ABSTRACT: Parametric Resonance Vibration in cables of cable-stayed bridges is mainly studied when the excitation frequency is close to or twice the cable natural frequency. It is, however, important to consider other cases for this frequency relationship, since among other factors, cable-parametric resonance vibrations are strongly depending on the displacement amplitude at the cable supports. Consequently, the present research work is focused on determining, by experimental and numerical analysis, the instability conditions for stay cables subjected to parametric resonance within a wide range of frequency ratios. This is accomplished, by finding the minimum displacement required at the cable supports in order to induce non-linear vibration of considerable amplitude at the cable. Once the cable characteristics (geometry, material properties, inherent damping and initial tensile preload) are known, the instability conditions are identified and expressed in a simplified and practical way in a diagram. Numerical results are compared to those obtained by experimental analysis carried out on a simplified scaled model (1:200) of the Öresund Bridge. A good agreement between numerical and experimental results is found.

1. INTRODUCTION

When periodical displacements are registered at the cable supports in the tower and/or the deck of a cable-stayed bridge, tensile oscillations are induced in the cables which can originate transversal vibrations characterized by large amplitudes. This phenomenon is known as Cable-Parametric Resonance Vibration or cable vibration due to parametric excitation and its mathematical background originally remains on the application of the Mathieu Equation [1]. Most specific numerical models describing the cable vibration under parametric excitation have been developed, showing a clear non-linear relationship between the excitation amplitude at the cable support and the cable-vibration amplitude [2, 3].

It is commonly found that Cable-Parametric Resonance Vibration is evaluated when the excitation at the cable supports is at a frequency which is close to and/or twice the cable natural frequency; since in these cases, minimal excitation amplitudes could induce large amplitude vibration at the cables. However, this phenomenon can also occur under other conditions. In fact, cable parametric resonance vibration can occur within a broad range of frequencies when larger excitation amplitudes are registered at the cable supports [2].

The instability conditions for a system subjected to parametric excitation are generally represented on graphics by regions as functions of non-dimensional parameters [4, 5], being the frequency ratio, $R$ (Excitation frequency normalized to the lowest natural frequency of the system) one of the most important variables to be considered. Since the studied phenomena here depends on different factors, the instability conditions can be found as a function of parameters such as the excitation amplitude at the cable supports, the cable-inherent damping, ratio sag to span, etc. Therefore, to express in a general
way, the instability conditions for cables is rather complicated. It implies the involvement of several parameters and as a consequence, several graphics are shown depending on those variables.

Instead, in the present work, the instability conditions are determined specifically for a cable in particular. Once the cable characteristics (geometry, material properties, inherent damping and initial tensile preload) are known, the corresponding governing equation is evaluated and, consequently, the instability conditions are identified and expressed into one curve. In such analysis, the instability conditions are given by the minimum displacement required at the cable support in order to produce non-linear and significant amplitude vibration at the cable. This analysis is done in a rather short time thanks to a computational subroutine, offering the possibility to evaluate all cables from a cable-stayed bridge in a relative short time as well.

The results here are obtained from numerical and experimental analyses based on a scaled model (1:200) of the Öresund Bridge, which is part of an important link that joins Sweden and Denmark through the cities of Malmö and Copenhagen. In this bridge, large amplitude vibrations were found at the longest cable stays, and in a preliminary analysis, parametric excitation was considered as one of the vibration sources [6].

The numerical and experimental methodologies employed here are explained in section 2 and 3, respectively. The corresponding results are shown and discussed in chapter 4, while concluding remarks are presented in chapter 5.

2. MODELLING CABLE-PARAMETRIC RESONANCE VIBRATION.

2.1 Numerical Model

The numerical model employed here corresponds to the one proposed by Lilien and Pinto [2], in which, one cable end is kept fixed and the other one is free to move (See Figure 1).

![Figure 1. A Cable Stay model studied by Lilien and Pinto [2].](image)

The corresponding governing equation can be written as follows:

\[
\ddot{Y} + b \dot{Y} + \omega_1^2 \left[ 1 + \frac{X_{cs}}{X_0} + \frac{4}{3} \left( \frac{Y}{\sqrt{K}} \right)^2 \right] Y = 0
\]

(1)

where: \(Y\) - cable transversal vibration; \(b\) – inherent cable damping coefficient; \(\omega_1\) – first natural frequency of the cable; \(X_{cs}\) – displacement of cable support; \(X_0\) – initial stretching of cable.

