Master Thesis Mathematical Modeling and Simulation
On
Fuzzy linear programming problems solved with
Fuzzy decisive set method

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Abstract

In the thesis, there are two kinds of fuzzy linear programming problems, one of them is a linear programming problem with fuzzy technological coefficients and the second is linear programming problem in which both the right-hand side and the technological coefficients are fuzzy numbers. I solve the fuzzy linear programming problems with fuzzy decisive set method.
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Introduction

Before introducing fuzzy linear programming, we will review traditional linear programming (LP). Linear programming is an algebraic method used to solve sets of linear equations. The formal methodology was developed around 1947. The purpose of linear programming is to find optimal solutions for systems which are modeled by linear equations.

In LP, sharp constraints are combined to limit the space of feasible solutions to the linear problem being posed. The variable dimensions of the system being modeled assume the form of a vector. The objectives of a problem are also modeled with linear equations. The linearity of the constraints and the objectives enables straight-forward solution methods. Vertices of the solution space correspond to optimizing vectors. The vectors are optimizing in the sense that non-zero linear equations of the system variables, representing the objectives, achieve (either maximal or minimal) values at the vertices of the feasible solution space. There are various ways to visualize how the LP method achieves an optimal solution for a linear system.

Some problems are solved by making simple and obvious choices between a small set of options. Many important problems are more complicated; there can be multiple simultaneous goals, and dilemmas and compromises inherent in all possible solutions. We require a framework to identify optimal solutions. Constraints on actions and resources have to be identified. Objectives have to be defined. There has to be a way of evaluating the degree to which definable objectives are met. Realistic problems and objectives are often only vaguely defined. The problems given as examples above range from trivial to subtle and complex, and all are representative of real life problems.
Fuzzy linear programming (FLP) is a refinement of linear programming (LP) which was developed since the nineteen-seventies. The fuzzy objective function is characterized by its membership function, and so are the constraints. Since we want to satisfy (optimize) the objective function as well as the constraint, a decision in a fuzzy environment is defined in analogy to non fuzzy environment as the selection of activities which simultaneously satisfy the objective function and the constraints. The relationship between the constraints and the objective function in a fuzzy environment is therefore fully symmetric, i.e. there is no longer difference between the former and latter.

The decision maker may not actually want to maximize or minimize the objective function. Rather he might want to reach some aspiration levels which might not even be definable crisply. Thus he might want to “improve the present cost situation considerably”, and so on.

The constraints might be vague in one of the following ways. The $\leq$ sign might not be meant in the strict mathematical sense, but smaller violations might well be acceptable. This can happen if the constraints represent aspiration levels as mentioned above, or if for instance, the constraints represent sensory requirements (taste, color, smell, etc.) which cannot adequately be approximated by a crisp constraint. Of course, the coefficients of the vectors $b$ or $c$ or of the matrix $A$ itself can have a fuzzy character, either because they are fuzzy in nature or because perception of them is fuzzy. The role of the constraints can be different from that in classical linear programming where the violation of any single constraint by any amount renders the solution infeasible. The decision maker might accept small violations of different constraints. Fuzzy linear programming offers a number of ways for all those types of vagueness [7].
Approach and Methods

In fuzzy decision making problems, the concept of maximizing a decision was proposed by Bellman and Zadeh [3]. This concept was adopted to problems of mathematical programming by Tanaka et al. [9]. Zimmermann [10] presented a fuzzy approach to multiobjective linear programming problems. Fuzzy linear programming problem with fuzzy coefficients was formulated by Negoita [11] and called robust programming. Dubois and Prade [2] investigated linear fuzzy constraints. Tanaka and Asai [14] also proposed a formulation of fuzzy programming with fuzzy constraints and gave a method for its solution which bases on inequality relations between fuzzy numbers. Shaocheng [13] considered the fuzzy linear programming problem with fuzzy constraints and defuzzificated it by first determining an upper bound for the objective function. Further, the so-obtained crisp problem solved by the fuzzy decisive set method has been introduced by Sakawa and Yana [12]. First consider linear programming problems in which only technological coefficients are fuzzy numbers and then linear programming problems in which both technological coefficients and right-hand side numbers are fuzzy numbers. The first problem is converted into an equivalent crisp problem. There exists a problem of finding a point which satisfies the constraints and the goal (the goal can be comparable to an objective function) with the maximum degree. The idea of the approach is due to Bellman and Zadeh [3]. The crisp problems, obtained by such a manner, can be non-linear (even non-convex), where the non-linearity arises in constraints. For solving these problems we utilize the fuzzy decisive set method, as introduced by Sakawa and Yana [12]. In this method a combination with the bisection method and the simplex method of linear programming is used to obtain a feasible solution.
2.1 Linear programming with fuzzy technological coefficients

\[\max \sum_{j=1}^{n} c_j x_j\]

subject to \[\sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq b_i, \quad 1 \leq i \leq m\]

\[x_j \geq 0, \quad 1 \leq j \leq n\]

at least one \(x_j > 0\)

(2.1.1)

Assumptions 1.
\(\tilde{a}_{ij}\) is a function of the following fuzzy set.

\[\mu_{a_{ij}}(x) = \begin{cases} 
1 & \text{if } x < a_{ij}, \\
(a_{ij} + d_{ij} - x)/d_{ij} & \text{if } a_{ij} \leq x < a_{ij} + d_{ij}, \\
0 & \text{if } x \geq a_{ij} + d_{ij}, 
\end{cases}\]

