Uncertainty in area determination

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Abstract

This report gives a thorough description of the determination of two-dimensional areas and the estimation of area uncertainty. Considering the formulas for area determination, the basis is regular shapes, e.g. triangles, squares, rectangles, rhombi and regular (closed) polygons. The given formulas are valid as such; however, they also indicate how they can be simplified. The report presents strict as well as approximate formulas for area uncertainty estimates.

The approach is mainly from a statistical point of view. However, there are some geometric factors/ variations which have an influence on the size of the area. They are dealt with by mathematical means.

In addition, less mathematical, more argument-based (philosophical) descriptions of other “areal defects”, having an influence on the area uncertainty, are presented. Examples of these defects are given to illustrate their impact on uncertainty estimation.

The report is mainly an English translation of an earlier Swedish report [1]. Therefore, some minor parts refer to Swedish conditions – e.g. regarding coordinate systems, map projections etc.

**Key words:** area determination, uncertainty in measurement, errors due to elevation, map projection and inclination, areal defects.
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1. Background and purpose

Measurement uncertainty is rarely presented together with area determinations. Additionally, there is almost total ignorance about how sure/unsure area data really is. There is often an over-reliance on the accuracy of the data.

The purpose of this report is to give a reasonably detailed description of area determination and the estimation of area uncertainty associated with this. Both strict and approximate formulas for area uncertainty are presented.

In terms of area determination, the basis is regular shapes, e.g. triangles, squares, rectangles, rhombi and regular (closed) polygons. The given formulas are valid as such; however, they also indicate how they can be simplified.

Possible applications of uncertainty data include e.g.:
- attaching an uncertainty statement in connection with the delivery of an area determination
- setting tolerances for acceptable area uncertainties for terrain surfaces with different appearances, in different applications
- setting tolerances for the difference between two determinations of the same area, e.g. in connection with control measurement.

The approach is mainly from a statistical point of view. However, there are some geometric factors/ variations which have an influence on the size of the area. They are dealt with by mathematical means.

The report begins with a presentation of basic terms, concepts and formulas – with particular focus on area uncertainty. In order not to make the system of formulae too complicated, we assume uncorrelated variables (measurements, coordinates, etc.). Based on a number of typical cases, formulas are then developed for calculating the statistical area uncertainty for domains with different appearances.

This is followed by a treatment of errors due to elevation, map projection, and inclination respectively, as well as digital terrain models’ use in connection with area determination. In addition, less mathematical, more argument-based (philosophical) descriptions of other “areal defects”, having an influence on the area uncertainty, are presented. Various numerical examples are included as clarifications.

1.1. Basic terms and formulas

The following section presents basic terms, concepts and formulas regarding area uncertainty.

1.2. Surface vs. area

Surface is the physical surface in three dimensions, which can be expressed mathematically in the form

\[ H = f(N, E) \]  

where \( H \) is the elevation in the point with the coordinates \((N, E)\), Northing and Easting. Area usually refers to the size of the domain in the plane, demarcated by the projection of the surface's boundary line (see Figure 1). Often, the boundary line has the shape of an \( n \)-sided polygon with the vertices \((N_i, E_i)\), \( i = 1, n \). It provides the opportunity to determine the area's uncertainty using relatively simple means.

In some contexts, however, it is the real 3-dimensional surface area that is of interest, and in that case the aforementioned reasoning does not fully work. Errors will occur which, inter alia, depend on the difference between the physical surface and its projection, as well as the incompleteness of the model intended to represent the surface.
1.3. Uncertainty

A measurement should give two results: a measurement value and an estimation of the measurement uncertainty. Both bits of data are important if we want to ensure that the measurement is used in the right context, i.e., that it fulfils the quality requirements of the specific application.

GUM – Guide to the Expression of Uncertainty in Measurement – is an internationally developed standard for expressing measurement uncertainty. Behind the GUM concept is a consortium of seven organisations, including ISO [2]. It has been widely adopted within many industries – however not within geodesy and geodata, thus far. GUM provides clear quality declarations that can be understood by all. This way of analysing measurement uncertainty is both terminologically stringent and pragmatic, as well as practically targeted.

GUM is applied throughout this report and is described in more detail in [3].

Standard uncertainty

The standard uncertainty of \( x \) is denoted \( u(x) \), where \( u \) stands for uncertainty. In GUM, it replaces the traditional term standard error, \( \sigma \). The standard uncertainty is a function \( f \), of one or more variables \( x_i \), calculated using the law of propagation of uncertainty

\[
\begin{align*}
    u^2(f) &= \left( \frac{df}{dx_1} \right)^2 u^2(x_1) + \left( \frac{df}{dx_2} \right)^2 u^2(x_2) + \ldots + \left( \frac{df}{dx_n} \right)^2 u^2(x_n) \\
\end{align*}
\]  

It should, again, be emphasised that Formula (2) assumes that the variables \( x_i \) are uncorrelated. We will continue to make several other simplifications in order to obtain manageable calculations, as we are striving to attain orders of magnitude in this report, not exactness.

Example 1: Calculate the standard uncertainty of the area of a circle \( f(r) = \pi r^2 \) if the radius \( r \) is determined with the standard uncertainty \( u(r) \)?

Solution: The propagation formula (2) for standard uncertainty gives

\[
\begin{align*}
    u^2(\text{area}) &= \left( \frac{df}{dr} \right)^2 u^2(r) = \left( \frac{d(\pi r^2)}{dr} \right)^2 u^2(r) = (2\pi r)^2 u^2(r) = O^2 u^2(r) \\
    \Rightarrow u(\text{area}) &= O u(r) \\
\end{align*}
\]  

where \( O = \) the circumference of the circle. \( u(\text{area}) \) is represented by the shaded domain in Figure 2.

Figure 1. Physical surface vs. its projection.
Note that the circumference of a circle, therefore, is the derivative of its area

\[
\frac{dA}{dr} = \frac{d\pi r^2}{dr} = 2\pi r = O
\]  \hfill (4)

\[\text{Figure 2. The standard uncertainty of the area of a circle with respect to the standard uncertainty in the determination of its radius.}\]

The propagation formula (2) also gives a number of other useful expressions. E.g. the standard uncertainty of the sum of \(n\) measurements with the same standard uncertainty is

\[u(\text{sum}) = \sqrt{n} \, u(\text{single})\]  \hfill (5)

where \(u(\text{single})\) is the individual measurement’s – single measurement’s – standard uncertainty.

The standard uncertainty of the difference between two measurements with the same standard uncertainty is

\[u(\text{diff}) = \sqrt{2} \, u(\text{single})\]  \hfill (6)

In the case of different standard uncertainties and two measurements (\(a\) and \(b\)) we have

\[u(a + b) = u(a - b) = \sqrt{u^2(a) + u^2(b)}\]  \hfill (7)

The standard uncertainty of the mean of \(n\) measurements with the same standard uncertainty is

\[u(\text{mean}) = u(\text{single})/\sqrt{n}\]  \hfill (8)

The horizontal standard uncertainty (two-dimensional point standard error) is represented as

\[u(\text{point}) = \sqrt{u^2(\text{coordinate}) + u^2(\text{coordinate})} = \sqrt{2} \, u(\text{coordinate})\]

\[\Leftrightarrow u(\text{coordinate}) = u(\text{point})/\sqrt{2}\]  \hfill (9)

where \(u(\text{coordinate})\) is the standard uncertainty of a coordinate. Typically, the coordinates are Northing (\(N\)) or Easting (\(E\)), but can also be measured longitudinally and laterally (in the direction of measurement and perpendicular to that direction), see Example 2.

