Bonus-Focused Yatzy

A COMPARISON BETWEEN AN OPTIMAL STRATEGY AND A BONUS-FOCUSED STRATEGY

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Abstract

Yatzy is the Scandinavian version of Yahtzee, a popular dice game known worldwide. This report compares the effectiveness between two strategies used when playing Yatzy in real life, an optimal strategy and a bonus-focused strategy. To evaluate this problem and determine which performs better or if they are somewhat equivalent, a Java program was constructed with the purpose of simulating 100,000 rounds of Yatzy following a bonus-focused strategy and the expected values was compared.

As the results will show, this study proves that a bonus-focused strategy should be preferred over an optimal one (now debunked, as proven in section 5) since the average score is greater, 215.70 compared to 189.80.

Sammanfattning

Yatzy är den Skandinaviska versionen av Yahtzee, ett populärt tärningspel känt världen runt. Denna rapport jämför effektiviteten mellan två strategier som kan användas när man spelar Yatzy i verkligheten, en optimal strategi och en bonusfokuserad strategi. För att bedöma vilken av strategierna presterar bättre eller om de är någorlunda likvärdiga, konstruerades ett javaprogram med syftet att simulera 100,000 omgångar Yatzy som följer en bonusfokuserad strategi och deras väntevärde jämfördes.

Denna studie bevisar att en bonusfokuserad strategi är att föredra över en optimal strategi (tidigare optimal, bevisat i kapitel 5) då medelvärdet är högre, 215.70 jämfört med 189.80.
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1 Introduction

Yatzy, the Scandinavian version of Yahtzee, is a popular dice game known world-wide. It is played with five dice with the purpose of filling 15 different categories with a score, preferably higher than zero. The winner of each game is the one that scores the highest when all categories are filled.

There are many different ways to go about filling the scorecard and what strategy to use when doing it. Either you are going for the highest score, called the optimal strategy, where the goal is to maximize every hand and category you want to fill, or you try to get enough points in the Upper Section (a total of 63 points or more) to obtain the bonus worth 50 points.

Since the outcome of the dice is determined by chance, a game of Yatzy is rarely ever played the same way and it might be hard for a human to always use a fixed optimal strategy to maximize each hand and final score when all categories are filled. However, with the help of statistics, probability theory and large amounts of computing power the mathematically optimal strategy has been calculated [1] [2], but is not practical to apply when playing Yatzy in real life. Therefore, a safe bet is to go for the bonus-focused strategy, in hopes of making up the deficit at the end of the game.

1.1 Purpose

This report covers the theories and algorithm to design a bonus-focused strategy for the dice game Yatzy, mainly focusing on obtaining the bonus from the Upper Section of the scorecard, as often as possible.

Many players seem to focus, for lack of any other strategies, on obtaining the bonus from the Upper Section when they play Yatzy. Therefore, the aim of this report is to investigate if they have been playing a bad strategy all this time or if there is a valid reason to do so.

1.2 Problem Statement

The questions this study will answer is:

1. How much does the bonus-focused strategy differ from the optimal one, score wise?
2. Is the bonus-focused strategy a valid one, can it compete with the optimal strategy?
3. What is the potential loss of using this strategy?

1.2.1 Problem Statement Definition

To answer each of the questions above it is vital to define a way to do so.

To determine the difference between the strategies, compare the expected value of each strategy. To derive an expected value from the bonus-focused
strategy, construct and simulate a program based on the this strategy a sufficient amount of times so that the average score moves towards a fixed value.

To determine if the bonus-focused strategy is worth playing, its expected value can not differ more than 5% from the optimal strategy described in Olaf Vancura’s ”Advantage Yahtzee” [3] and adapted in Niklas Axelsson’s and Rickard Larsson’s work [4], mentioned in more detail in section 2.2.

The potential loss of the bonus-focused strategy is determined by comparing the different ways this strategy prioritizes filling the different categories compared to the optimal one.
2 Background

2.1 History of Yatzy

Yatzy is the Scandinavian version of the dice game called "Yahtzee" or "The Yacht Game". "The Yacht Game" was invented by a wealthy Canadian couple in 1954. It became popular amongst their friends and the couple later approached Edwin S. Lowe, a toy-maker, to see if he could produce game sets for them. Lowe saw the potential of the game and later acquired the rights to it and renamed it "Yahtzee" in exchange for the price of the first 1000 games sold [5].

The game did not sell in the beginning, the rules were perceived as complicated and could not be adequately described in an ad. This changed when Lowe introduced Yahtzee parties and hoped that it would spread due to word of mouth, which it did [5].

In 1973 the E. S. Lowe Company and rights to Yahtzee were sold to the Milton Bradley Company. It was later bought by Hasbro Inc in 1984 which still to this day owns the rights to Yahtzee. [5].

