\textbf{RT}-symmetric Laplace operators on star graphs: real spectrum and self-adjointness

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Abstract. In the current article it is analyzed how ideas of $\mathcal{PT}$-symmetric quantum mechanics can be applied to quantum graphs, in particular to the star graph. The class of rotationally-symmetric vertex conditions is analyzed. It is shown that all such conditions can effectively be described by circulant matrices: real in the case of odd number of edges and complex having particular block structure in the even case. Spectral properties of the corresponding operators are discussed.

1. Introduction

Writing this article we got inspiration from two rapidly developing areas of modern mathematical physics: $\mathcal{PT}$-symmetric quantum mechanics and quantum graphs. Both areas attract interest of both mathematicians and physicists for the last two decades with numerous conferences organized and articles published. The first area grew up from the simple observation that a quantum mechanical Hamiltonian “often” has real spectrum even if it possesses combined parity and time-reversal symmetry instead of self-adjointness [11, 12, 10, 13, 22, 26]. Considering such operators increases the set of physical phenomena that could be modeled and raises up new interesting mathematical questions [6, 14, 15, 16]. It appears that spectral theory of such $\mathcal{PT}$-symmetric operators (see precise definition below) can be well-understood using framework of self-adjoint operators in Krein spaces [21].

The theory of quantum graphs - differential operators on metric graphs - can be used to model quantum or acoustic systems where motion is confined to a neighborhood of a set of (one-dimensional) intervals [8, 24, 20]. Until now quantum graphs were mostly studied in the context of self-adjoint or dissipative operators. Our key idea is to look at differential operators on metric graphs under more general symmetry assumptions reminding those in $\mathcal{PT}$-symmetric theory. Surprisingly spectral properties of operators on graphs with symmetries have not been payed much attention. We mention here just two papers [9] and [7], where symmetries of graphs were used to

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construct counterexamples showing that inverse problems are not necessarily uniquely solvable.

In quantum graphs motion along the edges is described by ordinary differential equations, which are coupled together by certain vertex conditions connecting together values of the functions at the end points of the intervals building the underlined metric graph. The role of vertex conditions is two-fold: to describe how the waves are penetrating through the vertices and to make the differential operator self-adjoint. If the requirement of self-adjointness is waved then such conditions should instead ensure that the resolvent set is not empty, i.e. the resolvent for the corresponding differential operator exists for some $\lambda$. The later condition is not very precise and one of the goals of the current article is to understand it in the case of the simplest merit graph with symmetries - the star graph. It can be considered as a building block to define differential operators on arbitrary metric graphs. To avoid discussing properties of the differential operator we limit our studies to the Laplace operator. Moreover the graph formed by semi-infinite edges is considered in order to avoid influence from the peripherique vertices. We just mention here that star graphs formed by finite edges but with standard conditions (see (2.2)) at the central vertex were considered recently in the framework of $\mathcal{PT}$-symmetry [27, 28, 29]. Our focus will be precisely on the vertex conditions at the central vertex. We are not interested in generalizing formalism of quantum graphs for the sake of generalization, but we look for new spectral phenomena that can be observed.

Current paper grew up from the Master Thesis of Maria Astudillo written in 2008 [5] and already attracted attention of mathematical physics community (see references in [18]). Current paper is just the first step to understand spectral structure of differential operators on graphs with nonstandard symmetries.

The paper is organized as follows: in the first two sections we introduce basic notations and discuss how to generalize the notion of $\mathcal{PT}$-symmetry for the case of star graph. The main difficulty is that the notion of $\mathcal{PT}$-symmetry may be generalized in two different ways and we decided to follow the definition giving new spectral structure (Section 3). We call corresponding operators $\mathcal{RT}$-symmetric. All possible Robin conditions leading to $\mathcal{RT}$-symmetric Laplacians are described in Section 4. It appear that the structure of matrices $A$ in the Robin condition depends on whether the number of edges is odd or even. In the first case the matrices are real circulant, while in the second case they may be complex, but are block-circulant. Possibility to obtain non-self-adjoint operators with $N$ real eigenvalues is studied in the last Section.
2. Notations and Elementary Properties

Our goal is to generalize ideas originated from $\mathcal{PT}$-symmetry for the case of the star graph. More precisely we shall confine our studies to the case of the Laplace operator with the domain given by generalized Robin conditions at the central vertex. Consider the star graph $\Gamma_N$ formed by $N$ semi-infinite edges $E_n = [0, \infty)$ joined together at one central vertex. The corresponding Hilbert space $L_2(\Gamma_N)$ can be identified with the space of vector valued functions $u \equiv \vec{u}(x)$ on $x \in [0, \infty)$ with the values in $\mathbb{C}^N$: $L_2(\Gamma_N) = L_2([0, \infty), \mathbb{C}^N)$.

**Definition 2.1.** The operator $L_A = -\frac{d^2}{dx^2}$ is defined on the set of functions from the Sobolev space $W^2_2([0, \infty); \mathbb{C}^N)$ satisfying generalized Robin conditions

\[(2.1)\]
\[\vec{u}'(0) = A\vec{u}(0),\]

where $A$ is a certain $N \times N$ matrix.