A distance in the plane – between the points \(A\) and \(B\) – is calculated from coordinates using the formula

\[d_{AB} = \sqrt{(N_B - N_A)^2 + (E_B - E_A)^2}\]  \hfill (10)

and has the standard uncertainty
where $u(point)$ is the horizontal standard uncertainty for each endpoint (if we assume that both endpoints have the same measurement uncertainty$^1$).

**Example 2:** With what standard uncertainty should the endpoints be determined if the distance between them is to have the standard uncertainty 10 mm?

**Solution:** Each endpoint shall (according to Formula (11)) be determined with a horizontal standard uncertainty of max 10 mm, i.e., $u(distance) = u(point) = 10$ mm.

### Expanded uncertainty

From the field of statistics, we know that the standard deviation/standard uncertainty as such is not generally a good quality measure. In the case of normal distribution, expressions of the type $\hat{x} \pm u(\hat{x})$ contain the “true” value $x$ with only 68% probability.

Therefore, we usually multiply by a coverage factor $k$, which is a number > 1. In GUM, the fairly standard coverage factor of 2 is used, which gives a coverage probability (confidence level) of approx. 95%. The coverage factor, multiplied by the standard uncertainty, is termed expanded uncertainty and is written

$$U_{exp}(\cdot) = k_{exp} \cdot u(\cdot)$$  \hspace{1cm} (12)

The indices ”95” refer to the coverage probability, that might as well be expressed verbally in the adjoining text.

**Example 3:** Specify the expanded uncertainty with a coverage probability of 95 % for the distance in Example 2.

**Solution:** If we apply GUM standard, i.e. the coverage factor $k = 2$, we get $U_{exp} = 2 \cdot 10 = 20$ mm.

### How do we estimate measurement uncertainty?

Where do we obtain data on the standard uncertainties for the vertices? It depends of course on the method by which they are determined: Network RTK, photogrammetry, digitisation, etc. For the tried and tested methods there are good experience figures, but for new techniques we have to rely on information given e.g. in the manufacturers' datasheets.

In that respect, in Sweden there are great hopes with regard to *HMK – Handbook in Surveying and Mapping* [4], which is currently being revised. Some preliminary data on standard uncertainties, taken from HMK’s work material 2012, is presented in Table 1.

In connection with, for example, aerial photography and laser scanning, the regional measurement uncertainty is considered to be of greatest interest. It is defined as the expected uncertainty within, for example, a municipality or an infrastructure project, with distances in the magnitude of several kilometres or more.

At a detailed level, we are usually most interested in the local measurement uncertainty, which relates to the relative uncertainty between adjacent control points, between objects in a geodatabase, or between building parts in a facility. Distances in question are ≤ 100 m.

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$^1$ In itself, there is a difference between two points (standard uncertainty $\sqrt{2} \cdot u(point)$ according to Formula (6)), but because the distance only relates to one of two directions (longitudinally but not laterally) we shall, according to the last part of Formula (9), also divide by $\sqrt{2}$. I.e. they cancel each other out, leaving only the standard uncertainty in one end point, $u(\text{punkt})$. 
Table 1. Geodetic measurement – some horizontal standard uncertainties.

<table>
<thead>
<tr>
<th>Method/technique</th>
<th>Horizontal standard uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Network RTK within a net of permanent reference stations, e.g. the Swedish SWEPOS</td>
<td>5-10 mm (regionally)</td>
</tr>
<tr>
<td>Single-station RTK</td>
<td>10 mm + 1 ppm (relative to the reference station)</td>
</tr>
<tr>
<td>Traditional control points</td>
<td>5-10 mm (locally)</td>
</tr>
<tr>
<td>Detailed surveying</td>
<td>10-15 mm (locally)</td>
</tr>
</tbody>
</table>

1.4. Fundamental variables

The regular polygon will play a central role in this account, and based on such, a number of fundamental variables are defined in Figure 3.

The following relations are found between these variables

\[ r_n = \frac{k_n}{2\sin(\pi/n)} \]  \hspace{1cm} (13)

\[ s_n = \frac{k_n}{2\cos(\pi/n)} \]  \hspace{1cm} (14)

\[ r_n = \frac{s_n}{2\sin(\pi/n)} \]  \hspace{1cm} (15)

where index \( n \) indicates the number of vertices in the polygon.

Figure 3. A regular hexagon. \( s = \) side length (the distance between two adjacent vertices), \( r = \) the distance from the centre to a vertex (“radius”), and \( k = \) the distance from one vertex to the second-next vertex (“chord”).

The polygon’s area is given by the expressions

\[ A_n = \frac{n r_n^2}{2} \sin(\pi/n) = \frac{n s_n^2}{4\tan(\pi/n)} = \frac{n k_n^2}{8\sin(2\pi/n)} \]  \hspace{1cm} (16)

and its circumference by

\[ O_n = n s_n = \frac{n k_n}{2\cos(\pi/n)} = 2n r_n \sin(\pi/n) \]  \hspace{1cm} (17)

Example 4: Apply Formula 14-17 on a square with side length \( s_4 = 1 \).
Solution:

\[ k_4 = 1 \cdot 2 \cos(\pi/4) = 1 \cdot 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2} \quad (\text{diagonal}) \]

\[ r_4 = \frac{1}{2 \sin(\pi/4)} = \frac{1}{2 \cdot \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{2} \quad (\text{distance to the centre}) \]

\[ A_4 = \frac{4 \cdot 1}{4 \tan(\pi/4)} = \frac{1}{1} = 1 \quad (\text{area}) \]

\[ O_4 = 4 \cdot 1 = 4 \quad (\text{circumference}) \]

As a curiosity, we also calculate the limits when \( n \to \infty \)

\[ A_n = \lim_{n \to \infty} A_n = \frac{n r_o^2}{2} = \frac{2\pi}{n} \quad r_o^2 \]

\[ O_n = \lim_{n \to \infty} O_n = 2n \cdot \frac{\pi}{n} = 2\pi \quad r_o \]

i.e. when the polygon becomes a circle. This gives us the usual formulas for the circle's area and circumference.

1.5. The Surveyor's Area Formula

The formula

\[ A = \frac{1}{2} \sum_{i=1}^{n} \left[ N_i (E_{i+1} - E_{i-1}) - E_i (N_{i+1} - N_{i-1}) \right] \]

where the area of a polygon is calculated from the vertices' coordinates \( N_i = \text{Northing}_i \) and \( E_i = \text{Easting}_i \), is usually called the "Surveyor's Area Formula". Note that the first point must be double-numbered (it gets both number 0 and \( n \)). The formula is attributed to C.F. Gauss, who was a geodesist – or "surveyor" in the vernacular. In some references it is called "Elling's method".

