ABSTRACT

This paper presents a new high accuracy window design method based on semi–infinite linear programming by using the Dual Nested Complex Approximation (DNCA) algorithm. The paper presents a general window design specification, the formulation of the corresponding semi–infinite linear programming solution and several highly optimized window design examples. The paper illustrates the capability to enhance the sidelobe attenuation of the new upcoming ISO 18431-2 flattop window standard from the international organization for standardization. The procedure of enhancing the ISO flattop window can be directly applicable to any other existing windows. Flattop windows are commonly available in frequency analyzers for accurate amplitude measurements of harmonic tones in presence of noise. Moreover, it is possible to design high accuracy windows with more than 150 dB sidelobe attenuation with the proposed method as well. Such high sidelobe attenuation is for example needed in today’s 24bit equivalent 1bit-Sigma-Delta ADC:s, where high frequency noise is generated.

1. INTRODUCTION

Frequency domain calculations using the FFT algorithm, is very common for example in the field of sound and vibration analysis. The FFT algorithm practically requires that the sequence to be analyzed is periodic with an exact integer number of cycles within the captured data block. This requirement can be traced back to the periodic extension property of the discrete Fourier transform. Rarely, this condition is met and discontinuities appears at the boundaries when the captured (non-periodic) data block is periodically repeated according to the periodic extension property. These discontinuities will give rise to the so called spectral leakage problem, which can be explained as the power originating from one frequency bin has been spread out over the entire frequency spectra.

One method to reduce the spectral leakage is to smooth out the discontinuities by smoothly bring the endpoints of the captured data block to zero or close to zero. This can be performed by multiplying the captured data block with a window function in the time domain, i.e., windowing.

Throughout the years, many different window functions has been designed and evaluated [1], [2], [3]. The window functions described by Harris in [1] are often referred to as classic windows characterized by a narrow mainlobe and moderate sidelobe attenuation. Later work by Salvatore in [4] also discuss a new class of windows known as flattop windows characterized by a wide mainlobe and high sidelobe attenuation.

In year 2000, the international organization for standardization (ISO) initiated the work of ISO 18431-2, Mechanical Vibration and Shock - Signal Processing Part 2 - Rectangular, Hanning, and FlatTop Windows for Fourier Transform Analysis. The project aims to standardize the three most commonly used window functions for frequency analysis, the Rectangular, Hanning and a new flattop window. More information about the upcoming ISO 18431-2 standard can be found at the international organization for standardization homepage1.

A novel high accuracy window design method is presented where the design problem is formulated as a semi-infinite linear programming problem. The new extended active set strategy, the Dual Nested Complex Approximation (DNCA) scheme, is used to obtain the solution of the optimization problem. The DNCA design technique may be useful for complex approximation with any filter having linear structure, such as the digital Laguerre networks [5], narrowband as well as broadband beamformers [6], [7], [8] and digital FIR equalizers [9].

This paper includes several window design examples demonstrating the efficiency of using the DNCA optimization algorithm. Two examples considers an enhanced design of a flattop window with higher sidelobe attenuation than the upcoming ISO flattop window while preserving the passband properties. One example consider a high accuracy window with 150 dB sidelobe attenuation. Windows with such high sidelobe attenuation are for example needed in today’s 24bit-equivalent 1 bit-Sigma-Delta ADC:s, where high frequency noise is generated.

1http://www.iso.org
2. PROBLEM FORMULATION

A common way to construct windows is by using the summation of shifted Dirichlet kernels. A data window for DFT or FFT usage constructed in this way is defined as

\[ w(n) = \sum_{k=0}^{K-1} (-1)^k a_k c_k(n), \quad n = 0, \ldots, N-1 \]  

where \( K \) denotes the number of \( a_k \) coefficients, \( N \) denotes the length of the window and \( c_k(n) \) denotes the basis function given by

\[ c_k(n) = \cos \left( \frac{2\pi kn}{N} \right) \]  

The corresponding Fourier transform of a data window for DFT or FFT usage is given by

\[ W(\omega) = \sum_{n=0}^{N-1} w(n) e^{-j\omega n} \]  

Inserting equation (1)-(2) into equation (3) gives

\[ W(\omega) = \sum_{n=0}^{N-1} \left( \sum_{k=0}^{K-1} (-1)^k a_k \cos \left( \frac{2\pi kn}{N} \right) \right) e^{-j\omega n} \]  

