Project in Applied Physics:
Brightness optimization in Thomson backscattering processes

Andreas Ekstedt
Uppsala University, Sweden

Supervisor: Vitaliy Goryashko
Department of Physics and Astronomy, Uppsala University, Sweden
Abstract  The generation of high-brightness femtosecond x-ray pulses enables exploration of so far largely unexplored areas of atomic physics, and also enables high contrast images of biological tissue. The use of Thomson backscattering with high energy electrons provides a new way to produce high brightness x-ray pulses, which is considerably cheaper than other techniques. We present a general description of Thomson scattering and take into account laser and electron focus effects. We also consider the effect of energy spread within the electron bunches and consider collisions at arbitrary collision angles. We also investigate flattening and chirped laser pulses, the consequent effect on the brightness and total number of scattering events. The optimization is then performed with respect to electron bunch energy and energy spread; laser and electron focus parameters; limitation of bandwidth within the laser pulse. The optimization of the brightness is subsequently performed with the help of a genetic algorithm technique.

There is one simplification at least. Electrons behave ... in exactly the same way as photons; they are both screwy, but in exactly in the same way... — Richard P. Feynman—
# Contents

1 Introduction to Inverse-Compton scattering  
   1.1 Inverse-Compton scattering  
   1.2 Problem statement  
   1.3 Structure of the article  

2 Background and the FREIA experiment  
   2.1 FREIA experiment  
   2.2 Introduction to THz  
   2.3 Optical enhancement cavity  

3 Field description of Thompson Scattering  
   3.1 Scattering rate  
   3.2 Differential cross section in electron rest frame  
      Cross section  
      Differential cross section in terms of incoming polarization vector  
      Transformation between coordinate systems  
      Stationary coordinate system  
      Laser frame  
      Electron lab frame  
      Rotation matrices  
      Lorentz boost  
      Transformation of angles  
      Invariant quantities  
   3.4 Differential cross section in the stationary frame  
      Electric and Magnetic fields  

4 Scattering rate  
   4.1 Scattering density  
   4.2 Electron and Photon densities  
      Photon density  
      Electron density  
   4.3 Scattering rate  

5 Head-on Brightness  
   5.1 Laser pulse frequency spread  
   5.2 Energy spread  
   5.3 Emittance contribution
5.4 Peak on-axis brightness ........................................ 34

6 Non Head-on collision ........................................... 37
6.1 Nearly Head-on collision ...................................... 37
   Photon density .................................................. 39
6.2 Laser and electron focusing effects ......................... 42
   On-axis peak spectral brightness ............................ 42
   Laser focusing effects ........................................ 43
   Electron beam focus effects ................................ 44
6.3 Scattered frequency spread .................................. 44
6.4 General collision angle ....................................... 45

7 More general laser pulse ....................................... 47
7.1 Flattened laser pulse ......................................... 47
7.2 Chirped laser pulse ........................................... 49

8 Result ................................................................... 51
8.1 Experimental parameters ................................. 51
8.2 Simulation result ............................................... 53
8.3 Optimization ..................................................... 55

9 Discussion .......................................................... 57
9.1 Laser pulse focal radius ....................................... 57
9.2 Laser pulse duration ........................................... 59
9.3 Electron bunch duration ..................................... 60
9.4 Energy spread .................................................... 60
9.5 Angular dependence .......................................... 61
9.6 Flattening of laser pulse ...................................... 61

10 Conclusion .......................................................... 62

A Quantum electrodynamics corrections .................. 63
   A.1 Klein–Nishina formula .................................... 63
   A.2 Wave vector relation ....................................... 65

B Klein–Nishina formula derivation ......................... 67

C Integral table ....................................................... 75
   C.1 Dirac Delta function ....................................... 76

D MATLAB code ...................................................... 76
E References

References
1 Introduction to Inverse-Compton scattering

There is a large necessity for x-ray beams that have both a very limited bandwidth and a high intensity. The applications for such an x-ray source range from atomic physics to medical research and technology. In medicine there is a demand for high intensity x-ray sources that can produce radiation in the so called water window\(^1\), since radiation in this wavelength range does not interact with water and are thus not absorbed by the water in the biological tissue. This results in the possibility of high-contrast images of biological tissue. On the other side of the spectrum there exists a big potential for the use of such x-ray radiation for the study of the inner electron shells of atoms, this would allow insight into the motion of individual electrons on different scales, a largely unexplored territory in science. The most common way to generate x-rays is by the means of a synchrotron source. However, synchrotron facilities are quite expensive and only a handful exist worldwide.