Derivation of Formula (20) gives

\[ \left( \frac{dA}{dN_i} \right)^2 = \frac{1}{4} (E_{i+1} - E_{i-1})^2 \]

\[ \left( \frac{dA}{dE_i} \right)^2 = \frac{1}{4} (N_{i+1} - N_{i-1})^2 \]

which via the propagation formula (2) for standard uncertainty leads us to the general uncertainty measure

\[ u^2(A) = \sum_{i=1}^{n} \left[ \left( \frac{dA}{dN_i} \right)^2 u^2(N_i) + \left( \frac{dA}{dE_i} \right)^2 u^2(E_i) \right] = \]

\[ = \frac{1}{4} \sum_{i=1}^{n} \left[ (E_{i+1} - E_{i-1})^2 u^2(N_i) + (N_{i+1} - N_{i-1})^2 u^2(E_i) \right] \]

From Formula (23), various special cases can be distinguished.

If we assume the same measurement uncertainty in Northing and Easting, i.e. that \( u(N_i) = u(E_i) = u(\text{point})/\sqrt{2} \), we get
\[ u^2(A) = \frac{1}{8} \sum_{i=1}^{n} \left[ (E_{i+1} - E_{i-1})^2 u^2(P_i) + (N_{i+1} - N_{i-1})^2 u^2(P_i) \right] = \]
\[ = \frac{1}{8} \sum_{i=1}^{n} \left[ (E_{i+1} - E_{i-1})^2 + (N_{i+1} - N_{i-1})^2 \right] u^2(\text{point}_i) \]

Furthermore, if the point uncertainty is equal for all vertices, i.e. \( u(\text{point}_i) = u(\text{point}) \) \( \forall i \), this turns into
\[
\[ u^2(A) = \frac{u^2(\text{point})}{8} \sum_{i=1}^{n} \left[ (E_{i+1} - E_{i-1})^2 + (N_{i+1} - N_{i-1})^2 \right] = \frac{u^2(\text{point})}{8} \sum_{i=1}^{n} k_i^2 \]
\]
where \( k_i \) is the “chord” by analogy with Figure 3, i.e. the distance between the vertex before and after the \( i \)-th point. In a regular polygon, all “chords” are also equal in length, i.e., \( k_i = k \) \( \forall i \), which finally gives
\[
\[ u^2(A) = \frac{n}{8} k^2 u^2(\text{point}) \Leftrightarrow u(A) = \frac{k}{2} \sqrt{\frac{n}{2}} u(\text{point}) \]
\]
and also (from Formula 13)
\[
\[ u(A) = r \sin(\frac{2\pi}{n}) \sqrt{\frac{n}{2}} u(\text{point}) \]
\]
as well as (from Formula 14)
\[
\[ u(A) = s \cos(\frac{\pi}{n}) \sqrt{\frac{n}{2}} u(\text{point}) \]
\]
The limit when the polygon becomes a circle is
\[
\[ u(A_\infty) = \lim_{n \to \infty} r_n \sin(\frac{2\pi}{n}) \sqrt{\frac{n}{2}} u(\text{point}) = r_\infty 2\pi \sqrt{\frac{n}{2}} u(\text{point}) = 2\pi r_\infty \frac{1}{\sqrt{n}} \frac{u(\text{point})}{\sqrt{2}} = 2\pi r_\infty \frac{u(\text{coordinate})}{\sqrt{n}} \]
\]
Comparison with Formula 3 for the circle area's standard uncertainty gives
\[
\frac{u(\text{coordinate})}{\sqrt{n}} = \frac{u(\text{longitudinally})}{\sqrt{n}} = u(r) . \] This means that \( u(r) \) can be interpreted as the mean of \( n \) radius determinations (the \( n \) vertices), and using Formula 8 we get the standard uncertainty
\[
\frac{u(\text{longitudinally})}{\sqrt{n}} \] for this mean.
2. Some typical cases
In the following section, formulas are developed for calculating the (statistical) area uncertainty of domains with different appearances, based on a number of typical cases.

2.1. Quadrangles
The standard uncertainty of the area determination of an arbitrary quadrangle is (Formula (25))

\[ u^2(\text{quadrangle}) = \frac{u^2(\text{point})}{8} \sum_{i=1}^{4} k_i^2 = \frac{u^2(\text{point})}{8} (d_{24}^2 + d_{13}^2 + d_{24}^2 + d_{13}^2) = \]

\[ = \frac{u^2(\text{point})}{4} (d_{24}^2 + d_{13}^2) \]

\[ \Leftrightarrow \]

\[ u(\text{quadrangle}) = \frac{u(\text{point})}{2} \sqrt{d_{24}^2 + d_{13}^2} \]  

(31)

where \( u(\text{point}) \) is the four vertices' standard uncertainty in the plane, which is assumed equal for all points. \( d_{13} \) and \( d_{24} \) represent the two diagonals (or "chords" as per Figure 3).

The standard uncertainty of the area determination of a square domain with side length \( s \) is

\[ u(\text{square}) = \frac{u(\text{point})}{2} \sqrt{2s^2 + 2s^2} = s \ u(\text{point}) \]  

(32)

since both the diagonals = \( \sqrt{2} \ s \). The same result is obtained using Formula (28)

\[ u(\text{square}) = s \ \cos(\pi / 4) \sqrt{\frac{4}{2}} u(\text{point}) = s \ \frac{1}{\sqrt{2}} \ u(\text{point}) = s \ u(\text{point}) \]  

(33)

Example 5: Determine the standard uncertainty of a square property with side length 100 m if each corner point is assumed to have the standard uncertainty 10 mm. Also determine the property price and its standard uncertainty if the price/square metre is SEK\(^2\) 60,000 (inner city of Stockholm).

Solution: The standard uncertainty will be (according to Formula (32) and Formula (33))

\[ u(\text{square}) = 100\cdot0.01 = 1 \text{ square metre} \]

Converted into property price we get \( 0.06 \cdot 100 = \text{SEK} 6 \text{ million} \) with the standard uncertainty 0.06 million. By converting the measurement uncertainty into SEK, it becomes more intuitively understandable: it thus gives a standard uncertainty in the cost of SEK 60,000.

The standard uncertainty for the area determination of a rectangular domain is

\[ u^2(\text{rectangle}) = \frac{u^2(\text{point})}{8} \sum_{i=1}^{4} k_i^2 = \frac{u^2(\text{point})}{8} (4a^2 + 4b^2) = \frac{u^2(\text{point})}{4} (a^2 + b^2) \]

\[ \Leftrightarrow u(\text{rectangle}) = u(\text{point}) \sqrt{\frac{a^2 + b^2}{2}} \]  

(35)

where \( a \) and \( b \) respectively are the two side lengths. The same formula can be shown to also apply to a rhomb.

