Analysis of the Behaviour of Standing Acoustic Waves in a Cubically Nonlinear Medium

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Abstract:
The basic distinction between linear and nonlinear acoustics is given in the introductory part. The differences between nonlinear acoustic waves in quadratic and cubic media are briefly explained.

The behavior of standing waves, including the wave profile evolution, is described.

A perturbation analysis leading to the wave profile for a standing wave in a cubic medium is presented. Finally, interpretation of the solutions containing shock waves and verification of the matching of the inner and the outer solutions through an intermediate solution at the regions containing shock fronts are made. The plots were generated using MATLAB.

Keywords: Cubic nonlinear medium, perturbation solutions, shock regions, matching, inner and outer solutions, wave profile.
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1. **Notations**

\[ A \] Amplitude of vibration

\[ a \] Square root of the difference between the mean intensity and normalized discrepancy

\[ b \] Absorption coefficient

\[ c \] speed of sound

\[ D \] Dissipation coefficient

\[ d \] Dissipation parameter

\[ I \] Frequency of vibration

\[ J \] Mean intensity

\[ I \] Wave number

\[ L \] Length of resonator

\[ T \] ‘slow’ time

\[ t \] time

\[ u \] Particle velocity

\[ x \] Slow distance traversed by the wave

\[ y \] mean intensity

\[ \delta \] Normalized discrepancy

\[ \varepsilon \] Parameter of acoustic nonlinearity

\[ \xi \] ‘fast’ time
Δ Dimensionless discrepancy
τ Retarded time
ω Lowest resonance frequency
Γ Normalized dissipation coefficient
∈ ‘Small term’ used as epsilon in the perturbation solution
∞ Infinity

Subscripts
out outer
in inner
int intermediate
l longitudinal
t transverse

Superscript
- negative side
+ positive side

Abbreviations
NDA Non-destructive analysis
NDT Non-destructive testing
2 Introduction

The area of acoustics that is concerned with the study of the nonlinear propagation of sound waves is called nonlinear acoustics. The phenomenon of nonlinearity is based on the dependence of resonance frequencies on the amplitude of the vibration. According to the linear theory of acoustics, increasing the level of a sound source will only increase the intensity but the sound field remains the same. The theory also predicts that only frequency components radiated directly by the source can be present in the sound field. These principles however, do not hold in nonlinear acoustics because of their dependence on the amplitude of vibration.

The presence of defects, cracks and/or any other material/structural inhomogeneities could result in phase and/or amplitude variations of incoming signals while the frequency remains the same. With strong nonlinear effects, acoustic signals could experience significant wave distortion and changes in their frequency content as they propagate, giving rise to shock waves. These nonlinear effects occur in gases, liquids and solids as well and they are observed over a broad range of frequencies.

To demonstrate this nonlinear effect where the resonance frequencies depend on the amplitude of the vibration, we use the illustration in figure 2.1. It shows the difference between varying amplitudes in a linear and a nonlinear system. The nonlinearity could be the presence of a crack in a test structure or a test material. The crack would introduce the nonlinearity which results in frequency shifts.

2.1 Some common applications

The interest on the nonlinear acoustic effect in solids is growing due to promising applications of these effects for non-destructive testing and in characterization of materials and structures.

The principles of nonlinear acoustics is basic to the procedure involved at megahertz frequencies as is applied in medical ultrasound and radio transmission as well as in nondestructive evaluation of materials (NDE) [1]. NDE involves the probing and sensing of material structure and properties without causing damage to it. Recently, it has become diversified and is a
multidisciplinary technology, drawing on the fields of applied physics, artificial intelligence, electronics, biomedical engineering, mechanical engineering and more. Basically, NDE techniques have been used mainly for the detection of macroscopic defects (especially cracks) in materials and structures which have been manufactured or placed in service. This procedure is commonly referred to as nondestructive testing (NDT).

Furthermore, nonlinearity can induce changes in non fluctuating properties of the medium. These include acoustic streaming, which involves the steady flow produced by the absorption of sound, and radiation pressure, which results in a steady force exerted by the sound on its surroundings.

In general, nonlinear acoustics has been studied extensively and solutions of some model equations solved. However, in most cases, the nonlinearity is quadratic and the model equations are usually Burgers’ or generalized Burgers’ equation. It is therefore interesting to investigate how the methods used for quadratic nonlinearities are applied in cubic nonlinearities.

Figure 2.1. Illustration of linear versus nonlinear wave resonance behavior in an acoustic medium. In (a), the medium is linear and varying the amplitude of vibration does not yield frequency shift; only the
resonance modes are present. In (b), the medium is inhomogeneous (nonlinear) probably because of a crack in the material. The modal frequencies therefore depend on the amplitude of the vibration and there is a frequency shift with varying amplitude. This is a nonlinear effect: a change in wave frequency with wave amplitude [2].

2.2 Quadratic nonlinearity versus cubic nonlinearity

Having indicated the basic difference between linear and nonlinear acoustic waves, it is pertinent to further distinguish between nonlinear acoustic waves in quadratic medium and that in cubic medium. By this, we of course mean the response of the medium in which the sound propagates, and not to the nonlinear behavior of the sound source.

Cubic nonlinearity contains both shocks of compression and shocks of rarefaction for each period of vibration. This is not the case for quadratic nonlinearity which contains only shocks of compression. Cubic nonlinearity also alters the velocity of propagation in a wave front.

In addition, while waves in fluids and longitudinal waves in general are normally modeled by a quadratic nonlinear wave equation, transverse wave propagation is modeled by a cubically nonlinear wave equation. This is because the quadratic nonlinearity cancels. [3]

There is a recent interest in the studies of nonlinearity in a cubic medium. For instance, in shear wave excitation as applied in medicine, the absorption of acoustic wave gives rise to radiation pressure which is associated with momentum transfer from the wave to the medium. Subsequently, shear stresses are generated across the bulk of the medium and this result in a time-modulated ultrasound beam being a source of shear waves. Ultrasonic excitation of shear waves is a promising method of elasticity imaging of biological tissue.

In addition, shock waves have been recently used to interpret some geophysical phenomena in nature. One example is in the use of shock waves
dynamics to explain many aspects of volcanoes’ explosive eruptions. [1] Figure 2.2 illustrates a simple case of shock of compression.

Figure 2.2. An illustration of a simple case of shock of compression; there is an abrupt increase in the pressure before the expansion process begins. That can be seen from the jump from a low pressure to a high pressure.
3 Physical background and analysis of standing wave in cubic nonlinear medium

The simplified method of studying standing waves in quadratic resonators was by adopting the approach that relies on the linear superposition of two counter-propagating nonlinear waves. In a linear one-dimensional system, a standing wave is made up of two plane waves moving in opposite directions. This idea is adopted for nonlinear standing waves inside rigid immovable walls and the vibration in the layer is described as the sum of two Burgers traveling waves.

