{n}\sum_{l\neq k}\frac{c_k + c_l}{(\sqrt{c_k} - \sqrt{c_l})^2} \leq 0.$$

Choosing $c_k = M^k$ and letting $M \to \infty$, we get that each term in the double sum tends to 1. Hence

$$\frac{n^2}{6} - \frac{3n}{8} \leq 0,$$

which is a contradiction for $n \geq 3$. □

Remark. In the case $d = 3$ (simple zero of Q) we can proceed as follows. Let $z_k = \xi_k + ic_k\delta$, where $f(\xi_k) = w_0$, $c_k > 0$ and $\delta > 0$. We have

$$f(z) = w_0 + A_k(z - \xi_k)^{2/3} + o((z - \xi_k)^{2/3}).$$

Let $u_k = A_k(ic_k)^{2/3}$ and $\gamma_k = \sqrt{P(\xi_k)}\alpha_k/c_k$. We have

$$q_k Q'(w_k) + r_k Q(w_k) = \left(\frac{11}{108}\gamma_k^2 - \frac{1}{4}|\gamma_k|^2 + \frac{4}{9}\sum_{l\neq k} \gamma_k\gamma_l \frac{u_k^{3/2}u_l^{1/2}}{(u_k-u_l)^2} \right) \delta^{-2} + o(\delta^{-2}).$$

The first sum in (4.26) is $O(\delta^{-4/3})$ so we get the inequality

$$\operatorname{Re}\sum_{k=1}^{n} \left(\frac{11}{108}\gamma_k^2 - \frac{1}{4}|\gamma_k|^2 + \frac{4}{9}\sum_{l\neq k} \gamma_k\gamma_l \frac{u_k^{3/2}u_l^{1/2}}{(u_k-u_l)^2} \right) \leq 0. \tag{4.30}$$

If $p<2$ there do exist extremals with $\sum_1^m \beta_j^p > 1$, see (3.6). By Theorem 4.3, the corresponding Q must have a simple zero on Γ. Hence (4.30) holds for all complex γ_k and u_k with $(u_k/u_1)^3 > 0$. To get information from (4.26) we can consider $n=3$, $\gamma_k = 1$, $u_2 = e^{2\pi i/3}u_1$, $u_3 = e^{4\pi i/3}u_1$, which gives equality in (4.30). For this choice it turns out that the left-hand side of (4.26) is $O(1)$. The leading term is quite complicated, and the author is unable to derive any contradiction from its non-positivity.

Another way to use the second variational inequality (4.26) is to let $n=m$ and $z_k = x_k + ic_k\delta$, where $c_k > 0$ and $\delta > 0$. It turns out that the leading term of the left-hand side of (4.26) is

$$-\frac{4}{\delta^4}\sum_{k=1}^{n} \frac{\beta_k^p}{c_k^4}(\operatorname{Im}\alpha_k)^2.$$

We thus restrict to the case $\alpha_k \in \mathbb{R}$. Now the leading term of (4.26) is

$$\begin{aligned}\sum_{j=1}^{m} \beta_j^p \Bigg[&\left(-\frac{4}{15}\operatorname{Re}\frac{f^{(6)}(x_j)}{f''(x_j)} + \frac{2}{3}\frac{f^{(4)}(x_j)^2}{f''(x_j)^2} \right)\alpha_j^2 + 24\sum_{k\neq j}\frac{\alpha_k^2 + 2\alpha_k\alpha_j}{(x_j-x_k)^4} \\ &+ (16p-8)\left(\frac{\alpha_j}{6}\frac{f^{(4)}(x_j)}{f''(x_j)} - \sum_{k\neq j}\frac{\alpha_k}{(x_j-x_k)^2} \right)^2 \Bigg] \leq 0.\end{aligned} \tag{4.31}$$

(We have used (4.22)–(4.25).) For the special case $\alpha_j = 0$ for $j \neq l$, $\alpha_l = 1$ this gives

$$\operatorname{Re}\frac{f^{(6)}(x_l)}{f''(x_l)} \geq \frac{5(p+1)}{3}\left(\frac{f^{(4)}(x_l)}{f''(x_l)}\right)^2 + 60(p+1)\sum_{j\neq l}\frac{\beta_j^p}{(x_j-x_l)^4}. \tag{4.32}$$

Remark. A similar inequality holds for *any* univalent function $f : \mathbb{H} \to \mathbb{C}$ that satisfies $f'(x) = 0$, $f'''(x) = 0$, $f^{(4)}(x)/f''(x) \in \mathbb{R}$ and $f^{(5)}(x)/f''(x) \in \mathbb{R}$ for some $x \in \mathbb{R}$ (cf. (4.22)–(4.24)). Namely, the well-known estimate

$$|Sf(z)| \leq \frac{3}{2(\operatorname{Im} z)^2}, \qquad z \in \mathbb{H}$$

for the Schwarzian derivative gives

$$\operatorname{Re}\frac{f^{(6)}(x)}{f''(x)} \geq \frac{5}{2}\left(\frac{f^{(4)}(x)}{f''(x)}\right)^2.$$

CHAPTER 5

Integral means

1. A stronger conjecture

By Theorem 3.7 Brennan's conjecture can be stated: For every $f \in S$ and $t \le -2$

$$\int_0^{2\pi} |f'(re^{i\theta})|^t \,\mathrm{d}\theta \le \frac{C_t}{(1-r)^{|t|-1}}, \qquad 0 < r < 1.$$

This makes the following stronger conjecture irresistible.

Conjecture 5.1. *For every $f \in S$ and $t \le -2$*

$$\int_0^{2\pi} |f'(re^{i\theta})|^t \,\mathrm{d}\theta \le \int_0^{2\pi} |k'(re^{i\theta})|^t \,\mathrm{d}\theta, \qquad 0 < r < 1, \tag{5.1}$$

where $k(z) = z(1+z)^{-2}$ is the Koebe function.

This is akin to Baernstein's theorem [**Bae74**] that

$$\int_0^{2\pi} |f(re^{i\theta})|^t \,\mathrm{d}\theta \le \int_0^{2\pi} |k(re^{i\theta})|^t \,\mathrm{d}\theta, \qquad 0 < r < 1,$$

for $f \in S$ and $t \in \mathbb{R}$. Leung used Baernstein's theory to prove that (5.1) holds for *every* $t \in \mathbb{R}$ if f is close-to-convex, see [**Leu79**] or [**Dur83**, Section 7.5]. In favor of Conjecture 5.1 we give the following result, which almost proves that the Koebe function is a local maximum of the functional

$$f \mapsto \int_0^{2\pi} |f'(re^{i\theta})|^{-2} \,\mathrm{d}\theta$$

on S (r is fixed).

Theorem 5.2. *Let $v_\varepsilon(w)$ be defined for w in a complex neighbourhood of $[1/4, +\infty]$ and for small complex ε. Assume that $v_0(w) = w$ and that $v_\varepsilon(w)/w$ is analytic in w and ε. For small real ε, let f_ε be the conformal map of $\mathbb{D}$ onto $\mathbb{C} \setminus v_\varepsilon([1/4, +\infty])$ normalized by $f_\varepsilon(0) = 0$, $f_\varepsilon'(0) > 0$. Let $0.91 < r < 1$. Then*

$$\int_{|z|=r} \left| \frac{f_\varepsilon'(0)}{f_\varepsilon'(z)} \right|^2 \mathrm{d}\theta = \int_{|z|=r} \left| \frac{1}{k'(z)} \right|^2 \mathrm{d}\theta + V_2 \varepsilon^2 + O(\varepsilon^3),$$

where $V_2 \le 0$. If $V_2 = 0$ then

$$v_\varepsilon(w) = C_\varepsilon(w + \varepsilon\tau(w) + O(\varepsilon^2)),$$

where $C_\varepsilon \in \mathbb{C}$ and $\tau(w)$ is real for real w.

The statement about equality means that $V_2 = 0$ only when the varied domain $f_\varepsilon(\mathbb{D})$ modulo $O(\varepsilon^2)$ is the Koebe domain $\mathbb{C} \setminus [1/4, +\infty)$ scaled and rotated around 0. The restriction to radii $r > 0.91$ is an artifact of the proof that probably could be overcome by modifying the proof.

We note that if $-2 < t < 0$, the Koebe function is *not* a local maximum to the functional

$$f \mapsto \int_0^{2\pi} |f'(re^{i\theta})|^t \,\mathrm{d}\theta$$

on S (for r large enough). This follows from a consideration of functions mapping onto $\mathbb{C} \setminus [1/4, +\infty)$ minus a slit.

The proof of Theorem 5.2 will be given in the next section. In Section 5.3 we examine the relation between integral means and coefficients of functions $(f')^p$, where $f \in S$.

2. The proof of Theorem 5.2

The proof is based on formulas for the Golusin variation up to second order around the Koebe function $k(z) = z(1+z)^{-2}$, see Lemma 5.6 below. The second variation V_2 of the integral mean of $|f'_\varepsilon|^{-2}$ is thus written as a sum of integrals. In one of these integrals we do some ad hoc estimates, while the others are evaluated by residues. All this turns out to yield $V_2 \leq 0$. Since the algebra becomes very laborious we use the computer program Mathematica, see the Appendix. As a check we have redone the computations with the program Maple on a different computer.

To simplify the proof we need some definitions and lemmas.

Definition 5.3. If g is meromorphic on $\mathbb{T}$ we define

$$\hat{g}(z) = g(z^{-1}),$$
$$g^*(z) = \overline{g(\overline{z})},$$
$$\operatorname{symm} g = (\hat{g} + g)/2.$$

If $g(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ for $r_0 < |z| < 1$, we let

$$[g(z)] = \operatorname{Re} c_0 + \sum_{n=1}^{\infty} (c_n + \overline{c_{-n}}) z^n.$$

Definition 5.4. Let $\mathcal{R}$ denote the set of all functions analytic in a neighbourhood of $\overline{\mathbb{D}}$ having real MacLaurin coefficients. Let $\mathcal{I}$ be the set all functions analytic in $\mathbb{D}$ having imaginary MacLaurin coefficients. Let Λ be the set of functions

$$a\frac{z+1}{z-1} + b\frac{z-1}{z+1}, \qquad \text{where } a, b \in \mathbb{R}.$$

Lemma 5.5. *If g is analytic in $\overline{\mathbb{D}}$, except possibly for simple poles at 1 and -1, then*

$$[g - \hat{g}^*] \in \Lambda.$$

Proof. We have $g(z) = h(z) + \alpha/(z-1) + \beta/(z+1)$, where h is analytic in $\overline{\mathbb{D}}$. It is easily checked that

$$[\hat{h}^*] = [h],$$
$$\left[\frac{\alpha}{z-1} - \frac{\overline{\alpha}}{z^{-1}-1}\right] = \operatorname{Re} \alpha \cdot \frac{z+1}{z-1},$$
$$\left[\frac{\beta}{z+1} - \frac{\overline{\beta}}{z^{-1}+1}\right] = \operatorname{Re} \beta \cdot \frac{1-z}{1+z}.$$
□

We now give representations of the first and second variations around the Koebe function in terms of a function $u \in \mathcal{R}$. (The second variation, though, is given modulo $\mathcal{I}$.)

Lemma 5.6. *With the notation in Theorem 5.2 we have*

$$f_\varepsilon = k + \varepsilon f_1 + \varepsilon^2 f_2 + O(\varepsilon^3)$$

and

$$v_\varepsilon(w) = w + \varepsilon v_1(w) + O(\varepsilon^2),$$

where

$$f_1 = \alpha_1 k + 2izk'u, \tag{5.2}$$

$$f_2 = \alpha_2 k - 2zk'(\operatorname{Im}\alpha_1 \cdot u + uv + q) + \psi, \tag{5.3}$$

$$v_1 \circ k = \alpha_1 k + zk'(U - \hat{U}). \tag{5.4}$$

Here $\alpha_1, \alpha_2 \in \mathbb{C}$, $u \in \mathcal{R}$, $v = \phi u + zu'$,

$$\phi = 1 + \frac{zk''}{k'} = \frac{1 - 4z + z^2}{1 - z^2},$$

$q = [\hat{u}zu']$, $\psi \in \mathcal{I}$ *and* $U - iu \in \mathcal{R}$.

Proof. It follows from the proof of Satz 1 in [**Gol57**, p. 96-102] that

$$f_\varepsilon(z) = v_\varepsilon\left(k(ze^{\varphi_\varepsilon(z)})\right), \qquad r_1 < |z| < 1,$$

where $\varphi_\varepsilon(z)$ is analytic in $(\varepsilon, z) \in D(0, \varepsilon_1) \times \{z : r_1 < |z| < 1/r_1\}$ and $\varphi_\varepsilon(z)$ is imaginary for $z \in \mathbb{T}$ and real ε. Hence

$$v_\varepsilon(w) = w + \varepsilon v_1(w) + \varepsilon^2 v_2(w) + O(\varepsilon^3),$$

$$\varphi_\varepsilon(z) = \varepsilon\varphi_1(z) + \varepsilon^2\varphi_2(z) + O(\varepsilon^3),$$

$$f_\varepsilon(z) = k(z) + \varepsilon f_1(z) + \varepsilon^2 f_2(z) + O(\varepsilon^3),$$

where

$$f_1 = \varphi_1 zk' + F_1, \tag{5.5}$$

$$f_2 = \varphi_2 zk' + \frac{1}{2}\varphi_1^2(zk' + z^2k'') + \varphi_1 zF_1' + F_2 \tag{5.6}$$

and $F_j = v_j \circ k$. We can write $F_j(z) = k(z)(\alpha_j + S_j(z))$, where $\alpha_j = 4F_j(1)$, S_j is analytic on $\mathbb{T}$ and $S_j(1) = 0$. Since $\hat{k} = k$ we have $\hat{S}_j = S_j$. Hence $S_j = T_j + \hat{T}_j$, where T_j is analytic in $\overline{\mathbb{D}}$. Since $T_j(1) = 0$ we can write $T_j(z) = \frac{1-z}{1+z}U_j(z)$, where U_j is analytic in $\overline{\mathbb{D}}$. This gives

$$\frac{F_j - \alpha_j k}{zk'} = \frac{k}{zk'}S_j = \frac{1+z}{1-z}\left(\frac{1-z}{1+z}U_j + \frac{1-z^{-1}}{1+z^{-1}}\hat{U}_j\right) = U_j - \hat{U}_j. \tag{5.7}$$

Together with equation (5.5) this gives

$$\frac{f_1 - \alpha_1 k}{zk'} = \varphi_1 + U_1 - \hat{U}_1.$$

Since the left-hand side is analytic in $\mathbb{D}$, and since φ_1 is imaginary on $\mathbb{T}$, this implies that

$$\varphi_1 = \hat{U}_1 - U_1^* + i\gamma_1, \qquad \text{where } \gamma_1 \in \mathbb{R}.$$

Writing $U = U_1 + i\gamma_1/2$ we get

$$\varphi_1 = \hat{U} - U^* \quad \text{and} \quad f_1 = \alpha_1 k + zk'(U - U^*).$$

Introducing $u = (U - U^*)/2i \in \mathcal{R}$ we get (5.2) and (5.4).