2.2. Arbitrary polygons
For an arbitrary polygon, we apply the approximate formula

\[ u(\text{area}) = s \ u(\text{point}) \ \frac{\sqrt{n}}{2} \]  

(36)

---

\(^2\) 100 SEK = 11 EUR = 14 USD
Control: If we insert \( n = 4 \) into Formula (17), we get the expression for a square
\[
u(square) = s \ u(point) \frac{\sqrt{4}}{2} = s \ u(point)
\]
i.e. Formula (32) and Formula (33).

The derivations above indicate that the sides – as well as the standard uncertainties – should be handled in a quadratic way. If the sides are of different lengths and/or if the standard uncertainties for the vertices vary in size, it is therefore natural to first calculate the root mean square values (RMS)
\[
\bar{s} = \sqrt{\frac{\sum_{i=1}^{n} s_i^2}{n}} = \sqrt{\frac{s_1^2 + s_2^2 + ... + s_n^2}{n}} \tag{37}
\]
and
\[
\bar{u}(point) = \sqrt{\frac{\sum_{i=1}^{n} u_i^2(point)}{n}} = \sqrt{\frac{u_1^2(point) + u_2^2(point) + ... + u_n^2(point)}{n}} \tag{38}
\]
and insert these into Formula (36). This usually provides a fairly good approximation of the area’s standard uncertainty – often better than if we, instead, would have used the arithmetic (ordinary) mean values.

The Formulas (37) and (38), are primarily applied for educational reasons, i.e. to illustrate the “quadratic behaviour” of both side lengths and standard uncertainties. Using a computer programme, one can of course directly apply Formula (23). Either way, rough calculations using the Formulas (36)-(38) normally give a satisfactory result.

Example 6: Calculate the approximate standard uncertainty for the area of the irregular decagon below.

<table>
<thead>
<tr>
<th>Vertices</th>
<th>Point</th>
<th>N [m]</th>
<th>E [m]</th>
<th>( u(point) ) [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>120</td>
<td>100</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>180</td>
<td>100</td>
<td>3</td>
</tr>
<tr>
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<tr>
<td></td>
<td>10</td>
<td>120</td>
<td>140</td>
<td>1</td>
</tr>
</tbody>
</table>
Solution: Using Formula (37) and Formula (38) we get

\[ \bar{s} = 52.915 \text{ m} \]

\[ u(point) = 9.5 \text{ mm} \]

Formula (36) then gives the approximation

\[ u(area) = 52.915 \cdot 0.0095 \cdot \frac{\sqrt{10}}{2} = 0.79 \text{ square metres} \]

The correct value is 0.85 square metres. The area is 1 ha.

2.3. Area differences

The difference between two area determinations is interesting to study for several reasons. Firstly, it can be done in order to verify that area data is correct, and secondly it can aim to establish whether the area – of, for example, coniferous forest, desert, etc. – has increased or decreased.

For the difference between two independent area determinations, we get, from Formula (7), the standard uncertainty

\[ u(A_1 - A_2) = \sqrt{u^2(A_1) + u^2(A_2)} \] (39)

or, if \( u(A_1) = u(A_2) = u(A) \)

\[ u(A_1 - A_2) = \sqrt{2} u(A) \] (40)

according to Formula (6). A tolerance \( T \) for the difference can then be calculated from Formula (12)

\[ T_{95}(A_1 - A_2) = U_{95}(A_1 - A_2) = k_{95} u(A_1 - A_2) = 2 u(A_1 - A_2) \] (41)

if we apply GUM standard and set the coverage probability at 95 % and the coverage factor at 2. It gives

\[ T_{95}(A_1 - A_2) = 2\sqrt{2} u(A) \] (42)

if Formula (40) applies.

Example 7: In connection with the sale of a property, an area of 3.00 ha was determined. During an inspection measurement afterwards, the area was measured at 2.97 ha. Is the difference considered to be significant, i.e., does the farmer risk reprisals for having submitted false information? After all, the difference is 0.03 ha = 300 square metres.

The area has 18 breakpoints and an average side length of 60 metres (RMS, Formula 37). The breakpoints’ positions are assumed to have a horizontal standard uncertainty \( u(point) = 1 \) metre.
Solution: The standard uncertainty becomes (according to Formula (36))

\[ u(\text{area}) = 60 \cdot 1 \cdot \frac{\sqrt{18}}{2} = 127 \text{ square metres} \]

i.e., the expanded uncertainty of the difference between the two determinations of the area becomes \( 2\sqrt{2} \cdot 127 = \pm 360 \text{ square metres} \) at 95% confidence level.

Since the measured difference is less than this value, it is not significant, so the farmer does not need to be worried. Moreover, the coverage factor of 2 is probably the lowest value we can imagine; the value of 3 is just as common when setting tolerances. With the given conditions, it is only for differences of over 400-500 square metres (0.04-0.05 ha) that we can justifiably suspect cheating. The uncertainty is thus on the level of an ordinary small garden plot.

Note that the conditions must be the same in both the area determinations. This applies, for example, to the geometric relationships in the form of coordinate systems and map projection systems (see next section), but also thematic/cartographic aspects such as object definitions and generalisation. The latter is addressed in Section 4, and can provide additional uncertainties beyond that in the measurement itself.
3. Errors due to elevation, map projection, and inclination

This section addresses errors due to elevation, map projection, and inclination respectively, as well as the use of digital terrain models in connection with area determination.

3.1. Overview

Using the “Surveyor's Area Formula” (20), the area is calculated from coordinates in a plane coordinate system. This means, among other things, that the coordinates have first been reduced down to the reference ellipsoid and then projected out to the projection plane. The deviation between the real and the obtained area depends on the choice of ellipsoid and projection method.

If the “true” area is important to recreate – e.g. in connection with precision setting-out – we must therefore take into account:

- errors due to elevation and
- errors due to map projection.

If the surface area to be determined is not flat, we must also correct for:

- errors due to inclination or use a
- Digital Elevation Model (DEM).

In Sweden, we currently have SWEREF 99 as the national system, in which the ellipsoid GRS80 and Gauss' conformal projection Transversal Mercator (TM) are used [5].

3.2. Error due to elevation

The elevation reduction of a distance \( l \) (1D) is obtained from Figure 4 as

\[
\ell_{\text{ellipsoid}} = \frac{R}{R + h} \ell
\]  

(43)

where \( R \) is the radius of the earth and \( h \) is the elevation. If we want to re-calculate the real distance from the distance determined with the use of coordinates, this requires a correction with the factor

\[
\text{lengthcorr}_{\text{elevation}} = \frac{R + h}{R} = 1 + \frac{h}{R}
\]  

(44)

i.e. a magnification.

As a surface in space is two-dimensional (2D), the area correction factor becomes approximately

\[
\text{areacorr}_{\text{elevation}} \approx (1 + \frac{h}{R})^2 \approx 1 + \frac{2h}{R}
\]  

(45)

which means that also the area determined from coordinates is too small and has to be magnified. Otherwise we get a relative area error due to elevation of the magnitude \( 2h/R \).

A little more stringently, \( R \) is the curvature radius of the reference ellipsoid (\( \approx 6390 \) km in Sweden) and \( h \) is the elevation above this ellipsoid. I.e. \( h = H + N \), where \( H \) is the elevation above the geoid and \( N \) is the geoid height.

