Symplectic geometry and Calogero-Moser systems

Lukas Rødland
Supervisor: Luigi Tizzano
Subject reader: Maxime Zabzine
Bachelor of Science Degree in Physics
Uppsala University

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Abstract

We introduce some basic concepts from symplectic geometry, classical mechanics and integrable systems. We use this theory to show that the rational and the trigonometric Calogero-Moser systems, that is the Hamiltonian systems with Hamiltonian $H = \sum_{i=1}^{n} p_i^2 - \sum_{j\neq i} \frac{1}{(x_i-x_j)^2}$ and $H = \sum_i p_i + \sum_{i\neq j} \frac{1}{4\sin^2((x_i-x_j)/2)}$ respectively are integrable systems. We do this using symplectic reduction on $T^* \operatorname{Mat}_n(\mathcal{C})$.

Sammanfattning

Vi presenterar några grundläggande idéer från symplektisk geometri, klassisk mekanik och integrabla system. Vi använder denna teori för att visa att rationella och trigonometriska Calogero-Moser system, det vill säga hamiltonska system med hamiltonoperator $H = \sum_{i=1}^{n} p_i^2 - \sum_{j\neq i} \frac{1}{(x_i-x_j)^2}$ respektive $H = \sum_i p_i + \sum_{i\neq j} \frac{1}{4\sin^2((x_i-x_j)/2)}$ respektive är integrerbara system. Vi gör detta genom att använda symplektisk reduktion på $T^* \operatorname{Mat}_n(\mathcal{C})$.

Contents

1 Introduction

One of the big turning points in the history of classical mechanics came when Poincaré showed that most systems cannot be solved exactly. Even simple systems, as the three-body problem in three dimensions does not have an exact solution. Those systems that have a exact solution is called Liouville integrable systems, or just integrable systems. Even if most realistic systems are not integrable, there is still interesting to study them.

One group of these systems is the Calogero-Moser systems. They are one-dimensional N-body problems with a pairwise potential. Francesco Calogero showed in 1971 that the quantum mechanical system with Hamiltonian function

$$H = \sum_{i=1}^{N} p_i^2 - \sum_{j \neq i} \frac{1}{(x_i - x_j)^2}.$$

is integrable in the quantum mechanical sense. This is the system with N particles on a line with a pairwise potential which depends of the distance squared between each pair of particles and is known as the rational Calogero-Moser system. In 1971 Sutherland showed that the system with potential proportional to $\sin^{-2}(x_i - x_j)$ also is integrable as a quantum mechanical system. This system is called the Sutherland system or the trigonometric Calogero-Moser system and can be thought of as N particles on a circle.

In 1975 Moser showed that the rational and the trigonometric Calogero-Moser systems also are integrable in the classical sense. Both the rational and the trigonometric systems are special cases of the elliptic Calogero-Moser systems.

Calogero-Moser systems plays a role in many areas of physics such as statistical mechanics, condensed matter physics, quantum field theory and string theory.

The goal of this thesis is to show that the rational and trigonometric Calogero-Moser systems are integrable. To do this Iuse a method developed by Kazhdan, Kostant, Stenberg [7]. This method uses symplectic reduction on the space of matrices to show that the system is in fact integrable.

The structure of this thesis is the following: In section 2 I go through some basic theory about symplectic geometry. In section 2.1 I go through the basic definitions and results. In section 2.2 I introduce the concept of the flow of a vector field and Hamiltonian vector fields.

In section 3 I start with going through the connection between Hamiltonian mechanics and symplectic geometry and then I define what an integrable system is, and discuss how I can solve them in general.

In section 4 I go through the Hamiltonian action of a Lie group on a symplectic manifold, I use this to define the generalization of momentum called a momentum map, which is closely related to symmetries.

Using moment maps I can define the concept of symplectic reduction, that is what I do in section 5.

In section 6.1 I introduce the Calogero-Moser space. I use the Calogero-Moser space to define the rational Calogero-Moser system in 6.2. I show that the rational Calogero-Moser system is just the system of N particles on a line. I then show that this system is integrable and find a solution. In 6.3 I show that the trigonometric Calogero-Moser system is integrable.

For most of section 2, 3, 4 and 5 I will follow [4]. The interested reader can find a short and good introduction to symplectic geometry in [4]. In section 6 I follow the first two chapters in [1].

I only assume that the reader know some basic differential geometry such as some knowledge about manifolds, tangent spaces, differential forms etc. For a Introduction on differential geometry the reader might look at [3]. For some more theory about the relation of Hamiltonian mechanics and symplectic geometry [2] is a good source.

2 Symplectic geometry

I start by giving an introduction of the general concepts of symplectic geometry and in the next section I will use them to define Hamiltonian mechanics. Here I will be brief, the proof of all the theorems presented here can be found in [4].

2.1 Symplectic manifolds

A symplectic form on a vector space, V is a non-degenerate antisymmetric bilinear form, that is a map $\Omega : V \times V \to \mathbb{R}$ such that

$$\begin{aligned}\Omega(v, w) &= -\Omega(w, v) \quad \forall v, w \in V \\ \Omega(v, w) &= 0 \quad \forall v \in V \Rightarrow w = 0 \\ \Omega(au + bv, w) &= a\Omega(u, w) + b\Omega(v, w) \quad \forall v, w \in V, \quad a, b \in \mathbb{R}.\end{aligned}$$

A vector space V with a symplectic form Ω is called a symplectic vector space.

Theorem 2.1. *Let (V, Ω) be a symplectic vector space. Then we have a basis*

$e_1, \ldots, e_n, f_1, \ldots f_n$ *such that*

$$\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0 \qquad \textit{for all i,j and}$$
$$\Omega(e_i, f_j) = \delta_{ij} \qquad \textit{for all i,j}$$

Such a basis $e_1, \ldots, e_n, f_1, \ldots f_n$ is called a symplectic basis of V. It follows from this that the dimension of a symplectic vector space is even.

Example 2.1. Let (V, Ω) be a symplectic vector space with a symplectic basis $e_1, \ldots, e_n, f_1, \ldots f_n$. With respect to this basis we have

$$\Omega(u, v) = (-u-) \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} \begin{pmatrix} | \\ v \\ | \end{pmatrix}$$

Definition 2.1. Two symplectic vector spaces (V, Ω) and (W, Θ) is symplectomorphic if there exists an isomorphism $\phi : V \to W$ such that $\Omega(v, w) = \Theta(\phi(v), \phi(w))$. The map ϕ is called a symplectomorphism.

Let ω be a two-form on a manifold M, that is for each point $p \in M$, the map $\omega_p : T_pM \times T_pM$ is a skew-symmetric bilinear form on the tangent space at p such that ω_p varies smoothly in p. We say that ω is closed if $d\omega = 0$, where d is the exterior derivative.

Definition 2.2. A two-form ω on a manifold M is called symplectic if ω is closed, and ω_p is symplectic for all $p \in M$.

If ω is a symplectic form on M then we have that $\dim T_pM = \dim M$ is even.

Definition 2.3. A symplectic manifold (M, ω) is a pair where M is a manifold and ω is a symplectic form.

Since ω is a non-degenerate two-form we have that $\omega^n = \omega \wedge \omega \wedge \cdots \wedge \omega$ is non-vanishing. Since ω^n is a non-vanishing top-form, i.e. a volume form, any symplectic manifold is orientable. $\omega^n/n!$ is called the Liouville volume of (M, ω).

Example 2.2. The prototype of a symplectic manifold is $M = \mathbb{R}^{2n}$ with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. The form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is symplectic because

$$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \dots, \left(\frac{\partial}{\partial y_n}\right)_p$$

forms a symplectic basis of T_pM.