We adopt the method described in [4] and choose the model wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = -\frac{2\epsilon}{3c^4} \frac{\partial^2 u}{\partial t^2}$$

(3.1)

where $\epsilon$ is the parameter of acoustic nonlinearity, ‘$u$’ is the particle velocity and ‘$c$’ is the speed of sound. Equation (3.1) contains the cubic nonlinearity and is the simplest differential equation of the 2nd order. From the method described in [5], the simplified evolution equation of 1st order can be derived from (3.1):

$$\frac{\partial u}{\partial x} - \frac{\epsilon}{c^3} u \frac{\partial u}{\partial \tau} = 0$$

(3.2)

Equation (3.2) describes Riemann waves traveling through the cubic nonlinear medium, where, $x$ represents the ‘slow’ distance traversed by the wave and $\tau = t - x/c$ is the retarded time. The solution to equation (3.1) as described in [3] can be written as

$$u = F \left[ \omega t - \frac{\omega}{c} (x - L) + \frac{\epsilon \omega}{c^3} (I + F^2) (x - L) \right] -$$
The boundary conditions for a resonator at \( x = L \) correspond to a fixed right-hand boundary of the resonator and also the vibration velocity (3.3) equal to zero at \( x = L \). Let the other end of the resonator \( x = 0 \) vibrate according to the law
\[
u(x = 0, t) = Af(\omega t)
\] (3.4)

where \( f \) is a periodic function with the period \( 2\pi \); its mean value is zero, \( \langle f \rangle = 0 \). For harmonic vibration of the boundary, we will use the equation \( f = \sin(\omega t) \). Now, using the boundary condition (3.4), the equation (3.3) can be reduced to a nonlinear functional equation

\[
F\left[\omega t + kL - \frac{\varepsilon}{c^2}kL(I + F^2)\right] - F\left[\omega t - kL + \frac{\varepsilon}{c^2}kL(I + F^2)\right] = Af(\omega t)
\] (3.5)

where \( k = \omega/c \) is the wave number. Consequently, the harmonic boundary vibration is considered as \( f = \cos(\omega t) \).

Using the method described in [3], the functional equation (3.5) is reduced to the differential one:

\[
\frac{\partial U}{\partial T} + \left(\Delta - \pi \varepsilon J - \pi \varepsilon U^2\right)\frac{\partial U}{\partial \xi} - D \frac{\partial^2 U}{\partial \xi^2} = -\frac{M}{2}\cos \xi
\] (3.6)

\( \Delta \) is the dimensionless discrepancy and is taken to be a small quantity.

Noting that the same dimensionless notations as used in [3] which are:

\[
U = \frac{F}{c}, M = \frac{A}{c}, J = \frac{I}{c^2}, \xi = \omega t, T = \frac{\omega t}{\pi}, \Delta = \frac{\omega - \omega_0}{\omega_0}, D = \frac{b \omega^2}{2 c^3 \rho}
\] (3.7)

were also used here. We will now briefly describe the characteristics of standing acoustic waves in a cubic nonlinear medium.

There are cases when the standing waves are free, forced or forced-shocked. A brief discussion of each of them will be made.
3.1 Free standing waves

This is the simplest case of free vibrations. For a resonator that is fixed at both ends so that \( u(x = 0, t) = u(x = L, t) = 0 \). At the moment when \( t=0 \), the strong vibration is excited between immovable walls and changes in the acoustic field in without inflow of additional energy. In this case, the homogeneous equation is considered so that the right hand side of equation (3.6) becomes zero.

\[
\frac{\partial U}{\partial T} + \left( \Delta - \pi \varepsilon I - \pi \varepsilon U^2 \right) \frac{\partial U}{\partial \xi} - D \frac{\partial^2 U}{\partial \xi^2} = 0
\]  

(3.8)

3.2 Forced standing waves

In this case [6], the inhomogeneous equation is considered and the weakly nonlinear solution to equation (3.6) is in the form

\[
U = A(T) \cos \xi + B(T) \sin \xi
\]  

(3.9)

When terms are separated at \( \sin \xi, \cos \xi \), we derive a system of two ordinary coupled equations:

\[
\frac{dB}{dT} - \left[ \Delta - \frac{3\pi \varepsilon}{4} \left( A^2 + B^2 \right) \right] A + DB = 0,
\]

\[
\frac{dA}{dT} + \left[ \Delta - \frac{3\pi \varepsilon}{4} \left( A^2 + B^2 \right) \right] B + DA = -\frac{M}{2}
\]  

(3.10)

An analytical solution that corresponds to the steady-state regime of vibration is reached at \( T \rightarrow \infty \) when temporal derivatives are zero: \( dA/dT = dB/dT = 0 \). For steady-state regime of vibration, the following equation can be derived from the system (3.10)

\[
\left[ \Delta - \frac{3\pi \varepsilon}{4} \left( A^2 + B^2 \right) \right]^2 + D^2 = \frac{M^2}{4(A^2 + B^2)}
\]  

(3.11)
Using the following notations

\[ \delta_i = \frac{\Delta}{C}, d = \frac{D}{C}, y = \frac{3}{2} \pi \varepsilon \frac{J}{C}, C = \left( \frac{3 \pi \varepsilon M^2}{16} \right)^{1/3}, J = \frac{1}{2} (A^2 + B^2) \]  

(3.12)

The equation (3.11) is reduced to the following simple form

\[ \delta_i = y \pm \sqrt{\left( \frac{1}{y} - d^2 \right)} \]  

(3.13)

Figure 3.1 [7] below shows the resonant curves for the mean intensity \( y \), as a function of discrepancy \( \delta_i \) for different values of dissipation parameter: \( d = 2, 1.25, \) and \( 0.75 \). It can be clearly seen that as the dissipation decreases, the frequency response distorts in its shape.

![Resonant curves for the mean intensity of vibrations in a cubically nonlinear resonator for different values of dissipation parameter 'd'.](image)

**Figure 3.1.** Resonant curves for the mean intensity of vibrations in a cubically nonlinear resonator for different values of dissipation parameter 'd'.
3.3 **Forced shocked waves**

The equilibrium state is reached at $T \to \infty$, is governed by the ordinary differential equation following from (3.6) for $\frac{\partial U}{\partial T} = 0$:

$$
(\Delta - \pi \varepsilon J - \pi \varepsilon U^2) \frac{dU}{d\xi} - D \frac{d^2U}{d\xi^2} = -\frac{M}{2} \cos \xi \tag{3.14}
$$

After integration, the first-order equation is obtained

$$
D \frac{dU}{d\xi} + \frac{\pi \varepsilon}{3} U^3 + (\pi \varepsilon J - \Delta)U = \frac{M}{2} \sin \xi \tag{3.15}
$$

where $\Delta$ is the discrepancy and is equal to

$$
\Delta = \left( \omega - \omega_0 \right) \omega_0 \tag{3.16}
$$

and $\omega = \frac{\pi L}{c}$ (the lowest resonance frequency of resonator).