Changing the summation order of equation (4) and rearranging the variables gives

\[ W(\omega) = \sum_{k=0}^{K-1} (-1)^k a_k \sum_{n=0}^{N-1} \cos \left( \frac{2\pi kn}{N} \right) e^{-j\omega n} \]  

By using the formula of Euler, equation (5) can be written as

\[ W(\omega) = \sum_{k=0}^{K-1} (-1)^k a_k \sum_{n=0}^{N-1} \left( e^{j2\pi kn/N} + e^{-j2\pi kn/N} \right) e^{-j\omega n} \]  

\[ = \sum_{k=0}^{K-1} (-1)^k a_k \left( D\left(\omega - \frac{2\pi k}{N}\right) + D\left(\omega + \frac{2\pi k}{N}\right) \right) \]  

where \( D(\omega) \) is known as the Dirichlet kernel defined as

\[ D(\omega) = e^{-j\omega \left( \frac{N-1}{2} \right) \sin \left( \frac{\omega N}{2} \right) \sin \left( \frac{\omega F}{2} \right) } \]  

Equation (7) shows that any window constructed using equation (1) has a Fourier Transform which consist of a summation of shifted Dirichlet kernels. Some of the most well known windows constructed in this way are the Hanning, Hamming and Blackman window.

Using vector notations, equation (5) can be written as

\[ W(\omega) = \phi^T(\omega) a \]  

where

\[ \phi(\omega) = \left[ \sum_{n=0}^{N-1} e^{-j\omega n} \sum_{n=0}^{N-1} (-1)^1 c_1(n) e^{-j\omega n} \sum_{n=0}^{N-1} (-1)^2 c_2(n) e^{-j\omega n} \cdots \sum_{n=0}^{N-1} (-1)^K c_{K-1}(n) e^{-j\omega n} \right] \]  

and

\[ a^T = [ a_0 \ a_1 \ a_2 \ \ldots \ \ a_{K-1} ] \]  

3. THE DESIGN SPECIFICATION

Consider the following window design specification

\[ \left\{ \begin{array}{l} |E(\omega)| \leq \sigma(\omega) \quad \omega \in \Omega_s = [\omega_s, \pi] \\ |E(\omega)| \leq \sigma_p(\omega) \quad \omega \in \Omega_p = [0, \omega_p] \end{array} \right. \]  

where \( E(\omega) \) denotes the complex error function defined as

\[ E(\omega) = \phi^T(\omega) a - W_d(\omega) \]  

and \( \sigma_s(\omega) \) is a strictly positive magnitude bound in the stopband \( \Omega_s \), \( \sigma_p(\omega) \) the corresponding strictly positive magnitude bound in the passband \( \Omega_p \) and \( W_d(\omega) \) the desired complex window specification. The variables \( \omega_p \) and \( \omega_s \) denotes the passband edge and stopband edge respectively. The specification (12) is equivalent to

\[ \left\{ \begin{array}{l} \max_{\omega \in \Omega_s} \frac{1}{\sigma(\omega)} |E(\omega)| \leq 1 \\ |E(\omega)| \leq \sigma_p(\omega) \quad \omega \in \Omega_p \end{array} \right. \]  

which leads into the minimax optimization design formulation

\[ \left\{ \begin{array}{l} \min \max_{\omega \in \Omega_s} v_s(\omega) |E(\omega)| \\ |E(\omega)| \leq \sigma_p(\omega) \quad \omega \in \Omega_p \end{array} \right. \]  

where \( v_s(\omega) = 1/\sigma_s(\omega) \). It is concluded that a solution to the design specification (12) exists if and only if the optimal objective value in (15) is less or equal to one. Hence, the optimization formulation (15) will give the answer if there exists a feasible solution to (12) and furthermore, if a solution to (12) exists we will obtain the solution which is furthest away from the upper bound in a "logarithmic" minimax sense. The optimal solution to (15) yields the smallest \( \delta \) such that

\[ |E(\omega)| \leq \delta \sigma_s(\omega) \quad \omega \in \Omega_s \]  

or,

\[ 20 \log |E(\omega)| \leq 20 \log \delta + 20 \log \sigma_s(\omega) \quad \omega \in \Omega_s \]  

where \( \delta \) denote the objective value in (15).
3.1. The Semi–Infinite Linear Programming Solution

The optimal solution to the minimax formulation (15) is given by the equivalent formulation

\[
\min \delta, \text{ subject to } \begin{align*}
|v_s(\omega)|E(\omega)| - \delta & \leq 0, \quad \omega \in \Omega_s, \\
|E(\omega)| & \leq \sigma_p(\omega), \quad \omega \in \Omega_p
\end{align*}
\tag{18}
\]