Yatzy differs from the original Yahtzee in many ways. The categories One pair and Two pair can not be found on the Yahtzee scorecard. The bonus awarded from the Upper Section is only 35 points compared to 50 points when playing Yatzy, even if the 63 point limit still applies to both versions. Almost all of the categories in Yatzy is scored differently in Yahtzee. When a category scores a fixed value in one version, the other scores the same category by summing the dice in the hand and vice versa.

2.2 Existing Research

From the pool of already existing work that examines the optimal Yatzy strategy, it is vital to find the work that correlates and is comparable to this study. Marcus Larsson and Andreas Sjöberg implemented the optimization technique described in James Glenn’s work "An Optimal Strategy of Yahtzee" [1] and adjusted it to the Scandinavian version of the game. To solve the problem as mathematically optimal as possible, with the help of a large amount of computer power, a game of Yatzy is represented by a graph in which each node corresponds to a different state in the game. This strategy solves the problem well and results in an average score of 248.92 [2].

Niklas Axelsson and Rickard Larsson implemented a strategy based on Olaf Vancura’s book "Advantage Yahtzee" that is suitable to use when playing Yatzy in real life, and is as optimal as possible when not using a computer [3]. Each combination has a priority in the three different stages in the game (early-, mid- and end-game). Their average score is 189.80 [4].

Since the problem statement mentioned in section 1.2 does not differ much from Niklas Axelsson and Rickard Larsson work, both studies examine strategies to use when playing real-life Yatzy, the foundation this study’s algorithm will be based on their work.
The work mentioned above provides an interval to compare the bonus-focused strategy’s results to. The lower limit, the expected value 189.80, is the value the bonus-focused strategy should get as close as possible to, this to obtain a favourable result and conclusion. The mathematically optimal expected value, the upper limit of 248.92, is a value the bonus-focused strategy can not exceed since it is simplified to be able to use in real life. The bonus-focused strategy can not perform better or as good as this strategy.

2.3 Rules of Yatzy

A game of Yatzy is played by one to an unlimited number of players. Each player rolls five dice and tries to fill as many of the 15 categories, also named slots or boxes, (explained in subsection 2.4) on the Yatzy scorecard, or game protocol, with a score above zero. See Appendix, section 6, for an example of a Yatzy scorecard.

The dice can be rolled three times in total each round and the player chooses which dice to keep after each throw. If a player scores 63 points or more in the Upper Section of the scorecard (categories Ones-Sixes) that player is rewarded a bonus of 50 points.

If a player is not able to obtain a valid combination after three throws, the player must place a zero in the category of his or her choosing.

The winner is the player who scores the highest when all scorecards are filled.

The maximum score when playing Yatzy is 374.

There are many ways to go about filling the scorecard when playing Yatzy. A version, called "Forced Yatzy", forces the participants to fill the categories in the order they appear on the scorecard. A version of "Forced Yatzy" is "Half-Forced Yatzy, where the players must fill all the Upper Section categories before filling the Lower Section categories [6].

2.4 Dice Combinations

Below are the categories to fill during a game of Yatzy, listed by order of appearance in the game protocol [6].

**Ones-Sixes:** Only the total of all the dice that show the same value are counted, for example: rolling 1-1-1-2-5 scores three points in the Ones category. The hand can also be placed in the Twos category for two points or in Fives for five points.

**Bonus:** A player who scores 63 points or more in the Upper Section of the scorecard is awarded 50 bonus points.

**One pair/Two pairs:** The total of the pair(s) is counted. Throwing 4-4-4-4-6 does not count as two pairs. Both pairs must be different.
**Three-of-a-kind/Four-of-a-kind:** Only the total of all the dice which are the same is counted, for example: three 4s (= 12 points) or four 2s (= 8 points).

**Small straight:** Throw 1-2-3-4-5. This throw scores 15 points.

**Large straight:** Throw 2-3-4-5-6. This throw scores 20 points.

**Full house:** Throw any Pair and any Three-of-a-kind, for example: 2-2-5-5-5. The Pair and the Three-of-a-kind must be different. Count the total of the dice and enter it on the scorecard.

**Chance:** Count the total of the dice and enter it on the scorecard.

**Yatzy:** All dice must show the same value. This throw scores 50 points.
3 Method

To obtain a reliable result (in this case an expected value) to answer the problem statement mentioned in section 1.2, it is vital to play enough rounds of Yatzy following a bonus-focused strategy so that an average can be computed. The "Law of Large Numbers" states that the average derived from a large number of observations, where each observation is independent and does not depend on the values of the other observations, will be close to the expected value only if the number of observations is enough [7].