Definition 2.4. Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds, and let $\phi : M_1 \to M_2$ be a diffeomorphism. Then ϕ is a symplectomorphism if $\phi^*\omega_2 = \omega_1$. If there exists a symplectomorphism then M_1 and M_2 is called symplectomorphic.

Theorem 2.2 (Darboux). *Let (M, ω) be a symplectic manifold of dimension $2n$. Then there exists local coordinates $(\mathcal{U}, x_1, \dots, x_n, y_1, \dots, y_n)$ such that*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

$(x_1, \dots, x_n, y_1, \dots, y_n)$ is called symplectic coordinates.

The Darboux theorem is one of the most important results in symplectic geometry. It tells us that a symplectic manifold locally looks like our prototype symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$.

Example 2.3. Let X be a n dimensional manifold with local coordinates $(\mathcal{U}, x_1, \dots, x_n)$. At any point $x \in \mathcal{U}$, the differentials $(dx_1)_x, \dots, (dx_n)_x$ forms a basis of T_x^*X. An element $\xi \in T_x^*X$ can be written as $\xi = \sum_{i=1}^{n} \xi_i (dx_i)_x$ for some real coefficients $\xi_1, \dots, \xi_n$.

The cotangent bundle of X is defined as

$$T^*X := \bigcup_{x \in X} T_x^*X$$

This induces a map

$$\begin{aligned} &T^*\mathcal{U} \to \mathbb{R}^{2n} \\ &(x, \xi) \mapsto (x_1, \dots, x_n, \xi_1, \dots, \xi_n) \end{aligned}$$

The chart $(T^*\mathcal{U}, x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ forms a coordinate chart for T^*X. Given two charts $(\mathcal{U}, x_1, \dots, x_n)$ and $(\mathcal{U}', x_1', \dots, x_n')$, and $x \in \mathcal{U} \cap \mathcal{U}'$, if $\xi \in T_x^*X$, then

$$\xi = \sum_{i=1}^{n} \xi_i (dx_i)_x = \sum_{i,j} \xi_i \left(\frac{\partial x_i}{\partial x_j'}\right) (dx_j')_x = \sum_{i=1}^{n} \xi_j' (dx_j')_x$$

where $\xi'_j = \sum_i \xi_i \left(\frac{\partial x_i}{\partial x'_j}\right)$ is smooth. Hence T^*X is a $2n$ dimensional manifold with local coordinates $(T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$.

Let

$$\begin{aligned} \pi : T^*X &\to X \\ p = (x, \xi) &\mapsto x \end{aligned}$$

be the natural projection. The tautological one-form α may be defined pointwise as

$$\alpha_p = (d\pi_p)^*\xi \quad \in T_p^*(T^*X)$$

where $(d\pi)^*$ is the transpose of $d\pi$, that is, $(d\pi_p)^*\xi = \xi \circ d\pi_p$. Where the map

$$d\pi_p : T_p(T^*X)) \to T_xX$$

sends $\frac{\partial}{\partial x_j}$ to $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial \xi_j}$ to 0.

We can define the canonical symplectic form on a cotangent bundle in a coordinate-free way as $\omega = -d\alpha$.

In the coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ the tautological form is on the form

$$\alpha = \sum_{i=1}^{n} \xi_i dx_i.$$

We can now see that the symplectic $\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i = -d\alpha$ which means that $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ are symplectic coordinates on T^*X.

2.2 Isotopies and vector fields

The concept of a flow on a manifold is of high importance in classical mechanics, because the state of the system follows the flow of the Hamiltonian vector field.

Definition 2.5. Let M be a manifold, a map $\rho : M \times \mathbb{R} \to M$ is an isotopy if $\rho_t(p) := \rho(p, t)$ is a diffeomorphism for all $t \in \mathbb{R}$ and $\rho_0(p) = p$. Given an isotopy ρ we get a time-dependent vector field v_t such that

$$v_t \circ \rho_t = \frac{d\rho_t}{dt}$$

So the time derivative of the isotopy defines the vector field v_t, that means that for each ρ_t we can define a time-dependent vector field. If M is a compact manifold we have one-to-one correspondence of isotopies and time-dependent vector fields on M.

Definition 2.6. When v is a time-independent vector field the corresponding isotopy is called the flow or the exponential map of the vector field. It is often written as $\rho_t = \exp tv$ and it is a smooth family of diffeomorphisms on M.

For a fixed point $p \in M$ we call the map $\rho(p,t) : \mathbb{R} \to M$ for the integral curve through p. The elements of the corresponding vector field at points along the curve is clearly tangent vectors to the curve. We also have that $\rho_t \circ \rho_s = \rho_{t+s}$ therefore ρ_t forms an abelian group called the one parameter group.

Definition 2.7. The Lie derivative by v_t is the operator

$$\mathcal{L}_{v_t} : \Omega^k(M) \to \Omega^k(M)$$

defined by

$$\mathcal{L}_{v_t} = \frac{d}{dt}(\rho_t)^*\omega|_{t=0}$$

The Lie derivative of a r-form along a vector field tells us something about how much the form changes along the flow of the vector field.

Proposition 2.1 (Cartan magic formula)**.** *Let ω be a differential form. For a given time-independent vector field v we find that*

$$\mathcal{L}_v\omega = d \circ \iota_v\omega + \iota_v \circ d\omega. \tag{1}$$

We also have that when v_t be a time-dependent vector field, then

$$\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{v_t}\omega.$$

ι_v is the interior product with respect to v defined as $(\iota_v\omega)(X_1, X_2, ..., X_{r-1}) = \omega(v, X_1, ..., X_{r-1})$, the interior product sends r-forms to $(r-1)$-forms.

3 Hamiltonian mechanics

3.1 Hamiltonian and symplectic vector fields

For a given smooth function $H \in C^\infty(M)$ on a symplectic manifold (M,ω) we can define a unique vector field X_H, called the Hamiltonian vector field to the Hamiltonian function H as $\iota_{X_H}\omega = dH$.

Proposition 3.1. *Let ρ_t be the flow generated by the Hamiltonian vector field X_f. Then we can see that the symplectic form is preserved by the one parameter group, that is $\rho_t^* \omega = \omega$.*

Proof.

$$\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{X_f}\omega = \rho_t^*(d \circ \iota_{X_f}\omega + \iota_{X_f} \circ d\omega) = \rho_t^* d \circ df = 0$$
$$\Rightarrow \rho_t^*\omega = const. \quad \text{since} \quad \rho_0^*\omega = \omega \quad \Rightarrow \rho_t^*\omega = \omega \quad \forall t \in \mathbb{R}$$

□

Because of this the family of diffeomorphisms ρ_t generated by a function is symplectomorphisms on (M, ω).

Definition 3.1. A vector field X on (M, ω) is a symplectic vector field if ω is preserved along the flow generated by X. This happens exactly when $\iota_X \omega$ is closed.

It is clear that all Hamiltonian vector fields are symplectic because all exact forms are closed.

3.2 Brackets

Definition 3.2. A Lie algebra is a vector space $\mathfrak{g}$ with a Lie bracket $[\cdot, \cdot]$, i.e. a bilinear map

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

such that

$$[X, Y] = -[Y, X] \quad \forall X, Y \in \mathfrak{g}$$
$$0 = [X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] \quad \forall X, Y, Z \in \mathfrak{g}$$

Definition 3.3. The Lie bracket of a vector field Y along the flow of a vector field X is defined as

$$[X, Y] := \mathcal{L}_X Y = \frac{d}{dt}(\rho)_* Y|_{t=0}$$

The set of vector fields on a manifold M with the Lie bracket defined as above defines a Lie algebra.