In equation (1), \(K\) is defined as follows:

\[
K = \frac{4}{\pi} \sqrt{\frac{X_0 l}{2}}
\]

(2)

where: \(X_0\) - cable initial stretching; \(l\) - original cable length.

The 1st-cable natural frequency, \(\omega_1\) is obtained from the theory of a taut string [7], with \(n=1\), as follows:
\[ \omega_n = \frac{n\pi}{l} \sqrt{\frac{T_0}{\rho A}} \]  

(3)

where: \( T_0 \)-the initial tensile load of the cable; \( \rho \)-the cable density; \( A \)-cable cross section area.

The initial stretching of the cable, \( X_0 \), could be determined by Hooke’s Law by knowing the initial tensile load, \( T_0 \), and its axial stiffness expressed on terms of the original cable length, \( l \), its cross section area, \( A \), and its Young Modulus, \( E \), as follows:

\[ X_0 = \frac{T_0 L}{AE} \]  

(4)

2.2 Cable characteristics, loading and excitation conditions

In order to carry out a numerical analysis based on the governing equation expressed by equation (1), it must be known first, the cable characteristics (material, inherent damping and geometrical properties), its loading condition, which is given by the initial tension force, \( T_0 \), and the characteristics of the excitation at the cable support, \( X_{cs} \). Parameters as the 1st-natural frequency of the cable, \( \omega_1 \), and its initial stretching \( X_0 \) are related to the cable characteristics and to the initial tension, \( T_0 \), according to equations (3) and (4).

The geometrical, material properties and inherent-relative damping, \( \zeta_c \), of the cable are taken from an experimental model utilized here. These characteristics are shown in Table 1. Notice that, the cable-inherent damping is determined as an average of the corresponding values found from experimental modal analysis.

Table 1. Cable Characteristics according to Experimental Model

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cable density, ( \rho ) [kg/m³]</td>
<td>7800</td>
</tr>
<tr>
<td>Cable Diameter [m]</td>
<td>3x10⁻⁴</td>
</tr>
<tr>
<td>Cable Elasticity Modulus, ( E ) [N/m²]</td>
<td>205x10⁹</td>
</tr>
<tr>
<td>Cable Length, ( L ) [m]</td>
<td>1.33</td>
</tr>
<tr>
<td>Cable Relative Damping, ( \zeta ) [%]</td>
<td>0.03</td>
</tr>
</tbody>
</table>

In the scaled model of the bridge, large displacement amplitude is expected at the cable support located near the top of the tower when the tower is vibrating under its 1st-bending vibration mode (in-plane), which is at 44.5 Hz, according to experimental modal analysis. Therefore, in order to conveniently have large enough displacements at the mentioned cable support, it was chosen to excite the bridge tower at its vibration mode. Such frequency represented the main excitation frequency for the parametric-resonance vibration of the cable when developing the experiments.

Since the cable support motion, \( X_{cs} \), is assumed to be periodical with a simple frequency, it can be written as:

\[ X_{cs} = X_d \sin(\omega_d t) \]  

(5)

where: \( X_d \)-the excitation amplitude; \( \omega_d \)-the excitation frequency; \( t \)-the time.

The displacement amplitude, \( X_d \), at the cable support is chosen to be swept from zero to an upper limit, which will depend on the dynamic characteristics of the bridge tower. This is in order to identify, within an amplitude range, the minimum required excitation at the cable support capable to induce a significant vibration at the cable. This is also a good choice when detecting non-linear behaviour between the excitation amplitude and the cable vibration.

The estimation of the frequency ratio, \( R \), given by the excitation frequency and the natural frequency of the cable is also considered before carrying out the numerical analysis. Such ratio is as:

\[ R = \frac{\omega_d}{\omega_1} \]  

(6)

where: \( \omega_d \) - excitation frequency; \( \omega_1 \)- the cable-1st natural frequency.
Since in the experiments the excitation frequency is chosen to be the first natural frequency (bending), \( \omega_{b1} \), of the bridge tower, then, the only way to experimentally vary the ratio \( R \) will be by changing the 1\textsuperscript{st}-natural frequency of the cable, which can be reached by changing its initial tensile load, \( T_0 \), and keeping fixed its geometrical and material properties. However, it is important to indicate that in a practical situation, it will be expected a frequency ratio \( R \), varying due to several excitation frequencies (depending on the dynamic properties of the bridge and the environmental surrounding) and not because of changes at the cable natural frequency, as it was conveniently chosen for the experimental analysis carried out here.