Where \(x \in \mathbb{R}\) and \(d_{ij} > 0\) for all \(i = 1, \ldots, m, j = 1, \ldots, n\). For defuzzification of this problem, we first fuzzify the objective function. This is done by calculating the lower and upper bounds of the optimal values first. The bounds of the optimal values, \(z_l\) and \(z_u\) are obtained by solving the standard linear programming problems.
\[ Z_1 = \max \sum_{j=1}^{n} c_j x_j \]

subject to \[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad 1, \ldots, m, \]
\[ x_j \geq 0, \quad j = 1, \ldots, n, \]

(2.1.2)

and

\[ Z_2 = \max \sum_{j=1}^{n} c_j x_j \]
\[ \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j \leq b_i \]
\[ x_j \geq 0. \]

(2.1.3)

The objective function takes values \( z_1 \) and \( z_2 \) technological coefficients between \( a_{ij} \) and \( a_{ij} + d_{ij} \). Let \( z_l = \min (z_1, z_2) \) and \( z_u = \max (z_1, z_2) \). Then \( z_l \) and \( z_u \) are called lower and upper bounds of the optimal values.

Assumption 2.

The linear crisp problems (2.1.2) and (2.1.3) have finite optimal values. In this case the fuzzy set \( G \) of optimal values, which is a subset of \( \mathbb{R}^n \), is defined as (see Klir and Yuan [5])
The fuzzy set of the ith constraint \( C_i \), which is a subset of \( R^m \), is defined by

\[
\mu_{C_i}(x) = \begin{cases} 
0 & \text{if } x < a_{ij}, \\
\left( \sum_{j=1}^{n} c_j x_j - Z_i \right) / (Z_u - Z_i) & \text{if } Z_i \leq \sum_{j=1}^{n} c_j x_j < Z_u, \\
1 & \text{if } \sum_{j=1}^{n} c_j x_j \geq Z_u,
\end{cases}
\]

(2.1.4)

According to the definition of the fuzzy decision proposed by Bellman and Zadeh [3] we obtain the fuzzy decision set \( D \), characterized by the membership function stated as the minimum over the fuzzy goal \( G \)-set and some constraints \( C_i \). Provided that the goal and the constraints lie over the same universe space the function is formalized due to the formula

\[
\mu_D(x) = \min(\mu_G(x), \min_{i}(\mu_{C_i}(x)))
\]

(2.1.5)
After adapting (2.1.6) to the purpose of finding the membership degree of the optimal fuzzy decision we determine a solution of the problem as

$$\max_{x > 0} \mu_D(x) = \max_{x > 0} \min(\mu_G(x), \min_{i} (\mu_{C_i}(x)))$$  \hspace{1cm} (2.1.7)$$

consequently, the problem (2.1.1) becomes to the following optimization problem

$$\max \lambda$$

$$\mu_G(x) \geq \lambda$$

$$\mu_{C_i}(x) \geq \lambda, \quad 1 < i < m$$

$$x \geq 0, \quad 0 \leq \lambda \leq 1$$  \hspace{1cm} (2.1.8)$$

In which $\lambda$ is a parameter from $[0, 1]$.

By using (2.1.4) and (2.1.5), the problem (2.1.8) can be written as

$$\max \lambda$$

$$\lambda(Z_1 - Z_2) - \sum_{j=1}^{n} c_j x_j + Z_2 \leq 0,$$

$$\sum_{j=1}^{n} (a_{ij} + \lambda d_{ij}) x_j - b_i \leq 0, \quad 1 < i < m$$

$$x_j \geq 0, \quad j = 1, \ldots, n,$$

$$0 \leq \lambda \leq 1.$$  \hspace{1cm} (2.1.9)$$

Notice that, the constraints in problem (2.1.9) containing the cross product terms $\lambda x_j$ are not convex. Therefore the solution of this problem requires the special approach adopted for solving general nonconvex optimization problems.
2.2 Lp problems with fuzzy technological coefficients and Fuzzy Right-hand-side numbers.

\[
\begin{align*}
\max & \quad \sum_{j=1}^{n} c_j x_j \\
\sum_{j=1}^{n} \tilde{a}_{ij} x_j & \leq \tilde{b}_i, \quad 1 \leq i \leq m \\
x_j & \geq 0,
\end{align*}
\]  \hspace{1cm} (2.2.1)

Where at least one \( x_j > 0 \)

**Assumption 3.**

\( \tilde{a}_j \) and \( \tilde{b}_i \) are fuzzy numbers with the following linear membership functions:

\[
\mu_{a_j}(x) = \begin{cases} 
1 & \text{if } x < a_{ij}, \\
(a_{ij} + d_{ij} - x) / d_{ij} & \text{if } a_{ij} \leq x < a_{ij} + d_{ij}, \\
0 & \text{if } x \geq a_{ij} + d_{ij},
\end{cases}
\]

and

\[
\mu_{b_i}(x) = \begin{cases} 
1 & \text{if } x < b_i, \\
(b_i + p_i - x) / p_i & \text{if } b_i \leq x < b_i + p_i, \\
0 & \text{if } x \geq b_i + p_i,
\end{cases}
\]