Based on the theorem mentioned above and the fact that both Vancura, Axelsson and Larsson simulated Yatzy 100 000 times [3] [4], this study needs to conduct just as many tests. To speed up the process and save some time, it was necessary to construct a Java program that simulates a game of solitaire Yatzy 100 000 times. The program simulated a game of Yatzy based on the optimal strategy, theories and algorithm described in "Advanced Yahtzee" [3] that were adapted to the Scandinavian rules and to the bonus-focused strategy. This strategy is summarized in the sections below.

The bonus-focused algorithm should, given the current result and the current values of the dice, determine which dice to save or, if all three throws have been used, determine where to put the score in consideration to filling the Upper Section of the scorecard and obtaining the bonus. The Upper Section categories will be filled if placing the hand there does not risk the quest of obtaining the bonus. This is an important rule when trying to acquire the bonus, called staying "on par" [3], and will be further explained in section 3.2.2. When the bonus is acquired, or the possibility of obtaining it is nonexistent, the focus is shifted to filling the Lower Section of the scorecard.

When 100 000 simulations were done, all final scores were saved and the expected value was calculated, see section 4 for the results.

3.1 Basic Strategy

One thing a player should remember throughout the game is that each and every scorecard-entry, with the exception of Chance, can be considered either an "X-of-a-kind" or a "Straight". Understanding that a game of Yatzy consists of these two types of hands and that the scorecard contains twelve X-of-a-kind categories and two Straights, it seems logical that a winning strategy should be bias toward trying for the X-of-a-kind hands [3].

The Chance entry is unique in that sense that it is neither an X-of-a-kind nor a Straight. It is just the sum of all five dice and can be used at any time, and can never be scored with a zero. When playing Yatzy, it is never proper (except at the very end of the game) to purposely try for Chance. Instead, this slot should serve as a catchall in case of being dealt a bad hand. When faced with the choice of placing a hand in Chance or any other Lower Section category, always fill the latter. The other categories have some kind of restriction on filling them where as Chance, which has no restrictions, should be reserved as a safety mechanism only for an unwanted outcome [3].
There is a special relationship between the Three-of-a-kind and Four-of-a-kind, and between the One pair, Two pair and Full house. The primary reason for not filling the Three-of-a-kind and pairs too early is their frequency of occurrence. Since they are common hands there is no rush to complete them but rather keeping them open when their counterparts are among the more difficult categories to fill [3].

Consider the pairs and Full house. As long as the Two pair is unfilled, there is no downside in trying for the Full house. When trying for the Full house, the hand will often hold a Two pair. If the Full house is not rolled after the third and final throw, the option to fill the Two pair or One pair works as a good fall-back. However, once the Two pair is filled, the quest for the Full house becomes more problematic. When trying for the Full house with the Two pair and One pair filled, one is forced to sacrifice something else if going in for the Full house and failing. Maintain maximum flexibility by not filling the lower categories until their higher counterparts are filled.

3.2 Opening Strategy

The first part of the game is represented by all the hands up until the bonus is acquired or no longer possible to obtain.

The proper strategy to follow in the beginning can be summarized by three major themes:

- Follow the dice.
- Get ahead (or at least stay on par in the Upper Section).
- Be opportunistic with "difficult" hands, such as Yatzy, Four-of-a-kind and the two Straights.

3.2.1 Follow the Dice

The rule in the beginning of the game is to take what the dice give. Since the scorecard is empty at this point it is not sound to force the issue and trying to complete a specific category, but rather taking the good scores that presents themselves.

If a roll contains three or more of any particular value, it is generally best to hold those dice. If the dice on the other hand contains a broken Straight (the dice 1-2-3-4, 2-3-4-5 or 3-4-5-6), one should not waste the opportunity to go for a Straight, especially if the broken Straight is "open-ended" (can be a complete Straight with either one 1 or one 6) [3].

3.2.2 Staying On Par

One of the important strategies mentioned throughout Vancura’s book is to stay ahead or at least "on par" in the Upper Section during the game. If a player exceeds 63 points in the Upper Section that player is rewarded a 50-point bonus.
The value 63 represents an average of a three-die tally entry for each of the categories in the Upper Section. That is, three 1s, three 2s, three 3s, three 4s, three 5s and three 6s \((3+6+9+12+15+18 = 63)\).

If placing a below-average score in one of the categories, one must achieve a better-than-average score in one of the others in order to make up the deficit and obtain the 50-point bonus. Therefore, a player should not place below-average hands in the higher Upper Section categories such as Fours, Fives and Sixes [3].

Seeing that this report will examine the benefits of a bonus-focused strategy, investing higher X-of-a-kind hands in the Upper Section and staying on par is vital in this study. Any hand where the highest frequency of the dice is less than three after the final roll is considered a “bad hand” until the bonus is obtained, even if it can be placed in a Lower Section slot.