2.3. Determining the instability conditions.

The cable parametric vibration described by equation (1) is a Non-Linear Ordinary Differential Equation of second order. A methodology for solving this type of equations is based on recognizing the corresponding linear system and the non-linear components, which, consequently, will be considered as external forces acting into the system [8].

Then, equation (1) can be re-organized and written as:

\[
\ddot{y} + h \dot{y} + \omega_0^2 y = \left[ \frac{X_{\text{inc}}}{X_0} + \frac{4}{3} \left( \frac{y}{R} \right)^3 \right] y
\]  

(7)

In equation (7), the first member is clearly defining a mechanical-linear system with a Single Degree of Freedom (SDOF), subjected to what is assumed to be an external force given by the right member of this equation.

![Figure 2. Numerical Results when ratio \( R \) is 0.75. (a) Time history simulation of Excitation Amplitude at Cable Support. (b) Time history simulation of Cable displacement at its mid point. (c) Relationship between excitation amplitude and cable displacement amplitude.](image)
The response, \( Y \), to a given input, \( X_{cs} \), is calculated from (7) using a routine where the SDOF system is described by a digital filter. In each time step, a nonlinear equation has to be solved [8].

Once the corresponding cable midpoint response is obtained, the next step is to identify the instability conditions for cable parametric resonance vibration. In this research, such conditions are represented by the minimum displacement required at the cable support in order to induce a non-linear cable vibration of significant amplitude and the corresponding frequency ratio \( R \).

In Figure 2 is shown as example the numerical results when the ratio \( R \) is equal to 0.75, the excitation frequency is at 44.5 Hz and the geometrical and material properties of the cable are corresponding to those given in Table 1.

As seen in figure 2.a, the displacement at the cable support presents an amplitude swept from zero to an upper limit. After reaching an excitation-amplitude of \( 1.55 \times 10^{-3} \) m. is noticed how the cable displacement rises significantly and suddenly from almost zero displacement (see figures 2.b and 2.c). Therefore, this excitation amplitude is considered as the minimum displacement required at the cable support for inducing the instability, when the ratio \( R \) is 0.75.

Then, by repeating this process for each value of the frequency Ratio, \( R \), the instability conditions can be determined for a frequency range in particular.

3. EXPERIMENTAL SET UP AND METHODOLOGY

The experimental set up used in this research is shown in figure 3. It is based on a simplified scaled model (1:200) of the Öresund Bridge, made of aluminum, whose cables were replaced by strings made of steel. Since, in the original bridge the largest vibration amplitudes were registered at the longest cables [6], only this cable was mounted on the experimental set up.

The experimental set up is also constituted by a shaker, a signal generator, an amplifier and a force transducer, in order to introduce and measure the excitation force applied on one of the bridge towers; an accelerometer to estimate the cable support displacement on the bridge tower; and a laser vibrometer in order to estimate the displacements at the cable midpoint.

Initially, an experimental modal analysis was carried out on the scaled model. Most of the information obtained is used as a data and reference for the numerical analysis.

The experiments for studying the cable-parametric resonance vibration basically consisted in exciting the bridge tower where the cable is connected, by introducing a sinusoidal force, swept in amplitude and with a simple frequency. Then, the displacements of the cable midpoint and cable support at the tower were estimated by using the laser vibrometer and an accelerometer, respectively. The cable displacements are measured in plane of the cable and not out of plane as represented in figure 3, which corresponds to a preliminary analysis [9].
The goal of the test is to identify the displacement amplitude at the cable support capable to induce a significant and non-linear vibration at the cable when a particular relationship between the excitation frequency and the corresponding 1\textsuperscript{st} natural frequency of the cable is obtained. This is possible by tuning the cable before the experiments to a particular frequency, while the excitation frequency is chosen to be the corresponding one to the 1\textsuperscript{st}-bending vibration mode of the bridge tower, for reasons that will be explained later.