Where \( X \in \mathbb{R} \) \( d_g > 0 \) for defuzzification of the problem (2.1.1), we first calculate the lower and upper bounds of the optimal values. The optimal values \( z_i \) and \( z_u \) can be defined by solving the following standard linear programming problems, for which we assume that all they have the finite optimal values.
\[ Z_1 = \max \sum_{j=1}^{n} c_j x_j \]
\[ \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j \leq b_i, \quad 1 \leq i \leq m \]
\[ x_j \geq 0, \]

and

\[ Z_2 = \max \sum_{j=1}^{n} c_j x_j \]
\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i + p_i, \quad 1 \leq i \leq m \]
\[ x_j \geq 0, \]

and

\[ Z_3 = \max \sum_{j=1}^{n} c_j x_j \]
\[ \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j \leq b_i + p_i, \quad 1 \leq i \leq m \]
\[ x_j \geq 0, \]

and

\[ Z_4 = \max \sum_{j=1}^{n} c_j x_j \]
\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad 1 \leq i \leq m \]
\[ x_j \geq 0, \]
Let \( z_i = \min(z_1, z_2, z_3, z_4) \) and \( z_u = \max(z_1, z_2, z_3, z_4) \) The objective function takes values between \( z_i \) and \( z_u \) while technological coefficients take values between \( a_{ij} \) and \( a_{ij} \cdot d_{ij} \) and the right-hand side numbers take values between \( b_i \) and \( b_i + p_i \). The fuzzy set of optimal values, \( G \), which is subset of \( R^n \), is defined by

\[
\mu_G(x) = \begin{cases} 
0 & \text{if} \quad \sum_{j=1}^{n} c_j x_j < Z_i \\
\left(\sum_{j=1}^{n} c_j x_j - Z_i\right) / (Z_u - Z_i) & \text{if} \quad Z_i \leq \sum_{j=1}^{n} c_j x_j < Z_u \\
1 & \text{if} \quad \sum_{j=1}^{n} c_j x_j \geq Z_u 
\end{cases}
\]

(2.2.6)

The fuzzy set of the \( i^{th} \) constraint \( C_i \) which is subset of \( R^n \) is defined by

\[
\mu_{C_i}(x) = \begin{cases} 
0 & \text{if} \quad b_i < \sum_{j=1}^{n} a_{ij} x_j \\
\left(b_i - \sum_{j=1}^{n} a_{ij} x_j\right) / \left(\sum_{j=1}^{n} d_{ij} x_j + p_i\right) & \text{if} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i < \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j + p_i , \\
1 & \text{if} \quad b_i \geq \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j + p_i . 
\end{cases}
\]

(2.2.7)
By using the method of defuzzification as for the problem (2.1.8) the problem (2.2.1) is reduced to the following crisp problem:

\[
\text{max } \lambda \\
\lambda(Z_2 - Z_1) - \sum_{j=1}^{n} c_j x_j + Z_1 \leq 0 \\
\sum_{j=1}^{n} (a_{ij} + \lambda d_{ij})x_j + \lambda p_i - b_i \leq 0, \quad 1 < i < m \\
x_j \geq 0, \quad 0 \leq \lambda \leq 1.
\] 

(2.2.8)

Notice that, the problem (2.2.8) is also a non convex programming problem, similar to the problem (2.1.9).

The algorithm of the fuzzy decisive set method

This method is based on the idea that, for a fixed value of \( \lambda \), the problems (2.1.9) and (2.2.8) are linear programming problems. Obtaining the optimal solution \( \lambda' \) to the problems (2.1.9) and (2.2.8) is equivalent to determining the maximum value of \( \lambda \) for which the feasible set is nonempty. The algorithm of this method for the problem (2.1.9) and (2.2.8) is as follows.

Algorithm

Step1
Set \( \lambda = 1 \) and test whether a feasible set satisfying the constraints of the Problem (2.1.9) exists or not by using the optimality criterion of the simplex method. If a feasible set exists, set \( \lambda = 1 \), otherwise, set \( \lambda' = 0 \) and \( \lambda^R = 1 \) and go to the next step.

Step2
For the value of \( \lambda = (\lambda' + \lambda^R)/2 \) updated the value of \( \lambda' \) and \( \lambda^R \) using the bisection method as follows:
\( \lambda' = \lambda \) if feasible set is nonempty for \( \lambda \)
\( \lambda^R = \lambda \) if feasible set is empty for \( \lambda \)
Consequently, for each \( \lambda \) test whether a feasible set of the problem (2.1.9) exists or not by using the optimality criterion of the Simplex method and determine the maximum value \( \lambda' \) satisfying the constraints of the problem (2.1.9)
3. Example 1.

Solve the optimization problem [8].

\[
\begin{align*}
\text{max } & 2x_1 + 3x_2 \\
2 \leq & x_1 + x_2 \\
2 \leq & x_1 + 2x_2 \\
x_1, x_2 & \geq 0,
\end{align*}
\] (3.1)

Take fuzzy parameters as

\[
\begin{align*}
\tilde{2} &= L(2,2), \quad \tilde{1} = L(1,2) \\
\tilde{1} &= L(1,1), \quad \tilde{2} = L(2,1), \quad \tilde{b}_1 \quad \tilde{4} = L(4,1)
\end{align*}
\]

and \( \tilde{b}_2 \quad \tilde{5} = L(5,2) \) as used by Shaocheng[13].