The operator adjoint to $L_A$ is again the Laplace operator but is defined by vertex conditions (2.1) with the matrix $A$ substituted by the matrix $A^*$ i.e. the operator $L_{A^*}$

\[(2.3)\]
\[(L_A)^* = L_{A^*}.\]

This can be proven by integration by parts for $\vec{u} \in \text{Dom}(L_A), \vec{v} \in \text{Dom}(L_{A^*})$:

\[
\langle L_A \vec{u}, \vec{v} \rangle_{L_2(\Gamma_N)} = \int_0^\infty \langle -\vec{u}''(x), \vec{v}(x) \rangle_{\mathbb{C}^N} dx
\]
\[= \langle \vec{u}'(0), \vec{v}(0) \rangle_{\mathbb{C}^N} - \langle \vec{u}(0), \vec{v}'(0) \rangle_{\mathbb{C}^N} + \int_0^\infty \langle \vec{u}(x), -\vec{v}''(x) \rangle_{\mathbb{C}^N} dx
\]
\[= \langle \vec{u}(0), A^*\vec{v}(0) - \vec{v}'(0) \rangle_{\mathbb{C}^N} + \int_0^\infty \langle \vec{u}(x), -\vec{v}''(x) \rangle_{\mathbb{C}^N} dx,
\]
where we used that $\vec{u}$ satisfies (2.1). This formula defines a bounded functional with respect to $\vec{u}$ if and only if $A^* \vec{v}(0) - \vec{v}'(0) = 0$ and $v \in W^2_2([0, \infty); \mathbb{C}^N)$.

It follows that the operator $L_A$ is self-adjoint if and only if $A$ is a Hermitian matrix $A^* = A$. In this paper we are not interested in the case where $L_A$ is self-adjoint.

The spectrum of the operator $L_A$ may contain up to $N$ isolated eigenvalues. The corresponding eigenfunction is a solution to the differential equation

$$-u''(x) = \lambda u(x)$$

satisfying Robin conditions (2.1). Any square integrable solution to the differential equation is given by

$$\vec{u}(x) = \vec{a} \exp(ikx), \quad k^2 = \lambda, \quad \vec{a} \in \mathbb{C}^N$$

with $\Im k > 0$. This function satisfies Robin condition if and only if

$$\det(A - ik) = 0.$$ 

The last equation has at most $N$ distinct solutions (in the correct half-plane). Observe that not all solutions lead to eigenfunctions, since one needs to meet the condition $\Im k > 0$.

As the theory of $\mathcal{PT}$-symmetric operators indicates the most interesting case is when the spectrum of the operator is pure real, but the operator itself is not self-adjoint. We are going to look closer at such operators. If the operator $L_A$ has $N$ real eigenvalues, then the matrix $A$ has $N$ negative eigenvalues, but it does not imply that it is Hermitian.

Note that possible vertex conditions are not limited to those described by (2.1). More generally one may consider the Laplace operator $L_{B,C} = -\frac{d^2}{dx^2}$ defined on the domain of functions from $W^2_2([0, \infty); \mathbb{C}^N)$ satisfying the following vertex conditions

$$B \vec{\mu}'(0) + C \vec{\mu}(0) = 0,$$

where $B, C$ are certain $N \times N$ matrices, such that $\text{rank} (B, C) = N$. Here we follow ideas from [19]. The following theorem shows that all Laplace operators described by the vertex condition of the form (2.7) and having $N$ real eigenvalues can either be described by the vertex condition of the form (2.1) or its spectrum is the whole complex plane.

**Theorem 2.2.** Consider the operator $L_{B,C}$ defined by vertex condition (2.7). If $L_{B,C}$ has $N$ real eigenvalues (counting multiplicities), then either $B$ is invertible or the spectrum of $L_{B,C}$ is the whole complex plane.
Proof. A function \( u \) is an eigenfunction of \( L_{B,C} \) if and only if it satisfies the differential equation (2.4), which has solution (2.5). Substituting in (2.7), we get that for a certain \( \vec{a} \neq 0 \)

\[
(C + ikB)\vec{a} = 0.
\]

Conversely, if this holds for some \( k \) with \( \text{Im} \, k > 0 \) then \( k^2 \) is an eigenvalue of \( L_{B,C} \).

The last equation has non trivial solution if and only if

\[
\text{det} (C + ikB) = 0.
\]

Let us now consider two invertible matrices \( S \) and \( T \). If (2.8) is satisfied then also

\[
\text{det} (SCT + ikSBT) = 0.
\]

Let us take \( S \) and \( T \) such that

\[
(SBT)_{i,k} = \begin{cases} 1, & i = k \leq \text{rank } B, \\ 0, & \text{otherwise}. \end{cases}
\]