4. RESULTS & DISCUSSION

4.1 Numerical results

The instability conditions numerically estimated for a cable with characteristics according to table 1 are shown in figures 4 and 5. In figure 4 the frequency ratio, $R$, was varied through the 1\textsuperscript{st} natural frequency of the cable and keeping the excitation frequency fixed at 44.5 Hz. Since in a practical situation it would be expected to have a frequency ratio, $R$, varying because of the excitation frequency, the instability conditions were also numerically estimated under this situation. Then, two cases were studied, when the cable natural frequency was fixed to 70 Hz and 100 Hz (Figure 5).

![Figure 4](image1.png)

Figure 4. Estimated Instability Conditions for string (L=1.33m, Ø0.3mm) when excitation frequency is kept fixed at 44.5 Hz.

![Figure 5](image2.png)

Figure 5. Estimated Instability Conditions for string (L=1.33m, Ø0.3mm) when cable natural frequency is kept at 70 Hz and 100 Hz.
As it was expected, according to figures 4 and 5, the cable is found to be more prone to vibrate under parametric resonance when the frequency ratio, $R$, is equal to 2 than in any other condition given by this frequency ratio. Under this case, the parametric resonance vibration is induced by the lowest excitation amplitude within the range of frequency in consideration. In figures 4 and 5 the instability conditions curve also shows dropping points around frequency ratios equals to $R=2$, $R=1$, $R=0.67$, $R=0.5$, $R=0.4$ and $R=0.33$. These values correspond to the frequency ratios which define the unstable regions for parametric resonance vibration [2].

By obtaining a curve for the instability conditions as shown in figures 4 and 5, it could be possible to determine if such cable would vibrate due to parametric excitation, once the characteristics (frequencies and amplitudes) of the excitation at the cable supports are known.

Firstly, by considering the different vibration mode frequencies of the bridge, it will lead to determine the corresponding relationships (frequency ratios, $R$) between the natural frequency of the cable and the different possible excitation frequencies [10, 11]. Then, by estimating a dynamic response of the bridge according to its modal properties and the expected external forces to be acting on (wind, traffic loads), the excitation displacement at the cable supports could be known. Therefore, a curve as shown in figure 5 will be a useful tool, since what remains is to check if the excitation amplitude at the cable support is large enough for inducing parametric resonance vibration of the cable, under the corresponding frequency ratio $R$.

One important advantage of this method, is that offers, by simply looking at one curve, the possibility to evaluate a cable under a broad range of frequency ratios and not only when $R$ is equal to $R=2$, $R=1$ or $R=0.5$. The determination of the instability condition curve for each cable is rather fast, since it takes approximately 30 minutes by running a script in Matlab on a regular PC. Then, the checking process for all cables in a bridge could also be done in a rather short time.

4.2 Experimental Results

Experimental evaluations under conditions defined by a frequency ratio, $R$ equal to 0.48, 0.5, 0.59, 0.64, and 0.66, were carried out in order to confirm the corresponding numerical results. The instability conditions could be determined from the relationship between the cable response and the corresponding excitation at the cable support.

In all cases, except when the frequency ratio was 0.48 and 0.50, the cable vibration could reach the instability at certain moment, characterized by a large amplitude displacement. However, when $R$ was 0.48 and 0.50, the cable could also reach large amplitude vibration, but in direct proportion to the excitation amplitude. That occurred at excitation amplitudes much smaller than those numerically estimated in order to induce the instability of the cable vibration. In the case of $R=0.5$, for example, the numerical calculation gives 1.65 mm as minimum excitation amplitude in order to induce such instability (See figure 4). However, in this case, large amplitude vibration occurred by applying about 3 times smaller excitation amplitudes.

This could be explained by looking at the spectrum of the cable response. In general for the experiments where the instability was reached, it was observed a first stage previous to the instability, where the cable used to vibrate, in proportion to the excitation amplitude, at the same frequency of the excitation and its corresponding harmonics. Then, when the instability was already reached, the vibration was characterized by multiple frequencies (See figure 6, in which $R=0.66$). In the cases where $R$ was 0.48 and 0.50, the cable vibrates, in proportion to the excitation amplitude, at the excitation frequency and at its 1st harmonic which is coincident with the 1st natural frequency of the cable. That could explain why the cable could easily reach large amplitude vibration with much smaller excitation amplitudes than those required in order to induce the parametric resonance vibration.
Figure 6. Experimental Cable Response Spectrum when $R=0.66$. (a) Before the instability occurs. (b) When the instability already occurs.