Suppose that e.g. \( \tilde{a}_{ij} = L(2,2) \) means a fuzzy set with \( a_{ij} = 2 \) (peak) and \( d_{ij} = 2 \) (spread). The membership function of \( L(2,2) \) is

\[
L(2,2) = \begin{cases} 
1 & \text{for } x < 2 \\
\frac{4-x}{2} & \text{for } 2 \leq x \leq 4 \\
0 & \text{for } x > 4
\end{cases}
\]
\[
(a_{ij}) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (d_{ij}) = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \Rightarrow (a_{ij} + d_{ij}) = \begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix}
\]

\[
(b_{i}) = \begin{bmatrix} 4 \\ 5 \end{bmatrix} (p_{i}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow (b_{i} + p_{i}) = \begin{bmatrix} 5 \\ 7 \end{bmatrix}
\]

First solve following two sub-problems:

\[
Z_1 = \text{Max} 2x_1 + 3x_2 \\
4x_1 + 3x_2 \leq 4 \\
2x_1 + 3x_2 \leq 5 \\
x_1, x_2 \geq 0,
\]

and

\[
Z_2 = \text{Max} 2x_1 + 3x_2 \\
2x_1 + x_2 \leq 5 \\
x_1 + 2x_2 \leq 7 \\
x_1, x_2 \geq 0,
\]

Optimal solutions of these sub problems are:

\[
x_1 = 0 \quad x_1 = 1 \\
x_2 = 1.34 \quad x_2 = 3 \\
Z_1 = 4.02 \quad Z_1 = 11
\]

By using these optimal values, the problem can be reduced to following equivalent non-linear programming problem:
\[
\begin{align*}
\text{Max} & \quad \lambda \\
\frac{2x_1 + 3x_2 - 4.02}{11 - 4.02} & \geq \lambda \\
\frac{4 + 2x_1 - x_2}{2x_1 + 2x_2} & \geq \lambda \\
\frac{5 - x_1 - 2x_2}{x_1 + x_2} & \geq \lambda \\
0 & \leq \lambda \leq 1 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

That is

\[
\begin{align*}
\text{Max} & \quad \lambda \\
2x_1 + 3x_2 & \geq 4.02 + 6.98\lambda \\
(2 + 2\lambda)x_1 + (1 + 2\lambda)x_2 & \leq 4 - 1\lambda \\
(1 + \lambda)x_1 + (2 + \lambda)x_2 & \leq 5 - 2\lambda \\
0 & \leq \lambda \leq 1 \\
x_1, x_2 & \geq 0,
\end{align*}
\] (3.2)

Let solve (3.2) by using the fuzzy decisive set method

For \( \lambda = 1 \),

\[
\begin{align*}
2x_1 + 3x_2 & \geq 11 \\
4x_1 + 3x_2 & \leq 3 \\
2x_1 + 3x_2 & \leq 3 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

Since feasible set is empty, taking \( \lambda^l = 0 \) and \( \lambda^r = 1 \) the new value of \( \lambda = (0+1)/2 = 1/2 \) is applied.

For \( \lambda = 1/2 = 0.5 \)

\[
\begin{align*}
2x_1 + 3x_2 & \geq 7.51 \\
3x_1 + 2x_2 & \leq 3.5 \\
1.5x_1 + 2.5x_2 & \leq 4 \\
x_1, x_2 & \geq 0,
\end{align*}
\]
Since feasible set is empty, taking $\lambda^l = 0$ and $\lambda^u = 1/2$ the new value of $
exists (0+1/2)/2 = 1/4$ is applied.

For $\lambda = 1/4 = 0.25$, the problem can be written as

\[
\begin{align*}
2x_1 + 3x_2 &\geq 5.765 \\
2.5x_1 + 1.5x_2 &\leq 3.75 \\
1.25x_1 + 2.25x_2 &\leq 4.5 \\
x_1, x_2 &\geq 0,
\end{align*}
\]

Since feasible set is nonempty, taking $\lambda^l = 1/4$ and $\lambda^u = 1/2$ the new value of $
exists (1/4+1/2)/2 = 3/8$ is applied.

For $\lambda = 3/8 = 0.375$

\[
\begin{align*}
2x_1 + 3x_2 &\geq 6.6375 \\
2.75x_1 + 1.75x_2 &\leq 3.625 \\
1.375x_1 + 2.375x_2 &\leq 4.25 \\
x_1, x_2 &\geq 0,
\end{align*}
\]

Since feasible set is empty, taking $\lambda^l = 1/4$ and $\lambda^u = 3/8$ the new value of $
exists (1/4+3/8)/2 = 5/16$ is applied.

For $\lambda = 5/16 = 0.3125$

\[
\begin{align*}
2x_1 + 3x_2 &\geq 6.20125 \\
2.625x_1 + 1.625x_2 &\leq 3.6875 \\
1.3125x_1 + 2.3125x_2 &\leq 4.375 \\
x_1, x_2 &\geq 0,
\end{align*}
\]

Since feasible set is nonempty, taking $\lambda^l = 5/16$ and $\lambda^u = 3/8$ the new value of $
exists (5/16+3/8)/2 = 11/32$ is applied.