Equations (5.6) and (5.7) give

$$\frac{f_2 - \alpha_2 k}{zk'} = \varphi_2 + \frac{1}{2}\phi\varphi_1^2 + \varphi_1\frac{F_1'}{k'} + U_2 - \hat{U}_2, \tag{5.8}$$

where

$$\phi = 1 + \frac{zk''}{k'} = \frac{1 - 4z + z^2}{1 - z^2}.$$

We now apply the operation [ ] on both sides of equation (5.8). Since the left-hand side is analytic in $\mathbb{D}$, it is only shifted by an imaginary constant under this operation. We examine the terms on the right-hand side one by one. The first term φ_2 is imaginary on $\mathbb{T}$, so we have $[\varphi_2] = 0$. Since $\hat{\phi} = -\phi^*$ and $\hat{\varphi}_1 = -\varphi_1^*$ we have for the second term

$$2[\phi\varphi_1^2] = [\phi\varphi_1^2 - (\phi\varphi_1^2)^{\hat{}*}] \in \Lambda$$

by Lemma 5.5. To compute the third term we differentiate (5.4). We get

$$\frac{F_1'}{k'} = \alpha_1 + \phi(U - \hat{U}) + z(U' + z^{-2}\widehat{U'}) = \alpha_1 + V + \hat{V},$$

where $V = \phi U + zU'$. Thus

$$\left[\varphi_1 \frac{F_1'}{k'}\right] = \left[\alpha_1 \hat{U} - \alpha_1 U^* + \varphi_1 V + \varphi_1 \hat{V}\right]$$

Using Lemma 5.5 this can be written

$$[\overline{\alpha_1} U^* - \alpha_1 U^* + \varphi_1 V - \varphi_1 V^*] + \lambda_1,$$

where $\lambda_1 \in \Lambda$. Introducing $v = (V - V^*)/2i = \phi u + zu'$ and $\psi_j \in \mathcal{I}$ we can simplify this further:

$$\begin{aligned} & -2\,\mathrm{Im}\,\alpha_1 \cdot u + \psi_1 + \left[(\hat{U} - U^*)2iv\right] + \lambda_1 \\ = & -2\,\mathrm{Im}\,\alpha_1 \cdot u + \psi_1 + [-2(\hat{u} + u)v] + \psi_2 + \lambda_1 \\ = & -2\,\mathrm{Im}\,\alpha_1 \cdot u - 2uv - 2\,[\hat{u}v] + \psi_3 + \lambda_1. \end{aligned}$$

Since $(\hat{u}\phi u)^{\hat{}} = -(\hat{u}\phi u)^*$, Lemma 5.5 gives $[\hat{u}\phi u] \in \Lambda$. Thus we can simplify $[\hat{u}v]$ and get

$$\left[\varphi_1 \frac{F_1'}{k'}\right] = -2\,\mathrm{Im}\,\alpha_1 u - 2uv - 2\,[\hat{u}zu'] + \psi_3 + \lambda_2,$$

where $\lambda_2 \in \Lambda$. For the fourth term we have $[U_2 - \hat{U}_2] = [U_2 - U_2^*] \in \mathcal{I}$. Thus equation (5.8) implies that

$$\frac{f_2 - \alpha_2 k}{zk'} = -2\,\mathrm{Im}\,\alpha_1 u - 2uv - 2q + \psi_4 + \lambda_3,$$

where $q = [\hat{u}zu']$, $\psi_4 \in \mathcal{I}$ and $\lambda_3 \in \Lambda$. To get (5.3) it remains to show that $\lambda_3 = 0$. From (5.8) it follows that the residue of $(f_2 - \alpha_2 k)/zk'$ at $z = 1$ is

$$\frac{1}{2}\varphi_1(1)^2 = \frac{1}{2}(U(1) - \overline{U(1)})^2 = -2u(1)^2,$$

which is the same as the residue of $-2uv$ at $z = 1$. Similarly, the residues at $z = -1$ coincide. This implies that $\lambda_3 = 0$. □

We now begin the proof of Theorem 5.2. By a scaling of $v_\varepsilon(w)$ we may assume that $f_\varepsilon'(0) = 1$. By Lemma 5.6 we get

$$\int_{|z|=r} |f_\varepsilon'|^{-2}\,\mathrm{d}\theta = \int_{|z|=r} |k'|^{-2}\,\mathrm{d}\theta + V_1\varepsilon + V_2\varepsilon^2 + O(\varepsilon^3),$$

where

$$V_1 = -2\,\mathrm{Re}\int_{|z|=r} |k'|^{-2} \frac{f_1'}{k'}\,\mathrm{d}\theta$$

and

$$V_2 = \int_{|z|=r} |k'|^{-2} \left(2\,\mathrm{Re}\left(\left(\frac{f_1'}{k'}\right)^2 - \frac{f_2'}{k'}\right) + \left|\frac{f_1'}{k'}\right|^2\right)\,\mathrm{d}\theta.$$

Equations (5.2) and (5.3) together with $f_1'(0) = f_2'(0) = 0$ give

$$\begin{aligned} \frac{f_1'}{k'} &= 2i(v - u(0)), \\ \frac{f_2'}{k'} &= 4u(0)v - 2u(0)^2 - 2\phi uv - 2z(uv)' - 2\phi q - 2zq' + 2q(0) + \psi_1, \end{aligned}$$

where $\psi_1 \in \mathcal{I}$. Let $R = r^2$. We evaluate the integrals above by residues using

$$|k'(z)|^{-2} = \frac{1}{k'(z)k'(R/z)} = \frac{(1+z)^3(z+R)^3}{(1-z)(z-R)z^2} = Q_1(z) \quad \text{for } |z| = r.$$

Since $f_1'/k' \in \mathcal{I}$ we get $V_1 = 0$. Similarly ψ_1 does not contribute to V_2. We get

$$\begin{aligned} V_2 &= \int_{|z|=r} Q_1 \Big\{ \operatorname{Re}\Big(-8(v-u(0))^2 - 8u(0)v + 4u(0)^2 \\ &\quad + 4(\phi uv + z(uv)') + 4(\phi q + zq' - q(0))\Big) + 4|v-u(0)|^2 \Big\}\, \mathrm{d}\theta \qquad (5.9) \\ &= \int_{|z|=r} Q_1 \left\{ -8v^2 + 4(\phi uv + z(uv)') + 4(\phi q + zq' - q(0)) + 4|v|^2 \right\} \mathrm{d}\theta, \end{aligned}$$

where we have used that u, v and ϕ have real coefficients. Since Q_1 has a pole at R it will be useful to decompose $u = u_a + u_h$, where

$$u_a(z) = a_0 + a_1 z + a_2 z^2, \quad u_h(z) = P_1(z)h(z), \quad h(z) = \sum_{n=0}^{\infty} h_n z^n \in \mathcal{R}$$

and $P_1(z) = (z-R)^3$. Thus $v = v_a + v_h$, where $v_a = \phi u_a + z u_a'$ and $v_h = \phi u_h + z u_h'$. We partition $V_2 = I_1 + I_2 + I_3 + I_4$ according to the last integral in (5.9). The first integral is

$$I_1 = -8 \int_{|z|=r} Q_1 v^2 \,\mathrm{d}\theta = -16\pi \left(\operatorname*{Res}_{z=0} \frac{Q_1}{z} v^2 + \operatorname*{Res}_{z=R} \frac{Q_1}{z} v_a^2 \right),$$

where we have used that $v_h(R) = 0$. The residues are evaluated with Mathematica, see (A.1) in the Appendix. Note that when calculating the first residue we can replace v (or u or h) by its MacLaurin polynomial of degree 2.

The second integral is

$$\begin{aligned} I_2 &= 4 \int_{|z|=r} Q_1 (\phi uv + z(uv)')\, \mathrm{d}\theta \\ &= 8\pi \left(\operatorname*{Res}_{z=0} \frac{Q_1}{z} (\phi uv + z(uv)') + \operatorname*{Res}_{z=R} \frac{Q_1}{z} (\phi u_a v_a + z(u_a v_a)') \right), \end{aligned}$$

where we have used that $u_h(R) = u_h'(R) = v_h(R) = v_h'(R) = 0$. The residues are again evaluated with Mathematica, see (A.2). When calculating the first residue we may replace h by its MacLaurin polynomial of degree 2.

To compute I_3 we first derive an integral formula for q. Writing $u = \sum_{n=0}^{\infty} u_n z^n$ we have

$$q(z) = [\hat{u} z u'] = \sum_{m,n=0}^{\infty} u_m u_n n z^{|n-m|}, \qquad |z| < 1.$$

Since

$$\frac{1}{2\pi} \int_{\mathbb{T}} \hat{u}(s) s u'(s) s^j \,|\mathrm{d}s| = \sum_{\substack{m,n=0 \\ n-m=-j}}^{\infty} u_m u_n n$$

this yields

$$\begin{aligned} q(z) &= \frac{1}{2\pi} \int_{\mathbb{T}} \hat{u}(s) s u'(s) \left(\sum_{j=0}^{\infty} z^j s^{-j} + \sum_{j=1}^{\infty} z^j s^j \right) |\mathrm{d}s| \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} \hat{u}(s) s u'(s) K(z,s)\, |\mathrm{d}s|, \end{aligned}$$

where

$$K(z,s) = \frac{1}{1 - zs^{-1}} + \frac{zs}{1 - zs}.$$

Thus the third integral is

$$\begin{aligned} I_3 &= 4\int_{|z|=r} Q_1(\phi q + zq' - q(0))\,\frac{|\mathrm{d}z|}{r} \\ &= 4\int_{\mathbb{T}} \hat{u}(s)su'(s)\frac{1}{2\pi}\int_{|z|=r} Q_1\left(\phi K + z\frac{\partial K}{\partial z} - K(0,s)\right)\,\frac{|\mathrm{d}z|}{r}\,|\mathrm{d}s|. \end{aligned} \tag{5.10}$$

Let

$$Q_2(z,s) = Q_1\left(\phi K + z\frac{\partial K}{\partial z} - 1\right).$$

A computation gives (see (A.3))

$$Q_2(z,s) = \frac{(s-1)^2(1+z)^3(z+R)^3(2sz^2-(s-1)^2z+2s)}{(1-z)(z-R)z(z-s)^2(sz-1)^2}.$$

Evaluating the inner integral in (5.10) by residues we get

$$I_3 = \int_{\mathbb{T}} \hat{u}(s)su'(s)Q_3(s)\,|\mathrm{d}s|, \tag{5.11}$$

where

$$Q_3(s) = 4\left(\operatorname*{Res}_{z=0}\frac{Q_2}{z} + \operatorname*{Res}_{z=R}\frac{Q_2}{z}\right).$$

With Mathematica it is verified that $\hat{Q}_3 = Q_3$, see (A.4). We partition (5.11) as $I_3 = I_{3a} + I_{3b}$, where

$$I_{3a} = \int_{\mathbb{T}} Q_3(z)\overline{u}zu_a'\,\mathrm{d}\theta + \int_{\mathbb{T}} Q_3(z)\hat{u}_a zu_h'\,\mathrm{d}\theta$$

and

$$I_{3b} = \int_{\mathbb{T}} Q_3(z)\overline{u_h}zu_h'\,\mathrm{d}\theta.$$

Since the first integral in I_{3a} is real it can be written

$$\int_{\mathbb{T}} \overline{Q_3(z)}u\overline{zu_a'}\,\mathrm{d}\theta = \int_{\mathbb{T}} Q_3(z)u\widehat{zu_a'}\,\mathrm{d}\theta.$$

Since $Q_3(z)$ has double poles at $z = R$, $z = 0$ and $z = R^{-1}$ (see (A.3)), this gives

$$I_{3a} = 2\pi\left(\operatorname*{Res}_{z=R}\frac{Q_3}{z}u_a\widehat{zu_a'} + \operatorname*{Res}_{z=0}\frac{Q_3}{z}(u\widehat{zu_a'} + \hat{u}_a zu_h')\right),$$

where we have used that $u_h(R) = u_h'(R) = u_h''(R) = 0$. These residues are again evaluated with Mathematica, see (A.5). Note that when calculating the second residue we may replace h by its MacLaurin polynomial of degree 4. We have

$$I_{3b} = \int_{\mathbb{T}} Q_3\hat{P}_1\overline{h}z(P_1'h + P_1h')\,\mathrm{d}\theta.$$

Since I_{3b} is real we can rewrite this as

$$I_{3b} = \int_{\mathbb{T}} (Q_4|h|^2 + Q_5\overline{h}zh')\,\mathrm{d}\theta,$$

where

$$Q_4 = \operatorname{symm} Q_3\hat{P}_1zP_1', \qquad Q_5 = Q_3\hat{P}_1P_1.$$

Finally, the fourth integral I_4 is

$$I_4 = 4\int_{|z|=r} Q_1|v|^2\,\mathrm{d}\theta = I_{4a} + I_{4b},$$

where

$$I_{4a} = 4\int_{|z|=r} Q_1(\overline{v}_a v_a + 2\overline{v}_a v_h),\mathrm{d}\theta$$

and

$$I_{4b} = 4\int_{|z|=r} Q_1|v_h|^2\,\mathrm{d}\theta.$$

Evaluating I_{4a} by residues we get

$$I_{4a} = 8\pi \left(\operatorname*{Res}_{z=0} \frac{Q_1(z)}{z} v_a(R/z)(v_a(z) + 2v_h(z)) + \operatorname*{Res}_{z=R} \frac{Q_1(z)}{z} v_a(R/z) v_a(z) \right),$$

where we have used that $v_h(R) = 0$. Note that when evaluating the first residue we may replace v_h (or h) by its MacLaurin polynomial of degree 4. Again the residues are evaluated with Mathematica, see (A.6). To simplify I_{4b}, we use that

$$r \left| \frac{z-1}{z-R} \right| = 1 \quad \text{for } |z| = r.$$

This gives

$$I_{4b} = 4 \int_{|z|=r} \left| \frac{v_h}{k'} \right|^2 \mathrm{d}\theta = \int_{|z|=r} \left| Q_6(\phi u_h + z u_h') \right|^2 \mathrm{d}\theta,$$

where

$$Q_6(z) = R \left(\frac{z-1}{z-R} \right)^2 \frac{2}{k'(z)}.$$

We can write this

$$I_{4b} = \int_{|z|=r} \left| P_2 h + z(P_3 h)' \right|^2 \mathrm{d}\theta,$$

where

$$P_2 = (\phi Q_6 - z Q_6') P_1 = 2R(z-1)(z+1)^2(3z^2 - 5Rz - 3z + R)$$

and

$$P_3 = Q_6 P_1 = 2R(z-1)(z+1)^3(R-z),$$

see (A.6). (The factor $R((z-1)/(z-R))^2$ was introduced in Q_6 to make P_2 and P_3 polynomials.)

To prove that $V_2 \le 0$ we would like to estimate I_{4b} with an integral over $\mathbb{T}$ of the same type as I_{3b}. To do this we introduce the MacLaurin series

$$P_2 h = \sum_{n=0}^{\infty} p_n z^n \quad \text{and} \quad P_3 h = \sum_{n=0}^{\infty} q_n z^n.$$

We have

$$I_{4b} = 2\pi \sum_{n=0}^{\infty} (p_n + n q_n)^2 R^n.$$

For $n > 4$ we use the estimates

$$(p_n + n q_n)^2 \le 2p_n^2 + 2n^2 q_n^2 \quad \text{and} \quad nR^n \le \frac{1}{e(1-R)}.$$

This gives

$$\begin{aligned} I_{4b} &\le 2\pi \sum_{n=0}^{4} (p_n + n q_n)^2 R^n + 2\pi \sum_{n=5}^{\infty} \left(2p_n^2 + \frac{2}{e(1-R)} n q_n^2\right) \\ &= I_{4c} + \int_{\mathbb{T}} \left(2|P_2 h|^2 + \frac{2}{e(1-R)} \overline{P_3 h} z (P_3 h)' \right) \mathrm{d}\theta, \end{aligned} \tag{5.12}$$

where

$$I_{4c} = 2\pi \sum_{n=0}^{4} \left((p_n + n q_n)^2 R^n - 2p_n^2 - \frac{2}{e(1-R)} n q_n^2 \right).$$

The last integral in (5.12) is real and can thus be written

$$\int_{\mathbb{T}} \left(2\hat{P}_2 P_2 + \frac{2}{e(1-R)} \operatorname{symm}(\hat{P}_3 z P_3') \right) |h|^2 + \frac{2}{e(1-R)} \hat{P}_3 P_3 \overline{h} z h' \, \mathrm{d}\theta,$$

which is of the same type as I_{3b}.

Summing the four integrals I_1, I_2, I_3, I_4 we get the estimate

$$V_2 \le I_{sum} + \int_{\mathbb{T}} \left(Q_7|h|^2 + Q_8\overline{h}zh' \right) \, d\theta, \tag{5.13}$$

where

$$\begin{aligned} I_{sum} &= I_1 + I_2 + I_{3a} + I_{4a} + I_{4c}, \\ Q_7 &= Q_4 + 2\hat{P}_2 P_2 + \frac{2}{e(1-R)} \operatorname{symm}(\hat{P}_3 z P_3'), \\ Q_8 &= Q_5 + \frac{2}{e(1-R)} \hat{P}_3 P_3 \end{aligned}$$

are computed with Mathematica, see (A.7). We note that I_{sum} is a quadratic form in the variables $a_1, a_2, h_0, \ldots, h_4$, and that Q_7 and Q_8 are Laurent polynomials in z.