Then denote by \( q(k) := \text{det} (SCT + ikSBT) \). Because of (2.9), \( q \) is a polynomial in \( k \) of degree at most \( \text{rank } B \). If \( q \equiv 0 \), then any \( k \) with \( \text{Im} \, k > 0 \) gives an eigenvalue \( k^2 \) and as a result the spectrum is the whole complex plane. Otherwise \( q \) has at most \( \text{rank } B \) zeros. Suppose now that each eigenvalue has multiplicity 1, then if \( \text{rank } B \neq N \), the operator will have less than \( N \) eigenvalues. Therefore we must have that \( B \) is invertible. To complete the proof, the case of multiple eigenvalues has to be considered. Let us assume that \( k_0^2 \) is an eigenvalue of multiplicity \( m \). To see that \( B \) must be invertible if \( L_{B,C} \) has \( N \) eigenvalues, it is enough to prove that \( q(k) \) has a zero of order at least \( m \) at \( k_0 \). To prove this, we consider the eigenfunctions \( \vec{u}^1 = \vec{a}^1 e^{ik_0x}, \ldots, \vec{u}^m = \vec{a}^m e^{ik_0x} \). Here, all constant vectors \( \vec{a}^j, j = 1, \ldots, m \leq N \) are linearly independent. Let us then choose vectors \( \vec{a}^{m+1}, \ldots, \vec{a}^N \) such that \( D = [\vec{a}^1, \vec{a}^2, \ldots, \vec{a}^N] \) has determinant equal to 1. It then follows that

\[
\text{det} (C + ikB) = \text{det} ([C + ikB]D)
\]

\[
= \text{det} ([C + ikB]\vec{a}^1, (C + ikB)\vec{a}^2, \ldots, (C + ikB)\vec{a}_N],
\]

has a zero of order at least \( m \) at \( k = k_0 \). \( \square \)

Our method to obtain non-trivial operators with non-standard symmetries is a certain generalization of the method of point interactions originally developed in the framework of self-adjoint Hamiltonians \([1, 3]\). The phenomenon described in Theorem 2.2 in connection with \( PT \)-symmetric point interaction was first observed in \([4]\), following \([2]\).
The later Theorem implies that the class of operators $L_A$ defined by Robin vertex conditions (2.1) is rather wide, therefore in what follows we focus our attention on this class only.

3. PSEUDO-HERMITIAN AND PSEUDO-REAL OPERATORS

Our studies are inspired by recent papers devoted to investigation of so-called of $\mathcal{PT}$-symmetric operators in one dimension. An operator $L$ is called $\mathcal{PT}$-symmetric if it satisfies the following relation

$$(3.1) \quad \mathcal{PT}L = L\mathcal{PT},$$

where $\mathcal{P}$ is the spacial symmetry operator (parity symmetry)

$$(3.2) \quad (\mathcal{P}u)(x) = u(-x),$$

and $\mathcal{T}$ is the antilinear operator of complex conjugation (time-reversal symmetry)

$$(3.3) \quad (\mathcal{T}u)(x) = \overline{u(x)}.$$

$\mathcal{PT}$-symmetry (like usual operator symmetry) is not enough to guarantee that the corresponding operator is physically relevant: it might happen that its spectrum is empty (take for example the second derivative operator on the interval $[-a,a]$ with both Dirichlet and Neumann conditions imposed on both end-points of the interval). In conventional quantum theory the notion of a self-adjoint operator substitutes simple symmetry property. (Of course any self-adjoint operator is symmetric, but not the other way around.) Therefore it appears natural to substitute relation (3.1) with the following one

$$(3.4) \quad L^* = \mathcal{P}LP^{-1} = \mathcal{P}L\mathcal{P}^*,$$

where we use that $\mathcal{P}$ is unitary $\mathcal{P}^{-1} = \mathcal{P}^*$. In the case of conventional $\mathcal{PT}$-symmetric theory the operator $\mathcal{P}$ is not only unitary, but also self-adjoint $\mathcal{P}^* = \mathcal{P}$. It follows that the operator satisfying (3.4) is pseudo-self-adjoint, i.e. it is self-adjoint not in the original Hilbert space but in the Krein space with the sesquilinear form $[\cdot,\cdot]$ defined by $\mathcal{P}$ as Gram operator

$$[f,g] = \langle \mathcal{P}f, g \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $L_2$.

The goal of the current article is to generalize $\mathcal{PT}$-symmetry for the case of operators on metric graphs, more precisely for the star graph $\Gamma_N$. Operator on such a star graph can be considered as a building block to define
operators on arbitrary graphs. We are going to substitute the operator of space symmetry $\mathcal{P}$ with the rotation operator $\mathcal{R}$ defined as follows:

$$
\begin{bmatrix}
    u_1(x) \\
    u_2(x) \\
    u_3(x) \\
    \vdots \\
    u_{N-1}(x) \\
    u_N(x)
\end{bmatrix}
\begin{bmatrix}
    u_N(x) \\
    u_1(x) \\
    u_2(x) \\
    \vdots \\
    u_{N-2}(x) \\
    u_{N-1}(x)
\end{bmatrix} =
\begin{bmatrix}
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots
\end{bmatrix}
$$

We are not going to distinguish the rotation operators in $\mathbb{C}^N$ and in the space of vector-valued functions hoping that this will not lead to any misunderstanding. Observe that the operator $\mathcal{R}$ is unitary but not self-adjoint (of course provided $N \neq 2$).