The dissipation coefficient is defined as

$$
D = \frac{b \omega L}{2 c^3 \rho_0} << 1, \tag{3.17}
$$

where ‘b’ is the absorption coefficient

We now use the new notations below in order to simplify further calculations

$$
V = U \left( \frac{3}{2} \frac{M}{\pi \varepsilon} \right)^{-1/3}, \quad j = J \left( \frac{3}{2} \frac{M}{\pi \varepsilon} \right)^{-2/3},
\Gamma = D \left( \frac{\pi \varepsilon}{12} M^2 \right)^{-1/3}, \quad \delta = \frac{\Delta}{3} \left( \frac{\pi \varepsilon}{12} M^2 \right)^{-1/3} \tag{3.18}
$$

where, $j$ is the mean intensity, $\delta$ is the normalized discrepancy, $\varepsilon$ is the cubic nonlinearity parameter and $\Gamma$ is the normalized dissipation.
coefficient. For the case of transverse waves in a homogeneous solid, the parameter ‘ε’ is, in the simplest case, given as [5]

$$\varepsilon = -\frac{3}{4} \frac{c_l^2}{c_t^2 - c_l^2}$$

(3.19)

where the subscripts ‘l’ and ‘t’ are the longitudinal and transverse direction respectively.

Using the above notations, equation (3.15) is further reduced to the following:

$$\Gamma \frac{dV}{d\xi} + V^3 + 3(j - \delta)V = \sin \xi$$

(3.20)

Equation (3.20) is the simplified equation containing the cubic nonlinearity.

It is more interesting to study media that have weak dissipation because the attributes of nonlinearity can be easily expressed. Moreover, in an ideal non-dissipative media, we put $\Gamma = 0$ in equation (3.20). This approximation is valid everywhere except the regions close to the shock fronts (where there are sudden and abrupt discontinuities). The approximation is equivalent to the removal of the derivative and consequently, the differential equation (3.20) is transformed to an algebraic equation:

$$f(V) = V^3 + 3(j - \delta)V = \sin \xi$$

(3.21)

To describe the wave profile in the ideal non-dissipative media, we use the approximation $\Gamma \rightarrow 0$, which is valid everywhere except small regions around the shock fronts. For this reason, $\Gamma \rightarrow 0$ is not a valid approximation when considering forced shocked waves. However, the approximation $\Gamma = 0$ results in the elimination of the derivative in equation (3.20). It is therefore transformed to the algebraic one.

$$V^3 + 3(j - \delta)V = \sin \xi$$

(3.22)

Solutions to equation (3.22) are generated based on certain defined situations.
3.3.1 The mean intensity is greater than discrepancy

That is to say that \( j - \delta \equiv a^2 > 0 \); where the \( j - \delta \) is equivalent to \( a^2 \). Equation (3.22) is therefore written as

\[
f(V) = V^3 + 3a^2V = \sin \xi \tag{3.23}
\]

The behavior of the solution is shown in figure 3.2. The simple graphical construction is used to determine the temporal form of the solution representing the profile of the wave. Following the equation, \( f(V) = \sin \xi \), the point slipping with increase in time \( \xi \) along the curve \( f(V) \) is first determined. Secondly, the projection of these points gives us the wave profile \( V(\xi) = f^{-1}[V(\xi)] \), where \( f^{-1} \) is the inverse function with respect to the function \( f \).

![Graphical analysis of the wave profile](image)

Figure 3.2. Graphical analysis of the wave profile when the mean intensity is greater than the discrepancy. The motion starts from \( \xi = 0 \) at point \( O \) and vibrates between \( P \) and \( Q \).
It can be seen that the profile \( V(\xi) \) for this case has no singularities because the function \( f(V) \) is a monotonic one. The period and the polarity of \( V(\xi) \) are same with the right hand side of the equation (3.23) given by the function \( \sin \xi \). However, the wave \( V(\xi) \) is distorted since its spectrum contains higher harmonics because the plot of the function \( f(V) = V^3 + 3a^2V \) does not yield a straight line. This difference between \( V(\xi) \) and \( \sin \xi \) which result in a nonlinear distortion increases with increase in amplitude of vibration.

### 3.3.2 The mean intensity is equal to the discrepancy, \( j - \delta = 0 \)

Equation (3.22) is rewritten as

\[
f(V) = V^3 = \sin \xi
\]  

(3.24)

Equation (3.24) has an exact analytical solution:

\[
V = \sin^{1/3}(\xi) = \sum_{n=1}^{\infty} B_{2n-1} \sin[(2n-1)\xi], j = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{5}{6}\right) \approx 0.64
\]  

(3.25)

The Fourier expansion contains only odd harmonics because the medium do not have quadratic nonlinearity but cubic one.

### 3.3.3 The positive discrepancy is greater than the mean intensity, case 1, \( 2a^3 > 1 \)

This means that \( j - \delta \equiv -a^2 < 0 \) and with \( 2a^3 > 1 \); equation (3.22) in this situation becomes

\[
f(V) = V^3 - 3a^2V = \sin \xi
\]  

(3.26)

The behavior of the solution to the equation (3.26) is analyzed in fig (3.3). It is obvious that the cyclic motion O P Q O of the points along the curve \( f(V) \) is as a result of the vibration of the right hand side of the equation (3.26). Like in case 1, the wave profile \( V(\xi) \) is nonlinearly distorted but has no shocks. The polarity of \( V(\xi) \) is inversed relative to \( \sin \xi \), which is different from the profile shown in figure 3.3.
3.3.4 The positive discrepancy is greater than the mean intensity, case 2, $2a^3 < 1$

This means that $j - \delta = -a^2 < 0$, but $2a^3 < 1$. The presence of singularities at definite sections of the nonlinear wave profile makes this case more difficult to analyze. Let the motion start from the point O, at initial moment when $\xi = 0$ as shown in figure (3.4). As the function $\sin \xi$, which represents the right hand side of equation (3.22) increases, there is a slipping of the representative point along the curve $f(V)$ to the point P. If the point P is reached at $\xi = \xi_s$, further increase in $f(V)$ with increase in time $\xi > \xi_s$ along the ending branch OP of cubic parabola is impossible. This leads to a discontinuous jump from point P to Q. Thereafter, the motion along the section QR remains continuous and is caused by further
increase of $\sin \xi$ from the value $\sin \xi_s$ to the unity. Section RS is as a result of a decrease in $\sin \xi$ from unity to $-\sin \xi_s$. Again, the second discontinuous jump ST occurs followed by two stages of continuous motion: TU and UP. The steady-state vibration follows the cycle PQRSTUP, and the initial point O is inside the loop. The fast jumps from one branch of the cubic parabola to other results in alternating discontinuities (compression and rarefaction shocks). Such jumps are possible if locations of the representative point in point P and S are unstable, and, on the contrary, its locations at Q and T are stable.