To see that the right member of (5.13) is ≤ 0, we use the decomposition

$$-(1-R)Q_8(z) = \sum_{j=1}^{4} b_j B_j(z^{-1}) B_j(z), \tag{5.14}$$

where

$$\begin{aligned} B_1(z) &= (z-1)(Rz-1) \\ B_2(z) &= (z-1)^2(Rz-1) \\ B_3(z) &= (z+1)(z-1)^2(Rz-1) \\ B_4(z) &= (z+1)^2(z-1)^2(Rz-1) \end{aligned}$$

and

$$\begin{aligned} b_1 &= 32R + (288 - 512/e)R^2 + 256R^3 - 64R^4 + 672R^5 - 224R^6 + 64R^7 \\ b_2 &= (128/e - 20)R^2 + 68R^3 + 184R^4 - 56R^5 + 92R^6 - 12R^7 \\ b_3 &= (32/e)R^2 + 8R^3 + 8R^4 - 40R^5 + 24R^6 \\ b_4 &= (8/e)R^2 + 12R^4 - 12R^5. \end{aligned}$$

The proof of (5.14) is a computation in Mathematica, see (A.8). Since $0 < R < 1$, the b_j are positive. For b_1, b_2 and b_4 this follows by comparing each negative term with its preceding positive term. For b_3 we have

$$b_3 > 11R^2 + 8R^3 - 16R^5 + 8R^4(1 - 3R + 3R^2) > 0.$$

Using the decomposition (5.14), the inequality (5.13) can be written (ρ is the radial coordinate)

$$\begin{aligned} -(1-R)V_2 &\ge -(1-R)I_{sum} + \int_{\mathbb{T}} \left(-(1-R)Q_7|h|^2 + \frac{1}{2}\sum_{j=1}^{4} b_j |B_j|^2 \frac{\partial}{\partial\rho}|h|^2 \right) \, d\theta \\ &= -(1-R)I_{sum} + \int_{\mathbb{T}} \left(Q_9|h|^2 + \frac{1}{2}\sum_{j=1}^{4} b_j \frac{\partial}{\partial\rho}|B_j h|^2 \right) \, d\theta, \end{aligned} \tag{5.15}$$

where

$$Q_9 = -(1-R)Q_7 - \sum_{j=1}^{4} b_j \operatorname{symm}(\hat{B}_j z B_j').$$

To see that the right member of (5.15) is ≥ 0 we do the following decomposition, which again is verified with Mathematica, see (A.9).

$$Q_9(z) = \sum_{j=1}^{5} c_j C_j(z^{-1}) C_j(z), \tag{5.16}$$

where

$$\begin{aligned} C_1(z) &= z - 1 \\ C_2(z) &= (z-1)^2 \\ C_3(z) &= (z-1)^3 \\ C_4(z) &= (z+1)(z-1)^3 \\ C_5(z) &= (z+1)(z-1)^4 \end{aligned}$$

and

$$\begin{aligned} c_1 =& 16(1-R)R(5 + (48 - 80/e)R + (67 + 16/e)R^2 \\ & - 114R^3 + 99R^4 + 28R^5 - 11R^6 + 6R^7) \\ c_2 =& 8R^2(72/e - 147 + (89 - 144/e)R + (245 - 24/e)R^2 \\ & - 95R^3 - 113R^4 + 131R^5 - 49R^6 + 3R^7) \\ c_3 =& 4R^2(72 - 20/e + (40/e - 27)R + (12/e - 57)R^2 \\ & - 2R^3 + 66R^4 - 67R^5 + 15R^6) \\ c_4 =& 8R^2(9 - 1/e + (2/e - 15)R + (9 + 1/e)R^2 - 13R^3 + 16R^4 - 6R^5) \\ c_5 =& 12(1-R)R^3(2 + R^2) \end{aligned}$$

By plotting with Mathematica (see (A.10)) it is verified that c_1, c_3, c_4 and c_5 are positive for $0 < R < 1$, and that $c_2 > 0$ for $0.827 < R < 1$. This is where the assumption $r > 0.91$ comes in. (By doing smarter decompositions one could probably get positive c_j and b_j in the entire interval $0 < r < 1$.)

Using the decomposition (5.16) in (5.15) we get

$$\begin{aligned} -\frac{1-R}{2\pi} V_2 &\geq -\frac{1-R}{2\pi} I_{sum} + \sum_{j=1}^{5} c_j \int_{\mathbb{T}} |C_j h|^2 \,\frac{\mathrm{d}\theta}{2\pi} + \sum_{j=1}^{4} b_j \int_{\mathbb{T}} \frac{1}{2}\frac{\partial}{\partial\rho} |B_j h|^2 \,\frac{\mathrm{d}\theta}{2\pi} \\ &\geq -\frac{1-R}{2\pi} I_{sum} + \sum_{j=1}^{5} c_j \sum_{n=0}^{4} (\text{Coefficient of } z^n \text{ in } C_j h)^2 \\ &\qquad + \sum_{j=1}^{4} b_j \sum_{n=0}^{4} n (\text{Coefficient of } z^n \text{ in } B_j h)^2. \end{aligned} \tag{5.17}$$

Let Q be the last expression times e/R. Q is a quadratic form in the variables $a_1, a_2, h_0, \ldots, h_4$. Using Mathematica (see (A.11)) it is verified the the corresponding symmetric 7x7 matrix M has characteristic polynomial

$$A_1 x + \cdots + A_7 x^7, \qquad \text{where } A_1 \neq 0 \text{ for } 0 \leq R < 1.$$

Thus M has an eigenvalue 0 of multiplicity 1 for $0 \leq R < 1$. For $R = 0$ it is verified (see (A.12)) that the A_js have alternating signs, and hence M has no negative eigenvalues. By continuity it follows that M has no negative eigenvalues when $0 \leq R < 1$. With Mathematica it is checked that the vector $(1 - 3R, 1, 1, 1, 1, 1, 1)$ is a null-vector of M, see (A.13). Hence $Q \geq 0$, with equality if and only if

$$(a_1, a_2, h_0, \ldots, h_4) = c(1 - 3R, 1, 1, 1, 1, 1, 1). \tag{5.18}$$

Thus $V_2 \leq 0$.

If $V_2 = 0$ we must have equality in the second inequality of (5.17). Hence $C_0 h$ must be a polynomial of degree at most 4, and so h has degree at most 3. This means that $c = 0$ in (5.18). Thus $a_1 = a_2 = h = 0$ and we get $u = a_0$. By (5.4) this means that $v_1(w) = \alpha_1 w + \tau(w)$, where $\tau(w)$ is real for real w.

3. Integral means and coefficients

In this section we give some relations between integral means and coefficients of functions $(f')^p$, where $f \in S$ and $p \in \mathbb{R}$. We will denote the nth MacLaurin coefficient of a function g by $c_n(g)$, that is

$$g(z) = \sum_{n=0}^{\infty} c_n(g) z^n.$$

The first relation of the above-mentioned type is given by Parseval's formula:

$$\int_{|z|=r} |g|^2 \, \mathrm{d}\theta = 2\pi \sum_{n=0}^{\infty} |c_n(g)|^2 r^{2n}, \qquad 0 < r < 1. \tag{5.19}$$

This leads to the following simple proposition.

Proposition 5.7. *Let g be analytic in $\mathbb{D}$, and let $\gamma > 0$. The following are equivalent:*

$$\int_{|z|=r} |g|^2 \, \mathrm{d}\theta = O\left(\left(\frac{1}{1-r} \right)^{\gamma} \right) \tag{5.20}$$

$$\sum_{n=0}^{N} |c_n(g)|^2 = O(N^{\gamma}). \tag{5.21}$$

Proof. Assume that (5.20) holds, and let N be a positive integer. Define $r = e^{-1/N}$. Then (5.19) gives

$$e^{-2} \sum_{n=0}^{N} |c_n(g)|^2 \le \sum_{n=0}^{N} |c_n(g)|^2 r^{2n} \le \frac{1}{2\pi} \int_{|z|=r} |g|^2 \, \mathrm{d}\theta = O(N^{\gamma}).$$

Conversely, suppose that (5.21) holds. Define

$$s_N = \sum_{n=0}^{N} |c_n(g)|^2.$$

Summation by part gives

$$\sum_{n=0}^{\infty} |c_n(g)|^2 r^{2n} = (1 - r^2) \sum_{N=0}^{\infty} s_N r^{2N} \le A(1 - r^2) \sum_{N=0}^{\infty} N^{\gamma} r^{2N} = O\left(\left(\frac{1}{1-r} \right)^{\gamma} \right),$$

and (5.20) follows from (5.19). □

It follows that Brennan's conjecture can be stated

$$\sum_{n=0}^{N} |c_n((f')^p)|^2 \le A_p N^{2|p|-1} \qquad \text{for } f \in S \text{ and } p \le -1.$$

Another relation comes from Cauchy's formula

$$c_n(g) = \frac{1}{2\pi} \int_{|z|=1-1/n} \frac{g(z)}{z^n} \, \mathrm{d}\theta.$$

This gives the estimate

$$|c_n((f')^p)| \le B \int_{|z|=1-1/n} |f'|^p \, \mathrm{d}\theta. \tag{5.22}$$

The following theorem gives a kind of reversed estimate. The case $p = 1$ of this theorem was proved in [**CaJo92**, Theorem 1]. Our proof is a modification of the proof in [**CaJo92**].

Theorem 5.8. *Let $f_0 : \mathbb{D} \to \mathbb{C}$ be univalent, let $p \in \mathbb{R} \setminus \{0\}$ and n be a positive integer. Then there is a function h_n univalent in $\mathbb{D}$ such that*

$$|c_n((h_n')^p)| \geq A \int_{|z|=1-1/n} |f_0'|^p \, \mathrm{d}\theta \tag{5.23}$$

$$h_n(0) = f_0(0) \tag{5.24}$$

$$h_n(\mathbb{D}) \subset f_0(\mathbb{D}) \tag{5.25}$$

$$|h_n'(0)| \geq A|f_0'(0)|. \tag{5.26}$$

A is a constant which may depend on p.

Proof. Let A_j and δ denote various positive constants which may depend on p. Let $r_n = 1 - A_0/n$ and $n > A_0$, where A_0 will be chosen later. Let

$$V(\theta) = \frac{|f_0'(r_n e^{i\theta})|^p}{f_0'(r_n e^{i\theta})^p}.$$

The distortion estimate

$$\left| \frac{f_0''(z)}{f_0'(z)} \right| \leq \frac{6}{1 - |z|^2}, \qquad |z| < 1 \tag{5.27}$$

implies that

$$|V'(\theta)| \leq \frac{6|p|n}{A_0}. \tag{5.28}$$

Let V_n be the nth Fejer mean of V, that is,

$$V_n(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(\alpha) V(\theta + \alpha) \, \mathrm{d}\alpha,$$

where

$$k_n(\alpha) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{ij\alpha}.$$

Since $k_n(\alpha) \geq 0$ and $\int_{-\pi}^{\pi} k_n(\alpha) \, \mathrm{d}\alpha = 2\pi$ we get

$$|V_n(\theta)| \leq 1.$$

It is well-known that

$$\int_{A_1/n \leq |\alpha| \leq \pi} k_n(\alpha) \, \mathrm{d}\alpha \leq \frac{\pi}{4}$$

if A_1 is large enough. Together with (5.28) this implies

$$|V_n(\theta) - V(\theta)| \leq \frac{1}{2} \tag{5.29}$$

provided A_0 is large enough. There is a polynomial $\psi(z)$ such that

$$\psi(r_n e^{i\theta}) = e^{i(n+1)\theta} V_n(\theta).$$

Obviously the degree of ψ is at most $2n + 1$, $\psi(0) = 0$ and

$$|\psi(z)| \leq 1 \qquad \text{for } |z| \leq r_n. \tag{5.30}$$

For $z = r_n e^{i\theta}$ the estimate (5.29) gives

$$\operatorname{Re} \left(\frac{r_n}{z}\right)^{n+1} \psi(z) f_0'(z)^p = \operatorname{Re} V_n(\theta) f_0'(z)^p \geq \frac{1}{2}|f_0'(z)|^p. \tag{5.31}$$

Let

$$f(z) = f_0(z) + \frac{\delta}{n}\psi(z) f_0'(z), \qquad |z| < 1, \tag{5.32}$$

where δ will be chosen later. Thus

$$\frac{f'}{f_0'} = 1 + \frac{\delta}{n}\left(\psi' + \psi\frac{f_0''}{f_0'}\right).$$

By (5.30) and Bernstein's theorem [**Dur83**, p. 195] we get

$$|\psi'(z)| \le \frac{2n+1}{r_n} \qquad \text{for } |z| \le r_n.$$

Together with (5.30) and (5.27) this gives

$$\left|\psi' + \psi\frac{f_0''}{f_0'}\right| \le 5n \qquad \text{for } |z| \le r_n,$$

provided $A_0 \ge 6$ and $n \ge 4A_0$. Hence

$$\left(\frac{f'}{f_0'}\right)^p = 1 + \frac{p\delta}{n}\left(\psi' + \psi\frac{f_0''}{f_0'}\right) + \delta^2 R,$$

where $|R(z)| \le A_1$ for $|z| \le r_n$ provided δ is chosen small enough. We can write this as

$$(f')^p - (f_0')^p = \frac{p\delta}{n}\left((\psi(f_0')^p)' + (1-p)\psi\frac{f_0''}{f_0'}(f_0')^p\right) + \delta^2 R(f_0')^p. \tag{5.33}$$

By (5.31) we have

$$\begin{aligned}\operatorname{Re} c_n\left((\psi(f_0')^p)'\right) &= (n+1)\operatorname{Re} c_{n+1}\left(\psi(f_0')^p\right)\\ &= (n+1)\operatorname{Re}\int_{|z|=r_n}\frac{\psi(f_0')^p}{z^{n+1}}\,\frac{\mathrm{d}\theta}{2\pi} \ge \frac{n+1}{2r_n^{n+1}}\int_{|z|=r_n}|f_0'|^p\,\frac{\mathrm{d}\theta}{2\pi}.\end{aligned}$$

Thus

$$\left|c_n\left(\frac{p\delta}{n}(\psi(f_0')^p)'\right)\right| \ge \frac{|p|\delta}{2r_n^n}\int_{|z|=r_n}|f_0'|^p\,\frac{\mathrm{d}\theta}{2\pi}. \tag{5.34}$$

For the remainder term

$$T = \left(\frac{p\delta}{n}(1-p)\psi\frac{f_0''}{f_0'} + \delta^2 R\right)(f_0')^p$$

we have

$$|T| \le \left(\frac{|p|\delta}{n}|1-p|\frac{6n}{A_0} + \delta^2 A_1\right)|f_0'|^p \le \frac{|p|\delta}{4}|f_0'|^p$$

for $|z| \le r_n$ provided A_0 is sufficiently large and δ is sufficiently small. Thus

$$|c_n(T)| \le \frac{|p|\delta}{4r_n^n}\int_{|z|=r_n}|f_0'|^p\,\frac{\mathrm{d}\theta}{2\pi}. \tag{5.35}$$

Now (5.33), (5.34) and (5.35) give

$$|c_n((f')^p) - c_n((f_0')^p)| \ge \frac{|p|\delta}{4r_n^n}\int_{|z|=r_n}|f_0'|^p\,\frac{\mathrm{d}\theta}{2\pi} \ge A_3\int_{|z|=1-1/n}|f_0'|^p\,\mathrm{d}\theta, \tag{5.36}$$

where the last estimate follows from the distortion theorem.

We now prove that f is univalent in $|z| < r_n - 8\delta/n$. It follows from (5.32) and (5.30) that

$$|f(z) - f_0(z)| \le |f_0'(z)|\frac{\delta}{n}, \qquad |z| \le r_n. \tag{5.37}$$

Let $|z_0| < r_n - 8\delta/n$. By (5.37) and the Koebe $\frac{1}{4}$ theorem there is a z_1 such that $|z_1 - z_0| \le 4\delta/n$ and $f_0(z_1) = f(z_0)$. Hence

$$f\left(D(0, 1-(A_0+8\delta)/n)\right) \subset f_0(\mathbb{D}).$$

Let ζ be any complex number on the circle $|\zeta| = r_n$. By the Koebe $\frac{1}{4}$ theorem

$$|f_0(z_1) - f_0(\zeta)| \ge \frac{1}{4}|f_0'(\zeta)||z_1 - \zeta|.$$

Together with (5.37) this gives

$$|f_0(z_1) - f_0(\zeta)| > |f(\zeta) - f_0(\zeta)|.$$

Thus, by Rouché's theorem the equations

$$f(z) = f_0(z_1) \quad \text{and} \quad f_0(z) = f_0(z_1)$$

have the same number of solutions in $|z| < r_n$. Hence the equation $f(z) = f(z_0)$ has exactly one solution for $|z| < r_n$. This means that f is univalent in $|z| < r_n - 8\delta/n$. The function

$$g(z) = f\big((r_n - 8\delta/n)z\big)$$

satisfies (5.24), (5.25), (5.26) and

$$|c_n((g')^p)| \geq A_4 |c_n((f')^p)|.$$

Thus, by (5.36) either $h_n = g$ or $h_n = f_0$ satisfies (5.23), provided n is large enough. For small n we can take $h_n(z) = f_0(0) + |f_0'(0)|\phi(z)/4$, where $\phi : \mathbb{D} \to \mathbb{D}$ is a univalent function with nonzero coefficients with the exception that $\phi(0) = 0$. □

It follows from (5.22) and Theorem 5.8 that Brennan's conjecture can be stated

$$|c_n((f')^p)| \leq C n^{|p|-1} \qquad \text{for } n \geq 1,\ f \in S,\ p \leq -2. \tag{5.38}$$

where C is a constant that may depend on p. One might even conjecture that

$$|c_n((f')^p)| \leq |c_n((k')^p)| \qquad \text{for } n \geq 0\ f \in S,\ p \leq -2. \tag{5.39}$$

CHAPTER 6

De Branges' method

De Branges' miraculous proof of Milin's conjecture [**deB85, FiPo85**] gives the uninitiated reader the impression that everything is so fine-tuned that it is hard to see how it could be applied to other problems. There have also been very few applications of the ideas used in de Branges' proof; the most noteworthy is [**deB86a**], which estimates the coefficients of powers of functions in S. The ideas put forward by de Branges have a potential of wider applicability, however. This was shown in [**deB87, deB86b**] and in more detail in [**VaNi91, VaNi92**]. These papers describe the underlying concepts in de Branges' proof in an operator-theoretic language. For an exposition more directed to system theorists, see [**HeWe96**]. In Section 6.1 we aim at a less sophisticated presentation of the principles underlying de Branges' proof of Milin's conjecture. In the following sections we apply this general framework to three problems: Milin's conjecture, Conjecture 5.1 and the conjecture (5.39).