The problem formulation in equation (18) corresponds to a non-linear optimization problem, which is difficult to solve as it stands. According to the real rotation theorem [10], a magnitude inequality in the complex plane can be expressed in the equivalent form

\[
|z| \leq \gamma \iff \Re \{z e^{i\theta}\} \leq \gamma, \quad \forall \theta \in [0, 2\pi]
\tag{19}
\]

where \(\Re\{\cdot\}\) denotes the real part of \(\cdot\) and the phase \(\theta\) belongs to the infinite set \(\Theta = [0, 2\pi]\). By making use of equation (19), the design problem can be formulated as

\[
\begin{align*}
\min \delta, \text{ subject to } \\
v_s(\omega)f(\omega, \theta) - \delta & \leq 0, \quad \forall \theta \in \Theta, \omega \in \Omega_s, \\
f(\omega, \theta) & \leq \sigma_p(\omega), \quad \forall \theta \in \Theta, \omega \in \Omega_p
\end{align*}
\tag{20}
\]

where

\[
f(\omega, \theta) = \Re \{E(\omega)e^{i\theta}\}
\tag{21}
\]

The linear program in equation (20) is called semi-infinite since the number of unknown variables are finite but the constraint set is infinite due to the continuous phase \(\theta \in [0, 2\pi]\). For practical purposes in the implementation of the algorithm, \(\Omega_p\) and \(\Omega_s\) are assumed to be a finite subsets of \(\Omega = [0, \pi]\), i.e. \(\omega_i = [\omega_1, \omega_2, \ldots, \omega_I], i = 1, \ldots, I\). In the discrete frequency domain the design problem can be formulated as

\[
\begin{align*}
\min \delta, \text{ subject to } \\
v_s(\omega_i)f(\omega_i, \theta) - \delta & \leq 0, \quad \forall \theta \in \Theta, \omega_i \in \Omega_s, \\
f(\omega_i, \theta) & \leq \sigma_p(\omega_i), \quad \forall \theta \in \Theta, \omega_i \in \Omega_p
\end{align*}
\tag{22}
\]

Observe that the approximation problem in equation (22) is with respect to the true complex error due to the continuous phase \(\theta \in [0, 2\pi]\) and is not affected by the discretization of the frequency domain.

4. THE DNCA-LP OPTIMIZATION ALGORITHM

The semi-infinite linear program in (20) can be solved using the Dual Nested Complex Approximation algorithm. The optimization is done by solving a sequence of subproblems of (22) with increasing minimum cost based on finite subsets, which can be described with the following basic steps:

1. Given a reference set

\[
\mathcal{R}_k = \mathcal{R}_k^p \cup \mathcal{R}_k^s
\]

\[
= \{(\omega_1, \theta_1), \ldots, (\omega_r, \theta_r)\} \subset \mathcal{D}
\tag{24}
\]

\[
= (\Omega_p \cup \Omega_s) \times \Theta
\tag{25}
\]

where \(\mathcal{R}_k^p\) is the reference set in the passband and \(\mathcal{R}_k^s\) is the reference set in the stopband. Let \((a_k, b_k)\) and \(W_k(\omega) = \phi_k^e(\omega) a_k\) denote the optimal solution to the k-th subproblem

\[
\begin{align*}
\min \delta, \text{ subject to } \\
v_s(\omega_i)f(\omega_i, \theta_i) - \delta & \leq 0, \quad (\omega_i, \theta_i) \in \mathcal{R}_k^s, \\
f(\omega_i, \theta_i) & \leq \sigma_p(\omega_i), \quad (\omega_i, \theta_i) \in \mathcal{R}_k^p
\end{align*}
\tag{26}
\]

2. We now maximize

\[
g_p(\omega, \theta) = \Re \{(W_k(\omega) - W_d(\omega)) e^{i\theta}\} - \sigma_p(\omega)
\tag{27}
\]

over the passband \(\omega \in \Omega_p\) and for every angle \(\theta \in [0, 2\pi]\) yielding \(g_p(\omega, \theta)\) yielding \(g_p, \text{max} = \omega_p, \text{max}, \theta_p, \text{max}\).