3.2.3 Be Opportunistic

Aside from Yatzy, the three toughest categories to fill in the Lower Section are the Four-of-a-kind and the two Straights. The probabilities of said categories can be found in section 3.5.

One might question the occasional abandonment of the bonus-quest when placing a Five-of-a-kind in the Yatzy entry and not the corresponding Upper Section category. This move can only be motivated by minimizing any possible point-losing damage the bonus-focused strategy may inflict. Just consider the move of placing a hand of five 2s in the Twos category instead of in Yatzy.

Since all Four-of-a-kinds will be placed in the Upper Section until the bonus is acquired, the Lower Section entry will remain empty until the mid-game state or it can be filled if the corresponding Upper Section slot is already filled. One should not squander this opportunity nor make the wrongful move of keeping the one die of an empty category and re-throwing a Four-of-a-kind when it can be placed elsewhere.

When the dice produce a broken Straight, it generally makes sense to go for it. Given a hand such as 2-3-4-5-5, the broken Straight 2-3-4-5 is superior to the 5-5 alone (it is "open-ended" and can be completed with either one 1 or one 6). When faced with a "close-ended" broken Straight (1-2-3-4 or 3-4-5-6) and a pair, the superior play is to keep the pair if the dice value is four or higher [3].

3.2.4 Filling the Scorecard

Except for going after the bonus from the Upper Section, the Lower Section categories will be filled in an opportunistic as-come basis. If the last roll results in a Yatzy, Straight or Full house, the corresponding category should be filled [3].

If the first roll does not result in a broken Straight, focus on the Upper Section slots and X-of-a-kinds. Given the current result and values of the dice, keep the highest value (given that the corresponding slot is not filled) with the highest frequency and roll again. If the three throws allowed in the game have been used, determine where to put the score, in order maximize the chance of
obtaining the bonus.

Any final hand where the highest frequency of the dice is less than three, check the Lower Section if the hand can be placed there instead with an above-average score, calculated in section [3.5.2]. If not, investigate if placing a bad hand in the Upper Section risks the bonus.

To assess the Upper Section status, if the player is on par or not, add the values in the boxes that are already filled and then calculate the sum for the unfilled boxes by assuming a minimum three-die entry in each category. If the total score is equal to or exceeds 63 points, the player is still on par or ahead and should place the hand in the appropriate slot. If not, the risk of not obtaining the bonus is too great and one should investigate if it is possible to place the hand in the Lower Section for a below-average score.

When the bonus is acquired, or the possibility of obtaining it is nonexistent, move on to the Lower Section of the scorecard and focus on filling those categories with the remaining Upper Section categories as fall-backs.

3.2.5 Placing Zero-Entries

Besides handling a bad hand as mentioned in the section above, that is, checking if it can be placed in the Upper Section without jeopardizing the bonus or in the Lower Section at all, one may be faced with entering a zero somewhere in the scorecard. In that case, knowing the proper order in which one should fill categories with zero is critical.

When necessary, fill the categories with zeros according to their expected score [3]. This has been calculated in section [3.5]. As the calculations will show, one should not hesitate to fill the Yatzy category with zero before the other categories, its expected value diminishes rapidly as the game proceeds [3].

As long as the bonus is not obtained, use the Lower Section categories as zero-entry trays if it comes to that. If the bonus is obtained and some categories in the Upper Section is still unfilled, use the remaining boxes as fall-backs when not achieving a desirable hand when trying to fill the Lower Section.

3.3 Mid-Game Strategy

The mid-game state begins after either acquiring the bonus, or when the bonus is no longer possible to obtain, up until round 13.

3.3.1 Transition and Planning

When entering the mid-game state the dice should still be allowed to dictate the game, but only to a point. With several slots already filled it is important to assess the still-empty categories and allowing a bias in the strategy to fill them.

Depending on if the hand consists of a pure X-of-a-kind or Two pair in the first throw, a priority will be given to either the Full house category or Yatzy (if said categories are empty). Since a failed Full house hand can be placed either in Two pairs, One pair or Three-of-a-kind and a failed Yatzy can be placed in
The main points can be summarized as below:

- Should the first throw of the dice result in a broken Straight one should roll the dice in hope of filling either the Small or Large Straight.

- The same rule apply for the X-of-a-kinds. Go for the Yatzy or Full house and if all else fails either the Four-of-a-kind, Three-of-a-kind, Two pair or One pair can be filled instead.

- One should also remember not to fill the Chance slot too early but rather save it for the higher hands that did not fill any of the categories mentioned above.

3.4 End-Game Strategy

The End-Game is represented by hands 13 to 15. When only a few categories remain unfilled it is vital to aggressively try to make those specific hands.