1. The general framework

What we call de Branges' method is a method for solving certain problems of the following type: Let $f \in S$ and form a function $G(f)$ analytic in $\mathbb{D}$ by

$$G(f)(z) = \Phi(z, f(z), f'(z), f''(z), \ldots, f^{(\nu)}(z)),$$

where Φ is some fixed analytic function. Let $H_1, H_2, \ldots$ be real constants. Show that the functional

$$f \mapsto \sum_{n=0}^{\infty} H_n |c_n(G(f))|^2 \tag{6.1}$$

has maximum when f is the Koebe function $k(z) = z(1+z)^{-2}$. As before, $c_n(g)$ denotes the nth MacLaurin coefficient of g.

Of course, we need some condition on H_n for the sum in (6.1) to converge. Let us suppose that $|H_n| \le A\rho^n$ for some constants $A > 0$, $\rho < 1$. This implies that the functional (6.1) is continuous. (We use the topology of locally uniform convergence on S.) Hence it suffices to consider functions f in the following dense subclass of S. Let $\Gamma : [0, +\infty) \to \mathbb{C}$ be a parametrization of a Jordan arc such that, for some $T > 0$, the arc $\Gamma([T, +\infty))$ is an interval $[\Gamma(T), +\infty)$ on the positive real axis. Let f_0 be a conformal map of $\mathbb{D}$ onto $\mathbb{C} \setminus \Gamma([0, +\infty))$. The set of such mappings f_0 is dense in S [**Dur83**, p. 81]. Hence it suffices to prove

$$\sum_{n=0}^{\infty} H_n |c_n(G(f_0))|^2 \le \sum_{n=0}^{\infty} H_n |c_n(G(k))|^2$$

for such $f_0 \in S$. For $t > 0$, let f_t be the Riemann mapping of the unit disc onto the complement of the arc $\Gamma([t, +\infty))$, normalized so that $f_t(0) = 0$ and $f_t'(0) > 0$. We can choose the parametrization Γ so that $f_t'(0) = e^t$. Löwner's differential equation [**Ahl73**, Chapter 6] then relates the mappings f_t:

$$\dot{f}_t(z) = \frac{1 + \omega(t)z}{1 - \omega(t)z} z f_t'(z) \qquad \text{for } t \ge 0, \tag{6.2}$$

where ω is a complex-valued continuous function with $|\omega(t)| = 1$, and the dot denotes differentiation with respect to t. We have the initial condition

$$f_T(z) = e^T k(z). \tag{6.3}$$

Note that the choice $\omega \equiv 1$ corresponds to $f_t = e^t k$. Löwner's theory has thus transformed our problem to the following optimal control problem:

Let $\omega : [0, T] \to \mathbb{T}$ be a continuous function, and let f_t be the solution of the initial value problem (6.2), (6.3). Show that

$$\sum_{n=0}^{\infty} H_n |c_n(G(f_0))|^2 \tag{6.4}$$

is maximal when $\omega \equiv 1$.

In more geometric terms, we start with the slit domain $f_T(\mathbb{D}) = \mathbb{C} \setminus [e^T/4, +\infty)$, and then prolong the slit and get a new domain $f_0(\mathbb{D})$. We have to show that the resulting quantity (6.4) is greatest if the slit is prolonged along the real axis. It is natural to study the functions $G(e^{-t} f_t)$, or more generally $g_t = G_t(f_t)$, where G_t are differential operators such that $G_0 = G$. A common way to estimate (6.4) is to study how the quantity

$$\sum_{n=0}^{\infty} H_n |c_n(g_t))|^2$$

evolves in t. This seldom works for our problem, though. The brilliant idea of de Branges was to study such a quantity

$$Q(t) = \sum_{n=0}^{\infty} h_n(t) |c_n(g_t))|^2$$

where the real-valued weights $h_n(t)$ depend on t in a cunningly chosen way (but they do not depend on ω). We will see that there is nothing mysterious about the choice of h_n, since they can be computed as the solution of a system of linear differential equations. The idea is to choose h_n such that the following two conditions are satisfied. Let $\hat{Q}(t)$ be the value of $Q(t)$ obtained in the case $\omega \equiv 1$.

$$h_n(0) = H_n, \tag{6.5}$$

$$\dot{Q}(t) \geq \dot{\hat{Q}}(t) \quad \text{for all choices of } \omega, \text{ for } t \geq 0. \tag{6.6}$$

The problem is solved as soon as these conditions are established, since we get

$$\begin{aligned}\sum_{n=0}^{\infty} H_n |c_n(G(f_0))|^2 &= Q(0) = Q(T) - \int_0^T \dot{Q}(t)\,\mathrm{d}t \\ &\leq \sum_{n=0}^{\infty} h_n(T) |c_n(G_T(e^T k))|^2 - \int_0^T \dot{\hat{Q}}(t)\,\mathrm{d}t,\end{aligned}$$

with equality if $\omega \equiv 1$.

Let us examine the quantity $\dot{Q}(t)$:

$$\dot{Q}(t) = \sum_{n=0}^{\infty} \dot{h}_n |c_n(g_t)|^2 + h_n 2 \operatorname{Re} \overline{c_n(g_t)} c_n(\dot{g}_t). \tag{6.7}$$

To justify this we assume that $|h_n| \leq A\rho^n$ and $|\dot{h}_n| \leq A\rho^n$ for some constants $A > 0$ and $\rho < 1$. To get a compact notation, let us introduce the scalar product

$$(u, v) = \sum_{n=0}^{\infty} c_n(u) \overline{c_n(v)}$$

whenever $u, v \in A(\mathbb{D})$ and the sum is absolutely convergent. Here, $A(\Omega)$ denotes the set of analytic functions in Ω. Let $D_t : A(\mathbb{D}) \to A(D(0, \rho^{-1}))$ be the linear operator which is diagonal in the basis $1, z, z^2, \ldots$, with diagonal elements $h_n(t)$, that is,

$$(D_t u)(z) = \sum_{n=0}^{\infty} h_n(t) c_n(u) z^n.$$

Now (6.7) can be written

$$\dot{Q}(t) = (\dot{D}_t g_t, g_t) + 2\,\mathrm{Re}(D_t g_t, \dot{g}_t). \tag{6.8}$$

From (6.2) follows that $\dot{g}_t(z)$ can be expressed in terms of $t, \omega, z, f_t(z), f_t'(z), \ldots$. (From now on we write ω instead of $\omega(t)$.) In order to prove (6.6) it is desirable that $\dot{Q}(t)$ is quadratic g_t. Let us therefore assume that $\dot{g}_t(z)$ can be expressed in terms of $t, \omega, z, g_t(z), g_t'(z), \ldots, g_t^{(m)}(z)$, and that the dependence on the $g_t^{(j)}(z)$ is linear:

$$\dot{g}_t(z) = \sum_{j=0}^{m} \psi_j(t, \omega, z) g_t^{(j)}(z) + \varphi(t, \omega, z).$$

It is easy to see that we must have $m = 1$ and $\psi_1 = \frac{1+\omega z}{1-\omega z} z$, so that

$$\dot{g}_t = L_{t,\omega} g_t + \varphi_{t,\omega}, \tag{6.9}$$

where $L_{t,\omega}$ is the linear differential operator

$$L_{t,\omega} = \psi_{t,\omega}(z) + \frac{1+\omega z}{1-\omega z} z \frac{\mathrm{d}}{\mathrm{d}z}$$

and $\varphi_{t,\omega}(z)$ and $\psi_{t,\omega}(z)$ are analytic for $z \in \mathbb{D}$.

Remark. One can show that the operators G_t for which $g_t = G_t(f_t)$ satisfies (6.9) can be written $G_t(f) = \lambda_t H(f) + \mu_t$, where $\lambda_t(z)$ and $\mu_t(z)$ are analytic functions and $H(f)$ is one of the following differential operators:

$$(\eta \circ f)(f')^p, \qquad \log f' + \eta \circ f, \tag{6.10}$$

$$\frac{f''}{f'} + (\eta \circ f) f', \qquad \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 + (\eta \circ f)(f')^2,$$

where $p \in \mathbb{C}$ and η is an analytic function. Thus our assumption is quite restrictive, but not too restrictive. The proof of this is based on the observation that (6.9) implies that the operator G satisfies a certain "chain rule". To see that this implies that G_t has to be one of the mentioned operators, one uses the fact that a finite-dimensional connected complex Lie group of conformal maps must be conjugate to the Möbius group, the affine group or the translation group.

Let $L^*_{t,\omega} : A(\overline{\mathbb{D}}) \to A(\overline{\mathbb{D}})$ be the adjoint operator, which satisfies

$$(L^*_{t,\omega} u, v) = (u, L_{t,\omega} v) \qquad \text{for } u \in A(\overline{\mathbb{D}}),\ v \in A(\mathbb{D}).$$

Now (6.8) can be written

$$\dot{Q}(t) = (P_{t,\omega} g_t, g_t) + 2\,\mathrm{Re}(g_t, D_t \varphi_{t,\omega}), \tag{6.11}$$

where $P_{t,\omega} = \dot{D}_t + L^*_{t,\omega} D_t + D_t L_{t,\omega}$. To "complete the square" we assume that there exists a $\phi_{t,\omega} \in A(\mathbb{D})$ such that

$$P_{t,\omega} \phi_{t,\omega} = D_t \varphi_{t,\omega}. \tag{6.12}$$

Together with $P^*_{t,\omega} = P_{t,\omega}$ this gives

$$\begin{aligned} \dot{Q}(t) = (P_{t,\omega} g_t, g_t) + (P_{t,\omega} g_t, \phi_{t,\omega}) + (P_{t,\omega} \phi_{t,\omega}, g_t) \\ = (P_{t,\omega}(g_t + \phi_{t,\omega}), g_t + \phi_{t,\omega}) - (P_{t,\omega} \phi_{t,\omega}, \phi_{t,\omega}). \end{aligned}$$

Since the function $g_t + \phi_{t,\omega}$ is difficult to keep track of, the only reasonable way to get condition (6.6) is to require that

$$(P_{t,\omega} u, u) \geq 0 \qquad \text{for all } u \in A(\mathbb{D}), \tag{6.13}$$

$$(P_{t,\omega}(g_t + \phi_{t,\omega}), g_t + \phi_{t,\omega}) = 0 \qquad \text{if } \omega \equiv 1 \tag{6.14}$$

and

$$-(P_{t,\omega}\phi_{t,\omega}, \phi_{t,\omega}) \geq -(P_{t,1}\phi_{t,1}, \phi_{t,1}) \qquad \text{for } \omega \in \mathbb{T}. \tag{6.15}$$

We will now show that these conditions determine the weight functions $h_n(t)$. In case $\omega \equiv 1$ we have $g_t = G_t(e^t k) = \hat{g}_t$. Thus condition (6.14) is

$$(P_{t,1}(\hat{g}_t + \phi_{t,1}), \hat{g}_t + \phi_{t,1}) = 0.$$

By (6.13) this means that

$$P_{t,1}(\hat{g}_t + \phi_{t,1}) = 0,$$

that is,

$$\dot{D}_t \hat{g}_t + L^*_{t,1} D_t \hat{g}_t + D_t L_{t,1} \hat{g}_t + D_t \varphi_{t,1} = 0. \tag{6.16}$$

By (6.9) we have $\dot{\hat{g}}_t = L_{t,1}\hat{g}_t + \varphi_{t,1}$, so (6.16) can be written

$$\dot{y}_t = -L^*_{t,1} y_t, \tag{6.17}$$

where $y_t = D_t \hat{g}_t$. Finally, condition (6.5) gives

$$y_0 = \sum_{n=0}^{\infty} H_n c_n(\hat{g}_0) z^n \tag{6.18}$$

Hence y_t is uniquely determined, and so are the weights h_n, provided $c_n(\hat{g}_t) \neq 0$.

In conclusion, for a particular problem the procedure is as follows. Choose the differential operators G_t so that $G_0 = G$ and $g_t = G_t(f_t)$ satisfies (6.9). (We will always take $G_t(f) = G(e^{-t} f)$, since this makes $\hat{g}_t$ independent of t.) Solve the differential equation (6.17) with initial condition (6.18). Then compute the weights $h_n(t) = c_n(y_t)/c_n(\hat{g}_t)$ and the operator $P_{t,\omega}$. Finally, try to show that (6.12), (6.13) and (6.15) hold. If we succeed, the problem is solved. To simplify the last step, we can often use that $\psi_{t,\omega}$ and $\varphi_{t,\omega}$ depend on ω in the following special way:

$$\psi_{t,\omega}(z) = \tilde{\psi}_t(\omega z), \qquad \varphi_{t,\omega} = \omega^d \tilde{\varphi}_t(\omega z). \tag{6.19}$$

This implies that

$$L_{t,\omega} = U_\omega L_{t,1} U_{\omega^{-1}},$$

where U_ω is the operator that takes a function $u(z)$ to the function $u(\omega z)$. This gives

$$P_{t,\omega} = U_\omega P_{t,1} U_{\omega^{-1}}, \tag{6.20}$$

so that (6.13) can be written

$$(P_{t,1}u, u) \geq 0 \qquad \text{for all } u \in A(\mathbb{D}). \tag{6.21}$$

Moreover, by (6.16) we have

$$P_{t,1}\hat{g}_t + D_t \varphi_{t,1} = 0, \tag{6.22}$$

so that (6.12) is satisfied with $\phi_{t,\omega} = -\omega^d U_\omega \hat{g}_t$. This also shows that (6.15) is satisfied with equality.

2. Milin's conjecture

In this section we show how de Branges' proof of Milin's conjecture fits into the framework of the preceding section. Some of the material is taken from [**VaNi92**, Chapter E].

Let

$$G(f)(z) = \log \frac{f(z)}{z} \qquad \text{for } f \in S.$$

Let N be a positive integer and

$$\begin{aligned} H_n &= n(N+1-n) && \text{for } n = 1, \dots, N, \\ H_n &= 0 && \text{for } n > N. \end{aligned}$$

Milin's conjecture states that

$$\sum_{n=1}^{\infty} H_n |c_n(G(f))|^2$$

is maximized when f is the Koebe function $k(z) = z(1+z)^{-2}$.