We would like to understand whether ideas from $\mathcal{PT}$-symmetric theory may lead to an interesting new class of operators $L_A$, which are not self-adjoint and not $\mathcal{PT}$-symmetric with a suitably defined self-adjoint spacial symmetry operator $\mathcal{P}$. For example if $N$ is even then such a spacial symmetry operator can be defined as

$$\mathcal{P} = \mathcal{R}^{N/2}.$$  

Using $\mathcal{P}$ we may introduce $\mathcal{PT}$-symmetric operators on $\Gamma_N$, but this would be just a vector version of conventional one-dimensional theory (see Lemma 3.2 below).

Formulas (3.1) and (3.4) suggest us to look closer at the following two possible generalizations of the notion of $\mathcal{PT}$-symmetry:

- **pseudo-real** operators

$$\mathcal{RT} L = L \mathcal{RT},$$

and

- **pseudo-Hermitian** operators

$$L^* = \mathcal{RLR}^{-1} = \mathcal{RLR}^*.$$  

We are going to reserve the term pseudo-self-adjoint for operators which are pseudo-Hermitian with respect to a self-adjoint operator $\mathcal{P}$ as in (3.4).

Surprisingly pseudo-Hermitian operators do not define any new interesting class as follows from the following two Lemmas.

**Lemma 3.1.** Let $N$ be an odd number, than the operator $L_A$ is $\mathcal{R}$-pseudo-Hermitian only if it is self-adjoint i.e., $A = A^*$.  

**Proof.** Iterating formula (3.7) one gets the following set of equations

$$
\begin{align*}
L_A &= \mathcal{R}^m L_A \mathcal{R}^{-m}, & m \text{ is odd}; \\
L_A &= \mathcal{R}^m L_A \mathcal{R}^{-m}, & m \text{ is even}.
\end{align*}
$$
Taking into account that $R^N$ is the identity operator we arrive at the following relation for any odd $N$

$$L_A^* = L_A$$

proving our statement. \qed

**Lemma 3.2.** Let $N$ be an even number.

- If in addition $N/2$ is an odd number, then the operator $L_A$ is $R$-pseudo-Hermitian only if it is pseudo-self-adjoint with respect to $\mathcal{P} = R^{N/2}$.
- If in addition $N/2$ is an even number, then the operator $L_A$ is $R$-pseudo-Hermitian only if it commutes with the self-adjoint rotation $\mathcal{P} = R^{N/2}$ and therefore is unitarily equivalent to an orthogonal sum of Laplace operators on $\Gamma_{N/2}$ with Robin conditions at the central vertices.

**Proof.** We just apply formula (3.8) to get

$$L_A^* = R^{N/2} L_A R^{-N/2}, \quad N/2 \text{ is odd;}$$

$$L_A = R^{N/2} L_A R^{-N/2}, \quad N/2 \text{ is even.}$$

In the first case the operator $L_A$ is pseudo-self-adjoint with respect to $\mathcal{P} = R^{N/2}$.

In the second case ($N = 4n$, $n \in \mathbb{N}$) the operator $L_A$ commutes with the self-adjoint operator $\mathcal{P} = R^{N/2}$. Since $\mathcal{P}^2 = R^N = I$ is the identity operator, its spectrum is $\pm 1$. The corresponding subspaces coincide with the sets of functions satisfying $u_{j+N/2}(x) = \pm u_j(x)$. Each of the eigensubspaces can be identified with the Hilbert space $L_2([0, \infty); \mathbb{C}^{N/2})$. Since $L_A$ commutes with $\mathcal{P}$ it can be written as an orthogonal sum of two Laplace operators $L_{\pm} = -\frac{d^2}{dx^2}$, each acting in $L_2([0, \infty); \mathbb{C}^{N/2})$ simply because $L_A \mathcal{P} = \mathcal{P} L_A$ implies that $\tilde{u} \in \text{Dom}(L_A) \iff \mathcal{P} \tilde{u} \in \text{Dom}(L_A)$. The operators $L_{\pm}$ are then defined by Robin conditions of the form $\tilde{u}_{\pm}^2(0) = A_{\pm} \tilde{u}_{\pm}(0)$, where $\tilde{u}_{\pm} \in W_2^2([0, \infty); \mathbb{C}^{N/2})$. Finally, the operators $L_{\pm}$ can be seen as Laplace operators on start graphs $\Gamma_{N/2}$ with $N/2$ edges with Robin conditions at the central vertices. \qed

The operators $L_{\pm}$ appearing in the later Lemma satisfy symmetry properties similar to (3.7), but with the “rotation” operators $R_{\pm}$ of lower size ($N/2$ instead of $N$). If $R_+$ is the standard rotation operator in the space $\mathbb{C}^{N/2}$, the operator $R_-$ is a certain modified rotation operator:

$$R_- = \text{diag}(1, 1, \ldots, 1, -1) \ R_+.$$ 

It might be interesting to understand the symmetry of $L_-$ in more detail, but we may conclude already now that in most cases the operator $L_A$ is
pseudo-Hermitian only if it is also pseudo-self-adjoint (with respect to another symmetry operator). Therefore in what follows we focus on the studies of pseudo-real realisations of the Laplace operator on the star graph $\Gamma_N$. Therefore we are going to use the following definition:

**Definition 3.3.** An operator $L$ in $L^2(\Gamma_N)$ is called $\mathcal{RT}$-symmetric if and only if it satisfies the following relation

$$(3.9) \quad \mathcal{RT} L = L \mathcal{T},$$

where $\mathcal{T}$ is the antilinear operator of complex conjugation.