Because only a few slots are left open, one may on occasion end up with a hand that does not correspond to any of the available categories. This is not a sign of poor play, it is to be expected. Even when playing optimally, it is very likely that one or more categories will be filled with zero.

3.4.1 Forcing the Issue

At this point, instead of "taking what the dice give", it is necessary to force the issue. Desperate measures will sometimes become necessary.

For example, left with only a Large straight and Four-of-a-kind to fill and the first roll presents the hand 1-2-4-5-6, keep the 2-4-5-6 in hopes of rolling a 3. Early in the game, one should never make this move. Needing only one more die to make the Large Straight forces the player to try for it at this point.

If only the Ones and Fours slots are left and the first roll presents the dice 2-4-5-5-6, keep only the 4. This is a horrible play in the early stages but in the end game, in desperate need of fours, it is the only proper play.

When left with only Twos and Yatzy, and the final roll present the hand 2-3-4-5-5, keep the 2 making Twos the fall-back if not succeeding on acquiring the not so likely Yatzy.

Should Chance be the only slot left, the goal is to roll high numbers.

3.4.2 End-Game Considerations

Presented with a situation in which only the slots Four-of-a-kind and Three-of-a-kind are unfilled, and one of them must be filled with a zero, what should one do?

Assume that hand 14 just ended and the final outcome was 3-3-4-4-5. This can neither be placed in the Four-of-a-kind nor the Three-of-a-kind, but one of
these must be filled with zero. To maximize the expected score of the final round, the correct answer is to fill the Four-of-a-kind with zero, hoping to achieve a Three-of-a-kind during the last round [3].

Entering zero in a slot that may yield a higher score than the one left might be difficult to do. However, it is not merely the potential value of an entry which is important to consider, but rather the likelihood that one might achieve it. That is why it is more valuable to preserve the Three-of-a-kind than the Four-of-a-kind this late in the game. As the game progress and the player are left with with fewer unfilled slots, the expected value of each category decrease [3].

3.5 Expected Values

As mentioned in section 3.2.5 and 3.4.2, during some games one or more zero-entries will be made, and a category will be sacrificed. Since the logical thing to do is to enter a zero in the category with the lowest expected score [3], this section will cover the necessary calculations to determine the correct order of sacrifice.

The expected value, denoted by the expression $E(X)$ where $X$ stands for any given event, is the probability-weighted average of all possible outcomes. It is calculated by multiplying each outcome of the sample space with the probability for that outcome. The resulting products are then summed [7].

When calculating the probability for a category, to simplify things, the event $X$ will stand for trying to achieve a category on the first throw. It could also be solved by using Markov chains or conditional probabilities, but seems excessive when solving this type of problem. Even if it might affect the order of sacrifice, using Vancura’s tips as a reference should ensure that the calculations will not differ much.

3.5.1 Calculating Probabilities

A probability is how likely an event or outcome may occur. This is denoted by the expression $P(X)$ where $X$ stands for an event [7] and corresponds to a category in the Yatzy scorecard. The probability for each category is calculated by:

$$\frac{\text{Number of favorable combinations}}{\text{Number of possible combinations}}$$

Since the hand consists of five dice and six different values, the number of possible combinations is given by six to the power of five.

$$\text{Number of possible combinations} = 6^5 = 7776$$

The following probabilities are the odds for obtaining a given category on the very first throw. Since it is possible to place an overachieving hand in a lower category, like placing a Three-of-a-kind in the One pair category, the probability
of obtaining "at least" One pair is determined by adding the probability of obtaining Two pairs, Three-of-a-kind, Four-of-a-kind, Full house and Yatzy as these too can be placed in the Pair category. Therefore, these calculations will be done in two steps. First, the probability of achieving exactly a category and second, the probability of achieving at least a category.

The probability of obtaining exactly One pair is determined by choosing two dice out of five, multiplied with the number of possible values that pair can take. The three dice left must be distinct from each other and from the pair. This is divided by the total number of possible combinations.

\[
P(\text{One pair}) = \frac{\binom{5}{2} \cdot 6 \cdot 5 \cdot 4 \cdot 3}{7776} = \frac{3600}{7776}
\]

The probability of obtaining exactly Two pairs is determined by first choosing two dice out of five and then two dice out of three, multiplied by the 15 different combinations that the two pairs can take. The die left must be distinct from the two pairs and can be chosen in four different ways. This is divided by the total number of possible combinations.

\[
P(\text{Two pairs}) = \frac{\binom{5}{2} \cdot \binom{3}{2} \cdot 15 \cdot 4}{7776} = \frac{1800}{7776}
\]

The probability of obtaining exactly a Three-of-a-kind is determined by choosing three dice out of five, multiplied with the number of possible values that the Three-of-a-kind can take. The two dice left must be distinct from the Three-of-a-kind and each other.