In Sweden, \( N \) is in the range of 20-37 m, which means that we get an extra area error of a negligible 3-6 ppm (a few square centimetres per hectare) if we do not take into account the geoid height and put \( h = H \).

---

3 Note that we use the term “error” in this context, not “uncertainty”. The reason for that is that the type of “areal defects” described here are purely geometrical/mathematical – with a known true value, and, thus, a known error.
3.3. Error due to map projection

When we transform a distance or an area from the ellipsoid to the projection plane, a *map projection error* occurs. This section presents elements – as a very good approximation – of the Swedish conditions, i.e., SWEREF 99 and Gauss projection (TM).

*Map projection correction* of a distance on the ellipsoid ($l_{\text{ellipsoid}}$) is performed according to

$$l_{\text{map projection}} = (1 + \frac{E^2}{2R^2}) l_{\text{ellipsoid}}$$

(46)

where $E$ is the Easting coordinate (without any additions) and $R$ is the radius of curvature of the ellipsoid. This results in positive, relative errors as described in upper curve in Figure (5).

If we, also here, want to re-calculate the real distance, this requires a “reverse correction” of

$$\text{lengthcorr}_{\text{map projection}} = \frac{1}{(1 + \frac{E^2}{2R^2})} \approx 1 - \frac{E^2}{2R^2}$$

(47)

and the corresponding area correction factor becomes approximately

$$\text{areacorr}_{\text{map projection}} \approx (1 - \frac{E^2}{2R^2})^2 \approx 1 - \frac{E^2}{R^2}$$

(48)

i.e. the *relative area error due to map projection* (before re-calculation) is $\approx E^2 / R^2$.

Figure 4. Elevation correction of a distance, i.e., reduction down to the ellipsoid.

Figure 5. Relative map projection errors in distances for different E coordinates. 
In the upper curve, the scale factor is $k_s = 1$, and in the lower $k_s = 0.9996$. 
The reverse corrections for errors due to elevation and map projection respectively, as seen, go in different directions. The “de-projection” decreases the area and the reverse elevation correction increases it. The expression

\[
(1 + \frac{2h}{R})(1 - \frac{E^2}{R^2}) = 1 \iff \frac{2h}{R} = \frac{E^2}{R^2} \iff 2Rh = E^2 \iff |E| = \sqrt{2Rh} \text{ km}
\]  

(49)

represents the equation for when the two corrections cancel each other out. The left curve in Figure 6 shows that this is true for e.g. \(h = 0.2 \text{ km } (200 \text{ m})\), \(E = \pm 50 \text{ km}\) (and \(R = 6390 \text{ km}\)). If \(h\) is given in metres, Formula (49) can – with the Swedish conditions – be written

\[
|E| \approx 3.57\sqrt{h} \text{ km}
\]

(50)

The above formulas assume that the map projection uses the scale factor (the magnification factor along the central meridian) \(k_o = 1\), which, for example, the local projection zones in SWEREF 99 do.

\[\text{Figure 6. The figure shows when the product of the elevation and map projection corrections} = 1. \text{ In the left curve the scale factor is } k_o = 1 \text{ and in the right } k_o = 0.9996.\]

However, in SWEREF 99 TM, \(k_o = 0.9996\), which means that the map projection errors will be both negative and positive (see the lower curve in Figure 5) and that the maximum error becomes smaller.

The map projection errors then become 0 (zero) at \(|E| \approx 180 \text{ km}\) and the reverse corrections become

\[
\text{lengthcorr}_{\text{map projection}} = \frac{1}{k_o(1 + \frac{E^2}{2R^2k_o^2})} \approx \frac{1}{k_o} \left(1 - \frac{E^2}{2R^2k_o^2}\right)
\]

(51)

and

\[
\text{areacorr}_{\text{map projection}} \approx \frac{1}{k_o^2} \left(1 - \frac{E^2}{2R^2k_o^2}\right) \approx \frac{1}{k_o^2} \left(1 - \frac{E^2}{R^2k_o^2}\right)
\]

(52)
Formula (49) in turn becomes

\[
(1 + \frac{2h}{R})(1 - \frac{E^2}{R^2k_o^2}) = k_o^2 \quad \iff \quad 1 + \frac{2h}{R} - \frac{E^2}{R^2k_o^2} = k_o^2
\]

\[
E^2 = R^2k_o^2(1 - k_o^2 + \frac{2h}{R}) \quad \iff \quad |E| = Rk_o\sqrt{1 - k_o^2 + \frac{2h}{R}}
\] (53)

With \( h \) in metres and with Swedish conditions we get (cf. Formula (50))

\[
|E| \approx 3.57\sqrt{2560 + h} \text{ km (right curve in Figure 6)}
\] (54)

Figure 6 shows that for \( k_o = 1 \), there are combinations of elevations and E-coordinates (distance from the central meridian) that give the correct area scale. But for \( k_o = 0.9996 \), in e.g. SWEREF 99 TM, this only applies for E-coordinates whose absolute value (without additions) is just over 180 km – i.e. the Easting value at which the map projection error becomes equal to 0 (zero).

Why should we then make elevation and map projection corrections when it can sometimes be quite wrong for area calculations? It is because it is impossible to create unambiguity in any other way than to make different types of representations of the real terrain surface, on a spherical earth. We could, instead, insert the real area as an attribute in the database, but then the objects must be standardised, such as properties. The Real Property Register in Sweden has this solution – that the area is an attribute – but instead we then get a "gap" between register and map/geometry in terms of areas.

Example 8: What will be the total correction factor for areas along the central meridian at the elevation 400 metres when the scale factor is 0.9996?

Solution: Thus we have \( E = 0 \text{ km} \), \( h = 0.4 \text{ km (400 m)} \) and \( k_o = 0.9996 \). Formula (45) and Formula (52) then give the total correction factor

\[
totalcorr = (1 + \frac{2h}{R})\frac{1}{k_o^2}(1 - \frac{E^2}{R^2k_o^2}) = 1.000125 \cdot 1.000800 \cdot 1 = 1.000925
\]

i.e. an error of 925 ppm or approx. 10 square metres/hectare. Compare this with Example 5 where the standard uncertainty (the statistical uncertainty) was only 1 square metre for one hectare.

3.4. Error due to inclination

The projection of a flat surface inclined at an angle of \( \alpha \) from the horizontal plane has the area (see Figure 7)

\[
area_{proj} = area_{real} \cos(\alpha)
\] (55)

I.e. if we want the real area – on the ground – we must impose the inclination correction

\[
areacorr_{declination} = \frac{1}{\cos(\alpha)}
\] (56)

which completely derails for vertical surfaces (slope \( \alpha = 90^\circ \) ) and which becomes = 1 for horizontal surfaces. If the surface is not really flat, but has a few breakpoints along the way, it is of course also possible to divide it into sections, each of which is calculated in this way. Then we are heading towards digital terrain models, see Section 3.5.
Example 9: Determine the projected area of a steep ski slope, which is 50 m wide, 1,000 m long and inclines at 35 (old) degrees.