1.1 Inverse-Compton scattering

The utilization of inverse-Compton scattering is an alternative to synchrotron radiation, as inverse-Compton sources provide a cheaper way to produce high intensity coherent x-ray radiation. Inverse-Compton scattering is the process of a photon scattering off an electron and gaining energy. So this means that high energy electrons are colliding with photons and in the process transferring energy to the colliding photons. There are plans to build a Compton source at the FREIA laboratory of Uppsala university, in such a source \(10-15\) Mev electron bunches will collide with IR laser pulses to produce X-ray. To achieve a high intensity the IR laser pulses first enter an optical enhancement cavity where the pulses are recirculating thus providing very high intra-cavity power of laser pulses. This technique is considerably cheaper than that of synchrotron radiation sources, as it does not require huge facilities to accelerate the electrons in a circular trajectory.

1.2 Problem statement

In this report we are mainly interested in optimizing the brightness of an inverse-Compton scattering setup, which can be thought of as the laser intensity. We wish to optimize with respect to several different parameters, ranging from the energy spread to the electron bunch size. The main goal of the subsequent sections is to derive a analytic expression for the brightness in terms of the parameters

\(^1\)3.3-4.4 nm
of interest. The dependence on these parameters may then be studied, and possibilities for optimization of the brightness can be considered. We will mainly deal with the classical version of inverse-compton scattering, the so called Thomson scattering, the validity of this choice is then analyzed in section A in the appendix.

1.3 Structure of the article

The subsequent chapters are organized as follows:

- Section 2 will give a short introduction to the planned facility at FREIA, and will give a short introduction to optical enhancement cavities.

- Section 3 will derive the classical Thomson differential cross section in an arbitrary geometry.

- Section 4 will deal with the description of the electron bunches and laser pulses, and will present an expression for the total number of scattered photons.

- Section 5 will present a simplified model of the peak-brightness for the case of a head-on collision.

- Section 6 addresses the case of a non-head on collision for both the simplified model and the more general model used in section 3.

- Section 7 deals with the case of broadened and chirped laser pulses.

- Section 8 will present all the numerical simulations, created with the techniques presented in the earlier sections.

- Section 9 will discuss the result in section 8, and explore the possibilities for maximizing the brightness.

2 Background and the FREIA experiment

In the past there have existed several obstacles for inverse-compton x-ray sources, for instance it was only recently that sufficiently efficient methods were developed to allow for the creation of high intensity x-ray beams. In this section we explore some of the techniques and methods that is required in order for a inverse-compton x-ray source to be viable\(^2\).

\(^2\)See [12]
2.1 FREIA experiment

The FREIA laboratory at Uppsala university, Sweden is currently constructing a combined THz/X-ray source, where the THz-source enables generations of THz-pulses with a bandwidth of 0.01 %, and generations of short pulses with several cycles in duration. The X-ray source will operate by letting IR-laser pulses scatter of high brightness electrons with inverse-Compton scattering. Such an X-ray source would operate from 1-4 keV with output intensity comparable to second generation synchrotron sources\(^3\).

![Diagram of a combined THz/X-ray source]

Figure 1: Schematic of a THz FEL complemented with an X-ray source.

The outline of the combined THz/X-ray source is shown in fig.??, a SC linac is used to accelerate electron bunches to relativistic energies, these are then scattered of IR-laser pulses to produce X-ray radiation. The electron bunches are then pass through an undulator to produce FEL THz radiation.

2.2 Introduction to THz

Photons with energies in the THz spectrum range match many of the excitation in matter, such as low frequency vibrations of molecules, molecular rotations and vibrations, internal excitation of electron-hole pairs. THz radiations also finds applications in biophysics, for instance an application of THz radiation is connected to the study of chiral molecules which are found throughout biology, for instance in ammino acids.