Figure 3.4. Showing the graphical analysis of the wave profile for large positive discrepancies $\delta = j + a^2$, and $2a^3 < 1$. 
4 Perturbation solutions to the wave equation with cubic nonlinearity and the conditions given in chapter 3.3.4

We study the equation (3.20) with $j - \delta = -a^2$:

$$\Gamma \frac{dV}{d\xi} + V^3 - 3a^2V = \sin \xi, \quad 2a^3 \leq 1 \quad (4.0)$$

Perturbation theory involves mathematical methods that are used to find an approximate solution to a problem which cannot be solved exactly. This is achieved by starting from the exact solution of a related problem. Perturbation method is applicable if the problem at hand can be formulated by adding a small term to the mathematical description of the exactly solvable problem. The perturbation method was used in deriving the solutions described here.

4.1 Graphical analysis of the equation with cubic nonlinearity

To begin with the analysis of the solutions to the equation (3.20), we considered an ideal non-dissipative media and make an approximation by putting $\Gamma = 0$ in equation (3.20). This approximation is valid for the outer solutions where there are no shock fronts. For the inner solutions, such approximations are not valid because they are much closer to the areas of discontinuities in the wave profile.

By putting $\Gamma = 0$ in equation (3.20), we obtain [8]

$$V - 3a^2V = \sin \xi \quad (4.1)$$

The real roots to equation (4.1), called $V_1, V_2, V_3$ are: For $|\sin \xi| > 2a^3$

$$V_1 = \frac{1}{2} \sin \xi + \sqrt{\frac{1}{4} \sin^2 \xi - a^6} + \frac{1}{2} \sin \xi - \sqrt{\frac{1}{4} \sin^2 \xi - a^6} \quad (4.2)$$

For $0 < \sin \xi < 2a^3$, there are three roots:
The outer solutions take the form of equations (4.2 to 4.5) with jumps from one form to another. Figure (4.1) shows a graphical analysis of the wave profile for large positive discrepancies. If the motion starts with the point O where $V = 0$ at $\xi = 0$. At $\xi = 0$, the solution $V_3$ is used within the interval $0 \leq \xi < \xi_s$ and noting that $\sin \xi_s = 2a^3$.

At $\xi = \xi_s$, the solution jumps from $V_3(\xi_s^-) = -a$ to $V_1(\xi_s) = 2a$ and continues according (4.3). This corresponds to a jump from position P to the position Q. Between the values $\xi_s$ and $\pi - \xi_s$, the real solution $V_1(\xi)$ reaches a maximum at $\xi = \pi/2$. That is,

$$V(\pi/2) = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - a^6}} + \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - a^6}} \qquad (4.6)$$

Continuing to increase the value of $\xi$ from $\pi - \xi_s$, the solution $V_1$ changes its form from (4.2) to (4.3) with $V_1(\pi - \xi_s) = 2a$. Furthermore, at $\xi = \pi$, the sign of the function $\sin \xi$ changes so that there is a shift from $V_1(\pi^-) = \sqrt{3}a$ to $V_1(\pi^+) = -\sqrt{3}a$; according to equation (4.1). Since the solution $V$ must be
continuous at \( \xi = \pi \), it should change from \( V_1 \) to \( V_2 \) or \( V_3 \). From (4.4) and (4.5) we find the right choice, bearing in mind the change of sign at \( \xi = \pi^+ \). Therefore, at \( \xi = \pi \), the solution changes its form to

\[
V = V_2(\pi^+) = a \cos \frac{\pi}{6} + \sqrt{3}a \sin \frac{\pi}{6} = a \sqrt{3}
\]  

(4.7)

The alternative solution:

\[
V = V_3(\pi^+) = a \cos \frac{\pi}{6} - \sqrt{3}a \sin \frac{\pi}{6} = 0
\]

(4.8)

is therefore excluded.

If we move further from \( \xi = \pi \) to \( \xi = \pi + \xi_s \), and still using \( V_2 \) but remembering the change of sign, the solution yields:

\[
V_2(\pi^- + \xi_s) = a \cos 0 + \sqrt{3}a \sin 0 = a
\]

(4.9)
Figure 4.1. Showing the graphical analysis of the wave profile for large positive discrepancies $\delta = j + a^2$, and $2a^3 < 1$. Figure (4.1A) shows the plot of the right hand side of equation (4.1) and the left hand side. Figure (4.1B) shows the plot of the perturbation solutions to equation (4.1). It describes the wave profile. The solutions at the shock regions around $\xi = \xi_s$ and $\xi = \pi + \xi_s$ are studied in chapter 4.2.
Further from the point $\xi = \pi + \xi_s$, there is a jump from $V = V_2(\pi^- + \xi_s) = a$ in 4.9 which corresponds to a discontinuity at that point to

$$V_o = V_1(\pi + \xi_s) = -2a \cos 0 = -2a$$

(4.10)

At $\xi = 1.5\pi$, the solution $V = V_1$ reaches its maximum. This is seen at the point U in fig (4.1). This maximum is the same as in (4.6) but with the opposite sign.

Further increase up to the value $\xi = 2\pi^-$, will yield using equation (4.3),

$$V_1(2\pi^-) = -2a \cos\left(-\frac{1}{3} \arctan \infty\right) = -2a \cos \frac{\pi}{6} = -a\sqrt{3}$$

(4.11)

Further to $2\pi - \xi_s$, the function $\sin \xi$ changes its sign resulting in the change of the solution to

$$V_2(2\pi^+) = -a \cos\left(\frac{1}{3} \arctan \infty\right) - \sqrt{3}a \sin\left(\frac{1}{3} \arctan \infty\right) = -a\sqrt{3}$$

(4.12)

At this stage, another jump occurs at $2\pi + \xi_s$ and is a repetition of the jump from $V = -a$ to $V = 2a$. In this case, it is from $V_2(2\pi^- + \xi_s) = -a$ to $V_1(2\pi^+ + \xi_s) = 2a$. At this stage, a full description of first period of the vibration is made. We now move on to the analysis at regions that are close to the discontinuities. These are regions close to the shock front.