Solution: The real area – without projection – is of courses \(50 \times 1000 = 50,000\) square metres. Formula (55) gives: \(\text{area}_{\text{proj}} = \text{area}_{\text{real}} \cos(\alpha) = 50,000 \cos(35^\circ) = 40,958\) square metres. The real area of the land is thus 22 % larger than the projected – something to consider if you have the task of sowing grass seed on the slope.

3.5. Digital Elevation Models
A digital database consisting of elevation data is usually called a DEM (Digital Elevation Model). A DEM can be stored either as vectors or as a raster. In cases where the surface is represented by a DEM, point density (resolution) and the representation method are, among other elements, crucial for the area determination of the surface.

A DEM is primarily for presenting elevation conditions and terrain features as well as for performing mass calculations, see Figure 8. A detailed description can be found in, for example, reference [6]. DEM’s can however also be used for accurate area calculations, of the real terrain surface! This means that also the terrain undulation is taken into consideration – which, in itself, might be considered as an additional error source.

Contour lines
In the transition from analogue to digital mapping, drawn contour lines constituted a natural basis for constructing elevation models. Today, the contour lines as such can in some cases act as a good illustration aid, but they do not represent an elevation model in the strict sense. Figure 9 presents the digital storage of analogue contour lines in both vector and raster form, where the effect of different point densities or pixel sizes is clearly shown.
**Figure 9.** A Digital Elevation Model of a terrain surface represented as analogue data (a), vector data (b), and raster data (c).

**Elevation model in vector form**

When elevation data is stored as vectors, the representation form of a *Triangulated Irregular Network, TIN*, is standard. Figure 10 shows a section of a DEM stored as a TIN. Each point with a known elevation represents a corner point of two or more triangles. This means that the elevation values are in the triangle vertices, which are chosen so that all triangles together depict the terrain as well as possible. With a simple supplement to the calculation algorithm, the surface area of each triangle can also be calculated, and summed up to a total area.

*Heron’s Formula* for the area of a plane triangle with an arbitrary gradient reads

\[ A = \sqrt{p(p-a)(p-b)(p-c)} \]  \hspace{1cm} (57)

where \(a, b, c\) are the side lengths and \(p = \frac{1}{2}(a + b + c)\).

**Figure 10.** Digital Elevation Model (DEM) in the form of a Triangulated Irregular Network (TIN). Raster density in each triangle is proportional to the steepness.

If we have both the projected area and the real surface area (according to Formula (57)), the angle of inclination \(\alpha\) can be calculated “back-wards” from Formula (55)

\[ \alpha = \arccos\left(\frac{\text{area}_{\text{proj}}}{\text{area}_{\text{real}}}\right) \]  \hspace{1cm} (58)

By doing this for each triangle, a slope map can be generated, cf. Figure 15.

**Elevation model in raster form**

Elevation data in raster form is often generated by interpolating point data or line data to a raster-based DEM. Each cell in the raster shows the elevation of the midpoint of the surface that the cell represents [6], see Figure 11.
By studying Figure 11, we realise how important the point density of the data is for the elevation model's ability to reflect the topography. The elevation in the centre of the surface, as is evident from the contour lines in the left image, cannot be discerned in the right image where the cell values represent the elevation of the corresponding cell's midpoint.

Raster data has, due to its regular form, a higher uncertainty (lower accuracy) than corresponding data stored as vector data (TIN). The best representation of a surface shall include the entire surface, no more no less. Raster data does not meet this criterion, except in exceptional cases, as it is seldom that a surface, with its typically irregular form, coincides with the raster data's regular grid (see Section 4.6 below).

If the data is stored is raster form, the area is best calculated by dividing up each cell into several triangles, which are then treated using Heron's formula (Formula (57)). One method that several computer programmes use is to, for each cell, generate eight three-dimensional triangles that are considered to represent the cell's area in a better way than the plane (projected) cell [7].

Suppose we have access to a DEM in raster form, see Figure 12a. To describe the method, we will calculate the area of the cell with elevation value 138. The area is calculated using the elevation value of said cell and the eight adjacent cells' elevation values. These nine cells are also illustrated three-dimensionally in Figure 12b as a set of blocks where the height of the blocks represent each cell's elevation.
We generate the eight triangles by connecting the midpoint of the cell in question (in this case, the cell with the elevation value 138 in Figure 12a) with the midpoints of the eight adjacent cells. All the side lengths in all of these triangles are calculated as the hypotenuse (c) from the Pythagorean Theorem (see also Figure 13)

\[ c = \sqrt{a^2 + b^2} \]  

(59)

where \( a \) is the horizontal distance between P (the midpoint of the cell in question) and Q (the midpoint of the adjacent cell; eight of them), \( b \) is the height difference between P and Q, and \( c \) is the slope distance between P and Q.

The distance \( a \) is easy to calculate because it is either the side length of the cell or \( a=\sqrt{2} \) sidelenth for distances in a diagonal line. The distance \( b \) is the difference between the two cell values (the height difference).

In this way, we can calculate all the side lengths of the eight triangles. The combined surface of these triangles however extends outside the cell in question (with elevation value 138) and therefore represents an area larger than the cell's corresponding size (triangles I-VIII in Figure 14). The triangles must therefore be “trimmed” in to the cell's boundary line.

We do this by halving all side lengths that we have obtained from Formula (59). We thus obtain triangles (i-viii; Figure 14) which only extend within the cell's surface. The area of each triangle is then calculated from Heron's formula (Formula (57)). Finally, we add together the areas of the eight triangles to obtain the total cell area.

Example 10: Determine the real area, using the method described, for the selected cell in Figure 12 with the elevation value 13, if the cells' side length is 10 m and the specified elevation values are in metres.

Solution: Start by calculating the 16 triangle sides using Formula (59). Trim the triangle sides to obtain the triangles i-viii as shown in Figure 14. Use these triangle sides to obtain each triangle's real area using Formula (57). Finally, add up the eight triangles' areas to obtain the real area of the cell. The real area is 108,0 m². This is 8 m² more than the horizontal (projected) area (10 m \( \cdot \) 10 m = 100 m²).
In addition, customary elevation and map projection correction is performed as per Sections 3.2 and 3.3.

As mentioned above, raster data has a higher level of uncertainty, in the case of, e.g. area determination, than that of corresponding data stored as TIN. Thus we ask ourselves: why do we not always store elevation data in the form of a TIN? The answer is that there are many other advantages with raster data since it, among other things, allows for the use of all the digital image processing's filtering algorithms, see the example in Figure 15. It is however outside the scope of this report to elaborate on these aspects. The next section provides, inter alia, a comprehensive description of raster data's properties, which DEM's in raster form thereby also possess.

\[ \text{Figure 15. Different areas of application for a Digital Elevation Model; hill shading (a); slope (b) [8].} \]
4. Other “areal defects”

The following section involves an argument-based (philosophical) description of other “areal defects” that affect the uncertainty in area determinations.