Correspondingly we maximize

\[
g_s(\omega, \theta) = \Re \{(W_k(\omega) - W_d(\omega)) e^{i\theta}\}
\tag{28}
\]

over the stopband \(\omega \in \Omega_s\) for every angle \(\theta \in [0, 2\pi]\) yielding \(g_s, \text{max} = \omega_s, \text{max}, \theta_s, \text{max}\).

Define the entering reference set \(\mathcal{R}_e = (\omega_e, \theta_e)\) as \(\omega_p, \text{max}, \theta_p, \text{max}\) if \(g_p, \text{max} > g_s, \text{max}\), otherwise \((\omega_e, \theta_e)\) are selected as \((\omega_s, \text{max}, \theta_s, \text{max})\).

3. Stop if

\[
\max\{|v_s(\omega)|W_k(\omega) - W_d(\omega)|\} < \delta_0(1+\epsilon), \omega \in \Omega_s
\tag{29}
\]

and

\[
\max\{|W_k(\omega) - W_d(\omega)|\} < \sigma_0(1+\epsilon), \omega \in \Omega_p
\tag{30}
\]

where \(\epsilon\) is a predefined optimization tolerance parameter normally related to the actual underlying optimization engine and machine precision.

4. Define the leaving reference set

\[
\mathcal{R}_l = \{(\omega_l, \theta_l) \in \mathcal{R}_k | \lambda_l = 0\}
\tag{31}
\]

where the Lagrange multipliers \(\lambda_l = 0\) indicates the inactive constraints in the reference set. The leaving indices essentially consists of the inactive constraints in (23).

5. Define a new reference set

\[
\mathcal{R}_{k+1} = (\mathcal{R}_k \setminus \mathcal{R}_l) \cup \mathcal{R}_e
\tag{32}
\]

and return to step 1.

The DNCA-LP algorithm has been implemented in MATLAB™ and is available at [http://www.bth.se/its/dnca/](http://www.bth.se/its/dnca/).
5. DESIGN EXAMPLES

In this section the flexibility in design using the proposed Dual Nestled Complex Approximation algorithm is illustrated by numerical examples. The algorithm is able to solve huge design problems with simultaneous mainlobe and sidelobe control. All figures and examples corresponds to the problem formulation, design specification and optimization described in section 2-4. All windows presented in this section are normalized. The sample domain representation is normalized according to \( ||w(n)||_\infty = 1 \). The frequency domain representation is normalized according to a DC level equal to 0dB, i.e. \( |W(0)| = 1 \).

The examples consider the design of an Enhanced ISO (E-ISO) flattop window (-89dB), a Modified ISO (M-ISO) flattop window with higher sidelobe attenuation (-96dB) and the DNCA-150 high accuracy Dual Nestled Complex Approximation optimized flattop window (-150dB).

5.1. E-ISO Flattop Window

The upcoming ISO 18431-2 standard states a new flattop window for mechanical vibration and shock analysis. The ISO 18431-2 flattop window is constructed by (1), where the five \((K = 5)\) coefficients is given by

\[
a_{ISO} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.933 \\ 1.286 \\ 0.388 \\ 0.0322 \end{bmatrix}
\] (33)

The sample domain (1) and frequency domain spectrum of (3) for the coefficients (33) using \( N=256 \) is illustrated in Figs. 1-2. The peak sidelobe level \( \sigma_s \) for the ISO flattop window is approximately -84 dB.

The stopband edge \( \omega_s \) is determined as the frequency on the mainlobe where the amplitude is equal to the peak sidelobe level \( \sigma_s \). The design specification is derived from the ISO flattop window as follows; we first find the passband limit as the angular frequency \( \omega_p \) where

\[
W_{ISO}(\omega_p) = 2 - ||W_{ISO}(\omega)||_\infty
\] (34)

We then define

\[
\begin{align*}
\sigma_p(\omega) & = \sigma_p = ||W_{ISO}(\omega)||_\infty - 1, \ \omega \in \Omega_p \\
\sigma_s(\omega) & = \sigma_s = ||W_{ISO}(\omega)||_\infty, \ \omega \in \Omega_s \\
W_d(\omega) & = \begin{cases} e^{j\omega W_{ISO}(\omega)}, & \omega \in \Omega_p \\
0, & \omega \in \Omega_s \end{cases}
\end{align*}
\] (35-37)

The E-ISO window is obtained by using the optimization described in section 4, see Figs. 3. The E-ISO window is slightly (not observable) flatter in the passband compared...
to the ISO flattop window, see Fig. 4. The optimized amplitude margin is \(20 \log \delta = -5\) dB, i.e. the E-ISO window’s peak sidelobe attenuation is -89 dB compared with the ISO flattop window peak sidelobe attenuation of -84 dB, see Fig. 5.