Let $G_t(f) = G(e^{-t}f)$. (This corresponds to the first operator in (6.10) with $\eta = \log$, $p = 0$, $\lambda_t(z) = 1$, $\mu_t(z) = -\log z - t$.) The differential equation (6.2) implies that $g_t = G_t(f_t)$ satisfies

$$\dot{g}_t = \frac{1+\omega z}{1-\omega z} z g_t' + \frac{2\omega z}{1-\omega z},$$

so that (6.9) and (6.19) are satisfied with

$$L_{t,1} = \frac{1+z}{1-z} z \frac{\mathrm{d}}{\mathrm{d}z}, \qquad \tilde{\varphi}_1(z) = \frac{2z}{1-z}, \qquad d = 0.$$

Since $g_t(0) = 0$ it suffices to consider $L_{t,1}$ as defined for functions $u \in A(\mathbb{D})$ with $u(0) = 0$. The matrix of $L_{t,1}$ in the basis $z, z^2, z^3, \dots$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 0 & 0 & 0 & 0 & \dots \\ 2 & 4 & 3 & 0 & 0 & 0 & \dots \\ 2 & 4 & 6 & 4 & 0 & 0 & \dots \\ 2 & 4 & 6 & 8 & 5 & 0 & \dots \\ 2 & 4 & 6 & 8 & 10 & 6 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Thus the transposed matrix defines an adjoint operator $L_{t,1}^*$ on the space of functions $u \in A(\overline{\mathbb{D}})$ with $u(0) = 0$. The equation (6.17) therefore reads

$$\dot{y}_n(t) = -n y_n(t) - 2n \sum_{j=n+1}^{\infty} y_j(t), \qquad n \geq 1, \tag{6.23}$$

where $y_n(t) = c_n(y_t)$. Since

$$\hat{g}_t(z) = G_t(e^t k)(z) = -2\log(1+z) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n} z^n,$$

the initial condition (6.18) is

$$\begin{aligned} y_n(0) &= 2(-1)^n (N+1-n) && \text{for } n = 1, 2, \dots, N, \\ y_n(0) &= 0 && \text{for } n > N. \end{aligned} \tag{6.24}$$

The solution of the initial value problem (6.23), (6.24) is of the form

$$y_n(t) = \sum_{j=n}^{N} a_{nj} e^{-jt}, \qquad n = 1, 2, \dots, N, \tag{6.25}$$

$$y_n(t) = 0, \qquad n > N. \tag{6.26}$$

Note also that (6.23) implies that

$$\frac{\dot{y}_n(t)}{n} - \frac{\dot{y}_{n+1}(t)}{n+1} = -y_n(t) - y_{n+1}(t), \qquad n \geq 1.$$

Thus the weight functions $h_n(t) = y_n(t)/c_n(\hat{g}_t) = (-1)^n n y_n(t)/2$ satisfy

$$\frac{\dot{h}_n(t)}{n^2} + \frac{\dot{h}_{n+1}(t)}{(n+1)^2} = -\frac{h_n(t)}{n} + \frac{h_{n+1}(t)}{n+1}, \qquad n \geq 1. \tag{6.27}$$

This means that the functions $h_n(t)/n$ coincide with the weight functions $\sigma_n(e^t)$ in [**deB85**, p.141].

To prove Milin's conjecture it remains to show that the operator $P_{t,1} = \dot{D}_t + D_t L_{t,1} + L_{t,1}^* D_t$ satisfies (6.21), or rather

$$(P_{t,1}u, u) \geq 0 \quad \text{for all } u \in A(\mathbb{D}) \text{ with } u(0) = 0.$$

Equivalently, choosing a basis $e_1, e_2, \ldots$ we have to prove that the symmetric matrix M with elements

$$M_{nj} = (P_{t,1}e_n, e_j) = (\dot{D}_t e_n, e_j) + (D_t L_{t,1} e_n, e_j) + (D_t e_n, L_{t,1} e_j)$$

is positive semidefinite. The most convenient choice of basis is

$$e_n = \frac{z^n}{n} - \frac{z^{n+1}}{n+1}, \qquad n = 1, 2, \ldots,$$

since this gives $L_{t,1}e_n = z^n + z^{n+1}$ a simple form. This immediately gives $M_{nj} = 0$ if $|n-j| > 1$, so that M is tridiagonal:

$$M = \begin{pmatrix} \lambda_1 & \mu_2 & 0 & 0 & 0 & \ldots \\ \mu_2 & \lambda_2 & \mu_3 & 0 & 0 & \ldots \\ 0 & \mu_3 & \lambda_3 & \mu_4 & 0 & \ldots \\ 0 & 0 & \mu_4 & \lambda_4 & \mu_5 & \ldots \\ 0 & 0 & 0 & \mu_5 & \lambda_5 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

We compute

$$\lambda_n = M_{nn} = \frac{\dot{h}_n}{n^2} + \frac{\dot{h}_{n+1}}{(n+1)^2} + \frac{2h_n}{n} - \frac{2h_{n+1}}{n+1},$$

$$\mu_n = M_{n,n-1} = -\frac{\dot{h}_n}{n^2}.$$

Hence (by (6.26)) we can consider M as a $N \times N$ matrix. Equation (6.27) shows that we have the relation $\lambda_n = \mu_n + \mu_{n-1}$. Thus the corresponding Hermitian form can be written

$$(P_{t,1}u, u) = \sum_{n,j=1}^{N} M_{nj} u_n \overline{u_j} = \mu_1 |u_1|^2 + \sum_{n=2}^{N} \mu_n |u_n + u_{n-1}|^2,$$

where $u = \sum_{n=1}^{N} u_n e_n$. It remains to prove that

$$\mu_n \geq 0 \qquad \text{for } t \geq 0,\ n = 1, 2, \ldots, N. \tag{6.28}$$

A computation of the a_{nj} in (6.25) shows that μ_n can be expressed in terms of a generalized hypergeometric function (polynomial). A known inequality for such polynomials [**AsGa76**, Theorem 3 and equation (3.1)] now proves (6.28). For a more enlightening proof of (6.28), see [**VaNi92**, pp. 1232-1236].

3. Integral means

In this section we apply the method described in Section 6.1 to Conjecture 5.1. That is, we let $G(f) = (f')^p$, $H_n = r^{2n}$, where $p \leq -1$ and $0 < r < 1$. We want to show that

$$\sum_{n=0}^{\infty} H_n |c_n(G(f))|^2 \leq \sum_{n=0}^{\infty} H_n |c_n(G(k))|^2, \qquad \text{for } f \in S. \tag{6.29}$$

Let $G_t(f) = G(e^{-t} f)$. (This corresponds to the first operator in (6.10) with $\eta = 1$, $\lambda_t = e^{-pt}$, $\mu_t = 0$.) Differentiation of (6.2) shows that $g_t = G_t(f_t)$ satisfies

$$\dot{g}_t = 2p \left(\frac{1}{(1-\omega z)^2} - 1 \right) g_t + \frac{1+\omega z}{1-\omega z} z g_t'.$$

Thus (6.9) and (6.19) hold with

$$L_{t,1} = 2p \left(\frac{1}{(1-z)^2} - 1 \right) + \left(\frac{2}{1-z} - 1 \right) z \frac{\mathrm{d}}{\mathrm{d}z}, \qquad \varphi_{t,\omega} = 0.$$

To compute the adjoint $L_{t,1}^*$, we first compute the adjoint of $1/(1-z)$. For $u \in A(\overline{\mathbb{D}})$ and $v \in A(\mathbb{D})$ we have by partial summation

$$(u, \frac{1}{1-z} v) = \sum_{n=0}^{\infty} c_n(u) \overline{(c_0(v) + \cdots + c_n(v))} = \sum_{n=0}^{\infty} \left(\sum_{j=n}^{\infty} c_j(u) \right) \overline{c_n(v)},$$

so that

$$\left(\frac{1}{1-z} \right)^* u = \sum_{n=0}^{\infty} \left(\sum_{j=n}^{\infty} c_j(u) \right) z^n = \sum_{j=0}^{\infty} c_j(u) \frac{1-z^{j+1}}{1-z} = \frac{u(1) - zu}{1-z}.$$

This gives

$$\begin{aligned}
\left(\frac{1}{(1-z)^2} \right)^* u = \left(\frac{1}{1-z} \right)^* \left(\frac{1}{1-z} \right)^* u &= \frac{1}{1-z} \left(\frac{\mathrm{d}}{\mathrm{d}z} (zu)|_{z=1} - z \frac{u(1) - zu}{1-z} \right) \\
&= \frac{1-2z}{(1-z)^2} u(1) + \frac{u'(1)}{1-z} + \frac{z^2}{(1-z)^2} u.
\end{aligned}$$

Together with $\left(z \frac{\mathrm{d}}{\mathrm{d}z}\right)^* = z \frac{\mathrm{d}}{\mathrm{d}z}$ this yields

$$\begin{aligned}
L_{t,1}^* u &= 2p \left(\frac{1-2z}{(1-z)^2} u(1) + \frac{u'(1)}{1-z} + \frac{2z-1}{(1-z)^2} u \right) + z \frac{\mathrm{d}}{\mathrm{d}z} \left(2 \frac{u(1) - zu}{1-z} - u \right) \\
&= 2p \left(\frac{1-2z}{(1-z)^2} u(1) + \frac{u'(1)}{1-z} + \frac{2z-1}{(1-z)^2} u \right) + \frac{2z}{(1-z)^2} u(1) - \frac{1+z}{1-z} z u' - \frac{2z}{(1-z)^2} u.
\end{aligned}$$

The next step is to solve the differential equation

$$-\dot{y}_t = L_{t,1}^* y_t, \tag{6.30}$$

with the initial condition

$$y_0(z) = \sum_{n=0}^{\infty} C_n H_n z^n = \sum_{n=0}^{\infty} C_n r^{2n} z^n = \hat{g}(r^2 z), \tag{6.31}$$

where $C_n = c_n(\hat{g})$ and $\hat{g} = G_t(e^t k) = G(k) = (k')^p$. To do this we first solve (6.30) with the initial condition

$$\tilde{y}_0(z) = \sum_{n=0}^{\infty} \zeta^n z^n = \frac{1}{1 - \zeta z},$$

where $|\zeta| = r$. This makes it natural to try a solution of the type

$$\tilde{y}_t(z) = \frac{a_t}{1 - s_t z},$$

where $a_t \in \mathbb{C}$ and $s_t \in \mathbb{D}$. Putting this into (6.30) gives

$$-\frac{a\dot{s}z}{(1-sz)^2} - \frac{\dot{a}}{1-sz} = 2p\left(\frac{a}{1-s}\frac{1-2z}{(1-z)^2} + \frac{as}{(1-s)^2}\frac{1}{1-z} + \frac{2z-1}{(1-z)^2}\frac{a}{1-sz}\right) + \frac{a}{1-s}\frac{2z}{(1-z)^2} - \frac{1+z}{1-z}z\frac{as}{(1-sz)^2} - \frac{2z}{(1-z)^2}\frac{a}{1-sz}.$$

Since $L_{t,1}^* : A(\overline{\mathbb{D}}) \to A(\overline{\mathbb{D}})$, both sides are analytic for $z \in \overline{\mathbb{D}}$, and therefore we need only ensure that the principal parts at $z = 1/s$ agree. This gives the equations

$$\dot{s} = s\frac{s+1}{s-1}, \tag{6.32}$$

$$\frac{\dot{a}}{a} = -2p\frac{(2-s)s}{(1-s)^2}. \tag{6.33}$$

The solution of these equations with initial conditions $s_0 = \zeta$ and $a_0 = 1$ is

$$\frac{s_t}{(1+s_t)^2} = \frac{\zeta}{(1+\zeta)^2}e^{-t}, \tag{6.34}$$

$$a_t = \left(\frac{1-\zeta}{(1+\zeta)^3}\frac{(1+s_t)^3}{1-s_t}\right)^p. \tag{6.35}$$

Now note that

$$\hat{g}(r^2 z) = \int_{|\zeta|=r} \tilde{y}_0(z)\overline{\hat{g}(\zeta)}\,\frac{|\mathrm{d}\zeta|}{2\pi r}.$$

By linearity this means that the solution of (6.30) with initial condition (6.31) is

$$y_t(z) = \int_{|\zeta|=r} \tilde{y}_t(z)\overline{\hat{g}(\zeta)}\,\frac{|\mathrm{d}\zeta|}{2\pi r} = \int_{|\zeta|=r} \frac{a_t}{1-s_t z}\overline{\hat{g}(\zeta)}\,\frac{|\mathrm{d}\zeta|}{2\pi r},$$

and the weight functions are

$$h_n(t) = \frac{c_n(y_t)}{C_n} = \int_{|\zeta|=r} \frac{a_t s_t^n}{C_n}\overline{\hat{g}(\zeta)}\,\frac{|\mathrm{d}\zeta|}{2\pi r}.$$

(We have $C_n > 0$, see Lemma 6.6 below.) By (6.34) and (6.35), the Taylor polynomial of degree $n-1$ of $a_t s_t^n$ around $\zeta = 0$ vanishes. Hence we may replace $\hat{g}(\zeta)$ with $T_n(\zeta) = \sum_{j=n}^{\infty} C_j \zeta^j$ in the last integral:

$$h_n(t) = \int_{|\zeta|=r} \frac{a_t s_t^n}{C_n}\overline{T_n(\zeta)}\,\frac{|\mathrm{d}\zeta|}{2\pi r}. \tag{6.36}$$

We compute the derivative using (6.32) and (6.33):

$$\dot{h}_n(t) = \int_{|\zeta|=r} \frac{a_t s_t^n}{C_n}\left(-2p\frac{(2-s_t)s_t}{(1-s_t)^2} - \frac{1+s_t}{1-s_t}n\right)\overline{T_n(\zeta)}\,\frac{|\mathrm{d}\zeta|}{2\pi r}. \tag{6.37}$$

It remains to examine whether the operator

$$P_{t,1} = \dot{D}_t + D_t L_{t,1} + L_{t,1}^* D_t$$

satisfies (6.21), that is

$$(P_{t,1}u, u) \geq 0, \qquad \text{for all } u \in A(\mathbb{D}). \tag{6.38}$$

As a first test, let us compute

$$(P_{t,1}z^n, z^n) = (\dot{D}_t z^n, z^n) + (D_t L_{t,1} z^n, z^n) + (D_t z^n, L_{t,1} z^n)$$

for $t = 0$. Since

$$L_{t,1} z^n = 2p\left(\frac{1}{(1-z)^2} - 1\right)z^n + \frac{1+z}{1-z}nz^n$$

we get

$$(P_{0,1}z^n, z^n) = \dot{h}_n(0) + 2nh_n(0) \\ = \int_{|\zeta|=r} \frac{\zeta^n}{C_n}\left(-2p\frac{(2-\zeta)\zeta}{(1-\zeta)^2} - \frac{1+\zeta}{1-\zeta}n + 2n\right)\overline{T_n(\zeta)}\,\frac{|\mathrm{d}\zeta|}{2\pi r}.$$

Using the series for $(1-z)^p$ one can show that $\lim_{n\to\infty} C_n/n^{|p|-1} \in (0,+\infty)$. Thus the dominated convergence theorem gives

$$\lim_{n\to\infty}\frac{T_n(\zeta)}{C_n\zeta^n} = \lim_{n\to\infty}\sum_{j=0}^{\infty}\frac{C_{n+j}}{C_n}\zeta^j = \frac{1}{1-\zeta}.$$

This gives

$$\lim_{n\to\infty}\frac{(P_{0,1}z^n, z^n)}{nr^{2n}} = \int_{|\zeta|=r}\left(-\frac{1+\zeta}{1-\zeta}+2\right)\frac{1}{1-\overline{\zeta}}\,\frac{|\mathrm{d}\zeta|}{2\pi r} \\ = \int_{|\zeta|=r}\frac{1-3\zeta}{1-\zeta}\,\frac{1}{\zeta-r^2}\,\frac{\mathrm{d}\zeta}{2\pi i} = \frac{1-3r^2}{1-r^2}.$$

Thus, if $r > \frac{1}{\sqrt{3}}$, the operator $P_{0,1}$ does not satisfy (6.38), and therefore the method does not work.