### 4.2 Matching of the inner to the outer solutions using an intermediate solution at shock regions.

As mentioned earlier, at the regions close to the shock front, the approximation $\Gamma = 0$, which reduced the differential equation (3.20) to an algebraic equation is not valid. This is as a result of discontinuities (jumps) in the wave profile at these regions. In order to relate the inner solution to the outer solution, intermediate solutions are derived and are used in the matching of the inner and the outer solutions.

The lowest order outer solution of the equation (3.20)
\[ \Gamma \frac{dV}{d\xi} + V^3 + 3(j - \delta)V = \sin \xi \]

has two discontinuities in its period \(0 \leq \xi \leq 2\pi\): the jump from \(V_o = -a\) to \(V_o = 2a\) at \(\xi = \xi_s\) and the jump from \(V_o = a\) to \(V_o = -2a\) at \(\xi = \pi + \xi_s\). If we put \(\Gamma = \in\) and introduce an inner expansion in order to investigate the shock structure of the two discontinuities [8]. Considering first the region, \(V_o = -a\), \(\xi = \xi^-\);

Let the inner expansion be defined as:

\[ V_{in} = V_o^* + \in V_1^* \quad (4.13) \]

We replace \(\Gamma\) by \(\in\) and assume \(\in \ll 1\)

In order to study the discontinuity at \(\xi = \xi_s\) an inner coordinate \(\xi^*\) is introduced:

\[ \xi^* = \frac{\xi - \xi_s}{\in} \rightarrow \xi = \xi_s + \in \xi^* \quad (4.14) \]

An outer expansion is also required and is defined as:

\[ V_{out} = V_o + \in V_1 + ... \quad (4.15) \]

This is valid in the neighbourhood of

\[ V_o = -a, \quad \xi = \xi^- \quad \text{and} \quad V_o = 2a, \quad \xi = \xi^+ \]

With the method described in [8], the outer solution at \(\xi = \xi^-\) and \(V = -a\) for the equation:

\[ \in \frac{dV}{d\xi} + V^3 + 3(j - \delta)V = \sin \xi, \quad 2a^3 \leq 1 \]

and for

\[ \xi \rightarrow \xi_s, \sin \xi_s = 2a^3, \quad \xi < \xi_s \]

is:
\[ V_{\text{out}} = V_0 + \epsilon V_1 = -a \left\{ 1 - \frac{2}{3} \left( 1 - \frac{\sin \xi}{2a^3} \right) - \frac{1}{9} \left( 1 - \frac{\sin \xi}{2a^3} \right)^2 \right\} - \epsilon \frac{1}{24a^4} \frac{\cos \xi}{1 - \frac{\sin \xi}{2a^3}} \]  
\[ (4.16) \]

where the lowest order term is given by evaluating \( V_3 \) in (4.5) for \( \xi \) in the neighbourhood of \( \xi_s^- \).

The inner solution at \( \xi = \xi_s^- \) and \( V = -a \) was also derived:

\[ V_{\text{in}} = -a - \frac{1}{3a(\xi^* - \epsilon^{-1/3} \lambda^{-1} \alpha^{-1/3})} + \frac{\ln |\xi^* - \epsilon^{-1/3} \lambda^{-1} \alpha^{-1/3}|}{27a^3 \left( \xi^* - \epsilon^{-1/3} \lambda^{-1} \alpha^{-1/3} \right)^2} \]  
\[ (4.17) \]

\[ + \epsilon \frac{\sqrt{1 - 4a^6}}{2} \xi^* \]

where \( \xi^* = \frac{\xi - \xi_s^-}{\epsilon} \), \( \lambda = \frac{\sqrt{\pi}}{2^{1/3} 3^{2/3} \Gamma(\frac{7}{6})} \)

The intermediate solution which was used to match the inner solution to the outer solution was derived thus:

\[ V_{\text{int}} = -a - \epsilon^{1/3} \frac{d}{d\xi} \text{Bi}(-\alpha^{1/3} \xi) + \frac{\epsilon^{2/3}}{3a\text{Bi}(\alpha^{1/3} \xi)} + \frac{\epsilon^{1/3}}{27a^3 \text{Bi}^2(-\alpha^{1/3} \xi)} \int \left\{ \frac{d}{d\xi} \text{Bi}(\alpha^{1/3} \xi) \right\} d\xi \]

where, \( \xi = \frac{\xi - \xi_s^-}{\epsilon^{1/3}} \), \( \alpha = 3a\sqrt{1 - 4a^6} \)  
\[ (4.18) \]

The function ‘Bi’ is an Airy function and was evaluated.

The intermediate solution \( V_{\text{int}} \) is matched to the outer solution \( V_{\text{out}} \) by \( \xi \rightarrow \infty \) and is matched to the inner solution \( V_{\text{in}} \) by \( \xi \rightarrow 0 \). The plots of the solutions \( V_{\text{out}}, V_{\text{int}}, V_{\text{in}} \) against \( \xi \) is as shown in figure 4.2 and is used to verify the matching as solved.
Figure 4.2. Showing the matching of both the inner and the outer solution to the intermediate solution at \( V = -a, \xi = \xi_s^- \); for \( \varepsilon = 0.00001, a = 0.61 \)

- To make matching at \( \xi = \pi + \xi_s \), \( V = a \), the same procedure of finding an outer and inner expansions in the neighbourhood of

\[
V_{out} = -a, \xi = \pi + \xi_s \quad \text{and} \quad V_{in} = a, \xi = \pi + \xi_s ; \quad \text{with} \quad \xi^* = \frac{\xi - (\pi + \xi_s)}{\varepsilon}
\]

The equation: \( \varepsilon \frac{dV}{d\xi} + V^3 - 3a^2V = \sin \xi, \quad 2a^3 \leq 1 \) has the solutions

\[
V_{out}, V_{int}, V_{in} \quad \text{for} \quad \xi \to \pi + \xi_s, \quad \sin \xi_s = 2a^3 \quad \xi < \pi + \xi_s ;
\]

\[
V_{out} = a + \frac{4\sqrt{1-4a^6}}{3a} \left[ - (\xi - (\pi + \xi_s)) \right]^\frac{1}{2} + \frac{\sqrt{1-4a^6}}{18a^2} (\xi - (\pi + \xi_s)) - \varepsilon \frac{1}{12a(\xi - (\pi + \xi_s))}
\]