4.1. Geodata quality

The geometric uncertainty – positional uncertainty – is often the first thing we think of when we talk about data quality. But there are other equally important measures of quality. Before going into these, we will explore the concept of “data quality” in connection with geodata, i.e., geodata quality [4], [9].

The basis for the assessment of quality in databases is the database (or data product) specification. This is drawn up for a particular database and application and includes a description of the database’s content, level of detail, quality, etc. It also defines object classes and data models, along with how the data capture will be achieved.

Data quality can be defined as “the degree of consistency between the database’s overall, actual quality and the intentions set out in the database specification.” The quality assessment can either be done at the object level or at the aggregate level, i.e., at a detailed level or referring to all or large parts of the dataset.

One of the primary purposes of geographic data is to provide the opportunity for co-processing and analysis of data from various databases. Quality measures connected to output data are therefore needed to determine which analyses are possible, and the level of quality the end products can be expected to have. The need for a quality report is accentuated further by the fact that future users and applications are not fully known today.

4.2. From reality to database

Geodata’s route from reality to database and further on to application is not straightforward, see Table 2. The data capture is performed in the real world – but in the way that the database specification indicates.

This means that we must see the world through a “filter”, through which we get a form of nominal world – an abstraction of the real, which is simpler and more tailored to the needs. Data is then structured via a data model and is physically stored in a database. From the database, selection/retrieval can be done in the form of a geodata product.

Table 2. Data capture is the link between reality and database [10].

<table>
<thead>
<tr>
<th>Entities</th>
<th>Objects</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real world</td>
<td>Nominal world</td>
</tr>
<tr>
<td>Unstructured</td>
<td>As per database specification</td>
</tr>
</tbody>
</table>

During the data capture, the data quality is affected by, inter alia, the foundation (control network, maps, arial photographs etc.), the technique, the method, and the observer. In terrestrial surveying, the “foundation” is equal to “the nominal world”, which the observer has to interpret. In the case of automatic interpretation methods, the observer is fully or partially disconnected.

4.3. Quality themes

Obviously, there is a great need for clear information on the content and quality of databases. For such data, often stored in separate databases, the term metadata is used. The quality themes described in ISO 19157 [11] are usability, purpose, origin, completeness, positional accuracy, thematic accuracy, logical consistency and temporal accuracy.
Usually, the quality themes are divided up into two groups depending on whether the content is quantitatively measurable or not. Usability, purpose and origin refer to a linguistic description of data and are therefore qualitative rather than quantitative [12].

Below, a selection of the different quality themes are described:

- **Data's origin** documents how data has been collected, the supporting material that has been used, when the data collection was performed and which organisation is responsible.

- **Completeness** refers to the consistency of the information's content with the database specification. The theme describes the quantity of information specified, in relation to reality. Both *commission* (excess items) and *omission* (missing items) can occur.

- The *positional accuracy* (positional uncertainty) in geometric data is presented in the form of *standard errors* (standard uncertainty).

- **Thematic accuracy** (thematic uncertainty) indicates how correct the specified attribute information for an object is, e.g. *classification accuracy*. Not to be confused with completeness.

- **Logical consistency** refers to data's consistency with the logical rules that have been established in the database specification. This means, for example, that the relationships between objects in the database are correctly presented, e.g. that networks are connected and that surfaces are closed.

- **Temporal accuracy** (temporal uncertainty) describes the temporal consistency and the uncertainty in methodology for documentation of temporal data.

- For **attribute data**, a separate quality description may be justified. This consists of several components, such as attribute data's origin, uncertainty, accuracy, and completeness.

A further measure of quality that is sometimes used is **up-to-dateness**. This is not described in ISO 19157 as a quality theme, however, it is often associated with temporal accuracy. Up-to-dateness is declared through dating; more specifically the date the object was last noted as being correctly reported (last updated). The quality requirements represented in the different quality themes are imposed relative to a database specification, or a specification from an imaginary user. For example, a tourist map of an urban area does not impose the same requirements as a base map for a municipality.

### 4.4. Vagueness and classification accuracy

The difficulty in describing classes is usually called *vagueness* [13]. Vagueness occurs due to a lack of distinction between different (hard to define) classes or individual objects. Vagueness occurs when that which we measure is not well defined. Another phenomenon that has great impact on uncertainty during area determination is classification accuracy or **thematic uncertainty**. With thematic uncertainty, the classes are correctly defined – and well defined – but we have an uncertainty in what we are measuring. The “regular” measurement uncertainty could be analogously termed **geometric uncertainty**.

An example is as follows. If we want to determine the area of a coniferous forest, we have to find the border between deciduous and coniferous forest, which may be interpreted in different ways. But we also need to have clear definitions of what a forest is (e.g. how tall trees are), and how pure the forest must be, for example, the proportion of deciduous trees permitted in a coniferous forest (one birch, two birches, etc.).

This shows in part that the thematic uncertainty is sometimes considerably greater than the geometric, see Figure 16. This applies both to terrestrial surveying and to **remote sensing**, i.e. measurement in aerial photographs, satellite imagery interpretation, etc. The observer must in both cases make a decision, which then usually becomes subjective.
Figure 16. Different ways to delineate forest clumps ([13], page. 97).

An opposite example is *property* which – at least in Sweden – is 100% correctly demarcated on the ground (boundary marks apply). Therefore, it is only the measuring process that is associated with some uncertainty – measurement uncertainty or geometric uncertainty.

Often it can be difficult to distinguish between the thematic (qualitative) and the geometric (quantitative) measurement uncertainty. Or as Goodchild in reference [14] asks: “Correct attribute at wrong location or wrong attribute at correct location?” and “Accuracy of position or accuracy of attributes?” It is uncertainty, quite simply.

Sometimes vagueness is due to shortcomings in the database specification's definitions, which can be too narrow, overlapping between object classes or generally unclear. In the case of digitisation, there is also the problem of cartographic *generalisation*, i.e. that objects have been displaced from their proper position in order to increase the readability of the map.

4.5. **Vector data**

Vector data can consist of points, lines or polygons (surfaces). Line vectors are stored as coordinate pairs that represent start points, breakpoints and end points of the line (see Figure 9). To represent a polygon (surface), one stores the coordinates of the lines that encompass the surface.

Since a line consisting of points, a cartographic simplification is (almost) always performed where one tries to reproduce the shape of a surface in vector representation. For a property, it is true that the boundary lines are straight lines, but this is rarely the case for other surfaces. This problem is related to cartographic generalisation, but also applies to the database and data capture level.

4.6. **Raster data**

*Direct classification uncertainty* is due to the true surface having uncertainties in the class definition or being “fuzzy” in the outlines, e.g. “urban area”, “forest/pastureland”, etc. *Indirect classification uncertainty* is due to the model, or representation method, not being able to reproduce reality perfectly.

*Raster data* appears thusly to be the most important model to study in terms of uncertainty in area determination. Both the data capture and the storage form can be raster-based.

Within remote sensing, one mainly talks about three types of raster data – or *image data*:

- *Panchromatic* images, a picture taken of the entire visible wavelength range as well as part of the infrared. Panchromatic images and a DEM are both examples of a continuous raster image.