However, for small r we get a positive result:

Theorem 6.1. *Let $p \in \mathbb{R}$ be such that $C_n \neq 0$ for all $n \geq 0$, where*

$$\left(\frac{1-z}{(1+z)^3}\right)^p = \sum_{n=0}^{\infty} C_n z^n.$$

If r is sufficiently small (depending on p) then

$$\int_{|z|=r}|f'(z)|^p\,\mathrm{d}\theta \leq \int_{|z|=r}|k'(z)|^p\,\mathrm{d}\theta \qquad \text{for all } f \in S.$$

Note that we no longer require that $p \leq -1$. The condition $C_n \neq 0$ is fulfilled if $p > -\frac{1}{8}$, see Lemma 6.6. To prove Theorem 6.1, we first note that by (6.36), (6.34) and (6.35) we have

$$h_n(t) = \int_{|\zeta|=r} s_t^n\overline{\zeta}^n(1+O(r))\,\frac{|\mathrm{d}\zeta|}{2\pi r} = \int_{|\zeta|=r} s_t^n\overline{\zeta}^n\,\frac{|\mathrm{d}\zeta|}{2\pi r} + O(r^{2n+1}e^{-nt}) \\ = r^{2n}e^{-nt} + O(r^{2n+1}e^{-nt}), \qquad (6.39)$$

where the constants in the $O(\ )$ only depend on p. Here we have used that $|C_{n+j}/C_n| < Aj^b$ for some constants A and b. Similarly, (6.37) gives

$$\dot{h}_n(t) = -nr^{2n}e^{-nt} + O(nr^{2n+1}e^{-nt}). \qquad (6.40)$$

This implies that the operator $P_{t,1}$ maps $A(\mathbb{D})$ continuously into $A(D(0,r^{-2}))$ (in the topologies of locally uniform convergence). Hence $(P_{t,1}u, u)$ is continuous for $u \in A(\mathbb{D})$. Thus it suffices to prove (6.38) for

$$u = \alpha_0\hat{g} + \sum_{n=1}^{N}\alpha_n e_n,$$

where $e_1, e_2, e_3, \ldots$ is a basis for the polynomials with zero constant term. Since $P_{t,1}\hat{g} = 0$ by (6.22), this amounts to proving that the $N \times N$ symmetric matrix M with elements

$$M_{nj} = (P_{t,1}e_n, e_j) = (\dot{D}_t e_n, e_j) + (D_t L_{t,1}e_n, e_j) + (D_t e_n, L_{t,1}e_j)$$

is positive semidefinite. The obvious choice of basis is

$$e_n(z) = z^n(1-z)^2 = z^n - 2z^{n+1} + z^{n+2}, \qquad n = 1, 2, 3, \ldots,$$

since this makes

$$L_{t,1}e_n = nz^n + (4p-2)z^{n+1} + (-2p-n-2)z^{n+2}$$

a polynomial. This implies that M is five-diagonal:

$$M = \begin{pmatrix} u_1 & v_2 & w_3 & 0 & 0 & 0 & \dots & 0 \\ v_2 & u_2 & v_3 & w_4 & 0 & 0 & \dots & 0 \\ w_3 & v_3 & u_3 & v_4 & w_5 & 0 & \dots & 0 \\ 0 & w_4 & v_4 & u_4 & v_5 & w_6 & \dots & 0 \\ 0 & 0 & w_5 & v_5 & u_5 & v_6 & \dots & 0 \\ 0 & 0 & 0 & w_6 & v_6 & u_6 & \dots & w_N \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & v_N \\ 0 & 0 & 0 & 0 & 0 & w_N & v_N & u_N \end{pmatrix}$$

A computation gives

$$\begin{aligned} u_n = M_{nn} &= \dot{h}_n + 4\dot{h}_{n+1} + \dot{h}_{n+2} + 2nh_n \\ &\quad + (-16p+8)h_{n+1} + (-2n-4p-4)h_{n+2}, \end{aligned} \tag{6.41}$$

$$v_n = M_{n,n-1} = -2\dot{h}_n - 2\dot{h}_{n+1} + (-2n+4p-2)h_n + (2n+8p)h_{n+1}, \tag{6.42}$$

$$w_n = M_{n,n-2} = \dot{h}_n - 2ph_n. \tag{6.43}$$

By (6.39) and (6.40) we get

$$\begin{aligned} u_n &= nr^{2n}e^{-nt}(1+O(r)), \\ v_n &= O(nr^{2n}e^{-nt}), \\ w_n &= O(nr^{2n}e^{-nt}), \end{aligned}$$

where the constants in $O(\ )$ only depend on p. Consider the matrix $\tilde{M}$ gotten by multiplying M with the diagonal matrix $\operatorname{diag}(e^{t/2}r^{-1}, e^t r^{-2}, e^{3t/2}r^{-3}, \dots, e^{N/2}r^{-N})$ from both sides:

$$\tilde{M} = \begin{pmatrix} \tilde{u}_1 & \tilde{v}_2 & \tilde{w}_3 & 0 & 0 & 0 & \dots & 0 \\ \tilde{v}_2 & \tilde{u}_2 & \tilde{v}_3 & \tilde{w}_4 & 0 & 0 & \dots & 0 \\ \tilde{w}_3 & \tilde{v}_3 & \tilde{u}_3 & \tilde{v}_4 & \tilde{w}_5 & 0 & \dots & 0 \\ 0 & \tilde{w}_4 & \tilde{v}_4 & \tilde{u}_4 & \tilde{v}_5 & \tilde{w}_6 & \dots & 0 \\ 0 & 0 & \tilde{w}_5 & \tilde{v}_5 & \tilde{u}_5 & \tilde{v}_6 & \dots & 0 \\ 0 & 0 & 0 & \tilde{w}_6 & \tilde{v}_6 & \tilde{u}_6 & \dots & \tilde{w}_N \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \tilde{v}_N \\ 0 & 0 & 0 & 0 & 0 & \tilde{w}_N & \tilde{v}_N & \tilde{u}_N \end{pmatrix}$$

where

$$\tilde{u}_n = n(1+O(r)), \quad \tilde{v}_n = O(n\sqrt{r}), \quad \tilde{w}_n = O(nr).$$

If r is sufficiently small (depending on p), the matrix $\tilde{M}$ is diagonally dominant in the sense that

$$\sum_{\substack{j=1 \\ j\neq n}}^{N} |\tilde{M}_{nj}| < \tilde{M}_{nn} \qquad \text{for } n = 1, 2, \dots, N.$$

By Gerschgorin's theorem [**Lan69**, p. 226], this implies that all eigenvalues of $\tilde{M}$ are positive, and so M is positive definite. This completes the proof of Theorem 6.1.

4. Coefficients of $(f')^p$

In this section we apply de Branges' method to the conjecture (5.39). That is, we let $G(f) = (f')^p$, where $p \le -2$, and we want to show that $|c_N(G(f))| \le |c_N(G(k))|$ for $f \in S$. We will see that the method works if and only if $N \le 2|p|+1$. We get the following theorem.

Theorem 6.2. *Let $p < 0$ and $1 \le N \le 2|p| + 1$. If $f \in S$ then*

$$|c_N((f')^p)| \le |c_N((k')^p)|.$$

Moroever, except for the case $N = 2$, $p = -\frac{1}{2}$, we have equality only when $f(z)$ is a Koebe function $z(1 + \lambda z)^{-2}$, where $|\lambda| = 1$.

Before giving the proof, let us note the special cases $N = 1, 2, 3$ are known before. Namely, writing $a_n = c_n(f)$ we have

$$c_1((f')^p) = 2pa_2,$$

$$c_2((f')^p) = -3p(a_2^2 - a_3) + 2p(p + \frac{1}{2})a_2^2,$$

$$c_3((f')^p) = -4p(a_4 - 3a_2a_3 + 2a_2^3) + 6p(p + 1)a_2(a_2^2 - a_3) - \frac{4}{3}p(p + \frac{1}{2})(p + 1)a_2^3.$$

Thus the case $N = 1$ is Bieberbach's inequality $|a_2| \le 2$. The case $N = 2, p = -\frac{1}{2}$ is the elementary estimate $|a_2^2 - a_3| \le 1$ which follows from the area theorem, see [**Pom75**, Theorem 1.5]. Here the extremal functions are

$$f(z) = \frac{z}{(1 - \lambda z)(1 - \mu z)}, \qquad \text{where } |\lambda| = |\mu| = 1.$$

Concerning this, the author apologizes for the incorrect statement in [**Ber98**]. The case $N = 3$, $p = -1$ is Ozawa's inequality $|a_4 - 3a_2a_3 + 2a_2^3| \le 2$, which was proved in [**Oza64**] using Schiffer's variational method.

As in the previous section, let $\hat{g} = G(k) = (k')^p$ and $C_n = c_n(\hat{g})$. We may assume that $p \le -\frac{1}{2}$. (Otherwise $N = 1$, and in that case the value of p does not matter.) By Lemma 6.6 below, this ensures that $C_n > 0$ for all $n \ge 0$. By continuity, we may for the first part of the theorem assume that $N < 2|p| + 1$. We have the same situation as in the previous section, except that H_n are changed. Now

$$H_n = 1 \qquad \text{for } n = N,$$
$$H_n = 0 \qquad \text{for } n \ne N.$$

The functions $g_t = G(e^{-t} f_t)$ still satisfy

$$\dot{g}_t = 2p\left(\frac{1}{(1 - \omega z)^2} - 1\right) g_t + \frac{1 + \omega z}{1 - \omega z} z g_t', \tag{6.44}$$

so that (6.9) and (6.19) and satisfied with

$$L_{t,1} = 2p\left(\frac{1}{(1 - z)^2} - 1\right) + \frac{1 + z}{1 - z} z \frac{\mathrm{d}}{\mathrm{d}z}, \qquad \varphi_{t,\omega} = 0.$$

We have to solve the equation

$$\dot{y}_t = -L_{t,1}^* y_t \tag{6.45}$$

with the initial condition

$$c_n(y_0) = H_n / C_n, \qquad n \ge 0.$$

Thus it is best to write (6.45) as a system of differential equations for the coefficients $y_n(t) = c_n(y_t)$. To do this, we first calculate the matrix of $L_{t,1}$ in the basis $1, z, z^2, \ldots$. The matrix elements are

$$l_{nj} = c_n(L_{t,1} z^j) = \begin{cases} 2p(n - j + 1) + 2j & \text{for } 0 \le j < n; \\ n & \text{for } 0 \le j = n; \\ 0 & \text{for } 0 \le n < j. \end{cases}$$

The matrix of $L_{t,1}^*$ is the transpose of this matrix. Thus equation (6.45) can be written

$$\dot{y}_n(t) = -ny_n(t) + \sum_{j=n+1}^{\infty} (-2p(j - n + 1) - 2n) y_j(t), \qquad n \ge 0. \tag{6.46}$$

The solution of this system with initial conditions $y_n(0) = H_n/C_n$ is clearly of the type

$$y_n(t) = \sum_{j=n}^{N} a_{nj} e^{-jt}, \qquad n = 0, 1, 2, \ldots, N, \tag{6.47}$$

$$y_n(t) = 0, \qquad n > N. \tag{6.48}$$

Equations (6.46) can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}(y_n(t)e^{nt}) = e^{nt} \sum_{j=n+1}^{N} (-2p(j-n+1) - 2n) y_j(t), \qquad 0 \le n \le N, \tag{6.49}$$

where the coefficient of $y_j(t)$ is positive thanks to the assumption $N < -2p + 1$. Thus a descending induction over n shows that

$$y_n(t) > 0 \qquad \text{for } t > 0,\ n = 0, 1, \ldots, N. \tag{6.50}$$

Hence we get by (6.47) and (6.49)

$$a_{nn} = y_n(0) + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}(y_n(t)e^{nt})\,\mathrm{d}t > 0 \qquad \text{for } n = 0, 1, \ldots, N. \tag{6.51}$$

The weight functions are $h_n(t) = y_n(t)/C_n$.

It remains to prove that $(P_{t,1}u, u) \ge 0$ for $u \in A(\mathbb{D})$. As in the previous section, we introduce the basis $e_n = z^n(1-z)^2$, $n = 0, 1, 2, \ldots$. Since $P_{t,1}e_n = 0$ for $n > N$ we need only consider $u = \sum_{n=0}^{N} \alpha_n e_n$. We thus have to prove that the Hermitian form

$$\eta_t = (P_{t,1}u, u) = \sum_{n,j=0}^{N} (P_{t,1}e_n, e_j)\alpha_n \overline{\alpha_j} \tag{6.52}$$

is positive semidefinite for $t \ge 0$. As before, we get that the corresponding matrix is five-diagonal, and we can write

$$\eta_t = \sum_{n=0}^{N} u_n |\alpha_n|^2 + v_n 2\,\mathrm{Re}(\alpha_n \overline{\alpha_{n-1}}) + w_n 2\,\mathrm{Re}(\alpha_n \overline{\alpha_{n-2}}), \tag{6.53}$$

where $\alpha_{-1} = \alpha_{-2} = 0$ and

$$\begin{aligned} u_n &= \dot h_n + 4\dot h_{n+1} + \dot h_{n+2} + 2nh_n \\ &\quad + (-16p + 8)h_{n+1} + (-2n - 4p - 4)h_{n+2}, \end{aligned} \tag{6.54}$$

$$v_n = -2\dot h_n - 2\dot h_{n+1} + (-2n + 4p - 2)h_n + (2n + 8p)h_{n+1}, \tag{6.55}$$

$$w_n = \dot h_n - 2ph_n. \tag{6.56}$$

The Hermitian form η_t is singular. To see this, let A_n be defined by

$$\hat g(z)(1-z)^{-2} = \sum_{n=0}^{\infty} A_n z^n,$$

so that

$$\hat g = \sum_{n=0}^{\infty} A_n e_n.$$

Since $P_{t,1}\hat g = 0$ by (6.22), we have

$$\eta_t = 0 \quad \text{if } \alpha_n = A_n \text{ for } n = 0, 1, \ldots, N. \tag{6.57}$$

We want to complete the squares in (6.53) and write

$$\eta_t = \sum_{n=0}^{N} p_n(t) |\alpha_n + q_n(t)\alpha_{n-1} + r_n(t)\alpha_{n-2}|^2. \tag{6.58}$$

First, we prove that this is possible if t is large. Using (6.47) we get that u_n, v_n and w_n are polynomials in e^{-t}:

$$u_n = n\frac{a_{nn}}{C_n}e^{-nt} + \ldots,$$
$$v_n = (4p-2)\frac{a_{nn}}{C_n}e^{-nt} + \ldots,$$
$$w_n = (-n-2p)\frac{a_{nn}}{C_n}e^{-nt} + \ldots,$$

where the omitted terms are of smaller order as $t \to +\infty$. This shows that for large t we can successively complete squares in (6.53), beginning with terms containing α_N. In each step of this process the coefficient of terms of type $|\alpha_n|^2$ will have leading term $n\frac{a_{nn}}{C_n}e^{-nt}$. Thus we get (6.58) with $p_n(t) > 0$ for $n = 1, 2, \ldots, N$, for large t. By (6.57) we must have $p_0(t) = 0$ for large t.

The functions $p_n(t), q_n(t), r_n(t)$, now defined for large t, are rational functions of e^{-t}. We now want to prove that these functions have no poles for $0 < e^{-t} \leq 1$, and that $p_n(t) > 0$ for $t \geq 0$, $n = 1, 2, \ldots, N$. Thanks to (6.57), we can derive a one-step recursion formula for $p_n(t)$. Identifying coefficients in (6.53) and (6.58) we get, for large t,

$$u_n = p_n + p_{n+1}q_{n+1}^2 + p_{n+2}r_{n+2}^2, \quad 0 \leq n \leq N \tag{6.59}$$
$$v_n = p_n q_n + p_{n+1}q_{n+1}r_{n+1}, \quad 1 \leq n \leq N \tag{6.60}$$
$$w_n = p_n r_n, \quad 2 \leq n \leq N \tag{6.61}$$

where $p_{N+1} = p_{N+2} = q_{N+1} = r_{N+1} = r_{N+2} = 0$.
By (6.53) and (6.57) we get, for large t,

$$A_n + q_n A_{n-1} + r_n A_{n-2} = 0, \qquad 1 \leq n \leq N. \tag{6.62}$$

Solve (6.62) for q_n, plug this into (6.60) and use (6.61). This gives the recursion formula

$$p_n = -\frac{A_{n-2}}{A_n}w_n - \frac{A_{n-1}A_{n+1}}{A_n^2}w_{n+1} - \frac{A_{n-1}}{A_n}v_n - \frac{A_{n-1}^2}{A_n^2}\frac{w_{n+1}^2}{p_{n+1}} \tag{6.63}$$

for $1 \leq n \leq N-1$, still only for large t. In the next section we use this formula to prove the following lemma.

Lemma 6.3. *The meromorphic functions $p_n(t)$ are analytic for $t \geq 0$, and for $t \geq 0$ we have*

$$p_N = Nh_N > 0, \qquad p_0 = 0, \tag{6.64}$$
$$p_n > \frac{A_{n-2}}{A_n}w_n + \frac{A_{n-1}}{A_n}2(n+1)h_n \geq 0 \qquad \text{for } n = 1, 2, \ldots, N-1. \tag{6.65}$$

It follows from this lemma and the equations (6.61) and (6.62) that the functions r_n and q_n are also analytic for $t \geq 0$. Hence (6.58) holds for $t \geq 0$, and thus $\eta_t \geq 0$.

Remark: If $N > 2|p| + 1$ the proof breaks down: We get $u_{N-1}(t) < 0$ for large t; hence η_t is not positive semi-definite.