Theorem 3.1. *Let X and Y be symplectic vector fields on a symplectic manifold (M, ω), then $[X, Y]$ is Hamiltonian with Hamiltonian function $\omega(X, Y)$.*

Definition 3.4. A Poisson bracket is a Lie bracket that follows Leibniz rule, $\{X, YZ\} = \{X, Y\}Z + Y\{X, Z\}$. A manifold M is a Poisson manifold if the set of continuous functions on M, $C^\infty(M)$ has a Poisson bracket.

On any symplectic manifold (M, ω) we can define a Poisson bracket,

$$\{f, g\} = \iota_{X_f}\iota_{X_g}\omega = \omega(X_f, X_g) \quad \forall f, g \in C^\infty(M)$$

where X_f, X_g is the Hamiltonian vector field of the functions f, g. This implies that every symplectic manifold is a Poisson manifold. This Poisson bracket is often called the Poisson bracket.

Theorem 3.2. *Let f and g be smooth functions on a symplectic manifold (M, ω) and let $\exp tX_g$ be the flow generated by g, then*

$$\dot{f} := \frac{d}{dt}(f \circ \exp tX_g) = \{g, f\}$$

Proof.

$$\begin{aligned} \frac{d}{dt} f(\exp tX_g) = \exp(tX_g)^* \mathcal{L}_{X_g} f \\ = \exp(tX_g)^*(\iota_{X_g} df + d\iota_{X_g} f) = \exp(tX_g)^* \iota_{X_g} df \\ = \exp(tX_g)^* \iota_{X_g}\iota_{X_f}\omega = \exp(tX_g)^*\{f, g\} \Rightarrow \{f, g\} \circ \exp(tX_g) = \dot{f} \end{aligned}$$

□

3.3 Classical mechanics

We are going to look at the classical phase space as a differentiable manifold with a symplectic form. The Hamiltonian function of the system is just a function on the manifold, and from the corresponding vector field we get out Hamilton's equations. Usually the phase space is a $2n$ dimensional manifold M, such that $M = T^*X$ is the cotangent bundle for some n dimensional manifold that often is the configuration space of the system.

In this section we let (M, ω) be a $2n$ dimensional symplectic manifold with symplectic coordinates $q_1, \ldots q_n, p_1 \ldots, p_n$ and H the Hamiltonian function of the system. The fact that (q, p) are the symplectic coordinates of the system means that $\omega = \sum_{i=1}^n dq_i \wedge dp_i$. And X_H is the Hamiltonian vector field of H, that is $\iota_{X_H}\omega = dH$.

From section 3.2 we have that

$$\dot{F} := \frac{d}{dt}(F \circ \rho_t) = \{H, F\}$$

That means that the time evolution of a function F along the flow defined by H is defined by the Poisson bracket. We can use this to find Hamilton's equations. In symplectic coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ we have that

$$\iota_{X_H}\omega = dH$$

We also know that the differential dH is on the form

$$dH = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i\right)$$

Using theorem 3.2 we also have that

$$\dot{q}_i = \{H, q_i\} = \omega(X_H, X_{q_i}) = \iota_{X_{q_i}}dH$$

We know that the Hamiltonian vector field corresponding to q_i is of the form

$$X_{q_i} = \sum_{i=1}^{n}\left(a_i\frac{\partial}{\partial q_i} + b_i\frac{\partial}{\partial p_i}\right)$$

which implies

$$dq_i = \iota_{X_{q_i}}\omega = \sum_{j=1}^{n}(a_j dp - b_j dq) \Rightarrow b_i = -1, a_j = 0, b_j = 0 \quad \text{for} \quad i \neq j \Rightarrow X_{q_i} = \frac{\partial}{\partial p_i}$$

$$\Rightarrow \dot{q}_i = \iota_{X_{q_i}}dH = \frac{\partial H}{\partial p_i}$$

In the same way we can show that

$$\dot{p}_i = \{H, p_i\} = -\frac{\partial H}{\partial q_i}$$

It follows from this that $dH = \sum_{i=1}^{n}(\dot{q}_i dp_i - \dot{p}_i dq_i)$.

The solution to the differential equations

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}\right\} \quad \text{Hamilton's equations} \tag{2}$$

is the flow generated by X_H. For a given initial point $q(0), p(0)$ the solution is an integral curve. We also have that the Hamiltonian function is constant along its flow, that is, it is constant along the solution $(q(t), p(t))$ to equation (2).

3.4 Integrable systems

Definition 3.5. A Hamiltonian system is a triple (M, ω, H), where (M, ω) is a symplectic manifold and H is a smooth function on M called the Hamiltonian function.

Proposition 3.2. *We have $\{f, H\} = 0$ if and only if f is constant along the integral curves of X_H.*

Proof. It follows from theorem 3.2. □

A function f such that $\{f, H\} = 0$ is called an integral of motion. The functions $f_1, \dots f_n$ are said to be linearly independent if their differentials $(df_1)_p, \dots, (df_n)_p$ are linearly independent at each point $p \in M$.

Definition 3.6. A Hamiltonian system (M, ω, H) is integrable if it possesses $n = \frac{1}{2} \operatorname{Dim} M$ independent integrals of motion $H = H_1, H_2, \dots, H_n$, such that $\{H_i, H_j\} = 0$ for all i, j.

Lemma 3.1. *Let (M, ω, H), be an integrable system of dimension $2n$ with integrals of motion $H = f_1, \dots, f_n$. Let $c \in \mathbb{R}^n$ be a regular value of $f := f_1, \dots, f_n$. If the Hamiltonian vector fields $X_{f_1}, \dots, X_{f_n}$ are complete on the level $f^{-1}(c)$, then the connected components of $f^{-1}(c)$ are of the form $\mathbb{R}^{n-k} \times \mathbb{T}^k$ for some k, $0 \leq k \leq n$, where $\mathbb{T}^k$ is the k-dimensional torus.*

Any compact component $f^{-1}(c)$ must hence be a torus. These components, when they exist, are called Liouville tori.

Theorem 3.3 (Arnold-Liouville)**.** *Let (M, ω, H), be an integrable system of dimension $2n$ with integrals of motion $f_1, \dots, f_n$, and let $c \in \mathbb{R}^n$ be a regular value of f.*

1. *If the flows of $X_{f_1}, \dots, X_{f_n}$ starting at a point $p \in f^{-1}(c)$ are complete, then the connected component of $f^{-1}(c)$ containing p is a homogeneous space for $\mathbb{R}^n$. With respect to this structure, the component have coordinates $\phi_1, \dots, \phi_n$, known as angle coordinates, in which the flows of the vector fields $X_{f_1}, \dots, X_{f_n}$ are linear.*

2. *There are coordinates $\psi_1, \dots, \psi_n$, known as action coordinates, complementary to the angle coordinates such that ψ_i are integrals of motion and $\phi_1, \dots, \psi_n, \psi_1, \dots, \psi_n$ are symplectic coordinates.*

4 Moment maps

4.1 Lie groups

Definition 4.1. A Lie group is a smooth manifold which also have a group structure. That is a smooth manifold G with a group operation $\cdot$ such that

$$\begin{aligned} G \times G &\to G \\ (g_1, g_2) &\mapsto g_1 \cdot g_2 \end{aligned}$$

and

$$\begin{aligned} G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are smooth maps. We will denote the identity element of G as e.

Example 4.1.

- $\mathbb{R}$ with addition
- S^1 regarded as unit complex numbers with multiplications, represents rotations of the plane.
- $U(n)$, unitary linear transformations of $\mathbb{C}^n$.
- $SU(n)$, unitary linear transformations of $\mathbb{C}^n$ with $\det = 1$.
- $O(n)$, orthogonal linear transformations of $\mathbb{R}^n$.
- $SO(n)$, elements of $O(n)$ with $\det = 1$
- $GL(V)$, invertible linear transformations of a vector space V.