For $\lambda = 11/32 = 0.34375$

\[
\begin{align*}
2x_1 + 3x_2 &\geq 6.419375 \\
2.6875x_1 + 1.6875x_2 &\leq 3.65625 \\
1.34375x_1 + 2.34375x_2 &\leq 4.3125 \\
x_1, x_2 &\geq 0,
\end{align*}
\]

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Since feasible set is nonempty, taking $\lambda^l = 11/32$ and $\lambda^u = 3/8$ the new value of $\lambda = (11/32+3/8)/2 = 23/64$ is applied.

For $\lambda = 23/64 = 0.359375$
\[
\begin{align*}
2x_1 + 3x_2 &\geq 6.5284375 \\
2.71875x_1 + 1.71875x_2 &\leq 3.640625 \\
1.359375x_1 + 2.359375x_2 &\leq 4.28125 \\
x_1, x_2 &\geq 0,
\end{align*}
\]

Since feasible set is nonempty, taking $\lambda^l = 23/64$ and $\lambda^u = 3/8$ the new value of $\lambda = (23/64+3/8)/2 = 47/128$ is applied.

For $\lambda = 47/128 = 0.3671875$
\[
\begin{align*}
2x_1 + 3x_2 &\geq 6.58296875 \\
2.734375x_1 + 1.734375x_2 &\leq 3.6328125 \\
1.3671875x_1 + 2.3671875x_2 &\leq 4.265625 \\
x_1, x_2 &\geq 0,
\end{align*}
\]

Since feasible set is empty, taking $\lambda^l = 23/64$ and $\lambda^u = 47/128$ the new value of $\lambda = (23/64+47/128)/2 = 93/256$ is applied.

For $\lambda = 93/256 = 0.36328125$
\[
\begin{align*}
2x_1 + 3x_2 &\geq 6.555703125 \\
2.7265625x_1 + 1.7265625x_2 &\leq 3.63671875 \\
1.36328125x_1 + 2.36328125x_2 &\leq 4.2734375 \\
x_1, x_2 &\geq 0,
\end{align*}
\]

Since feasible set is empty, taking $\lambda^l = 23/64$ and $\lambda^u = 93/256$ the new value of $\lambda = (23/64+93/256)/2 = 185/512$ is applied.

For $\lambda = 185/512 = 0.361328125$
\[
\begin{align*}
2x_1 + 3x_2 &\geq 6.542070313 \\
2.72265625x_1 + 1.72265625x_2 &\leq 3.638672 \\
1.361328125x_1 + 2.361328125x_2 &\leq 4.2773437 \\
x_1, x_2 &\geq 0,
\end{align*}
\]
Since feasible set is empty, taking $\lambda^l = 23/64$ and $\lambda^u = 185/512$ the new value of $\lambda = (23/64+185/512)/2 = 369/1024$ is applied.

For $\lambda = 369/1024 = 0.36035156$

$$2x_1 + 3x_2 \geq 6.535253889$$
$$2.72070312x_1 + 1.72070312x_2 \leq 3.63964844$$
$$1.36035156x_1 + 2.36035156x_2 \leq 4.27929688$$

$x_1, x_2 \geq 0,$

Since feasible set is empty, taking $\lambda^l = 23/64$ and $\lambda^u = 369/1024$ the new value of $\lambda = (23/64+369/1024)/2 = 737/2048$ is applied.

For $\lambda = 737/2048 = 0.3598632813$

$$2x_1 + 3x_2 \geq 6.531845703$$
$$2.7197266x_1 + 1.7197266x_2 \leq 3.64013672$$
$$1.35986328x_1 + 2.35986328x_2 \leq 4.2802734$$

$x_1, x_2 \geq 0,$

Since feasible set is nonempty, taking $\lambda^l = 737/2048$ and $\lambda^u = 369/1024$ the new value of $\lambda = (737/2048+369/1024)/2 = 1475/4096$ is applied.

For $\lambda = 1475/4096 = 0.3601074219$

$$2x_1 + 3x_2 \geq 6.533549805$$
$$2.7202214844x_1 + 1.720214844x_2 \leq 3.63989258$$
$$1.3601074219x_1 + 2.3601074219x_2 \leq 4.2798516$$

$x_1, x_2 \geq 0,$

Since feasible set is nonempty, taking $\lambda^l = 1475/4096$ and $\lambda^u = 369/1024$ the new value of $\lambda = (1475/4096+369/1024)/2 = 2951/8192$ is applied.

For $\lambda = 2951/8192 = 0.3602294922$

$$2x_1 + 3x_2 \geq 6.534401856$$
$$2.7204589844x_1 + 1.7204589844x_2 \leq 3.639770508$$
$$1.3602294922x_1 + 2.3602294922x_2 \leq 4.27954102$$

$x_1, x_2 \geq 0,$
Since feasible set is empty, taking $\lambda^l = 1475/4096$ and $\lambda^u = 2951/8192$ the new value of $\lambda = (1475/4096+2951/8192)/2 = 5901/16384$ is applied.

For $\lambda = 5901/16384 = 0.360168457$

\[
\begin{align*}
2x_1 + 3x_2 & \geq 6.53397583 \\
2.720336914x_1 + 1.720336914x_2 & \leq 3.639831543 \\
1.360168457x_1 + 2.360168457x_2 & \leq 4.2796631 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

Since feasible set is nonempty, taking $\lambda^l = 5901/16384$ and $\lambda^u = 2951/8192$ the new value of $\lambda = (5901/16384+2951/8192)/2 = 11802/32768$ is applied.