This definition guarantees that the spectrum of the operator is symmetric with respect to the real axis. Really if $\psi(x)$ is an eigenfunction corresponding to the eigenvalue $\lambda$, then $\varphi(x) = \mathcal{RT} \psi$ is also an eigenfunction but corresponding to the eigenvalue $\bar{\lambda}$, provided (3.9) holds:

$$(3.10) \quad L \varphi = L \mathcal{RT} \psi = \mathcal{RT} L \psi = \mathcal{RT} \lambda \psi = \bar{\lambda} \mathcal{RT} \psi = \bar{\lambda} \varphi.$$

Hence with definition 3.3 we always have spectrum which is symmetric with respect to the real axis as in the classical $\mathcal{PT}$-symmetric theory.

### 4. $\mathcal{RT}$-Symmetry of Point Interactions

In the current section we are going to describe the structure of $\mathcal{RT}$-symmetric operators. It appears that the corresponding matrices $A$ belong to the class of circulant matrices which we describe now.

**Definition 4.1.** An $N \times N$ matrix $A = \{a_{ik}\}$ is called circulant\(^1\) if the value of the entry $a_{ik}$ depends only on the difference $(k - i) \mod N$, i.e.,

$$A = \begin{pmatrix}
  a_0 & a_1 & a_2 & \cdots & a_{N-1} \\
  a_{N-1} & a_0 & a_1 & \cdots & a_{N-2} \\
  a_{N-2} & a_{N-1} & a_0 & \cdots & a_{N-3} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_1 & a_2 & a_3 & \cdots & a_0 
\end{pmatrix} =: \text{circ}(a_0, a_1, \ldots, a_{N-1}).$$

**Definition 4.2.** An $N \times N$ matrix $A$ is called $k \times k$ block circulant if $A = \text{circ}(B_0, B_1, \ldots, B_{k-1})$ where $B_i$, $i = 0, \ldots, k-1$ are block matrices of the same size $n \times n$, $N = kn$.

The following theorem describes matrices $A$ leading to $\mathcal{RT}$-symmetric operators on $\Gamma_N$.

\(^1\)A circulant matrix is a special case of a Toeplitz matrix.
Theorem 4.3. Consider the operator $L_A$ determined by Definition 2.1.

If $N$ is odd, then the operator $L_A$ is $RT$-symmetric if and only if $A$ is a real circulant matrix, i.e.,

$$A = \text{circ} (a_0, a_1, \ldots, a_{N-1}), \quad a_j \in \mathbb{R}, \ j = 0, \ldots, N - 1. \quad (4.1)$$

If $N$ is even, then the operator $L_A$ is $RT$-symmetric if and only if $A$ is a complex $N/2 \times N/2$ block circulant matrix formed by the following $2 \times 2$ blocks:

$$A = \begin{pmatrix}
    a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-2} & a_{N-1} \\
    \overline{a}_{N-1} & \overline{a}_0 & \overline{a}_1 & \overline{a}_2 & \cdots & \overline{a}_{N-3} & \overline{a}_{N-2} \\
    a_{N-2} & a_{N-1} & a_0 & a_1 & \cdots & a_{N-4} & a_{N-3} \\
    \overline{a}_{N-3} & \overline{a}_{N-2} & \overline{a}_{N-1} & \overline{a}_0 & \cdots & \overline{a}_{N-5} & \overline{a}_{N-4} \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_2 & a_3 & a_4 & a_5 & \cdots & a_0 & a_1 \\
    \overline{a}_1 & \overline{a}_2 & \overline{a}_3 & \overline{a}_4 & \cdots & \overline{a}_{N-1} & \overline{a}_0
\end{pmatrix}, \quad (4.2)
$$

$$a_j \in \mathbb{C}, \ j = 0, \ldots, N - 1.$$

Proof. Let us consider the operators $L_A$ as defined in Definition 2.1. Suppose that the function $u \in \text{Dom} (L_A)$ and therefore satisfies the boundary condition (2.1). The boundary condition for the function $RTu$ is given by

$$RT\overrightarrow{u}'(0) = ART\overrightarrow{u}(0),$$

$$\Rightarrow \overrightarrow{u}'(0) = TRART\overrightarrow{u}(0),$$

$$\Rightarrow \overrightarrow{u}'(0) = R^{-1}AR\overrightarrow{u}(0).$$

This condition should be identical with (2.1) leading to

$$A = R^{-1}AR,$$

$$\iff A = RAR^{-1},$$

where we used that the rotation matrix $R$ has real entries.