4.2 Smooth actions

Definition 4.2. An action of a Lie group G on a manifold M is a group homomorphism

$$\begin{aligned} \psi : M \times G &\to M \\ (p, g) &\mapsto \psi_g(p). \end{aligned}$$

It is a smooth action if ψ is a smooth map.

Example 4.2. Let X be a complete vector field on M and let $G = \mathbb{R}$, then

$$\rho : M \times \mathbb{R} \to M$$
$$(p, t) \mapsto \rho_t(p) = \exp(tX)(p)$$

is a smooth action of $\mathbb{R}$ on M. So for each complete vector field on M we have a smooth $\mathbb{R}$ action, in fact it is a one to one correspondence.

Definition 4.3. A group action ψ of a group G on (M, ω) is called a symplectic action when all the diffeomorphisms ψ_g also are symplectomorphisms.

Example 4.3. If $G = \mathbb{R}$ then the smooth action $\psi_t = \exp tX$ is symplectic when X is a symplectic vector field.

Example 4.4. Let $M = \mathbb{R}^{2n}$, $\omega = \sum dx_i \wedge dy_i$ and $X = -\frac{\partial}{\partial y_1}$. The orbits of the action generated by X are lines parallel to the y_1-axis,

$$\{(x_1, y_1 - t, x_2, y_2, \ldots, y_n)|t \in \mathbb{R}\}$$

4.3 Orbit spaces

Definition 4.4. Let ψ be a action of G on M. The orbit of G through $p \in M$ is $\{\psi_g(p)|g \in G\}$.

The Stabilizer of $p \in M$ is the subgroup $G_p := \{g \in G|\psi_g(p) = p\}$.

Definition 4.5. We say that the action of G on M is respectively:

- transitive if there is just one orbit,
- free if all stabilizers are trivial $\{e\}$.

Let $\sim$ be the equivalence relation for points in $p, q \in M$ such that

$$p \sim q \Leftrightarrow \text{p and q is in the same orbit}$$

Definition 4.6. The quotient space $M/G := M/\sim$ is the space of all orbits of an action on M, we call it the orbit space. If the action is transitive then M/G is trivial.

4.4 Adjoint and coadjoint representation

Let G be a Lie group. Given $a \in G$ let

$$L_a : G \to G$$
$$g \mapsto ag.$$

be left multiplication by a. A vector field X is called left-invariant if $L_{a*}X|_g = X|_{ag} \quad \forall a \in G$. Let $\mathfrak{g}$ be the set of left-invariant vector fields on G.

The Lie bracket $[X, Y]$ is defined as the Lie derivative of Y along the flow of X:

$$\mathcal{L}_X Y = \frac{d}{dt}(\rho)_* Y|_{t=0} = [X, Y]. \tag{3}$$

$\mathfrak{g}$ with the Lie bracket $[\cdot, \cdot]$ forms a Lie algebra, called the Lie algebra of G.

For every element $V \in T_eG$ we can associate a unique left-invariant vector field X_V on G

$$X_V := L_{g*}V$$

and for every left-invariant vector field we get an element in $T_e(G)$

$$X|_e := V$$

It follows that T_eG and $\mathfrak{g}$ are isomorphic and hence $\dim(\mathfrak{g}) = \dim(T_eG)$.

The set of vector fields on a manifold M with the Lie bracket as in (3) is also a Lie algebra.

Next we define an action of a Lie group G on itself called the adjoint representation of G. The action is defined as:

$$\psi : G \times G \to G$$
$$(a, g) \mapsto \psi(a, g) = \mathrm{ad}_a(g) = aga^{-1}.$$

If we restrict the push-forward map $\mathrm{ad}_{a*} : T_gG \to T_{\mathrm{ad}_a g}G$ to $g = e$ then we get a map

$$Ad_a := \mathrm{ad}_{a*} |T_eG : T_eG \to T_eG$$

Since T_eG is isomorphic to $\mathfrak{g}$, $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ is an action of G on its Lie algebra $\mathfrak{g}$ called the adjoint map or the adjoint action.

If $\mathfrak{g}^*$ is the dual vector space of $\mathfrak{g}$ we can define the coadjoint action Ad^*. That is

$$\mathrm{Ad}^* : \mathfrak{g}^* \times G \to \mathfrak{g}^*$$

where Ad^* is defined as

$$\langle \mathrm{Ad}^*_g \eta, X \rangle = \langle \eta, \mathrm{Ad}_{g^{-1}} X \rangle \quad \forall X \in \mathfrak{g}, \eta \in \mathfrak{g}^*$$

Where $\langle \cdot, \cdot \rangle$ is the natural pairing of $\mathfrak{g}$ with its dual space.

Theorem 4.1. *Let G be a matrix group and $X, Y \in \mathfrak{g}$, then*

$$\frac{d}{dt} \mathrm{Ad}_{\exp tX} Y \bigg|_{t=0} = [X, Y].$$

4.5 Moment and comoment maps

If a Lie group G acts on a manifold M with $\psi_g : M \to M$ and on a manifold N with $\eta_g : N \to N$, then a smooth map $f : M \to N$ is called equivariant with respect to ψ and η if

$$f \circ \psi_g = \eta_g \circ f \quad \forall g \in G$$

i.e. if this diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle \phi_g}\big\downarrow & & \big\downarrow{\scriptstyle \eta_g} \\ M & \xrightarrow{f} & N \end{array}$$

Definition 4.7. Let (M, ω) be a symplectic manifold, and ϕ be a symplectic action of a Lie group G. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. Then ϕ is a Hamiltonian action if there exists a map

$$\mu : G \to \mathfrak{g}^*$$

such that:

1. μ is equivariant with respect to ϕ and the coadjoint action Ad^* of G on $\mathfrak{g}^*$, so this diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\mu} & \mathfrak{g}^* \\ {\scriptstyle \phi_g}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Ad}^*_g} \\ M & \xrightarrow{\mu} & \mathfrak{g}^* \end{array}$$

2. For all $X \in \mathfrak{g}$, we define the map $\mu^X(p) := \langle \mu(p), X \rangle$. μ^X is a map from M to $\mathbb{R}$, so it is a smooth function on M. We can also define a vector field

$$X^{\#} = \frac{d}{dt}\phi_{\exp tX}|_{t=0}$$

Here $\exp tX$ is a flow in G. If condition 1 holds and $d\mu^X = \iota_{X^{\#}}\omega$, then ϕ is a Hamiltonian action.

The map μ is called a moment map or a momentum map, and (M, ω, G, μ) is called a Hamiltonian G-space. If G is a connected Lie group then we have an equivalent definition to the one above. The pullback map of μ, $\mu^* : \mathfrak{g} \to C^\infty(M)$ have the properties that:

1. $d(\mu^*(X)) = \iota_{X^{\#}}\omega$
2. $\mu^*[X, Y] = \mu^*\mathcal{L}_X Y = \{\mu^*(X), \mu^*(Y)\} = \omega(X_{\mu^*(X)}, X_{\mu^*(Y)})$

Moment maps are a generalization of the concept classical linear and angular momentum.

Because μ is a equivariant map, we can often do the calculations with the action in $\mathfrak{g}^*$ with the coadjoint action and then use the inverse map μ^{-1} to get back to M. The most important example of this is the coadjoint orbit, that is the orbit of G through a point $\xi \in \mathfrak{g}^*$, i.e. $\{\mathrm{Ad}_g^*(\xi) | g \in G\}$.