For $\lambda = 11802/32768 = 0.3601989746$

\[
\begin{align*}
2x_1 + 3x_2 & \geq 6.534188843 \\
2.72039795x_1 + 1.72039795x_2 & \leq 3.6398010 \\
1.3601989746x_1 + 2.3601989746x_2 & \leq 4.279602 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

Since feasible set is nonempty, taking $\lambda^l = 11802/32768$ and $\lambda^u = 2951/8192$ the new value of $\lambda = (11803/32768+2951/8192)/2 = 11802/32768$ is applied.

For $\lambda = 23607/65536 = 0.3602142334$

\[
\begin{align*}
2x_1 + 3x_2 & \geq 6.534295349 \\
2.720397949x_1 + 1.720397949x_2 & \leq 3.6397857 \\
1.36021433x_1 + 2.360214233x_2 & \leq 4.279602 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

Since feasible set is nonempty, taking $\lambda^l = 23607/65536$ and $\lambda^u = 2951/8192$ the new value of $\lambda = (23607/65536+2951/8192)/2 = 47215/131072$ is applied.

For $\lambda = 47215/131072 = 0.36022$

\[
\begin{align*}
2x_1 + 3x_2 & \geq 6.534348602 \\
2.7204437256x_1 + 1.7204437256x_2 & \leq 3.639778137 \\
1.3602218628x_1 + 2.3602218628x_2 & \leq 4.279556274 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

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Since feasible set is nonempty, taking $\lambda^l = 47215/131072$ and $\lambda^u = 2951/8192$ the new value of $\lambda = (47215/131072 + 2951/8192)/2 = 94431/262144$ is applied.

For $\lambda = 94431/262144 = 0.36022$

$$
2x_1 + 3x_2 \geq 6.534375229 \\
2.720451355x_1 + 1.720451355x_2 \leq 3.639774332 \\
1.3602256677x_1 + 2.3602256775x_2 \leq 4.279548645 \\
x_1, x_2 \geq 0,
$$

Since feasible set is nonempty, taking $\lambda^l = 94431/262144$ and $\lambda^u = 2951/8192$ the new value of $\lambda = (94431/262144 + 2951/8192)/2 = 188863/524288$ is applied.

For $\lambda^* = 188863/524288 = 0.36022$

$$
2x_1 + 3x_2 \geq 6.534388542 \\
2.7204551696x_1 + 1.7204551696x_2 \leq 3.639772415 \\
1.3602257848x_1 + 2.3602275848x_2 \leq 4.27954483 \\
x_1, x_2 \geq 0,
$$

First set the value of $\lambda = 1$ put the $\lambda$ value in 3.2 equation after solving the equation you can see we have new function written below the 3.2 now first solve that function with simplex linear programming method if the solution value is less then with right hand side value then feasible set is empty taking the $\lambda^l = 0$ and $\lambda^u = 1$ or if the value is greater the feasible set is nonempty taking $\lambda^l = 1$ and $\lambda^u = 0$ this process continue until we obtain the optimal value of $\lambda$. Check the difference between the previous lambda and the next lambda and compare the absolute difference to an accuracy constant $\epsilon$ almost equal to zero. For instance, if $\epsilon = 0.0001$ then $0.36021 - 0.36022 = 0.00001 < 0.0001$. Thus the last lambda $0.36022$ can be treated as optimal. For update the $\lambda$ use this formula $\lambda = (\lambda^l + \lambda^u)/2$, 23
Example 2.

Solve the optimization problem

\[
\begin{align*}
&\text{Max } 2x_1 + 3x_2 \\
&x_1 + 3x_2 \leq 6 \\
&3x_1 + 2x_2 \leq 6 \\
&x_1, x_2 \geq 0, \\
\end{align*}
\]

(3.3)

Take fuzzy parameters as:

\[
\begin{align*}
1 &= L(1,1), 3 = L(3,1), \\
3 &= L(3,3), 2 = L(2,3), b_1 = 6 = L(6,3) \\
\text{and } b_2 &= 6 = L(6,2),
\end{align*}
\]

\[
\begin{align*}
(a_{ij}) &= \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}, (d_{ij}) = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \Rightarrow (a_{ij} + d_{ij}) = \begin{bmatrix} 2 & 2 \\ 6 & 6 \end{bmatrix} \\
(b_i) &= \begin{bmatrix} 6 \\ 6 \end{bmatrix}, (p_i) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow (b_i + p_i) = \begin{bmatrix} 9 \\ 8 \end{bmatrix}
\end{align*}
\]

First solve following two sub-problems;

\[
\begin{align*}
Z_1 &= \text{Max } 2x_1 + 3x_2 \\
2x_1 + 4x_2 &\leq 6 \\
6x_1 + 5x_2 &\leq 6 \\
x_1, x_2 &\geq 0,
\end{align*}
\]

and
\[ Z_2 = \text{Max} 2x_1 + 3x_2 \]
\[ x_1 + 3x_2 \leq 9 \]
\[ 3x_1 + 2x_2 \leq 8 \]
\[ x_1, x_2 \geq 0, \]

Optimal solutions of these sub problems are;

\[ x_1 = 0 \quad \text{and} \quad x_1 = 0.86 \]
\[ x_2 = 1.2 \quad \text{and} \quad x_2 = 2.71 \]
\[ Z_1 = 3.6 \quad \text{and} \quad Z_2 = 9.85 \]