In figure 7 are shown the experimental-cable responses when $R=0.59$, $R=0.64$ and $R=0.66$ and the excitation frequency was kept at 44.5 Hz.

Figure 7. Cable Response and excitation displacement for frequency ratios $R=0.59$, $R=0.64$ and $R=0.66$. 
As seen in figure 7, the excitation amplitude at the cable support was gradually swept. At certain moment and for a particular value of the excitation amplitude the instability in the cable vibration could be reached. Then, an experimental determination of the corresponding instability conditions could be done. These values were clearly obtained by establishing a relationship between the excitation amplitude and the cable amplitude displacement. Then what remains is look at the excitation amplitude capable to induce the sudden jump in the cable vibration amplitude (See figure 8).

Figure 8. Experimental Relationship between Excitation Amplitude and Cable Displacement when $R=0.59$, $R=0.64$ and $R=0.66$ (Cable: $L=1.33m$, Ø0.3mm).

4.3 Comparison between Numerical and Experimental Results

A comparison between the instability conditions obtained by numerical and experimental analysis is shown in table 2.

<table>
<thead>
<tr>
<th>Freq. Ratio, R</th>
<th>Excitation Amplitude by Experiments $1x10^{-3}$ [m]</th>
<th>Excitation Amplitude by Numerical Analysis $1x10^{-3}$ [m]</th>
<th>Difference [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.59</td>
<td>1.61</td>
<td>1.79</td>
<td>10.06</td>
</tr>
<tr>
<td>0.64</td>
<td>0.83</td>
<td>1.05</td>
<td>20.95</td>
</tr>
<tr>
<td>0.66</td>
<td>0.77</td>
<td>0.71</td>
<td>8.45</td>
</tr>
</tbody>
</table>

In practice, it is challenging to define exactly a frequency ratio when doing the experiments, since a small variation in the measured cable resonance frequency or even in the excitation frequency can lead to evaluate a slight different value for the expected frequency ratio. From the instability curves shown in figures 4 and 5, it is noticed that the curves drop and rise sharply around the critical frequency ratios (e.g., $R=2$, $R=1$, $R=0.67$, $R=0.5$, $R=0.4$, etc.). That could explain why a maximum difference of 21% between the numerical and experimental results was obtained when the ratio $R$ was 0.64.

As a complementary test, a case when $R=1$ was also evaluated. In this case a cable of 1 mm in diameter was used. As in the case of $R=0.48$ and $R=0.5$, the cable vibrated reaching large amplitudes in a direct proportion to the excitation. The main vibration occurred at the excitation frequency which matches the natural frequency of the cable. The excitation amplitudes applied were also much smaller than the numerically estimated in order to induce the instability of the cable vibration.
5. CONCLUSIONS

Instability conditions for a cable subjected to parametric resonance vibration could be numerically estimated for a broad range of frequency ratios and expressed into one curve. In this way, it could be known if any cable in a cable-stayed bridge would vibrate due to parametric excitation, once the cable characteristics are defined, as well as the conditions of the excitation at the cable support. The process for determining the instability conditions within a broad range of frequency ratios is rather fast and offers as an advantage the possibility of evaluating, in a practical situation, all frequency ratios under which each cable is really subjected to, and not only when such ratio is $R=2$, $R=1$ and $R=0.5$.

According to the experimental analysis, cables under frequency ratios equals to 0.5 and 1 could develop large vibration amplitudes without reaching any instability. In fact, a proportional relationship between the cable response and the excitation was observed where large vibration amplitudes could be obtained with a much lower excitation amplitude than that one required in order to induce the instability in the cable. This result could reinforce or justify the tendency of only evaluating cables under frequency ratios equal to 0.5 and 1, besides $R=2$, where is clearly known as the most critical condition for parametric resonance vibration at the cable.

Although the estimated instability conditions and the experimental results confirmed that cables are more prone to vibrate under frequency ratios close or equal to $R=2$, $R=1$ and $R=0.5$ than in other cases, it is still important to consider other conditions, since the excitation amplitudes at the cable support could be large enough for inducing the parametric resonance vibration of the cable for one or several frequency ratios $R$ in which the cable could be subjected to.

6. REFERENCES