5 Symplectic reduction

5.1 Marsden-Weinstein-Meyer theorem

Theorem 5.1 (Marsden-Weinstein-Meyer). *Let (M, ω, G, μ) be a Hamiltonian G-space for a compact Lie group G. Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map.Let $\pi : M \to M/G$ be the point-orbit projection, which sends points to its orbit. Assume that G acts freely on $\mu^{-1}(0)$. Then*

- *the orbit space $M_{red} = \mu^{-1}(0)/G$ is a manifold,*
- *$\pi : \mu^{-1}(0) \to M_{red}$ is a principal G-bundle, and*
- *there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \pi^*\omega_{red}$.*

Definition 5.1. The pair (M_{red}, ω_{red}) is called the reduction or the reduced space of (M, ω) with respect to G and μ.

The Marsden-Weinstein-Meyer theorem tells us that we can use symmetries in our system to reduce it to something easier. Again, the proof of the theorem can be found in [4].

5.2 Noether principle

Theorem 5.2 (Noether). *Let (M, ω, G, μ) be a Hamiltonian G-space. If $f : M \to \mathbb{R}$ is a G-invariant function, then μ is constant on the trajectories of the Hamiltonian vector field of f.*

Proof. Let v_f be the Hamiltonian vector field of f. Let $X \in \mathfrak{g}$ and $\mu^X = \langle \mu, X \rangle : M \to \mathbb{R}$. We have

$$\begin{aligned} \mathcal{L}_{v_f} &= \iota_{v_f} d\mu^X = \iota_{v_f} \iota_{X^\#} \omega \\ &= -\iota_{X^\#} \iota_{v_f} \omega = -\iota_{X^\#} df \\ &= -\mathcal{L}_{X^\#} f = 0 \end{aligned}$$

□

Definition 5.2. A G-invariant function $f : M \to \mathbb{R}$ is called an integral of motion of (M, ω, G, μ). If μ is constant on the trajectories of a Hamiltonian vector field v_f, then the corresponding one-parameter group of diffeomorphisms $\{\exp tv_f | t \in \mathbb{R}\}$ is called a symmetry of (M, ω, G, μ) .

This means that there is a one-to-one correspondence between symmetries and integrals of motions for the system.

5.3 Elementary theory of reduction

If we find a symmetry for a $(2n)$-dimensional problem we can reduce it to a $(2n-2)$-dimensional problem. We are now going to show how we can understand the trajectories in a $2n$-dimensional system in terms of the trajectories of the reduced space.

Example 5.1. Let (M, ω, H) be a Hamiltonian system and let f be an integral of motion. This implies that $\{f, H\} = \dot{f} = 0$. Suppose that we can choose symplectic coordinates

$$(q_1, \dots, q_n, p_1, \dots p_n)$$

on an open set in M with $p_n = f$.

In these coordinates the Hamiltonian is a function of $q_1, \dots, q_n, p_1, \dots p_n$. Since f is constant along the flow of X_H we have that

$$\begin{aligned} &\{p_n, H\} = 0 = \dot{p}_n = -\frac{\partial H}{\partial q_n} \\ &\Rightarrow \quad H = H(q_1, \dots, q_{n-1}, p_1, \dots p_n). \end{aligned}$$

Since $\dot{p}_n$ is constant we can set it to a fixed value $c \in \mathbb{R}$. The motion of the system is descibed by the following Hamilton equations:

$$\begin{aligned} \frac{dq_i}{dt} &= \frac{\partial H}{\partial p_i}(q_1, \dots, q_{n-1}, p_1, \dots, p_{n-1}, c) \quad \text{for} \quad i = 1, \dots, n-1 \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q_i}(q_1, \dots, q_{n-1}, p_1, \dots, p_{n-1}, c) \quad \text{for} \quad i = 1, \dots, n-1 \\ \frac{dq_n}{dt} &= \frac{\partial H}{\partial p_n} \\ \frac{dp_n}{dt} &= -\frac{\partial H}{\partial q_n} = 0 \end{aligned}$$

The reduced phase space is

$$\begin{aligned} \mathcal{U}_{red} = \{(q_1, \dots, q_{n-1}, p_1, \dots, p_{n-1}) \in \mathbb{R}^{2n-2} | \\ (q_1, \dots, q_{n-1}, a, p_1, \dots, p_{n-1}, c) \in \mathcal{U} \quad \text{for some } a\}. \end{aligned}$$

The reduced Hamiltonian is

$$\begin{aligned} &H_{red} : \mathcal{U}_{red} \to \mathbb{R} \\ &H_{red}(q_1, \dots, q_{n-1}, p_1, \dots, p_{n-1}) := H(q_1, \dots, q_{n-1}, p_1, \dots, p_{n-1}, c). \end{aligned}$$

We now have a reduced space M_{red} of dimension $2n-2$ which we need to find the trajectories in, but for $q_n(t)$ and $p_n(t)$ we have

$$q_n(t) = q_n(0) + \int_0^t \frac{\partial H}{\partial p_n} dt$$
$$p_n(t) = c.$$

If g is a integral of motion independent of f, then we can use g to reduce the phase space to a $(2n-4)$-dimensional phase space, and the trajectories of q_{n-1} and p_{n-1} can be found in the same way as for those of q_n and p_n. This means that if we have n independent integrals of motion we can find the trajectories for all of q_i and p_i, and that is why a system with n independent integrals of motion is called integrable.

5.4 Reduction at other levels

We are now going to look at reduction along a coadjoint orbit. Let G be a compact Lie group that acts on a symplectic manifold (M, ω) in a Hamiltonian way with moment map $\mu : M \to \mathfrak{g}^*$. Let $\xi \in \mathfrak{g}^*$, and let $\mathcal{O}$ be the coadjoint orbit through ξ.

The reduction of M along $\mathcal{O}$ is defined as $\mu^{-1}(\mathcal{O})/G$, i.e. it is the set of orbits in $\mu^{-1}(\mathcal{O}) \subset M$ generated by the action of G.

Definition 5.3. $\mu^{-1}(\mathcal{O})/G$ is called the symplectic reduction of M with respect to G along $\mathcal{O}$. We denote is by $R(M, G, \mathcal{O})$.

Lemma 5.1. *If the action of G on μ^{-1} is free, then $R(M, G, \mathcal{O})$ is symplectic and* $\dim(R(M, G, \mathcal{O})) = \dim(M) - 2\dim(G) + \dim(\mathcal{O})$

6 Calogero-Moser systems

There are several kinds of Calogero-Moser systems, all of them are one dimensional problems. The Hamiltonian of Calogero-Moser systems is of the form:

$$\sum_i p_i^2 + \sum_{i \neq j} U(x_i - x_j)$$

Where the potential U can have several forms. The two types of systems we will look at here is the rational and the trigonometric Calogero-Moser systems. That is when $U = \frac{1}{(x_i - x_j)^2}$ and $U = \frac{1}{4\sin^2((x_i - x_j)/2)}$ respectively. The trigonometric system is also called Sutherland system. The Sutherland systems might be viewed as N particles on a circle with a inverse square of the distance potential. And the rational system corresponds to particles on a line with a $1/d^2$ potential.

6.1 Calogero-Moser space

In the following sections we have a symplectic manifold $M = T^*Mat_n(\mathbb{C})$[1], with symplectic form $\omega = \text{tr}(dY \wedge dX) = \sum_{i,j=1}^{n} dY_{ij} \wedge dX_{ij}$. We can identify M with $Mat_n(\mathbb{C}) \oplus Mat_n(\mathbb{C})$, so an element in M is just a pair of complex matrices (X, Y). Thus $\dim M = 2n^2$.

Example 6.1.