To show that only the Koebe functions give equality in Theorem 6.2, we have to turn back to the considerations in Section 6.1. Let $\delta = 2 - |c_2(f_0)|$. Thus

$$|c_1(g_0)| = |c_1((f_0')^p)| = |2pc_2(f_0)| = 4|p| - 2|p|\delta. \tag{6.66}$$

The differential equation (6.44) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}c_1(g_t) = 4p\omega(t) + c_1(g_t).$$

Together with $|c_1(g_t)| \leq C_1 = 4|p|$ this yields

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}c_1(g_t)\right| \leq 8|p|. \tag{6.67}$$

The estimates (6.66) and (6.67) imply

$$|c_1(g_t)| \leq 4|p| - |p|\delta \qquad \text{for } 0 \leq t \leq \frac{\delta}{8}. \tag{6.68}$$

Now (6.11), (6.20), (6.52) and (6.58) give

$$\dot{Q}(t) = (P_{t,\omega} g_t, g_t) = (P_{t,1} U_{\omega^{-1}} g_t, U_{\omega^{-1}} g_t) = \sum_{n=1}^{N} p_n(t)|\alpha_n + q_n(t)\alpha_{n-1} + r_n(t)\alpha_{n-2}|^2,$$

where $U_{\omega^{-1}} g_t = \sum_{n=0}^{\infty} \alpha_n e_n$. Thus $\alpha_0 = 1$ and $c_1(U_{\omega^{-1}} g_t) = -2 + \alpha_1$. Since $q_1(t) = -A_1/A_0 = 4p - 2$ by (6.62), this yields

$$\dot{Q}(t) \geq p_1(t)|\alpha_1 + q_1(t)\alpha_0|^2 = p_1(t)|c_1(U_{\omega^{-1}} g_t) + 4p|^2.$$

Lemma 6.3 and (6.68) now give

$$\dot{Q}(t) \geq \frac{4}{A_1} h_1(t)(|p|\delta)^2 \qquad \text{for } 0 \leq t \leq \frac{\delta}{8}.$$

Thus

$$|c_N(g_0)|^2 = Q(0) \leq Q(T) - \frac{4|p|^2\delta^2}{A_1} \int_0^{\delta/8} h_1(t)\,\mathrm{d}t.$$

Since $Q(T) = C_N^2$, we get by continuity

$$|c_N((f')^p)|^2 \leq C_N^2 - \frac{4|p|^2\delta^2}{A_1} \int_0^{\delta/8} h_1(t)\,\mathrm{d}t,$$

for all $f \in S$ and for $1 \leq N \leq 2|p|+1$, where now $\delta = 2 - |c_2(f)|$. By (6.49) we have $h_1(t) > 0$ for $t > 0$, except when $N = 2$ and $p = -\frac{1}{2}$. Thus, if $|c_N((f')^p| = C_N$, we must have $\delta = 0$, and so f is a Koebe function by Bieberbach's theorem [**Pom75**, Theorem 1.5].

5. The proof of Lemma 6.3

For convenience, introduce the parameter $q = -2p+1 > N$. We need some lemmas concerning the constants C_n, A_n and $B_n = A_n - A_{n-1} = C_0 + \cdots + C_n$.

Lemma 6.4. *The following recursion formulas hold for all integers n:*

$$nC_n = (2q-2)C_{n-1} - (q-n+1)C_{n-2}, \tag{6.69}$$

$$nB_n = (2q-1)B_{n-1} - (q-n)B_{n-2}, \tag{6.70}$$

$$nA_n = 2qA_{n-1} - (q-n-1)A_{n-2}. \tag{6.71}$$

Here, $C_n = B_n = A_n = 0$ for $n < 0$.

Proof. These formulas follow from applying $(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z}$ on the generating functions

$$(1-z)^p(1+z)^{-3p} = \sum_{n=0}^{\infty} C_n z^n,$$

$$(1-z)^{p-1}(1+z)^{-3p} = \sum_{n=0}^{\infty} B_n z^n,$$

$$(1-z)^{p-2}(1+z)^{-3p} = \sum_{n=0}^{\infty} A_n z^n.$$

□

Lemma 6.5. *If $q \geq 3$, then $nC_n > qC_{n-1} > 0$ for $1 \leq n \leq q+1$.*

Proof. The case $n = 1$ is clear. Inductively, assume that $nC_n > qC_{n-1} > 0$, where $1 \leq n \leq q$. Together with equation (6.69) this implies

$$(n+1)C_{n+1} \geq (2q-2)C_n - (q-n)nC_n/q > qC_n.$$

□

Lemma 6.6. *If $p < -\frac{1}{8}$ then $C_n > 0$ for $n \geq 0$.*

Proof. By the previous lemma $C_n > 0$ if $4 \leq n \leq q+1$. Moreover, $C_0 = 1$, $C_1 = 2q-2$, $C_2 = (q-1)(2q-5/2)$ and $C_3 = (q-1)(4q^2-11q+9)/3$ are all positive, since $q > 5/4$. Equation (6.69) now implies that $C_n > 0$ for all $n \geq 0$. □

Lemma 6.7. *If $q \geq 3$, then $C_{n-1}/C_n < C_n/C_{n+1}$ for $0 \leq n \leq q$.*

Proof. The case $n = 0$ is trivial. Inductively, assume that $C_nC_{n-2} < C_{n-1}^2$, where $1 \leq n \leq q$. Equation (6.69) implies

$$(n+1)C_{n+1}C_{n-1} - nC_n^2 = (q+1-n)C_nC_{n-2} - (q-n)C_{n-1}^2.$$

But Lemma 6.5 implies $C_n > C_{n-2}$, so we get

$$(n+1)(C_{n+1}C_{n-1} - C_n^2) < (q-n)(C_nC_{n-2} - C_{n-1}^2) \leq 0.$$

□

Lemma 6.8. *If $q \geq 3$, then $A_{n-1}/A_n < B_{n-1}/B_n$ for $1 \leq n \leq q$.*

Proof. By Lemma 6.7 we have

$$\frac{B_{j-1}}{B_j} = \frac{C_0 + \cdots + C_{j-1}}{C_0 + \cdots + C_j} < \frac{C_0 + \cdots + C_j}{C_0 + \cdots + C_{j+1}} = \frac{B_j}{B_{j+1}} \quad \text{for } 1 \leq j \leq q.$$

Thus

$$\frac{A_{n-1}}{A_n} = \frac{B_0 + \cdots + B_{n-1}}{B_0 + \cdots + B_n} < \frac{B_{n-1}}{B_n} \quad \text{for } 1 \leq n \leq q.$$

□

Lemma 6.9. *If $q \geq 3$ and $1 \leq n < q$, then*

$$\frac{C_nB_{n+1}}{C_{n+1}B_n} < 1 + \frac{1}{q-n}. \tag{6.72}$$

Proof. The case $n = 1$ is easily checked. Inductively, assume that

$$\frac{C_{n-1}B_n}{C_nB_{n-1}} < 1 + \frac{1}{q+1-n},$$

where $2 \leq n < q$. Using $C_n = B_n - B_{n-1}$ and the recursion formula (6.70) we can write this as

$$\begin{aligned}-n(q-n+1)B_n^2 + (2q(q-n) + 2q - 1)B_nB_{n-1}& \\ - (q-n+2)(q-n)B_{n-1}^2 > 0.& \quad (6.73)\end{aligned}$$

We want to prove the inequality (6.72), which in a similar way can be written

$$\begin{aligned}(2q-1-(n+1)(q-n+1))B_n^2 + ((2q-2)(q-n)+1)B_nB_{n-1}& \\ - (q-n-1)(q-n)B_{n-1}^2 > 0.& \quad (6.74)\end{aligned}$$

It suffices to prove that the difference between the left-hand sides of (6.74) and (6.73),

$$\begin{aligned}(q+n-2)B_n^2 + (-4q+2n+2)B_nB_{n-1} + 3(q-n)B_{n-1}^2 =& \\ = (B_n - B_{n-1})((q+n-2)B_n - 3(q-n)B_{n-1})&\end{aligned}$$

is positive. Lemmas 6.7 and 6.5 imply

$$\frac{B_n}{B_{n-1}} = \frac{C_0 + \cdots + C_n}{C_0 + \cdots + C_{n-1}} > \frac{C_n}{C_{n-1}} > \frac{q}{n}.$$

Thus we need only prove that $(q+n-2)q/n - 3(q-n) \geq 0$, which is a simple verification. □

We also need the following facts. By (6.49), (6.50) and $y_N(t) = C_Ne^{-Nt}$ we have

$$w_n = (\dot{h}_n + nh_n) + (-2p-n)h_n > 0 \qquad \text{for } t \geq 0, \text{ if } 0 \leq n \leq N-1. \tag{6.75}$$

Together with (6.50), this proves the second inequality of (6.65). The first equation in (6.64) follows from (6.59) and (6.54):

$$p_N(t) = u_N(t) = \dot{h}_N(t) + 2Nh_N(t) = Ne^{-Nt} = Nh_N(t) > 0. \tag{6.76}$$

By (6.55) and (6.56) we have

$$v_n = -2w_n - 2w_{n+1} - 2(n+1)h_n - 2(q-n-1)h_{n+1}.$$

Thus we can write the recursion formula (6.63) as

$$p_n = \frac{A_{n-1}}{A_n}\left[\left(2 - \frac{A_{n-2}}{A_{n-1}}\right)w_n + \left(2 - \frac{A_{n+1}}{A_n}\right)w_{n+1} + 2(n+1)h_n \right.$$
$$\left. +2(q-n-1)h_{n+1} - \frac{A_{n-1}w_{n+1}^2}{A_n p_{n+1}}\right], \quad 1 \le n \le N-1. \tag{6.77}$$

We prove the first inequality in (6.65) by descending induction over n.

Induction base: For $t \ge 0$ and $2 \le N < q$,

$$p_{N-1} > \frac{A_{N-3}}{A_{N-1}}w_{N-1} + \frac{A_{N-2}}{A_{N-1}}2Nh_{N-1}.$$

Proof. By (6.77) and (6.76), this is equivalent to

$$2\left(1 - \frac{A_{N-3}}{A_{N-2}}\right)w_{N-1} + \left(2 - \frac{A_N}{A_{N-1}}\right)w_N + 2(q-N)h_N - \frac{A_{N-2}}{A_{N-1}}\frac{w_N^2}{Nh_N} > 0. \tag{6.78}$$

By (6.56) and (6.46) we have

$$w_{N-1} = (q-N)(2C_N C_{N-1}^{-1}h_N + h_{N-1}) \quad \text{and} \quad w_N = (q-N-1)h_N,$$

which substituted into (6.78) gives

$$2\left(1 - \frac{A_{N-3}}{A_{N-2}}\right)(q-N)h_{N-1} + \left[2\left(1 - \frac{A_{N-3}}{A_{N-2}}\right)(q-N)\frac{2C_N}{C_{N-1}} + \right.$$
$$\left. + \left(2 - \frac{A_N}{A_{N-1}}\right)(q-N-1) + 2(q-N) - \frac{A_{N-2}}{A_{N-1}}\frac{(q-N-1)^2}{N}\right]h_N > 0.$$

Since $A_{N-3} < A_{N-2}$ and $h_{N-1} \ge 0$, we only have to prove that the coefficient of h_N is positive. Using the recursion formula (6.71) with $n = N$, we can write this coefficient as

$$4(q-N)\left[\left(1 - \frac{A_{N-3}}{A_{N-2}}\right)\frac{C_N}{C_{N-1}} + 1 - \frac{q-1}{2N}\right]. \tag{6.79}$$

It follows from Lemma 6.5 and $C_2/C_1 = q - 5/4$ that

$$C_N/C_{N-1} > (q-1)/(2N).$$

Lemma 6.7 implies that

$$\frac{A_{N-3}}{A_{N-2}} = \frac{C_{N-3} + 2C_{N-4} + \cdots + (N-2)C_0}{C_{N-2} + 2C_{N-3} + \cdots + (N-2)C_1 + (N-1)C_0} < \frac{C_{N-1}}{C_N}.$$

Thus (6.79) is positive. □

Induction step: Assume that $1 \le n \le N-2$, $t \ge 0$ and

$$p_{n+1} > \frac{A_{n-1}}{A_{n+1}}w_{n+1} + \frac{A_n}{A_{n+1}}2(n+2)h_{n+1}.$$

Hence, by (6.75) and (6.50),

$$p_{n+1} > \frac{A_{n-1}}{A_{n+1}}w_{n+1} > 0. \tag{6.80}$$

We want to prove that $p_n > (A_{n-2}/A_n)w_n + (A_{n-1}/A_n)2(n+1)h_n$. By the recursion formula (6.77) and (6.80) it is enough to prove

$$2\left(1-\frac{A_{n-2}}{A_{n-1}}\right)w_n + 2\left(1-\frac{A_{n+1}}{A_n}\right)w_{n+1} + 2(q-n-1)h_{n+1} > 0. \tag{6.81}$$

Using the functions $s_n = \sum_{j=n}^{N}(j-n+1)y_j$ we can write the differential equation (6.46) as

$$\dot{y}_n = -ns_n + 2(q-1)s_{n+1} + (-q+1+n)s_{n+2}. \tag{6.82}$$

Remember that $h_n = y_n/C_n \geq 0$. Substituting this, $y_n = s_n - 2s_{n+1} + s_{n+2}$ and (6.82) into (6.56) we get

$$w_n = C_n^{-1}((q-1-n)s_n + ns_{n+2}).$$

Putting this into (6.81) and using $h_{n+1} \geq 0$, we see that it suffices to prove

$$(q-1-n)s_n + ns_{n+2} - \mu_n(q-2-n)s_{n+1} - \mu_n(n+1)s_{n+3} > 0, \tag{6.83}$$

where

$$\mu_n = \frac{\frac{C_n}{C_{n+1}}\left(\frac{A_{n+1}}{A_n}-1\right)}{1-\frac{A_{n-2}}{A_{n-1}}} = \frac{C_nB_{n+1}A_{n-1}}{C_{n+1}B_{n-1}A_n} > 0.$$

Note that $q > N \geq 3$. From Lemma 6.8 and Lemma 6.9 it follows that

$$\mu_n = \frac{C_nB_{n+1}}{C_{n+1}B_n}\frac{B_nA_{n-1}}{B_{n-1}A_n} < 1 + \frac{1}{q-n}.$$

Hence

$$\frac{s_n}{s_{n+1}} = \frac{y_n + 2y_{n+1} + \cdots + (N-n+1)y_n}{y_{n+1} + \cdots + (N-n)y_n} \geq \frac{N-n+1}{N-n} > \mu_n,$$

and similarly $s_{n+2} > \mu_n s_{n+3}$. Thus

$$(q-1-n)(s_n - \mu_n s_{n+1}) + n(s_{n+2} - \mu_n s_{n+3}) > 0,$$

which implies (6.83), since $s_{n+1} > s_{n+3}$.

CHAPTER 7

Generalized Schwarzian derivatives

In Section 7.1 we show how the coefficient inequalities of Theorem 6.2 lead to estimates for certain generalizations of the Schwarzian derivative. In Section 7.2 we give a simpler proof of these estimates, based on the estimates of Klouth and Wirths [**Klo89**] for Peschl's generalized Schwarzian derivatives. In Section 7.3 we use these estimates to produce new estimates for integral means of $|f'|^p$, where p is a negative integer and f is univalent in the unit disc.

1. Generalized Schwarzians connected with Theorem 6.2

Given a coefficent estimate for functions f univalent in $\mathbb{D}$, one may apply it to the composite function $f \circ \tau$, where τ is a conformal self-map of $\mathbb{D}$:

$$\tau(z) = \frac{z+\zeta}{1+\bar{\zeta}z}, \qquad \text{where } \zeta \in \mathbb{D}. \tag{7.1}$$

This gives an estimate of the type

$$|Tf(\zeta)| \le B(\zeta), \qquad \zeta \in \mathbb{D},$$

where T is a differential operator. This procedure is especially simple for the Schwarzian derivative

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

since it satisfies a "chain rule"

$$S(f\circ\tau) = ((Sf)\circ\tau)\,(\tau')^2 \quad \text{if } \tau \text{ is a Möbius transformation.}$$

Using this, it is easy to see that the well-known estimate

$$|a_2^2 - a_3| \le 1, \qquad f \in S$$

gives the estimate [**Dur83**, p. 263]

$$|Sf(\zeta)| \le \frac{6}{1-|\zeta|^2}, \qquad \zeta \in \mathbb{D} \tag{7.2}$$

for univalent functions $f : \mathbb{D} \to \mathbb{C}$.

Since $c_2((f')^{-1/2}) = \frac{3}{2}(a_2^2 - a_3)$, one might ask if a similar chain rule can be used for the more general estimates of Theorem 6.2:

$$|c_n((f')^p)| \le |c_n((k')^p)|, \qquad p \le -\frac{n-1}{2},\ f \in S. \tag{7.3}$$

The answer is yes, if $p = -\frac{n-1}{2}$. Consider the differential operator

$$S_n f = (f')^{\frac{n-1}{2}} D^n (f')^{-\frac{n-1}{2}},$$

where we choose the same branch of $\sqrt{f'}$ at both occurences. (D is the differentiation operator.) We will see that S_n satisfies

$$S_n(f\circ\tau) = ((S_n f)\circ\tau)\,(\tau')^n \quad \text{if } \tau \text{ is a Möbius transformation.} \tag{7.4}$$

For this reason we call S_n a generalized Schwarzian derivative. Note that $S_2 = -S/2$. The case $p = -\frac{n-1}{2}$ of the estimate (7.3) can be written

$$|S_n f(0)| \le |S_n k(0)|.$$

This holds for all univalent $f : \mathbb{D} \to \mathbb{C}$ since S_n is homogeneous (that is, $S_n(cf) = S_n f$ if c is a constant). Applying this to the function $f \circ \tau$, where τ is given by (7.1) and using (7.4) we get

$$|S_n f(\zeta)\tau'(0)^n| \le |S_n k(0)|.$$

Here $\tau'(0) = 1 - |\zeta|^2$ and $S_n k(0)$ can be computed in the following way: Let $\varphi(z) = z^2/2$. Then

$$S_n\varphi(1) = \left(-\frac{n-1}{2}\right)\left(-\frac{n-1}{2} - 1\right)\dots\left(-\frac{n-1}{2} - (n-1)\right)$$

and (7.4) with $\tau(z) = \frac{1-z}{1+z}$ gives

$$S_n k(0) = S_n(\varphi \circ \tau)(0) = S_n\varphi(1)(-2)^n = (n-1)(n+1)(n+3)\dots(3n-3).$$

We have proved the following generalizations of the estimate (7.2):

Theorem 7.1. *For univalent functions $f : \mathbb{D} \to \mathbb{C}$ we have the sharp estimate*

$$|S_n f(z)| = \left| f'(z)^{(n-1)/2} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n f'(z)^{-(n-1)/2} \right| \le \frac{K_n}{(1-|z|^2)^n},$$

where $K_n = (n-1)(n+1)(n+3)\dots(3n-3)$ and n is a positive integer.