Let us denote the entries of the matrix $A$ by $a_{i,k}$, then the last equality implies

$$a_{i,k} = \overline{a}_{i-1 \mod N, k-1 \mod N}. \quad (4.3)$$

The structure of the matrix $A$ is as follows: every next row in the matrix is equal to the previous one shifted to the right one step and conjugated.

It is clear then that $A$ is determined by $N$ complex numbers, for example those building the first row. If there would be no complex conjugation or the entries would be real, then $A$ would be circulant.

Now, we consider the cases when $N$ is odd and even separately.

$N$ is odd: We get the following chain of equalities:

$$a_{i \mod N, k \mod N} = \overline{a}_{i+1 \mod N, k+1 \mod N} = a_{i+2 \mod N, k+2 \mod N} = \cdots = \overline{a}_{i+N \mod N, k+N \mod N} = \overline{a}_{i \mod N, k \mod N}.$$

$N$ is even: We get the following chain of equalities:

$$a_{i \mod N, k \mod N} = \overline{a}_{i+1 \mod N, k+1 \mod N} = a_{i+2 \mod N, k+2 \mod N} = \cdots = \overline{a}_{i+N \mod N, k+N \mod N} = \overline{a}_{i \mod N, k \mod N}.$$
implying that
\[ a_{i,k} = a_{i,k} \implies a_{i,k} \in \mathbb{R}, \ i, k = 0, \ldots, N - 1. \]

Therefore
\[ a_{i,k} = a_{(i-1) \mod N,(k-1) \mod N} \]
and we see that the matrix \( A \) is a real circulant matrix. It is determined by \( N \) real parameters:
\[ a_j := a_{1,j+1}, \]
\[ A = \text{circ} \left( a_0, a_1, a_2, \ldots, a_{N-1} \right). \]

\( N \) is even:

We again consider (4.3) to obtain the following
\[ a_{i \mod N, k \mod N} = \overline{a}_{i+1 \mod N, k+1 \mod N} = a_{i+2 \mod N, k+2 \mod N} = \ldots = a_{i+N \mod N, k+N \mod N}. \]

We do not obtain any restriction on \( a_{i,k} \) and hence the entries of the first row can be chosen arbitrarily among complex numbers. But we still have the following property, which reminds of circulant matrices:
\[ a_{i,k} = a_{i',k'}, \]
provided \( i' = (i + 2s) \mod N \) and \( k' = (k + 2s) \mod N \), \( s = 1, \ldots, N/2 \).

Let us denote the first row in \( A \) by \( a_j, j = 0, \ldots, N - 1 \), then each row of the matrix \( A \) is the conjugated of the previous row shifted to the right, i.e.,
\[ A = \begin{pmatrix}
    a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-2} & a_{N-1} \\
    a_{N-1} & a_0 & a_1 & a_2 & \cdots & a_{N-3} & a_{N-2} \\
    a_{N-2} & a_{N-1} & a_0 & a_1 & \cdots & a_{N-4} & a_{N-3} \\
    a_{N-3} & a_{N-2} & a_{N-1} & a_0 & \cdots & a_{N-5} & a_{N-4} \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_2 & a_3 & a_4 & a_5 & \cdots & a_0 & a_1 \\
    a_1 & a_2 & a_3 & a_4 & \cdots & a_{N-1} & a_0
\end{pmatrix}, \]
\[ a_j \in \mathbb{C}, \ j = 0, \ldots, N - 1, \]
as claimed. We see that \( A \) is a \( 2 \times 2 \) block circulant matrix. The blocks forming \( A \) are not chosen arbitrarily, but depend on \( N \) complex parameters. \( \square \)
5. ON THE SPECTRUM OF THE $\mathcal{RT}$-SYMMETRIC OPERATORS

We can now look at the discrete spectra of the constructed $\mathcal{RT}$-symmetric operators on the star graph $\Gamma_N$. We have already observed that, as in the case of all $\mathcal{PT}$-symmetric operators, the nonreal eigenvalues of a $\mathcal{RT}$-symmetric operator always appear in conjugate pairs (3.10). That is, if $\lambda$ is an eigenvalue of an $\mathcal{RT}$-symmetric operator $A$, then $\bar{\lambda}$ is also an eigenvalue of the operator $A$. We noted that the operator $L_A$ may have at most $N$ distinct eigenvalues (counting multiplicities). The most interesting case is when the spectrum of the operator is real, also the operator itself is not self-adjoint.

The eigenvalues of $L_A$ are given by solutions of equation (2.6) with $\Im k > 0$. Negative eigenvalues correspond to $k$ on the upper part of the imaginary axis. In what follows we study the case where the operator has precisely $N$ (negative) real eigenvalues. Since the structures of the matrices are different in the cases when the number of edges $N$ is even or odd, these cases will be studied separately.

5.1. An odd number of edges. Before we study the discrete spectrum of the operator $L_A$, we recall some known results about the eigenvalues of a circulant matrix which can be nicely calculated as follows [17].