\[
P(\text{Three of a kind}) = \frac{\binom{5}{3} \cdot 6 \cdot 5 \cdot 4}{7776} = \frac{1200}{7776}
\]

The probability of obtaining exactly a Four-of-a-kind is determined by choosing four dice out of five multiplied with the number of possible values that the Four-of-a-kind can take. The remaining die must be distinct from the Four-of-a-kind and in order for the Four-of-a-kind to not to become a Yatzy.

\[
P(\text{Four of a kind}) = \frac{\binom{5}{4} \cdot 6 \cdot 5}{7776} = \frac{150}{7776}
\]

The Full house is a combination of a Three-of-a-kind and a One pair. Like the Three-of-a-kind, choose three dice out of five and multiply with the number of possible values that the Three-of-a-kind can take. The One pair must be distinct from the Three-of-a-kind and can be chosen in five different ways.

\[
P(\text{Full house}) = \frac{\binom{5}{4} \cdot 6 \cdot 5}{7776} = \frac{300}{7776}
\]
The probability to roll the Small straight is the same as the Large straight. No matter the order of the dice, all five distinct values must be rolled for each Straight and is done in $5!$ different ways.

$$P(\text{Small straight}) = P(\text{Large straight}) = \frac{5!}{7776} = \frac{120}{7776}$$

There are no restrictions on what can be placed in a Chance.

$$P(\text{Chance}) = 1$$

There are six different ways to roll a Yatzy. Five 1s, five 2s and so on.

$$P(\text{Yatzy}) = \frac{6}{7776}$$

Now that the "exact" probabilities are calculated, the "at least" probabilities can be determined. As mentioned above, this is done by adding the probabilities of categories that include another category.

$$P(\text{At least One pair}) = P(\text{Pair}) + P(\text{Two pairs}) + P(\text{Three of a kind})$$
$$+ P(\text{Four of a kind}) + P(\text{Full house}) + P(\text{Yatzy}) = \frac{7056}{7776}$$

$$P(\text{At least Two pairs}) = P(\text{Two pairs}) + P(\text{Full house}) = \frac{2100}{7776}$$

$$P(\text{At least Three of a kind}) = P(\text{Three of a kind}) + P(\text{Four of a kind})$$
$$+ P(\text{Full house}) + P(\text{Yatzy}) = \frac{1656}{7776}$$

$$P(\text{At least Four of a kind}) = P(\text{Four of a kind}) + P(\text{Yatzy}) = \frac{156}{7776}$$

3.5.2 Calculating Average Points

The next step is to investigate the average score for each category. In theory, this is calculated by setting up a sample space, the set of all possible outcomes, for each outcome in a given category. For example: the sample space for one pair is 2, 4, 6, 8, 10, 12. The average score is then calculated by multiplying each value from the sample space with the probability for that category [7].
However, this can be done much simpler by calculating the average score of a
die-throw and multiplying it with the number of dice required for each category.
This is done for each category that does not award the player with a fixed score
value, such as Small straight, Large straight and Yatzy.

\[
Average \text{ score of a die throw} = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = 3,5
\]

With the average die-score of 3,5 points, the average score are as follows:

Average score of One pair = 3,5 ∙ 2 = 7,0

Average score of Two pairs = 3,5 ∙ 4 = 14,0

Average score of Three of a kind = 3,5 ∙ 3 = 10,5

Average score of Four of a kind = 3,5 ∙ 4 = 14,0

Average score of a Full house = 3,5 ∙ 5 = 17,5

Average score of a Small straight = 15

Average score of a Large straight = 20

Average score of a Chance = 3,5 ∙ 5 = 17,5

Average score of Yatzy = 50

### 3.5.3 Calculating Expected Values

As shown in sections 3.5.1 and 3.5.2, both average score and probabilities for
each category is calculated, and it is now possible to determine the expected
score for each category.

\[
E(\text{One pair}) = P(\text{At least One pair}) \times (\text{Average score of One pair})
\]
\[
= \frac{7056}{7776} \times 7,0 = \frac{49392}{7776}
\]

\[
E(\text{Two pairs}) = P(\text{At least Two pairs}) \times (\text{Average score of Two pairs})
\]
\[
= \frac{2100}{7776} \times 14,0 = \frac{29400}{7776}
\]
\[
E(\text{Three of a kind}) = P(\text{At least Three of a kind}) \ast \\
(Average \ score \ of \ Three \ of \ a \ kind) = \frac{1656}{7776} \cdot 10.5 = \frac{17388}{7776}
\]