To prove the invariance property (7.4) we use the following lemma, which can be found in [**GuPe89b**] and [**Bol49**].

Lemma 7.2. *Let τ be a Möbius transformation. If two analytic functions are related by $\tilde{g} = (g \circ \tau)\,(\tau')^{-(n-1)/2}$, then their nth derivatives have the relation $\tilde{g}^{(n)} = (g^{(n)} \circ \tau)\,(\tau')^{(n+1)/2}$.*

Proof. Since τ is a composition of transformations of the types $z \mapsto z + a$, $z \mapsto bz$ and $z \mapsto 1/z$, it suffices to prove the lemma when τ is one of these. The first two cases are rather trivial. In the third case $\tau(z) = 1/z$, we can by continuity and linearity assume that $g(z) = z^j$, and then an easy calculation proves the lemma. □

Now let $g = (f')^{-(n-1)/2}$ and $\tilde{g} = ((f \circ \tau)')^{-(n-1)/2}$. Since $\tilde{g} = (g \circ \tau)\,(\tau')^{-(n-1)/2}$, the lemma shows that

$$S_n(f \circ \tau) = \frac{\tilde{g}^{(n)}}{\tilde{g}} = \frac{g^{(n)} \circ \tau}{g \circ \tau}(\tau')^n = ((S_n f) \circ \tau)\,(\tau')^n,$$

if τ is a Möbius transformation.

Remarks. The operators S_n appear in [**GuPe89a**, Example 1, $\mu = 1$]. In Section 3 of the same paper the authors study the linear space Σ_n of all homogeneous differential operators of the form

$$Tf = \frac{\text{Polynomial in } f', f'', f''', \dots}{(f')^m}$$

that satisfy the "chain rule"

$$T(f \circ \tau) = ((Tf) \circ \tau)\,(\tau')^n \quad \text{if } \tau \text{ is a Möbius transformation.}$$

(Actually they consider more general spaces. Our Σ_n corresponds to their $\mathcal{M}_\lambda$ with $\mu = 1$.) The authors prove that the operators $T \in \Sigma_n$ can be generated in the following way. Let $P(x_1, \dots, x_n) = \sum a_{k_1,\dots,k_n} x_1^{k_1} \dots x_n^{k_n}$ be a homogeneous polynomial of degree n that satisfies

$$P(x_1 + 1, \dots, x_n + 1) = P(x_1, \dots, x_n).$$

Let

$$Tf = (f')^{-n} \sum a_{k_1,\dots,k_n} \frac{f^{(k_1+1)}}{(k_1+1)!} \cdots \frac{f^{(k_n+1)}}{(k_n+1)!}.$$

This description of Σ_n implies that its dimension $p(n) - p(n-1)$, where $p(n)$ is the number of partitions of n, see the following table.

n	1	2	3	4	5	6	7	8	9	10
$p(n)$	1	2	3	5	7	11	15	22	30	42
$\dim \Sigma_n$	0	1	1	2	2	4	4	7	8	12

Moreover, it is easy to see that if $T \in \Sigma_n$, then Tf is a polynomial in $S_2 f$, $S_3 f$, $\ldots$,$S_n f$.

2. Peschl's generalized Schwarzians

Another starting point for generalization of the Schwarzian derivative is the invariance property

$$S(\tau \circ F) = SF,$$

where τ is a Möbius transformation. It follows that the operators $Q_n = D^{n-2} S$ also satisfy $Q_n(\tau \circ F) = Q_n F$. Other higher-order differential operators with this property were constructed in [**Aha69**] and [**Tam96**]. Let $\tilde{Q}_n(f) = Q_n(f^{-1}) \circ f$. We get the invariance property

$$\tilde{Q}_n(f \circ \tau) = Q_n(\tau^{-1} \circ f^{-1}) \circ f \circ \tau = Q_n(f^{-1}) \circ f \circ \tau = (\tilde{Q}_n f) \circ \tau.$$

Also, if c is a constant,

$$\tilde{Q}_n(cf) = Q_n(f^{-1}(c^{-1} \cdot)) \circ (cf) = (c^{-1})^n Q_n(f^{-1}) \circ f = c^{-n} \tilde{Q}_n f.$$

Thus the operators

$$P_n f = (f')^n \tilde{Q}_n f$$

are homogeneous and satisfy

$$P_n(f \circ \tau) = ((P_n f) \circ \tau)\, (\tau')^n \quad \text{if } \tau \text{ is a Möbius transformation.} \tag{7.5}$$

In other words, $P_n \in \Sigma_n$. The differential operators P_n where introduced by Peschl [**Pes74**]. In [**KlWi80**] Klouth and Wirths proved the coefficient estimate

$$|c_n(S(f^{-1}))| \leq c_n(S(k^{-1})), \qquad n \geq 0,\ f \in S. \tag{7.6}$$

The proof is similar to Löwner's proof of his estimates for coefficients of f^{-1}. The estimates (7.6) can be written

$$|P_n f(0)| \leq P_n k(0), \qquad n \geq 2, \tag{7.7}$$

which holds for any univalent $f : \mathbb{D} \to \mathbb{C}$, since P_n is homogeneous. As in the previous section, this implies

$$|P_n f(\zeta)| \leq \frac{P_n k(0)}{(1 - |\zeta|^2)^n}, \qquad \zeta \in \mathbb{D}.$$

See [**Klo89**] for related material.

We now examine the relation between the two sets of generalized Schwarzians S_n and P_n that we have discussed.

Theorem 7.3. *$S_n f = \Phi(P_2 f, P_3 f, \ldots, P_n f)$, where Φ is a polynomial with positive coefficients.*

This shows that our estimate $|S_n f(0)| \leq S_n k(0)$ is a consequence the estimates (7.7) of Klouth and Wirths. To prove Theorem 7.3 we use the following lemma.

Lemma 7.4. *If H and f are analytic functions with $f' \neq 0$, then*

$$(f')^{-\frac{n+1}{2}} D^n \left((f')^{-\frac{n-1}{2}} H \circ f \right) = \sum_{j=0}^{n} p_{n,j} H^{(j)} \circ f, \tag{7.8}$$

where $p_{n,j}$ is a polynomial in $\tilde{Q}_2 f$, $\ldots$,$\tilde{Q}_n f$ with positive coefficients.

Proof. In terms of $F = f^{-1}$, equation (7.8) can be written

$$(F')^{\frac{n+1}{2}} \left(\frac{1}{F'}D\right)^n \left((F')^{\frac{n-1}{2}} H\right) = \sum_{j=0}^{n} \tilde{p}_{n,j} H^{(j)}, \tag{7.9}$$

where $\tilde{p}_{n,j}$ is a polynomial in Q_2F, ... ,Q_nF with positive coefficients. Denote the left-hand side of (7.8) with $L_{n,F}H$. A computation shows that the linear differential operator $L_{n,F}$ satisfies

$$L_{n+2,F}H = \left(-\frac{n+1}{2}N_F + D\right) L_{n,F} \left(\frac{n+1}{2}N_F + D\right) H, \tag{7.10}$$

where $N_F = F''/F'$. We use this to prove (7.9) by induction over n. First note that (7.9) is satisfied for $n = 0$ and $n = 1$. Assume that (7.9) holds for a specific $n \geq 0$. By (7.10),

$$L_{n+2,F}H = \left(-\frac{n+1}{2}N_F + D\right) \sum_{j=0}^{n} \tilde{p}_{n,j} D^j \left(\frac{n+1}{2}N_F + D\right) H.$$

Using $DN_F = SF + \frac{1}{2}N_F^2$, this can be written

$$L_{n+2,F}H = N_F\phi + \sum_{j=0}^{n+2} \tilde{p}_{n+2,j} H^{(j)}, \tag{7.11}$$

where $\tilde{p}_{n+2,j}$ is a polynomial in $Q_2F, \dots, Q_{n+2}F$ with positive coefficients, and ϕ is a polynomial in $N_F, Q_2F, \dots, Q_{n+1}F, H, H', \dots, H^{(n+1)}$. We now use that $L_{n,F}$ has the following invariance property:

$$L_{n,\tau\circ F}H = L_{n,F}H \quad \text{if } \tau \text{ is a Möbius transformation.} \tag{7.12}$$

Together with $Q_j(\tau \circ F) = Q_jF$ and

$$N_{\tau\circ F} = N_F + (N_\tau \circ F)F'$$

this shows that the term $N_F\phi$ in (7.11) must vanish. The induction step is completed.

To prove (7.12), we write it in terms of $f = F^{-1}$ and $\sigma = \tau^{-1}$:

$$\left(((f\circ\sigma)')^{-\frac{n+1}{2}} D^n \left(((f\circ\sigma)')^{-\frac{n-1}{2}} H \circ f \circ \sigma\right)\right) \circ \tau \circ F =$$
$$= \left((f')^{-\frac{n+1}{2}} D^n \left((f')^{-\frac{n-1}{2}} H \circ f\right)\right) \circ F,$$

which simplifies to

$$(\sigma')^{-\frac{n+1}{2}} D^n \left(((f')^{-\frac{n-1}{2}} H \circ f) \circ \sigma \cdot (\sigma')^{-\frac{n-1}{2}}\right) = \left(D^n((f')^{-\frac{n-1}{2}} H \circ f)\right) \circ \sigma.$$

But this follows from Lemma 7.2. ☐

Proof of Theorem 7.3. The case $H = 1$ of the lemma gives

$$(f')^{-n} S_n f = \Phi\left((f')^{-2}P_2f, (f')^{-3}P_3f, \dots, (f')^{-n}P_nf\right),$$

where Φ is a polynomial with positive coefficents. Since S_nf and P_jf are homogenous, the powers of f' must cancel. ☐

3. Estimates for integral means

Pommerenke [**Pom92**, Theorem 8.5] used the estimate (7.2) for the Schwarzian derivative to deduce the estimate

$$\int_0^{2\pi} |f'(re^{i\theta})|^{-1}\, d\theta = O((1-r)^{-0.601}) \tag{7.13}$$

for univalent functions $f : \mathbb{D} \to \mathbb{C}$. We use the same method to deduce from Theorem 7.1 the following estimates for integral means:

Theorem 7.5. *Let E_n be the positive root of*

$$E(E+1)(E+2)\dots(E+2n-1) = K_n^2,$$

where $K_n = (n-1)(n+1)(n+3)\dots(3n-3)$, and $n > 1$ is an integer. If $f : \mathbb{D} \to \mathbb{C}$ is a univalent function, then

$$\int_0^{2\pi} |f'(re^{i\theta})|^{-n+1}\, \mathrm{d}\theta = O((1-r)^{-E_n-\epsilon}) \qquad \text{for all } \epsilon > 0.$$

In particular,

$$\int_0^{2\pi} |f'(re^{i\theta})|^{-2}\, \mathrm{d}\theta = O((1-r)^{-1.547}),$$

$$\int_0^{2\pi} |f'(re^{i\theta})|^{-3}\, \mathrm{d}\theta = O((1-r)^{-2.530}).$$

These estimates are small improvements of the known estimates

$$\int_0^{2\pi} |f'(re^{i\theta})|^{p}\, \mathrm{d}\theta = O\left(\left(\frac{1}{1-r}\right)^{|p|-0.399}\right) \qquad \text{for } p \le -1,$$

which follow from (7.13) and the elementary estimate $|f'(z)| > \frac{1}{8}|f'(0)|(1-|z|)$. This leads to an improvement of the best exponent in Brennan's problem (3.1). Namely, $B(-1) \le 0.601$, $B(-2) \le 1.547$ and convexity of B gives

$$B(t_0) \le 1, \qquad \text{where } t_0 = -1 - \frac{1-0.601}{1.547-0.601} < -1.421.$$

By Proposition 3.4 this shows that (3.1) holds for $4/3 < q < 3.421$.

Note that $E_n = n - \frac{3}{2} + o(1)$ as $n \to +\infty$. Thus the estimate in Theorem 7.5 is asymptotically worse that the the estimate (1.7) of Carleson and Makarov.

To prove Theorem 7.5 we need the following lemma.

Lemma 7.6. *If g is analytic in the unit disc and $m(r) = \int_0^{2\pi} |g(re^{i\theta})|^2\, d\theta$, then*

$$m^{(2n)}(r) \le 4^n \int_0^{2\pi} |g^{(n)}(re^{i\theta})|^2\, \mathrm{d}\theta.$$

Proof. Writing $g(z) = \sum_{k=0}^{\infty} b_k z^k$, we get $m(r) = 2\pi \sum_{k=0}^{\infty} |b_k|^2 r^{2k}$. The lemma is evident from a comparison of coefficients in

$$m^{(2n)}(r) = 2\pi \sum_{k=n}^{\infty} |b_k|^2 2k(2k-1)\dots(2k-2n+1) r^{2k-2n}$$

and

$$\int_0^{2\pi} |g^{(n)}(re^{i\theta})|^2\, \mathrm{d}\theta = 2\pi \sum_{k=n}^{\infty} |b_k k(k-1)\dots(k-n+1)|^2 r^{2k-2n}. \qquad \square$$

Proof of Theorem 7.5. Let $n > 1$ be an integer and let $f : \mathbb{D} \to \mathbb{C}$ be a univalent function. Using Lemma 7.6 with $g = (f')^{-(n-1)/2}$ we get

$$m^{(2n)}(r) \le 4^n \int_0^{2\pi} |g(re^{i\theta}) S_n f(re^{i\theta})|^2\, \mathrm{d}\theta.$$

Theorem 7.1 now gives the differential inequality

$$m^{(2n)}(r) \le 4^n \left(\frac{K_n}{(1-r^2)^n}\right)^2 \int_0^{2\pi} |g(re^{i\theta})|^2\, \mathrm{d}\theta \le \left(\frac{2}{1+r_0}\right)^{2n} \frac{K_n^2}{(1-r)^{2n}} m(r)$$

for $r_0 \le r < 1$. The corresponding differential equation

$$\tilde{m}^{(2n)}(r) = \left(\frac{2}{1+r_0}\right)^{2n} \frac{K_n^2}{(1-r)^{2n}} \tilde{m}(r)$$

has solutions $\tilde{m}(r) = C(1-r)^{-E(r_0)}$, where $E(r_0)$ is the positive solution of

$$E(E+1)\dots(E+2n-1) = \left(\frac{2}{1+r_0}\right)^{2n} K_n^2.$$

Choosing C large enough, we get

$$m^{(k)}(r_0) < \tilde{m}^{(k)}(r_0), \qquad k = 0, 1, \dots, 2n-1,$$

and so Proposition 8.7 of [**Pom92**] gives

$$m(r) \le \tilde{m}(r) \qquad \text{for } r_0 \le r < 1. \tag{7.14}$$

Another proof of (7.14): The function $\Delta(r) = m(r) - \tilde{m}(r)$ satisfies

$$\Delta^{(2n)}(r) \le \left(\frac{2}{1+r_0}\right)^{2n} \frac{K_n^2}{(1-r)^n} \Delta(r), \qquad r_0 \le r < 1 \tag{7.15}$$

and $\Delta^{(k)}(r_0) < 0$ for $k = 0, 1, \dots, 2n$. Let $r_1 \le 1$ be the largest number such that $\Delta^{(2n)}(r) < 0$ for $r_0 \le r < r_1$. If $r_1 < 1$, then $\Delta^{(2n)}(r_1) = 0$ and $\Delta(r_1) < 0$, which contradicts (7.15). Thus $r_1 = 1$, and (7.14) follows.

We thus have

$$m(r) = \int_0^{2\pi} |f'(re^{i\theta})|^{-n+1}\, d\theta = O\left((1-r)^{-E(r_0)}\right).$$

Since $E(r_0) \to E_n$ as $r_0 \to 1$, Theorem 7.5 is proved. □

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