(4.19)
where the lowest order term is given by evaluating in $V_2$ in (4.4), regarding its change of sign at $\xi = \pi^+$, in the neighbourhood of $\pi^- + \xi_s$. (Compare (4.9))

$$V_{\text{int}} = a + \frac{1}{3} \left( \frac{d}{d\xi} \operatorname{Ai}(\alpha^{-\frac{1}{3}} \xi) \right) - \frac{2}{3} \int \frac{d}{d\xi} \left\{ \frac{\operatorname{Ai}(\alpha^{-\frac{1}{3}} \xi)}{27a^3 (\operatorname{Ai}(\alpha^{-\frac{1}{3}} \xi))^2} \right\} d\xi$$

$$\xi = \xi - (\pi + \xi_s), \; \alpha = 3a\sqrt{1 - 4a^6}, \; \xi < 0$$

$$V_{\text{in}} = a + \frac{1}{3a(\xi^- - \xi^+ \lambda^{-\frac{1}{3}} \alpha^{-\frac{1}{3}})} - \frac{\ln |\xi^- - \xi^+ \lambda^{-\frac{1}{3}} \alpha^{-\frac{1}{3}}|}{27a^3 (\xi^- - \xi^+ \lambda^{-\frac{1}{3}} \alpha^{-\frac{1}{3}})^2}$$

The values of ‘$a$’ used was chosen between 0.5 to 0.7 and the epsilon values ‘$\epsilon$’ between 0.001 and 0.00001 as indicated for each plot.

As shown in figure 4.3, the intermediate solution $V_{\text{int}}$ is matched to the outer solution $V_{\text{out}}$ by $\xi \rightarrow \infty$ and is matched to the inner solution $V_{\text{in}}$ by $\xi \rightarrow 0$. The plots of the solutions $V_{\text{out}}, V_{\text{int}}, V_{\text{in}}$ against $\xi$ is as shown and is used to verify the matching as solved.
Figure 4.3. Showing the matching of both the inner and the outer solution to the intermediate solution at $V = a$, $\xi = \pi + \xi_s$ and for $\epsilon = 0.00001, a = 0.61$

- At $V = 2a$ and $\xi = \xi_s^+$, we start as usual by making an expansion to $V_o$ and $V_1$ in powers of $\xi - \xi_s$. The outer solution at the region was derived and equals:

$$V_{out} = 2a + \frac{\sqrt{1 - 4a^6}}{9a^2} (\xi - \xi_s) - \frac{2 + 19a^6}{9 \cdot 27a^5} (\xi - \xi_s)^2 +$$

$$\epsilon \left\{ - \frac{\sqrt{1 - 4a^6}}{81a^4} + \frac{1}{81 \cdot 27a^7} (8 + 22a^6)(\xi - \xi_s) \right\} - \epsilon^2 \frac{1}{3 \cdot 81a^6} (10 + 14a^6) \quad (4.22)$$

The inner solution at $V = 2a$ and $\xi = \xi_s^+$ is:
\[ V_{in} = 2a + \varepsilon \left\{ -\frac{\sqrt{1 - 4a^6}}{81a^4} + \frac{\sqrt{1 - 4a^6}}{9a^2} \xi^* \right\} + \varepsilon^2 \left\{ -\frac{10 + 14a^6}{81a^9} + \frac{8 + 22a^6}{9a^7} \xi^* - \frac{2 + 19a^6}{a^5} \xi^*^2 \right\} \]  

(4.23)

\[ \xi^* = \frac{(\xi - \xi^*)}{\varepsilon} \]

Unlike in the cases of \( V = \pm a \), the matching at \( V = 2a \) can be done without an intermediate boundary layer. However, the intermediate solution is derived:

\[ V_{int} = 2a + \varepsilon^2 \frac{\sqrt{1 - 4a^6}}{9a^2} \xi^* + \varepsilon \left\{ -\frac{\sqrt{1 - 4a^6}}{81a^4} - \frac{2 + 19a^6}{9 \cdot 27a^5} \xi^*^2 \right\} + \varepsilon^2 \frac{8 + 22a^6}{81 \cdot 27a^7} \xi^* - \varepsilon^2 \frac{10 + 14a^6}{3 \cdot 81^2 a^9} \xi \]

(4.24)

\[ \xi = \frac{\xi - (\pi + \xi_s)}{\varepsilon^2} \]

\[ \xi^* \to \infty \text{ in } V_{in} \text{ when it is matched to } V_{out}. \] The graph showing the matching of \( V_{in} \) to \( V_{out} \) is as shown in figure 4.4a and is plotted against \( \xi^* \).

It can be observed that there is a proper matching between the inner solution and the outer solution at \( V = 2a \), \( \xi = \xi_s^+ \) even without an intermediate solution.
As shown in figure (4.4a), the matching of the inner and the outer solutions matched properly without an intermediate boundary layer. Figure (4.4b), containing the intermediate boundary layer is plotted between the inner and the outer solutions is shown below. It shows how the matching could be achieved with or without an intermediate boundary layer.
Figure 4.4b. Showing the matching of the inner solution to the outer solution at $V = 2a$ and $\xi = \xi_s^+$ for $\varepsilon = 0.00001, a = 0.615$. An intermediate solution is plotted between the inner and the outer solutions.

- At the region, $V = -2a$, $\xi = \pi + \xi_s$

The solutions $V_{\text{out}}, V_{\text{int}}, V_{\text{in}}$ at $V = -2a, \xi = \pi + \xi_s$ are obtained by making the changes $a \rightarrow -a$ and $\sqrt{1 - 4a^6} \rightarrow -\sqrt{1 - 4a^6}$ from $V = 2a$.

We therefore have that:

$$V_{\text{out}} = -2a - \frac{\sqrt{1 - 4a^6}}{9a^2} (\xi - (\pi + \xi_s)) + \frac{2 + 19a^6}{9.27a^5} (\xi - (\pi + \xi_s))^2 +$$

$$\in \left\{ \frac{\sqrt{1 - 4a^6}}{81a^4} - \frac{1}{81.27a^3} (8 + 22a^6)(\xi - (\pi + \xi)) \right\} + \varepsilon^2 \frac{1}{3.81^2 a^9} (10 + 14a^6)$$

(4.25)
The plots of the inner, outer and the intermediate solutions are shown in figure 4.5. Like in $V = 2a$, it was observed that contrary to the cases of $V = \pm a$, the matching of $V_{in}$ to $V_{out}$ can be done without an intermediate boundary layer. The plots of $V_{in}$ and $V_{out}$ against $\xi^*$ is as shown in figure 4.5. It can be observed that there is a proper matching between the inner solution and the outer solution at $V = -2a$, $\xi = \pi + \xi_s$, even without an intermediate solution.
As shown in figure (4.5a), the matching of the inner and the outer solutions matched without an intermediate boundary layer. Figure (4.5b), containing the intermediate boundary layer plotted between the inner and the outer solutions are shown below. It shows how the matching could be achieved with or without an intermediate boundary later.
Figure 4.5b. Showing the matching of the inner solution to the outer solution at $V = -2a$ and $\xi = \pi + \xi_s$, for $\epsilon = 0.00001, a = 0.615$. An intermediate solution is plotted between the inner and the outer solutions.