**Proposition 5.1.** Let $A = \text{circ} (a_0, a_1, \ldots, a_{N-1})$ be a circulant matrix, then its eigenvalues are given by

$$\mu_j = \sum_{j=0}^{N-1} a_j e^{\frac{2\pi i j}{N}}, \quad j = 0, \ldots, N - 1.$$  

**Proof.** The key idea is to write arbitrary circulant matrix $A$ as a sum of powers of the rotation matrix $\mathcal{R}$. The rotation matrix $\mathcal{R}$ can be seen as an elementary circulant matrix

$$\mathcal{R} = \text{circ} (0, 1, 0, \ldots, 0).$$

We have, similarly

$$\mathcal{R}^j = \text{circ} (0, 0, \ldots, \underbrace{1}_{j+1}, \ldots, 0), \quad \mathcal{R}^N = \mathcal{I}.$$ 

Hence any circulant $A$ possesses the representation

$$A = a_0 \mathcal{I} + \sum_{j=1}^{N-1} a_j \mathcal{R}^j.$$
The eigenvalues of the rotation matrix $R$ are $e^{j \frac{2\pi i}{N}}$, $j = 0, 1, \ldots, N - 1$ with the eigenvectors

$$(1, e^{j \frac{2\pi i}{N}}, e^{2j \frac{2\pi i}{N}}, \ldots, e^{(N - 1)j \frac{2\pi i}{N}}).$$

It follows that (5.1) holds. □

**Theorem 5.2.** Assume that $N$ is odd, then any $\mathcal{RT}$-symmetric operator $L_A$ on the star-graph $\Gamma_N$ has $N$ real eigenvalues only if it is self-adjoint.

**Proof.** First, we note that $\lambda < 0$ is an eigenvalue of $L_A$ if and only if $\mu = -\sqrt{-\lambda}$ is an eigenvalue of the matrix $A$. If $N$ is odd, then in accordance to Theorem 4.3 $A$ is a circulant matrix with real entries. Then its eigenvalues are nothing else, than a discrete Fourier transform of $\{a_n\}_{n=0}^{N-1}$

$$\mu_j = \sum_{n=0}^{N-1} e^{nj \frac{2\pi i}{N}} a_n, \quad j = 0, 1, \ldots, N - 1,$$

$$a_m = \frac{1}{N} \sum_{j=0}^{N-1} e^{-mj \frac{2\pi i}{N}} \mu_j, \quad m = 0, 1, \ldots, N - 1.$$

It follows that $\overline{a_n} = a_{N-n}$, indeed

$$\overline{a_n} = \frac{1}{N} \sum_{j=0}^{N-1} \overline{\mu_j} e^{nj \frac{2\pi i}{N}} = \frac{1}{N} \sum_{j=0}^{N-1} \mu_j e^{-(N-n)j \frac{2\pi i}{N}} = a_{N-n},$$

where we used that $\mu_j$ are all real. Taking into account that $A$ is circulant we conclude that it is Hermitian, hence the operator $L_A$ is self-adjoint. □

Of course, if the number of real eigenvalues is less than $N$ the operator $L_A$ does not need to be self-adjoint. The proof of the later theorem may give an impression that whether the size of $A$ is even or odd does not play any essential role. The next subsection shows that the difference is tremendous.

**5.2. An even number of edges.** We recall from Theorem 4.3 that in case the star graph $\Gamma_N$ has even number of edges the operator $L_A$ is $\mathcal{RT}$-symmetric if and only if $A = \text{circ}(B_0, B_1, \ldots, B_K)$, where $K = \frac{N}{2}$, and

$$B_j = \begin{pmatrix} a_{2i} & a_{(2i+1) \mod N} \\ a_{(2i-1) \mod N} & a_{2i} \end{pmatrix}.$$

A vector $\vec{d} \neq 0$ is an eigenvector of $A$ with eigenvalue $\mu$ if and only if

$$A\vec{d} = \mu \vec{d}.$$
Following the ideas in [25], we will look for eigenvectors of $A$ of the following form

$$
\vec{d} = \vec{d}(\omega, v) = (v, \omega v, \omega^2 v, \cdots, \omega^{K-1} v),
$$

where $v$ is a non-zero two-dimensional vector and $\omega$ is a fixed $K^{th}$ root of the unity, i.e.,

$$
\omega \in \{ e^{\frac{2\pi i}{K}} \}, \quad j = 0, \ldots, K - 1.
$$

One may prove that all eigenvectors of $A$ are of this form, since $A$ is commuting with $R^2$.