- $\mathrm{GL_n}(\mathbb{C}) = \{M \in \mathrm{Mat_n}(\mathbb{C}) \,|\, \det M \neq 0\}$

  $\mathrm{GL_n}(\mathbb{C})$ is called the general linear group and it have dimension n^2.

- $\mathrm{SL_n}(\mathbb{C}) = \{M \in \mathrm{Mat_n}(\mathbb{C}) \,|\, \det M = 1\}$

  $\mathrm{SL_n}(\mathbb{C})$ is called the special linear group and it have dimension $n^2 - 1$.

- $\mathrm{PGL_n}(\mathbb{C})) := GL_n(\mathbb{C})/\sim$

  $\mathrm{PGL_n}(\mathbb{C})$ is the projective general linear group. $\sim$ is the equivalence relation such that $A \sim \alpha A$ for $\alpha \in \mathbb{C}$, i.e. we identify matrices that are equal up to multiplication by a non zero complex number.

  We can relate $\mathrm{PGL_n}(\mathbb{C})$ to $\mathrm{SL_n}(\mathbb{C})$ [2]by using the fact that two matrices A and λA in $\mathrm{PGL_n}(\mathbb{C})$ is equal. This means that we can choose λ such that

  $$\det \lambda A = 1.$$

  So we have a one-to-one correspondence between elements in $\mathrm{PGL_n}(\mathbb{C})$ and $\mathrm{SL_n}(\mathbb{C})$. That means that $\dim \mathrm{PGL_n}(\mathbb{C}) = \dim \mathrm{SL_n}(\mathbb{C}) = n^2 - 1$.

Let the projective general linear group $G = \mathrm{PGL_n}(\mathbb{C})$ act on M with the action

$$\psi_g(X, Y) := (g^{-1}Xg, g^{-1}Yg)$$

for $g \in \mathrm{PGL_n}(\mathbb{C})$. The Lie algebra of $\mathrm{PGL_n}(\mathbb{C})$ is $\mathfrak{sl}_n(\mathbb{C})$. The dual space of $\mathfrak{sl}_n(\mathbb{C})$ is of course also $\mathfrak{sl}_n(\mathbb{C})$.

[1]In this section all manifolds will be complex manifolds, rather than real manifolds which we used in the previous sections, and also symplectic forms are going to be holomorphic symplectic forms.

[2]We are later going to use this to show that the Lie algebra of $\mathrm{PGL_n}(\mathbb{C})$ is $\mathfrak{sl}_n(\mathbb{C})$.

The Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ is defined as the set of traceless $n \times n$ matrices, with the commutator $[X, Y] = XY - YX$ as its Lie bracket. We identify elements in the dual space by using the trace form.

We define a map

$$\begin{aligned} \mu : M &\to \mathfrak{g}^* = \mathfrak{sl}_n(\mathbb{C}) \\ (X, Y) &\mapsto [X, Y] = XY - YX \end{aligned}$$

We are now going to show that the action of G on M is a Hamiltonian action with respect to μ. So we have to show that μ is a moment map.

- We need to show that μ is a equivariant with respect to the action $\psi_g : M \to M$ of G, where $\psi_g(X, Y) = (g^{-1}Xg, g^{-1}Yg)$. That is we need to show that $\mu \circ \psi_g = \mathrm{Ad}_g^* \circ \mu$. We have that

  $$(\mathrm{Ad}_g^* \circ \mu)(X, Y) = \mathrm{Ad}_g^*([X, Y]) = g^{-1}[X, Y]g$$

  On the other hand we have

  $$\begin{aligned} &(\mu \circ \psi_g)(X, Y) = \mu(g^{-1}Xg, g^{-1}Yg) = g^{-1}Xgg^{-1}Yg - g^{-1}Ygg^{-1}Xg \\ &= g^{-1}[X, Y]g. \end{aligned}$$

  So we are done.

- For $\xi \in \mathfrak{g}$, we define $\mu^\xi(X, Y) = \langle \mu(X, Y), \xi \rangle = \mathrm{tr}([X, Y]\xi) = \mathrm{tr}(X[Y, \xi])$. We need to show that $d\mu^\xi = \iota_{\xi^\#}\omega$ Using Cartan magic formula, and the Tautological one-form $\omega = d\alpha$ we can simplify our condition to

  $$d\mu^\xi = \iota_{\xi^\#}\omega = \iota_{\xi^\#}d\alpha = \mathcal{L}_{\xi^\#}\alpha - d\iota_{\xi^\#}\alpha = -d\iota_{\xi^\#}\alpha.$$

  This means that we have to show that $\mathrm{tr}([X, Y]\xi) = -\iota_{\xi^\#}\alpha$. We have that

  $$\xi^\#(X, Y) = \frac{d}{dt}\left(\psi_{\exp t\xi}(X, Y)\right) = \frac{d}{dt}\left(\mathrm{Ad}_{\exp -t\xi} X, \mathrm{Ad}_{\exp -t\xi} Y\right) = ([X, \xi], [Y, \xi])$$

  We also have that

  $$\alpha = \mathrm{tr}(Y dX)$$

  Using this we find that

  $$-\iota_{\xi^\#}\alpha = -\iota_{\xi^\#}\,\mathrm{tr}\, Y dX = -\,\mathrm{tr}(Y[X, \xi]) = \mathrm{tr}([X, Y]\xi)$$

We have now shown that ψ is a Hamiltonian action with the moment map μ.

Example 6.2. $\mu^{-1}(0)$ is the subspace of M such that the pair of matrices (X, Y) commutes. The reduced space $M_{red} = \mu^{-1}(0)/G$ is the set of G-orbits on $\mu^{-1}(0)$ as usual.

Definition 6.1. The Calogero-Moser space is defined as the reduced space along the coadjoint orbit $\mathcal{O}$ that goes through the point $\gamma := \operatorname{diag}(-1, -1, \ldots, n-1) \in \mathfrak{sl}_n(\mathbb{C})$. The Calogero-Moser space is denoted $\mathcal{C}_n := R(M, G, \mathcal{O})$, and is the set of orbits of the action of $\mathrm{PGL_n}(\mathbb{C})$ on $\mu^{-1}(\mathcal{O}) \subset T^*Mat_n(\mathbb{C})$.

Lemma 6.1. *The orbit* $\mathcal{O} = \{g^{-1}\gamma g | g \in PGL_n(\mathbb{C})\} = \{T \in \mathfrak{sl}_n(\mathbb{C}) | \operatorname{rank}(T+1) = 1\}$ *is the set of traceless matrices* T *such that* $T + 1$ *has rank one.*

Proof.

$$\operatorname{tr}(g^{-1}\gamma g) = \operatorname{tr}\gamma = 0$$

$$\operatorname{rank}(g^{-1}\gamma g + 1) = \operatorname{rank}(g^{-1}\gamma g + g^{-1}1g) = \operatorname{rank}(g^{-1}(\gamma + 1)g) = \operatorname{rank}(\gamma + 1) = 1$$

□

$\mu^{-1}(\mathcal{O})$ is the pair of matrices (X, Y) such that $\operatorname{rank}(\mu(X, Y) + 1) = \operatorname{rank}([X, Y] + 1) = 1$.

Using lemma 5.1 we can show that $\dim \mathcal{C}_n = 2n$. We already know the dimension of M and G, so we only need to find the dimension of $\mathcal{O}$.