BY using these optimal values, the problem can be reduced to following equivalent non-linear programming problem;

\[
\begin{align*}
\text{Max} & \quad \lambda \\
\frac{2x_1 + 3x_2 - 3.6}{9.85 - 3.6} & \geq \lambda \\
\frac{6 - x_1 - 3x_2}{x_1 + x_2} & \geq \lambda \\
\frac{6 - 3x_1 - 2x_2}{3x_1 + 3x_2} & \geq \lambda \\
0 \leq \lambda & \leq 1 \\
x_1, x_2 & \geq 0,
\end{align*}
\]
That is

\[
\begin{align*}
\text{Max } & \lambda \\
2x_1 + 3x_2 & \geq 3.6 + 6.25\lambda \\
(1 + \lambda)x_1 + (3 + \lambda)x_2 & \leq 6 - 3\lambda \\
(3 + 3\lambda)x_1 + (2 + 3\lambda)x_2 & \leq 6 - 2\lambda \\
0 & \leq \lambda \leq 1 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

(3.4)

Let solve (3.4) by using the fuzzy decisive set method

For \( \lambda = 1 \),

\[
\begin{align*}
2x_1 + 3x_2 & \geq 9.89 \\
x_1 + 3x_2 & \leq 3 \\
6x_1 + 5x_2 & \leq 4 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

Since feasible set is empty, taking \( \lambda^l = 0 \) and \( \lambda^u = 1 \) the new value of \( \lambda = (0+1)/2 = 1/2 \) is applied.

For \( \lambda = 1/2 = 0.5 \),

\[
\begin{align*}
2x_1 + 3x_2 & \geq 6.725 \\
1.5x_1 + 3.5x_2 & \leq 4.5 \\
4.5x_1 + 3.5x_2 & \leq 5 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

Since feasible set is empty, taking \( \lambda^l = 0 \) and \( \lambda^u = 1/2 \) the new value of \( \lambda = (0+1/2)/2 = 1/4 \) is applied.

For \( \lambda = 1/4 = 0.25 \),

\[
\begin{align*}
2x_1 + 3x_2 & \geq 5.1625 \\
1.25x_1 + 3.25x_2 & \leq 5.25 \\
3.75x_1 + 2.75x_2 & \leq 5.5 \\
x_1, x_2 & \geq 0,
\end{align*}
\]
Since feasible set is nonempty, taking \( \lambda' = 1/4 \) and \( \lambda^r = 1/2 \) the new value of \( \lambda = (1/4 + 1/2)/2 = 3/8 \) is applied.

For \( \lambda = 3/8 = 0.375 \)

\[
\begin{align*}
2x_1 + 3x_2 & \geq 5.94375 \\
1.375x_1 + 3.375x_2 & \leq 4.875 \\
4.125x_1 + 3.125x_2 & \leq 5.25 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

Since feasible set is empty, taking \( \lambda' = 1/4 \) and \( \lambda^r = 3/8 \) the new value of \( \lambda = (1/4 + 3/8)/2 = 5/16 \) is applied.

For \( \lambda = 5/16 = 0.3125 \)

\[
\begin{align*}
2x_1 + 3x_2 & \geq 5.553125 \\
1.3125x_1 + 3.3125x_2 & \leq 5.0625 \\
3.9375x_1 + 2.9375x_2 & \leq 5.375 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

Since feasible set is nonempty, taking \( \lambda' = 5/16 \) and \( \lambda^r = 3/8 \) the new value of \( \lambda = (5/16 + 3/8)/2 = 11/32 \) is applied.

For \( \lambda = 11/32 = 0.34375 \)

\[
\begin{align*}
2x_1 + 3x_2 & \geq 5.7484375 \\
1.34375x_1 + 3.34375x_2 & \leq 4.96875 \\
4.03125x_1 + 3.03125x_2 & \leq 5.3125 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

Since feasible set is nonempty, taking \( \lambda' = 11/32 \) and \( \lambda^r = 3/8 \) the new value of \( \lambda = (11/32 + 3/8)/2 = 23/64 \) is applied.

For \( \lambda = 23/64 = 0.359375 \)

\[
\begin{align*}
2x_1 + 3x_2 & \geq 5.84609375 \\
1.359375x_1 + 3.359375x_2 & \leq 4.921875 \\
4.078125x_1 + 3.078125x_2 & \leq 5.28125 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

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Since feasible set is empty, taking $\lambda' = 11/32$ and $\lambda^r = 3/64$ the new value of $\lambda = (11/32 + 23/64)/2 = 45/128$ is applied.

For $\lambda = 45/128 = 0.3515625$

\[
\begin{align*}
2x_1 + 3x_2 & \geq 5.797265625 \\
1.3515625x_1 + 3.3515625x_2 & \leq 4.9453125 \\
4.0546875x_1 + 3.0546875x_2 & \leq 5.296875 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

Since feasible set is nonempty, taking $\lambda' = 45/128$ and $\lambda^r = 23/64$ the new value of $\lambda = (45/128 + 23/64)/2 = 45/128$ is applied.