\[
E(\text{Four of a kind}) = P(\text{At least Four of a kind}) \ast \\
(Average \ score \ of \ Four \ of \ a \ kind) = \frac{156}{7776} \cdot 14.0 = \frac{2184}{7776}
\]

\[
E(\text{Full house}) = P(\text{Full house}) \ast (Average \ score \ of \ Full \ house) \\
= \frac{300}{7776} \cdot 17.5 = \frac{5250}{7776}
\]

\[
E(\text{Small straight}) = P(\text{Small straight}) \ast (Average \ score \ of \ a \ Small \ straight) \\
= \frac{120}{7776} \cdot 15 = \frac{1800}{7776}
\]

\[
E(\text{Large straight}) = P(\text{Large straight}) \ast (Average \ score \ of \ a \ Large \ straight) \\
= \frac{120}{7776} \cdot 20 = \frac{2400}{7776}
\]

\[
E(\text{Chance}) = P(\text{Chance}) \ast (Average \ score \ of \ a \ Chance) \\
= \frac{7776}{7776} \cdot 17.5 = \frac{136080}{7776}
\]

\[
E(\text{Yatzy}) = P(\text{Yatzy}) \ast (Average \ score \ of \ Yatzy) = \frac{6}{7776} \cdot 50 = \frac{300}{7776}
\]

Thus, the categories will be sacrificed in the following order: Yatzy, Small straight, Four-of-a-kind, Large straight, Full house, Three-of-a-kind, Two pairs, and finally One pair. This order also seems to correspond to Vancura’s list \([3]\), taking into account that both pairs can not be found in Yahtzee and the Small Straight looks a bit different.

Even though Chance can be found in these calculations, it is never possible to enter a zero in that slot. The lowest entry that can be made in Chance is five.
3.6 Simulation of the Game and Compiling the Results

When the code was completed, compiled and executed it simulated solitaire Yatzy 100 000 times based of the strategies and calculations mentioned in the sections above. The output resulted in a file filled with the final scores for each simulation. See section 4 for the compiled results.

The expected value was derived from the simulation by calculating the average score of the 100 000 simulations, as stated in section 3 and [1].
4 Result

After running the simulation 100 000 times, the expected score of 215.70 was obtained. The graph below shows the distribution of all final scores.

Figure 1: This graph illustrates the distribution of final scores of 100 000 simulations of Yatzy that follows a bonus-focused strategy.

The x-axis represents the final score of one simulation of Yatzy. The y-axis represent the frequency of that final score.

<table>
<thead>
<tr>
<th>Average value, E(X)</th>
<th>215.70</th>
</tr>
</thead>
<tbody>
<tr>
<td>Most occurring final score</td>
<td>215</td>
</tr>
<tr>
<td>Highest score</td>
<td>335</td>
</tr>
<tr>
<td>Lowest score</td>
<td>66</td>
</tr>
<tr>
<td>Number of times the bonus from the Upper Section was obtained (out of 100 000)</td>
<td>83 925, 84%</td>
</tr>
<tr>
<td>Number of times the bonus from the Upper Section and Yatzy was obtained (out of 100 000)</td>
<td>26 725, 27%</td>
</tr>
<tr>
<td>Number of times only Yatzy was obtained (out of 100 000)</td>
<td>5 652, 5.7%</td>
</tr>
</tbody>
</table>

Table 1: This table contains the results of this study
5 Discussion

5.1 Result Analysis

Surprised at such a high expected value, it seems that this strategy surpasses Axelsson’s and Larsson’s and can almost be compared with the mathematically optimal strategy (a difference of 33.22 points or almost 13%). This seems strange, considering that this algorithm is so naive in comparison. Even if a higher expected score was not anticipated, it should not be discarded but rather analysed some more, as done in section 5.2.

The graph almost shows a normal distribution of the final scores, with a slight shift or bigger weight towards higher final scores. Judging by the almost uniform-like distribution the algorithm seem to be balanced and performs as good as it can, taking into consideration that the dice behaves randomly and does not always produce a desirable outcome.

Vexed at the occurrence of obtaining the bonus, 84%, it is hard to decide if the missing 16% depends solely on the dice randomized behaviour or due to the fact that the algorithm allows placing bad hands (the highest frequency of a dice value is less than three) in the Upper Section if it does not compromise the bonus too much. As the strategy states, a player should not place a bad hand in categories Fours, Fives and Sixes because the deficit will be too great to make up with the lower categories in the Upper Section (Ones, Twos and Threes) \[3\], but seems to happen at times. For example, if Ones is scored with 4 points and Fives with 20 points, the algorithm will allow placing 12 points in the Sixes category assuming a three-die tally for the remaining empty categories. If given time, this should be investigated some more and possibly changed.