Let $T \in \mathcal{O}$. We know that a matrix with rank one can be written on the form

$$\begin{pmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{pmatrix}.$$

Since $T + 1$ has rank one we have that

$$T = \begin{pmatrix} a_1b_1 - 1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 - 1 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n - 1 \end{pmatrix}$$

We also have two constraint on T, that is

$$\operatorname{tr} T = 0$$

and the gauge degree of freedom $a \mapsto \lambda a, b \mapsto b/\lambda$. Using this we see that

$$\dim \mathcal{O} = 2n - 2$$

$$\dim \mathcal{C}_n = \dim(M) - 2\dim(G) + \dim(\mathcal{O}) = 2n^2 - 2(n^2 - 1) + 2n - 2 = 2n$$

6.2 Rational Calogero-Moser system

We start this section by defining n functions $H_i = \operatorname{tr}(Y^i)$ on M. $\{H_i, H_j\} = 0$ because H_i does not depend on X. Since $H_i = \operatorname{tr}(Y^i)$ then we have that $dH_i = \sum_{i,j=1}^{n} a_{ij} dy_{ij}$ for some a_{ij} and where y_{ij} is the elements of Y. The Hamiltonian vector field will be of the form $X_{H_i} = \sum_{i,j=1}^{n} \left(b_{ij} \frac{\partial}{\partial y_{ij}} + c_{ij} \frac{\partial}{\partial x_{ij}} \right)$. We use the interior product $\iota_{X_{H_i}} \omega = \sum_{i,j=1}^{n} (b_{ij} dx_{ij} + c_{ij} dy_{ij})$ to show that $b_{ij} = 0$ and $a_{ij} = c_{ij}$ since $\iota_{H_i} \omega = dH_i$. So the Hamiltonian vector fields for the functions H_i is of the form $\sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_{ij}}$. This implies that

$$\{H_i, H_j\} = \iota_{H_j} \iota_{H_i} \omega = \iota_{H_j} dH_i = \sum_{i,j=1}^{n} a_{ij} dy_{ij} \left(\sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_{ij}} \right) = 0$$

Since $\dim(\mathcal{C}_n) = 2n$ our functions H_i are an integrable system on $\mathcal{C}_n$. The rational Calogero-Moser system is the phase space $\mathcal{C}_n$ with the Hamiltonian $H_2 = \operatorname{tr}(Y^2)$. Since H_2 is included in the the integrable system $H_1, \ldots H_n$ we know that the rational Calogero-Moser system is integrable. We will now show how to find a solution to this system, and show why it is the same as the system of n particles on a line.

Theorem 6.1. *We are now going to introduce a theorem called the Necklace bracket formula. It says that if $a_1, \ldots, a_r, b_1, \ldots, b_s$ is either X or Y. Then we have*

$$\{\operatorname{tr}(a_1 \cdot a_2 \cdots a_r), \operatorname{tr}(b_1 \cdots b_s)\} =$$
$$\sum_{\substack{(i,j) \\ a_i = Y, b_j = X}} \operatorname{tr}(a_{i+1} \cdots a_r a_1 \cdots a_{i-1} b_{j+1} \cdots b_s b_1 \cdots a_{j-1})$$
$$- \sum_{\substack{(i,j) \\ a_i = X, b_j = Y}} \operatorname{tr}(a_{i+1} \cdots a_r a_1 \cdots a_{i-1} b_{j+1} \cdots b_s b_1 \cdots a_{j-1}).$$

Using this formula it is easy to see that $\{\operatorname{tr} Y^i, \operatorname{tr} Y^j\} = \{H_i, H_j\} = 0$.

We are going to represent a point $P \in \mathcal{C}_n$ by a pair of matrices such that $X = \operatorname{diag}(x_1, \ldots, x_n)$ is a diagonal matrix such that $x_i \neq x_j$. We denote the entries of Y by y_{ij}.

We are now going to find some constraints for the entries of Y, to do that we are going to use the fact that $\operatorname{rank}([X,Y]+1) = 1$ and that we can write the entries of a matrix of rank 1 on the form $a_i b_j$.

We have that the elements of $[X,Y] = XY - YX = t_{ij}$ is 0 when $i = j$ and $(x_i - x_j)y_{ij}$ when $i \neq j$. The elements in $[X,Y]+1 = \kappa_{ij}$ is therefor 1 for $i = j$ and $(x_i - x_j)y_{ij}$ when $i \neq j$. Since the rank of $[X,Y]+1$ is 1 we also have that $\kappa_{ij} = a_i b_j$. That is

$$\begin{pmatrix} 1 & (x_1 - x_2)y_{12} & \cdots & (x_1 - x_n)y_{1n} \\ (x_2 - x_1)y_{21} & 1 & \cdots & (x_2 - x_n)y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (x_n - x_1)y_{n1} & (x_n - x_2)y_{n2} & \cdots & 1 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \cdots & a_2 b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1 & a_n b_2 & \cdots & a_n b_n \end{pmatrix}$$

Since $\kappa_{ii} = 1 = a_i b_i$ we have that $a_i^{-1} = b_i$ so $\kappa_{ij} = a_i a_j^{-1}$. By conjugating (X,Y) by $A = \operatorname{diag}(a_1, \ldots, a_n)$ we get that

$$[AXA^{-1}, AYA^{-1}] = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}$$

We can set a_i to 1, then we have that $(x_i - x_j)y_{ij}$=1 which means that the entries of Y is $y_{ij} = \frac{1}{x_i - x_j}$ for $i \neq j$. We denote the diagonal entries of Y as $y_{ii} = p_i$. This representation of a point in $\mathcal{C}_n$ is unique up to permutation of the diagonal elements of X.

Theorem 6.2. *Let $\mathbb{C}^n_{reg}$ be an open set of $(x_1, \ldots, x_n) \in \mathbb{C}^n$ such that $x_i \neq x_j$. And let U_n be an open subset of M. There exists an isomorphism of symplectic manifolds*

$$\xi : T^*(\mathbb{C}^n_{reg}/S_n) \to U_n$$

where S_n is the group of permutations. The isomorphism is given by $(x_1, \ldots, x_n, p_1, \ldots, p_n) \mapsto (X,Y)$ where $X = \operatorname{diag}(x_1, \ldots, x_n)$ and

$$Y = \begin{pmatrix} p_1 & \frac{1}{x_1 - x_2} & \cdots & \frac{1}{x_1 - x_n} \\ \frac{1}{x_2 - x_1} & p_2 & \cdots & \frac{1}{x_2 - x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_n - x_1} & \frac{1}{x_n - x_2} & \cdots & p_n \end{pmatrix}$$

Proof. Let $a_k = \operatorname{tr} X^k$ and $b_k = \operatorname{tr} X^k Y$. We can use the necklace bracket formula to show that we have

$$\{a_m, a_k\} = 0, \quad \{b_m, a_k\} = k a_{m+k-1}, \quad \{b_m, b_k\} = (k-m) b_{m+k-1}.$$

On the other hand, $\xi^* a_k = \sum x_i^k$, $\xi^* b_k = \sum x_i^k p_i$. Thus we see that

$$\{f, g\} = \{\xi^* f, \xi^* g\}$$

where f,g is either a_k or b_k. Since a_k, b_k form a local coordinate system near a generic point of U_n we are done. □

We will now look at how the Hamiltonian $H = \operatorname{tr}(Y^2)$ looks like in our new coordinates (x, p).

$$\begin{aligned}\operatorname{tr}(Y(x,p)^2) = \sum_{i=1}^{n} (Y^2)_{ii} = \sum_{i=1}^{n} \left(p_i^2 + \sum_{j \neq i} \left(\frac{1}{x_i - x_j} \frac{1}{x_j - x_i} \right) \right) \\ = \sum_{i=1}^{n} p_i^2 - \sum_{j \neq i} \frac{1}{(x_i - x_j)^2}\end{aligned}$$

We now see that this is the Hamiltonian for n particles on a line, which was the problem we wanted to show was integrable. We showed that it was integrable by finding n independent integrals of motion, i.e. conserved quantities. So we now know that the rational Calogero-Moser space is an integrable system. Now it is time to find the solution of this system.