- **Variation of the value of ‘epsilon’, $\epsilon$**

Different values of epsilon, $\epsilon$ between 0.00001 and 0.01 were used and the resulting graphs compared. In general, it was observed that the orientation of the graphs remained same though as the value of epsilon increases, there is decrease in the values along the $\xi^*$ axis. The plots of $V$ against $\xi^*$ with values of epsilon as indicated on the graph are shown below for $V = 2a$ and $V = -2a$. The epsilon value used for each of the graphs shown above is 0.00001.
Matching of the inner solution to the outer solution (with epsilon=0.0001) for $V=2a$.

Figure 4.6. Showing the matching of the inner to the outer solution with epsilon value $\varepsilon = 0.0001$ at $V = 2a$. 

$V_{in}$ $V_{out}$

$\xi^*$
Figure 4.7. Showing the matching of the inner to the outer solution with epsilon value $\varepsilon = 0.001$ at $V = -2a$.
5 Conclusion and Discussions

A brief introduction to the basic difference between linear and nonlinear acoustic waves phenomenon is presented. Distinction between standing waves in quadratic nonlinear medium and that in a cubic nonlinear medium is also highlighted.

An approach based on the linear superposition of two counter-propagating nonlinear waves is used in the derivation of the equation containing the cubic nonlinearity. Resonant properties of standing waves in a nonlinear medium are briefly presented and the analysis of standing waves in a cubic medium for free vibration, forced and shocked vibrations are explained with illustrations.

The wave profile describing the path of a standing acoustic wave in a cubically nonlinear medium is derived based on perturbation solutions to the equation containing the cubic nonlinearity. Study of the regions containing shock fronts (discontinuities) is made and a verification of matching between the inner and the outer solutions using an intermediate boundary layer at the shock regions is presented.

Plots illustrating the matching are generated using MATLAB. The plots of the matching reveals some closeness in accuracy for the method of analysis used. It can be noticed from figures (4.4, 4.5, 4.6 and 4.7) that the choice of different values of epsilon within the limit of 0.001 and 0.00001 did not make significant differences in the result of the matching. Also, from the different plots of $V$ against $\xi$, one can evaluate some physical values based on the notations used in equation (3.18).
6 References


2 Appendices

Matlab Scripts

inner solution at v=2a from eqn 152

% close all;
% clear all;
% clc;
%
% xin = 0.001:0.00001:0.01;
% ein=0.00001;
% a = 0.615;
% x1in = xin.*(0.95);
% x2in =(xin-x1in)./ein;
% %
% A = ein*(-((1-(4*a^6))^(0.5))/(81*a^4)+((1-(4*a^6))^(0.5))/(9*a^2).*x2in);
% B = (ein^2./(9*27)).*(-(10+(14*a^6))./(81*a^9)+(8+(22*a^6))/(9*a^7).*x2in-(2+(19*a^6))/(a^5).*x2in.^2);
% Vin = a*2 + A + B;
% %
% plot(x2in,Vin,'linewidth',2)
% hold on

intermediate solution at v=2a from eqn 153

xint = 0.0005:0.00001:0.005;
x1int = xint.*(0.95);
eint = 0.00001;
x2int = (xint-x1int)./eint;
xbint = x2int.*(eint^(1/2));
a = 0.615;
A = (eint^4(1/2))*((1-(4*a^6))^(1/2))/(9*a^2).*xbint;
B = eint^*(-((1-(4*a^6))^(1/2))/(81*a^4)-((2+(19*a^6))/(9*27*a^5).*xbint.^2));
C = (eint^4(3/2)).*(((81*27*a^7)^(-1)).*(8+(22*a^6))).*xbint;
D = eint^2*(((3*81^2*a^9)^(-1)).*(10+(14*a^6)));
Vint = (2*a)+A+B+C-D;

plot(x2int,Vint,'b--','linewidth',2)
grid on
hold on

outer solution at v=2a from eqn 142

% xout = 0.0001:0.00001:0.001;
% a = 0.615;
% x1out = xout.*(0.95);
% x2out = (xout-x1out)./eout;
% D = ((1-(4*a^6))^(1/2))/(9*a^2).*(xout-x1out);
% E = (2+(19*a^6))/(9*27*a^5).*(xout-x1out).^2;
% F = eout*((1-(4*a^6))^(1/2))/(81*a^4)+((81*27*a^7)^(-1).*(8+(22*a^6)).*(xout-x1out));
% G = eout^2*((3*81^2*a^9)^(-1)).*(10+(14*a^6));
% Vout = (2*a)+D-E+F-G;
% plot(x2out,Vout,'r','linewidth',2)
% hold on
inner solution at v=-2a from eqn 155
% close all;
% clear all;
% clc;

% xin = 0.001:0.00001:0.01;
% ein=0.00001;
% a = 0.615;
% x1in =((xin.*(0.95)- pi);
% x2in =((xin-(pi+x1in))./ein;

% A = ein^2*(((1-(4*a^6))^(0.5))/((81*a^4)-((1-(4*a^6))^(0.5))/(9*a^2).*x2in);
% B = ein^2./(9*27)).*((10+(14*a^6))/(81*a^9)-(8+(22*a^6))/(9*a^7).*x2in+(2+(19*a^6))/(a^5).*x2in.^2);%
% Vin = -(a^2) + A + B;
% plot(x2in,Vin,'linewidth',2)
% hold on

intermediate solution at v=-2a from eqn 156
%
xint = 0.0005:0.00001:0.005;
x1int = (xint.*(0.95)- pi);
eint = 0.00001;
x2int = (xint-(pi+x1int))./eint;
xbint = x2int.*(eint^(1/2));
a = 0.615;

A = (eint^4(1/2))*(((1-(4*a^6))^(1/2))/(9*a^2).*xbint;
B = eint^2 ((((1-(4*a^6))^(1/2))/(81*a^4)+((2+(19*a^6))/(9*27*a^5).*xbint.^2));
C = (eint^4(3/2)).*((81*27*a^7)^(-1).*xbint;
D = eint^2*((3*81^2*a^9)^(-1)).*(10+(14*a^6));

Vint = -(2*a)-A+B-C+D;
plot(x2int,Vint,'r--','linewidth',2)
grid on
hold on
outer solution at v=-2a from eqn 154