5.2 Comparing Strategies

<table>
<thead>
<tr>
<th></th>
<th>Results from this study</th>
<th>Result obtained by Axelsson and Larsson</th>
<th>Result obtained by Larsson and Sjöberg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Value, E(X)</td>
<td>215,70</td>
<td>189,80</td>
<td>248,92</td>
</tr>
<tr>
<td>Most commonly occurring result</td>
<td>215</td>
<td>156</td>
<td>244</td>
</tr>
<tr>
<td>Highest score</td>
<td>335</td>
<td>331</td>
<td>-</td>
</tr>
<tr>
<td>Lowest score</td>
<td>66</td>
<td>81</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2: This table contains the results of this study and the optimal strategies

Baffled at the average score of 215,70, a result that exceeds Axelsson’s and Larsson’s result with 25.9 points or 13.65%, the first impression says that something is wrong. How could a bonus-focused strategy perform better than an optimal strategy?
Instead of discarding this study’s result and assuming that something went wrong when implementing the code, some important observations can be made. To confirm that a bonus-focused strategy indeed was implemented and not another optimal strategy that just happens to perform better, some changes were made to Axelsson’s and Larsson’s code. When simulating their strategy again with an added bonus-counter the result was clear:

Of 100 000 simulations following an optimal strategy, they obtained the bonus 29 645 times, almost 30%. In addition to the bonus, the Yatzy was obtained 11 180 times. The case where only Yatzy, but not the bonus, was obtained occurred 25 895 times.

To clarify, a bonus-focused strategy was indeed implemented and a contributing reason to Axelsson’s and Larsson’s lower score might be that the bonus was obtained only 30% compared to the 84% the bonus-focused strategy obtained it.

If one would subtract the points corresponding to the bonus and how many times each strategy obtained it,

\[
215,70 - 50\cdot0,84 = 173,7 \\
189,80 - 50\cdot0,30 = 174,8
\]

it seems that the strategies do not differ much at all, which they should not considering that the foundation of each algorithm is derived from the same book. Taking this into account, it also makes it hard to determine potential losses of this bonus-focused strategy.

Another contributing factor to Axelsson’s and Larsson’s lower score might be that their code does not function as intended. When reusing parts of their code, it became clear that it did not take into consideration that a Three-of-a-kind could be placed in a One pair etc. It also seems that they did not consider the necessary moves that should be made when there is only a few open slots left on the scorecard, as discussed in section 3.4 and which they also mentioned in their report [4].

Taking this into consideration when determining if the result is reasonable or not, and the fact that the bonus-focused strategy does not perform better than the mathematically optimal Yatzy strategy (the highest score ever achieved is 335) it seems that this strategy is valid if not better than the optimal one Axelsson and Larsson implemented.

5.3 Source of Errors

Since some parts of the program was not written from scratch, some small errors existed in the code. So far there has been some discoveries and changes to the code that did not operate as it should but most, if not all, errors should have been resolved at this point.

As mentioned in section 3.5 Markov chains or conditional probabilities was never used when calculating the odds of each category which might influence the order of sacrifice. But, when consulting Vancura’s list, it seems that there
is little or no difference.

If anything could have affected obtaining the bonus from the Upper Section, and in turn the expected value, it might be that bad hands can be placed in the Upper Section’s higher categories (like placing two 6s in the Sixes category) after the initial calculations still points that it is possible to obtain the bonus without any risk, when in reality a deficit of six points might be too hard to make up with only the lower categories left. If given any more time, this should be investigated further.
6 Conclusion

As stated in section 5.2, the bonus-focused strategy results in an average score of 215,70 and exceeds Niklas Axelsson’s and Ricard Larsson’s score with 25,9 points or almost 14%. Based on the positive difference between these strategies, the given data used for comparison and the problem statement in section 1.2, the conclusion is that the bonus-focused strategy is a viable one if not better than the optimal one.

Based on the notes in section 5.2, changes to Axelsson’s and Larsson’s algorithm should be made to see if the statement above still holds or not.

Seeing as the bonus-focused strategy performed better than Axelsson’s and Larsson’s optimal strategy, it is hard to see if there are any potential losses to using this strategy.
References


Appendix

Yatzy scorecard

This is a version of a Yatzy scorecard.

<table>
<thead>
<tr>
<th>Ones</th>
<th>Twos</th>
<th>Threes</th>
<th>Fours</th>
<th>Fives</th>
<th>Sixes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sum</th>
<th>Bonus</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>One pair</th>
<th>Two pair</th>
<th>3 of a kind</th>
<th>4 of a kind</th>
<th>Small Straight</th>
<th>Large straight</th>
<th>Full house</th>
<th>Chance</th>
<th>Yatzy!!!</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Total sum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>
Program Code

All code files and raw data can be found in the following GitHub repository: 
https://github.com/darjah/OptimalYatzy