Extending (5.4) with $A = \text{circ} (B_0, B_1, \ldots, B_K)$, as explained above, the following set of $K$ equations is obtained

$$
\begin{align*}
(B_0 + B_1 \omega + B_2 \omega^2 + B_3 \omega^3 + \cdots + B_{K-1} \omega^{K-1}) v &= \mu v, \\
(B_{K-1} + B_0 \omega + B_1 \omega^2 + B_2 \omega^3 + \cdots + B_{K-2} \omega^{K-1}) v &= \omega \mu v, \\
& \vdots \\
(B_1 + B_2 \omega + B_3 \omega^2 + B_4 \omega^3 + \cdots + B_{K-1} \omega^{K-1}) v &= \omega^{K-1} \mu v.
\end{align*}
$$

Dividing the $j^{th}$ equation by $\omega^j$, $j = 1, \ldots, K - 1$, it reduces to the first one. Hence we have just one equation. Let us now rewrite it as an eigenvector equation

$$
(5.5) \quad H v = \mu v,
$$

where the square matrix $H = H(\omega)$ is

$$
(5.6) \quad H = B_0 + B_1 \omega + B_2 \omega^2 + B_3 \omega^3 + \cdots + B_{K-1} \omega^{K-1} = \\
\begin{pmatrix}
\frac{a_0}{a_0} & \frac{a_1}{a_0} \\
\frac{a_2}{a_1} & \frac{a_3}{a_2} \\
\vdots & \vdots \\
\frac{a_{N-2}}{a_{N-3}} & \frac{a_{N-1}}{a_{N-2}}
\end{pmatrix} \omega^{K-1} = \\
\begin{pmatrix}
\frac{a_0 + a_2 \omega + \cdots + a_{N-2} \omega^{K-1}}{a_{N-1}} & \frac{a_1 + a_3 \omega + \cdots + a_{N-1} \omega^{K-1}}{a_0} \\
\frac{a_{N-1}}{a_0} & \frac{a_{N-2}}{a_1}
\end{pmatrix}.
$$

Each eigenvector $v = v(\omega)$ of $H(\omega)$ will generate an eigenvector $\vec{d}$ of $A$ with the same eigenvalue. If for every $\omega$ the $2 \times 2$ matrix $H(\omega)$ has two negative eigenvalues, then the matrix $A$ has precisely $N$ negative eigenvalues and so the operator $L_A$.

The following theorem is a counterpart of Theorem 5.2 for the case when $N$ is even.

---

$^2$This is a natural suggestion given the structure of the eigenvectors for the case when $N$ is odd.
Theorem 5.3. Let the number $N$ of edges of the star-graph $\Gamma_N$ be even. Then among $RT$-symmetric operators $L_A$ (given by Definition 2.1) there are some with $N$ real eigenvalues that are not self-adjoint.

Proof. To prove the Theorem it is enough to present an example of such a matrix $A$ for arbitrary even $N$. It is enough to find $A$ such that all $H(\omega)$ have two negative eigenvalues. As by Theorem 4.3 if $N$ is even then $L_A$ is $RT$-symmetric if and only if $A$ is block circulant. Consider such block circulant matrix $A$ given by

$$
\begin{align*}
  a_0 &= -8K + 2 + i, \\
  a_1 &= a_3 = \cdots = a_{N-1} = 1 + i, \\
  a_2 &= a_4 = \cdots = a_{N-2} = 2 + i.
\end{align*}
$$

Equivalently the matrix $A$ can be presented as

$$
A = -4N + \text{circ}
\begin{pmatrix}
  2 + i & 1 + i & \cdots & 2 + i & 1 + i \\
  1 - i & 2 + i & \cdots & 2 + i & 1 + i \\
  2 + i & 1 + i & -4N + 2 + i & \cdots & 2 + i & 1 + i \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  2 + i & 1 + i & 2 + i & \cdots & -4N + 2 + i & 1 + i \\
  1 - i & 2 + i & 1 + i & \cdots & 1 - i & -4N + 2 - i
\end{pmatrix}
$$

Using entries in (5.7) the corresponding matrices $H$, appeared first in (5.6), are given by

$$
H(\omega) = \begin{cases}
  N \begin{pmatrix}
    -4 & 0 \\
    0 & -4
  \end{pmatrix}, & \omega \neq 1, \\
  \frac{N}{2} \begin{pmatrix}
    -6 + i & 1 + i \\
    1 - i & -6 - i
  \end{pmatrix}, & \omega = 1.
\end{cases}
$$

Here we just used that

$$
1 + \omega + \omega^2 + \cdots + \omega^{N/2-1} = \begin{cases}
  0, & \omega \neq 1, \\
  N/2, & \omega = 1.
\end{cases}
$$

Each of the matrices $H(\omega)$ given by (5.9) has two negative eigenvalues implying that the corresponding $A$ as well as $L_A$ has $N$ negative eigenvalues. □

The theorem implies that the class of $RT$-symmetric operators is much richer in the case of even $N$. 

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6. Conclusions

A family of $\mathcal{RT}$-symmetric non self-adjoint operators with purely real discrete spectra was constructed when the number of edges of the star graph $\Gamma_N$ is even. It should be emphasized that the constructed $\mathcal{RT}$-symmetric operators are not all possible $\mathcal{RT}$-symmetric Laplacians with Robin conditions.

Constructed $\mathcal{PT}$-symmetric operators possess the same absolutely continuous spectrum as the unperpturbed (self-adjoint) Laplacian with standard vertex conditions. It would be interesting to look at the properties of the corresponding scattering matrix, which can be calculated explicitly.

It should also be mentioned that we constructed a particular family of non self-adjoint $\mathcal{RT}$-symmetric operators with $N$ negative eigenvalues. It might be interesting to study other families or even obtain a full description of all such operators.

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