% xout = 0.0001:0.00001:0.001;
% eout=0.00001;
% a = 0.615;
% x1out =(xout.*(0.95)- pi);
% x2out =(xout-(pi+x1out))./eout;
%
% D = ((1-(4*a^6))^(1/2))/(9*a^2).*((xout-(pi+x1out)));
% E = (2+(19*a^6))/((9*27*a^5)).*(xout-(pi+x1out)).^2;
% F = eout*((1-(4*a^6))^(1/2))/(81*a^4)-((81*27*a^7)^(-1)).*(8+(22*a^6)).*(xout-(pi+x1out)));
% G = eout^2*((3*81^2*a^9)^(-1)).*(10+(14*a^6));
%
% Vout = -(2*a)-D+E+F+G;
% plot(x2out,Vout,'r','linewidth',2)
% hold on
inner solution at v=-a from eqn 95
\%
% close all;
% clear all;
% clc;

% xin = 0.0001:0.0001:0.001;
% e = 0.00001;
% a = 0.615;
% x1in = xin.*(0.95);
% x2in = (xin-x1in)./e;
% xbin = x2in.*(e^(1/3));
% r = 0.796;
% k = (pi^0.5)/((3^(2/3))*(2^(1/3))*r*(7/6));
%
% p=3*a*(1-(4*a^6))^0.5;
% A = 1./(3*a*(x2in-(e^(-1/3)).*(1/k)*(p^(-1/3))));
% B = log(abs(x2in-(e^(-1/3)).*(1/k).*(p^(-1/3))))/(27*a.^3.*(x2in-(e^(-
1/3)).*(1/k).*(p^(-1/3))).^2);
% C = e*((1-(4*a.^6)).^1/2).*x2in.^2;
%
% Vin = -a -A + B + (C/2);
%
% plot(xbin,-Vin,'linewidth',2)

intermediate solution at v=-a from eqn 94
% xint = 0.001:0.0001:0.01;
% x1int= xint.*(0.95);
% eint=0.00001;
% x2int=(xint-x1int)./eint;
% xbint=x2int.*(eint^(1/3));
% a = 0.61;
% r=0.796;
% k=((pi)^1/2)/(3^(2/3)*2^(1/3)*r*(7/6));
% b=3*a*(1-(4*a^6))^1/2;
% C = log(eint^(-1/3));
% A = -((3*a*(x2int-(eint^(-1/3)*k^(-1)*b^(-1/3))))).^(-1)+(eint*(1-
((a^6)*4)))^((1/2))*(x2int.^2))/2;
% B = (1/(27*a^3))*1./(x2int-(eint^(-1/3).*k^(-1)*b^(-
1/3))).^2).*log(abs((eint^((1/3)*x2int)-(k^(-1)*b^(-1/3)))^C);
% Vint = -a+A+B;
%
% plot(xbint,-Vint,'r','linewidth',2)

grid on
% hold on

outer solution at v=-a from eqn 93

xout = 0.01:0.00001:0.1;

eout=0.00001;
a = 0.58;
x1out = xout.*(0.95);
x2out=(xout-x1out)./eout;
xbout=x2out.*(eout^(1/3));

D = (((1-(4*a^6))^(1/4))/((3*a)^(1/2)))*(-(xout-x1out)).^(1/2);
E = (((1-(4*a^6))^(1/2))*(xout-x1out).*((18*a^2)^(-1));
F = eout*((xout-x1out).*12*a)).^(-1);
Vout = -a+D-E+F;
axis([0 2 0.2 0.7])
plot(xbout,-Vout,'linewidth',2)
hold on
inner solution at v=a from eqn 131
% close all;
% clear all;
% clc;
%
% xin = 0.0001:0.0001:0.001;
% e=0.00001;
% a = 0.615;
% x1in = (xin.*(0.95)- pi);
% x2in = (xin-(pi+x1in))./e;
% xbin = x2in.*(e^(1/3));
% r=0.796;
% k = (pi^0.5)/((3^2/3)*(2^1/3)*r*(7/6));
%
% p=3*a*(1-(4*a^6))^0.5;
%
% A = 1./(3*a*(x2in-(e^(-1/3).*((1/k)*((p^(-1/3))))));
% B = log(abs(x2in-(e^-1/3)).^(1/3)).*(i.e.;)
% C = e^-(((1-(4*a^6)).^1/2).*x2in.^2);
%
% Vin = a + A - B - (C/2);
%
% hold on

intermediate solution at v=a from eqn 130
%
% xint = 0.001:0.0001:0.01;
% x1int = (xint.*(0.95)- pi);
% eint = 0.00001;
% x2int = (xint-(pi+x1int))./eint;
% xbind = x2int.*(eint^(1/3));
% r = 0.796;
% k = (pi^0.5)/((3^2/3)*(2^1/3)*r*(7/6));
% a = 0.61;
% b = 3*a*(1-(4*a^6))^1/2;
% % A = eint*(1/3).*((3*a*(xbint-(k^(-1)*b^(-1/3))))).^(1/2)+(eint^1/3).*(1-(
% ((a^6)*4)).^(1/2)/2;
% % B = eint^2/3).*(1/(27*a^3)).*(1/(xbint-(k^(-1)*b^(-1/3))))^2).*(log(abs(xbint-
% (k^(-1)*b^(-1/3)))))+log(eint^1/3));
% A = ((3*a*(x2int-(eint^(-1/3)))*k^(-1)*b^(-1/3))).^(1/2)/2;
% % B = 1/(27*a^3)).*(1/(x2int-(eint^(-1/3)))*k^(-1)*b^(-1/3))).^2).*(log(abs((eint^1/3)*x2int-(k^(-1)*b^(-1/3))))+log(eint^1/3));
% Vint = a+A-B;
outer solution at \( v=a \) from eqn 129 on

\[
\begin{align*}
% 
xout &= 0.01:0.00001:0.1; \\
eout &= 0.00001; \\
a &= 0.59; \\
x1out &= (xout.*(0.95) - \pi); \\
x2out &= (xout-(\pi+x1out))./eout; \\
xbout &= x2out.*(eout^{(1/3)});
\end{align*}
\]

\[
\begin{align*}
D &= ((1-(4*a^6))_^(1/4))/((3*a)_^(1/2))^(xout-(\pi+x1out))._^(1/2); \\
E &= ((1-(4*a^6))_^(1/2))*(xout-(\pi+x1out)).*(18*a^2)^(-1); \\
F &= eout*((xout-(\pi+xout)).*12*a).^(-1); \\
Vout &= a+D+E-F;
\end{align*}
\]

axis([0 2 0.2 0.7])
% hold on
plot(xbout,Vout,'linewidth',2)
hold on