5.2. M-ISO Flattop Window

With the proposed method, it is possible to further improve all flattop windows, such as the ISO flattop window or the Hewlett Packard flattop window series P-301 and P-401. A simple way to perform this is to increase the number of coefficients \(K\) in equation (1), which will offer more degree of freedom for the optimization. The modified ISO flattop window M-ISO illustrates this by enhancing the peak sidelobe performance for the ISO 18431-2 flattop window. The number of coefficients is increased to \(K = 8\) and the peak sidelobe level is approximately -96 dB compared to -84 dB, see Fig. 6-8

5.3. DNCA-150 High Accuracy Window

High accuracy windows are needed in today’s Sigma-Delta ADC’s, where high frequency noise is generated. Suppression of such noise often need more than 150 dB suppression for a 24 bit-equivalent 1 bit-Sigma-Delta ADC. When a window with such high sidelobe suppression performance is required, the Kaiser window [11] can be used. However, by using the proposed design method, it is possible to design high accuracy windows based on (1), i.e. the principle of using the summation of shifted Dirichlet kernels. In this example the ripple in the passband flatter than most of the commonly used flattop windows, the mainlobe is wider and the peak sidelobe attenuation is more than 150dB, see Figs 12-9. The sample domain representation of the DNCA-150 high accuracy window is shown in Fig. 12 together with the sample domain representation of the ISO flattop window. The DNCA-150 high accuracy window has \(K = 9\) coefficients given by

\[
an_{DNCA-150} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} 1.00000000000000 \\ 1.99819039793566 \\ 1.91411546196872 \\ 1.49894094341492 \\ 0.78616787563880 \\ 0.23463000608392 \\ 0.0329371146806 \\ 0.00145385349501 \\ 0.0000322104626 \end{bmatrix} \tag{38} \]

6. SUMMARY AND CONCLUSION

A new high accuracy window design method based on semi-infinite linear programming by using the Dual Nested Com-

Fig. 3. Frequency domain representation of the enhanced ISO design E-ISO window \(|W_{E-ISO}(\omega)|\) for length N=256.

Fig. 4. Frequency domain representation of the enhanced ISO design E-ISO window \(|W_{E-ISO}(\omega)|\) and the ISO flattop window \(|W_{ISO}(\omega)|\) for length N=256 in the passband. Note: E-ISO and ISO curves are overlapping in this figure.

Fig. 5. Frequency domain representation of the enhanced ISO design E-ISO window \(|W_{E-ISO}(\omega)|\) and the ISO flattop window \(|W_{ISO}(\omega)|\) for length N=256 in the stopband.
Fig. 6. Frequency domain representation of the modified ISO window $|W_{M-ISO}(\omega)|$ for length $N=256$.

Fig. 7. Frequency domain representation of the modified ISO window $|W_{E-ISO}(\omega)|$ for length $N=256$ in the passband.

Fig. 8. Frequency domain representation of the modified ISO window $|W_{E-ISO}(\omega)|$ for length $N=256$ in the stopband.

Fig. 9. Frequency domain representation of the DNCA-150 high accuracy window $|W_{DNCA-150}(\omega)|$ for length $N=256$.

Fig. 10. Frequency domain representation of the DNCA-150 high accuracy window $|W_{DNCA-150}(\omega)|$ for length $N=256$ in the passband.

Fig. 11. Frequency domain representation of the DNCA-150 high accuracy window $|W_{DNCA-150}(\omega)|$ for length $N=256$ in the stopband.
Fig. 12. Sample domain representation of the DNCA-150 high accuracy window $w_{\text{DNCA-150}}(n)$ and the ISO flattop window $w_{\text{ISO}}(n)$ for length $N=256$.

plex Approximation (DNCA) algorithm is presented. Design examples of data windows for DFT or FFT usage illustrates the capability of the design method. One example shows how to enhance the sidelobe attenuation of the new upcoming ISO 18431-2 flattop window. The results indicates that 5dB higher sidelobe attenuation can be achieved without affecting the passband properties and an enhanced coefficient set is achieved. However, with the proposed method, it is possible to design specialized windows with 150 dB sidelobe attenuation. The proposed design method is advantageous due to its flexibility with respect to additional specifications in the frequency domain. During the optimization process, constraints such as lower passband ripple, null response at certain frequencies, etc. is taken into consideration.

7. REFERENCES