We have that the flow of $H = \operatorname{tr}(Y^2)$ is

$$g_t(X, Y) = (X + 2tY, Y)$$

This is just the motion of a free particle in the space of matrices.

The solution of the system with initial condition $(X_0, Y_0) = \xi(x(0), p(0))$ is the eigenvalues $x_i(t)$ of $X_0 + 2tY_0$, and the momentum $p_i(t) = \frac{d}{dt} x_i(t)$. $X_i(t) = X_0 + 2tY_0$ is just X_0 following the flow g_t that is generated by the Hamiltonian vector field of $\operatorname{tr}(Y^2)$. In other words it is a integral curve that goes through the point $(X_0, Y_0) \in M$ such that the Hamiltonian is constant.

6.3 Trigonometric Calogero-Moser system

The Trigonometric Calogero-Moser system is the Calogero-Moser space with the Hamiltonian $H^* = \operatorname{tr}((XY)^2)$. We will show that this system is integrable by showing that H^* can be included in a integrable system $H_1, \ldots, H_n$ where

$\{H_i, H_j\} = 0$. $H_i = \mathrm{tr}((XY)^i)$ defines such a system which we will show using the Necklace bracket formula.

$$\{\mathrm{tr}((XY)^r), \mathrm{tr}((XY)^s)\} = \{\mathrm{tr}(a_1 \cdot a_2 \cdots a_{2r}), \mathrm{tr}(b_1 \cdots b_{2s})\} =$$
$$\sum_{i=1}^{r}\sum_{j=1}^{s} \mathrm{tr}(XYXY \ldots XY)$$
$$-\sum_{i=1}^{r}\sum_{j=1}^{s} \mathrm{tr}(YXYX \ldots YX) =$$
$$rs \cdot \mathrm{tr}((XY)^{r+s-1}) - rs \cdot \mathrm{tr}((YX)^{r+s-1}) = 0.$$

We want to find an expression for H in our coordinates (x, p).

$$(XY)_{ij} = \begin{cases} x_i p_i & \text{for } i = j \\ \frac{x_i}{x_i - x_j} & \text{for } i \neq j \end{cases}$$

$$\Rightarrow \mathrm{tr}((XY)^2) = \sum_{i=1}^{n}((XY)^2)_{ij} = \sum_{i=1}^{n}(x_i p_i)^2 - \sum_{i \neq j} \frac{x_i x_j}{(x_i - x_j)^2}$$

If we do the coordinate change $p_{i*} = x_i p_i$, $x_{i*} = \log x_i$ we get that

$$H = \sum_i p_{i*} - \sum_{i* \neq j*} \frac{e^{x_{i*} + x_{j*}}}{(e^{x_i} - e^{x_j})^2}$$
$$= \sum_i p_{i*} - \sum_{i* \neq j*} \frac{1}{(e^{(x_{i*} - x_{j*})/2} - e^{-(x_{i*} - x_{j*})/2})^2}$$
$$= \sum_i p_{i*} - \sum_{i* \neq j*} \frac{1}{4\sinh^2((x_{i*} - x_{j*})/2)}$$

This is the Hamiltonian for the hyperbolic Calogero-Moser system. To get to the trigonometric system we will do the coordinate change $x_{i*} = i \cdot x_{i*}$.

$$H = \sum_i p_{i*} - \sum_{i* \neq j*} \frac{1}{4\sinh^2(i(x_{i*} - x_{j*})/2)}$$
$$= \sum_i p_{i*} + \sum_{i* \neq j*} \frac{1}{4i^2\sinh^2(i(x_{i*} - x_{j*})/2)}$$
$$= \sum_i p_{i*} + \sum_{i* \neq j*} \frac{1}{4\sin^2((x_{i*} - x_{j*})/2)}$$

This is the Hamiltonian for a system of n particles on a circle with the potential $\frac{1}{4\sin^2((x_{i*} - x_{j*})/2)}$. We now know that this system is integrable. We are not going to try to find the solution to this system.

The rational, hyperbolic and trigonometric Calogero-Moser systems are all just special cases of a more general system called the elliptic Calogero-Moser system.

7 Conclusions

In this thesis we showed the integrability for the rational and trigonometric Calogero-Moser systems. That is the Hamiltonian systems with with Hamiltonian $H = \sum_{i=1}^{n} p_i^2 - \sum_{j\neq i} \frac{1}{(x_i-x_j)^2}$ and $H = \sum_i p_i + \sum_{i\neq j} \frac{1}{4\sin^2((x_i-x_j)/2)}$ respectively. We did this using symplectic reduction. More specifically we acted smoothly on the cotangent bundle of $n\times n$ complex matrices $T^* \operatorname{Mat}_n(\mathbb{C})$ with the Lie group $\mathrm{PGL_n}(\mathbb{C})$ and defined the Calogero-Moser space as the reduced space $\mathcal{C}_n := R(M, G, \mathcal{O})$. That is the symplectic reduction of $T^* \operatorname{Mat}_n(\mathbb{C})$ with respect to $\mathrm{PGL_n}(\mathbb{C})$ along the coadjoint orbit $\mathcal{O}$, where $\mathcal{O}$ is the orbit through the point $\operatorname{diag}(-1, -1, \ldots, n-1) \in \mathfrak{sl}_{\mathrm{n}}(\mathbb{C})$.

We then look at the Calogero-Moser space as the phase space with symplectic form $\omega = \operatorname{tr}(dX \wedge dY)$ and a we show that a point in $\mathcal{C}_n$ can be represented as a pair of complex $n \times n$ matrices (X, Y) such that $\operatorname{rank}(XY - YX + 1) = 1$. We first look at the Hamiltonian system $(\mathcal{C}_n, \omega, H = \operatorname{tr} Y^2)$ and show that this is an integrable system with integral of motion $\operatorname{tr} Y^i$ for $i = 1, \ldots, n$. Then we can do a change of coordinates such that we can see that this is just the rational Calogero-Moser system.

Then we looked at the Hamiltonian system $(\mathcal{C}_n, \omega, H = \operatorname{tr}((XY)^2))$ and show that it is integrable with integrals of motion $\operatorname{tr}((XY)^i)$. We can then see that this is the trigonometric Calogero-Moser system which is what we wanted.

For future work one might consider to look at the elliptic Calogero-Moser system where the rational and trigonometric systems are just special limits. An other alternative is to go to the quantum mechanical case for which the rational system first was solved.

References

[1] P.I. Etingof *Calogero-Moser systems and representation theory* 2007, European Mathematical Society, Zürich

[2] J.E Marsden, T.S. Ratiu *Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems* 1994, Springer, New York; Berlin

[3] M. Nakahara *Geometry, topology, and physics* 2003, Institute of Physics, Bristol

[4] A. C. d. Silva *Lectures on Symplectic Geometry* 2001, Springer, New York; Berlin

[5] F. Calogero *Solution of the One-Dimensional N-Body Problems with Quadratic and/or Inversely Quadratic Pair Potentials* 1971, Journal of Mathematical Physics, vol. 12, no. 3, pp. 419-436.

[6] J. Moser *Three Integrable Hamiltonian Systems Connected with Isospectral Deformations* 1975, Advances in Mathematics, vol. 16, no. 2, pp. 197-220.

[7] D. Kazhdan, B.Kostant, S. Sternberg *Hamiltonian Group Actions and Dynamical Systems of Calogero Type* 1978, Communications on Pure and Applied Mathematics, vol. 31, no. 4, pp. 481-507.