For $\lambda = 91/256 = 0.35546875$

\[
\begin{align*}
2x_1 + 3x_2 & \geq 5.821679688 \\
1.35546875x_1 + 3.35546875x_2 & \leq 4.9335937 \\
4.06640625x_1 + 3.06640625x_2 & \leq 5.2890625 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

Since feasible set is nonempty, taking $\lambda' = 91/256$ and $\lambda^r = 23/64$ the new value of $\lambda = (91/256 + 23/64)/2 = 183/512$ is applied.

For $\lambda = 183/512 = 0.357421875$

\[
\begin{align*}
2x_1 + 3x_2 & \geq 5.833886719 \\
1.357421875x_1 + 3.357421875x_2 & \leq 4.927434375 \\
4.072265625x_1 + 3.072265625x_2 & \leq 5.28515625 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

Since feasible set is empty, taking $\lambda' = 91/256$ and $\lambda^r = 183/512$ the new value of $\lambda = (91/256 + 183/512)/2 = 365/1024$ is applied.

For $\lambda = 365/1024 = 0.3564453125$

\[
\begin{align*}
2x_1 + 3x_2 & \geq 5.827783203 \\
1.3564453125x_1 + 3.3564453125x_2 & \leq 4.93066406 \\
4.0693359x_1 + 3.06933594x_2 & \leq 5.28710975 \\
x_1, x_2 & \geq 0,
\end{align*}
\]
Since feasible set is empty, taking $\lambda^l = 91/256$ and $\lambda^u = 365/1024$ the new value of $\lambda = (91/256 + 365/1024)/2 = 729/2048$ is applied.

For $\lambda = 729/2048 = 0.3559570313$

\[
\begin{align*}
2x_1 + 3x_2 & \geq 5.824731446 \\
1.35595703x_1 + 3.35595703x_2 & \leq 4.9321289 \\
4.06787109x_1 + 3.06787109x_2 & \leq 5.28808594 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

Since feasible set is empty, taking $\lambda^l = 91/256$ and $\lambda^u = 729/2048$ the new value of $\lambda = (91/256 + 729/2048)/2 = 1457/4096$ is applied.

For $\lambda = 1457/4096 = 0.3557128906$

\[
\begin{align*}
2x_1 + 3x_2 & \geq 5.823205566 \\
1.35571289x_1 + 3.35571289x_2 & \leq 4.93286133 \\
4.06713867x_1 + 3.067133867x_2 & \leq 5.28857422 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

Since feasible set is empty, taking $\lambda^l = 91/256$ and $\lambda^u = 1457/4096$ the new value of $\lambda = (91/256 + 1457/4096)/2 = 2913/8192$ is applied.

For $\lambda = 2913/8192 = 0.3555908203$

\[
\begin{align*}
2x_1 + 3x_2 & \geq 5.822442627 \\
1.35559082x_1 + 3.35559082x_2 & \leq 4.93322754 \\
4.06677246x_1 + 3.06677246x_2 & \leq 5.28881836 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

Since feasible set is nonempty, taking $\lambda^l = 2913/8192$ and $\lambda^u = 1457/4096$ the new value of $\lambda = (2913/8192 + 1457/4096)/2 = 5827/16384$ is applied.

For $\lambda = 5827/16384 = 0.35565185$

\[
\begin{align*}
2x_1 + 3x_2 & \geq 5.822824097 \\
1.35566185x_1 + 3.35566185x_2 & \leq 4.93304443 \\
4.06695556x_1 + 3.066955567x_2 & \leq 5.28886763 \\
x_1, x_2 & \geq 0,
\end{align*}
\]

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Since feasible set is empty, taking $\lambda' = 2913/8192$ and $\lambda^r = 5827/16384$
The new value of $\lambda = (2913/8192 + 5827/16384)/2 = 11653/32768$ is applied.

For $\lambda = 11653/32768 = 0.35562133$

\[
2x_1 + 3x_2 \geq 5.82263362 \\
1.35562134x_1 + 3.35562134x_2 \leq 4.93313599 \\
4.06686401x_1 + 3.06686401x_2 \leq 5.288757324
\]

$x_1, x_2 \geq 0$,

Since feasible set is nonempty, taking $\lambda' = 11653/32768$ and $\lambda^r = 5827/16384$
The new value of $\lambda = (11653/32768 + 5827/16384)/2 = 23307/65536$ is applied.
For $\lambda = 23307/65536 = 0.35563$

\[
2x_1 + 3x_2 \geq 5.822728729 \\
1.355636597x_1 + 3.355636597x_2 \leq 4.93309021 \\
4.06690979x_1 + 3.06690979x_2 \leq 5.288726807
\]

$x_1, x_2 \geq 0$,

Since feasible set is empty, taking $\lambda' = 11653/32768$ and $\lambda^r = 23307/65536$
The new value of $\lambda = (11653/32768 + 23307/65536)/2 = 23307/65536$ is applied.
For $\lambda^* = 46613/131072 = 0.35562$

We obtain the optimal value of $\lambda$ at $\lambda^*$ by using the fuzzy decisive set method.

**Conclusion**

The objective of this thesis is to find the optimal solution of the problem using the fuzzy decisive set method. The method is based on the transformation of a fuzzy linear program to an equivalent non-fuzzy form containing a parameter $\lambda$. By utilizing the bisection method we bound both the objective function and the constrains when finding the optimal value of the parameter $\lambda$, which warrants the non-emptiness of the solution set.
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