- *Multispectral* images, with multiple image planes registered in different wavelength bands (e.g. aerial photographs in colour or infrared images).

- *Classified images*, which have undergone a classification and are presented on a nominal scale (e.g. vegetation maps or "land cover"). The classification is normally based on an analysis of multispectral images and something called spectral signatures, see Section 4.7.
The most common method for area determination using raster data is simply to count the number of *pixels* (picture elements) needed to cover the surface in question, and then multiply this number by the pixel size. Regardless of the type described above, there will be two more (indirect) uncertainty factors: geometric and thematic *representativeness*. These are intimately linked to this storage method and enhance the geometric and thematic measurement uncertainty (vagueness) as discussed in the previous section. The effect of both sources of uncertainty is reduced when the pixel size is reduced.

**Geometric representativeness**

The problem with raster data's geometric representativeness is illustrated in Figure 17. The area to be determined is represented in a rougher manner and will be different depending on how the boundary line lies in relation to the raster grid. The uncertainty in an area determination is naturally affected by this.

Figure 17 can also be interpreted as a *transformation*, i.e. change of coordinate system. It is clear that rotations, for example, affect the representation – and therefore the size of the area; the same applies to small lateral displacements (*translations*). One therefore should not be surprised if one gets slightly different areas before and after such an operation.

![Figure 17](image)

**Thematic representativeness**

The thematic representativeness has to do with each pixel only being able to assume one single value. This value (e.g. a grey tone value in a panchromatic image, an elevation in a raster-stored DEM, a type of land in a land cover map, etc.) should be representative of the area that the pixel covers. This of course becomes more difficult the larger the pixels are.

However, there is also an advantage with raster storage associated with area determination, which normally comes to the fore in the case of classified images. Since each pixel is processed separately, the areas do not need to be either large or connected. Small “islands” of a deviating object type can very well be interspersed here and there, see Figure 18.

![Figure 18](image)
4.7. Fuzzy Logic

The problem with each pixel only being able to have one value, or one object class, can be reduced using something that is usually called Fuzzy Logic.

An example. Usually, vegetation boundaries are not especially sharp. They often look like the left image in Figure 19, i.e. we have mixed vegetation in the boundary area. One solution can be to introduce the object class “Mixed forest” – another way is to resort to fuzzy logic and deviate from the principle that a pixel may only have one value, “zero” or “one” (deciduous or coniferous).

![Figure 19. Fuzzy logic in the transition between coniferous and deciduous forest; b) illustrates the possible values in the profile selected in a).](image)

In the mixed forest belt, we thus make a sliding transition (e.g. linear) from deciduous to coniferous forest and let the pixels there have a value between zero and one, see Figure 20. We then add up the pixels – with their decimal values – to determine the area.

Example 10: Determine the areas of coniferous and deciduous forest in Figure 20 if we assume that the pixel size is 4 ares (20 \* 20 metres).

Solution: The total area then becomes \( 6.7 \times 4 = 168 \) ares, or 1.68 hectares. Coniferous forest occupies 20.7 pixels, i.e. 82.8 ares or 0.83 hectares. The deciduous forest area thus becomes \( 168 - 82.8 = 85.2 \) ares or 0.85 hectares

![Figure 20. Fuzzy logic with decimal pixel values in the mixed forest, cf. Figure 19. A one indicates pure coniferous forest, a zero represents pure deciduous forest and the decimal values represent the proportion of coniferous trees in the mixed forest.](image)

In connection with remote sensing, the reflection from the ground is affected by the angle of incidence of the sunlight, atmospheric disturbances, but above all by the different objects' properties, such as colour, structure and moisture conditions. Therefore, different objects have their own spectral signature [6].

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This property is also called *spectral reflectance* and is stated as a percentage of the amount of incident light for each wavelength band. E.g. plants' chlorophyll produces a large effect in the infrared band.

In the interpretation of satellite imagery, a pixel – via its signature – is classified with a certain probability, which can be determined via *training surfaces*. These surfaces constitute a sort of “ground truth” from field visits and various types of measurements in the terrain. The probabilities can then be part of a treatment with fuzzy logic, for example to determine the areas of different types of land. In these applications, it is often the case of more than two options, e.g., “Which is the most likely tree species – birch, oak, pine?”

4.8. **Monte Carlo simulation**

Monte Carlo simulation can be an alternative to Fuzzy Logic. Using a random number generator, a large number of artificial images are generated based on the fuzzy logic's probabilities. Here, however, we intend to present two other areas of application for this method.

The first is “graphical representation”, i.e. the different simulations' results are graphically presented – drawn on top of each other. The purpose with this is to visualise the area uncertainty, which appears as a wide border around the area-determined surface. Figure 21 shows that this provides a more intuitive feeling than simply an uncertainty measure.

![Figure 21. Monte Carlo simulation. 20 simulations. Normally distributed point errors with standard deviation of 10 metres/point. Nominal area is 100·100 metres, i.e. 1 ha.](image)

The second application is to determine confidence intervals for calculated areas. The histogram (see Figure 22) is the result from 10,000 simulations of the area in Figure 21. As seen, this is consistent with the theoretical normal distribution curve, i.e., areas determined using the Surveyor's Area Formula follow the usual normal distribution. This means that we – at least if the position determinations of the corners can be considered to be uncorrelated – can use the normal distribution's coverage factors. (The normal distribution can also be shown analytically.)

For example, the range \( \pm 2 \, u(area) \) gives a coverage probability of almost exactly 95 % – and, at the same time, we adhere to GUM practice if we use a coverage factor of two (2).
Figure 22. Monte Carlo simulation. 10,000 simulations of the area in Figure 21. Normally distributed point errors with standard deviation of 10 metres/point. Empirical histogram compared with the theoretical normal distribution curve.
5. Summary

We hope that readers have gained a better understanding of measurement uncertainty in area determination, as well as a few simple tools for its estimation. We would prefer to see more uncertainty statements in future reports. However, the key insight is probably still that measurement uncertainty is just one of several – often completely dominant – “disruptive factors”.

Here we content ourselves with this insight. Getting to the root of the problem requires its own report, one more comprehensive in nature than this. To obtain a completely reliable estimate of the uncertainty, we probably need to repeat the entire geodata process – inter alia including data collection/measurement, definitions/classification, storage, retrieval/selection and usage.

So, to all GIS users: Do not be blinded by the computers' Double Precision and three decimal places on the metre. The position determination methods do not provide this, and other “defects” mean that the real uncertainty – in, for example, area determinations – is greater than that. You would certainly react and think it remarkable if the TV meteorologist would report temperature forecasts with eight decimal places [15]. Of course, the same applies to geodata.

Moreover, we should remember that the uncertainty does not just contaminate area determinations – many other GIS analyses are also affected. Every analysis that does not take into account the uncertainty in geodata – both the geometric uncertainty and other “defects” – may greatly limit the usefulness of the analysis. Even if decision-makers and GIS users are not interested in data quality per se, there should be an awareness regarding uncertainty in geodata in order to be able to evaluate the reliability of the analyses performed and the decisions made based on this data.

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6. References