\(^3\)See [12]
2.3 Optical enhancement cavity

An optical enhancement cavity is a resonator in which a laser pulse is overlapped in phase on each turn. This means that by using an optical enhancement cavity it is possible to amplify the laser energy by at least two order of magnitude. This is achieved by forming a high-power pulse within the cavity that is circulating inside the cavity at the same repetition rate as the incoming laser pulses. The power amplification of an optical enhancement cavity is mainly limited by losses within the cavity and frequency dispersion effects. In the past optical enhancement cavities have been limited in their use due to the fact energy transfer from the incident laser to the cavity resulted in damage to the cavity, this effect severely limited the amplification provided by the cavity.

3 Field description of Thompson Scattering

3.1 Scattering rate

The main goals of this section are to first give an introduction on the main quantities of interest and to find an expression for the cross section, $\sigma$, which can be viewed as an effective area that governs the probability for a scattering to occur, a bigger area gives a bigger probability for the process to occur.

The main quantity of interest in this report is the brightness defined as

$$B = \frac{dN_s}{d\omega_s d\Omega dt dA},$$

(3.1)

where $N_s$ is the number of scattered photons, $\omega_s$ is the frequency of the scattered photons and $\Omega$ is the solid angle and $dA$ is a small area element. The brightness can be viewed as an intensity of photons, more photons per area or time gives a bigger brightness. We will begin by finding an expression for the brightness and see what quantities we need to acquire in order to describe it.

We will make the assumption that the frequency is low enough so that it is valid to use the classical formula for scattering, this assumptions means that the cross section, $\sigma$, do not depend on the frequency of the scattered photons, $\omega_s$. If we begin by considering the number of scattered photons, $N'_s$, in the electron rest frame, then the total number of scattered photons would simply be the

---

4We will use the notation of only using one d for derivatives, another common notation is to write $B = \frac{dN_s}{d\omega_s d\Omega dt dA}$.

5This assumption is explored in section A.
integral over the flux of incoming photons times the density of electrons and the Thomson cross section,

\[ N_s' = \int c\sigma n'_e(r', t') n'_\gamma(r', t') d^3r' dt'. \]  (3.2)

We will denote the electron and photon densities in the electron rest system as \( n'_e(r', t') \), \( n'_\gamma(r', t') \) respectively, and \( \sigma \) denotes the total Thomson cross section.

If we want to generalize this to an arbitrary system, we have to introduce the photon four flux, \( \Phi^\mu = c^2 n'_\gamma k^\mu/\omega \), and the electron beam four current\(^6\) \( j^\mu = \rho_0 \gamma(u)(c, u) \) \( = (\rho, J) \) \(^7\).

Where \( k^\mu((k^\mu) = (\omega/c, \vec{k})) \) is the four wave vector and we have defined \( \rho = \rho_0 \gamma(u) \), and \( J = \rho \vec{u} \). From definition we have \( \beta_e = \vec{u}/c \) and the charge density is given by \( \rho = en_e(r, t) \), using this we can rewrite the four current as

\[ j^\mu = ecn_e(1, \beta_e). \]  (3.3)

The total number of scattered photons in an arbitrary system is then given by\(^8\)

\[ N_s = \frac{\sigma}{ce} \int j^\mu \Phi^\mu d^4x \]

\[ = \sigma c \int (1 - \beta_e \cdot \frac{\vec{k} c}{\omega}) n_e(r, t) n_\gamma(r, t) d^4x, \]  (3.4)

and differentiating with respect to solid angle, \( \Omega \), spacetime coordinates, \( d^4x = d(ct)dx dy dz \), and scattering frequency, \( \omega_s \), we obtain the scattering rate per frequency and solid angle as

\[ \frac{dN_s}{d\Omega d\omega_s dx dy dz dt} = \frac{d\sigma}{d\Omega d\omega_s} c(1 - \beta_e \cdot \frac{\vec{k} c}{\omega}) n_\gamma(r_e(t), t) n_e(r_e(t), t). \]  (3.5)

We now restrict ourselves the case when the photon energy is much smaller than the electron rest mass, this means that the electron is approximately at rest both before and after the scattering in its rest frame, and then for all kinetically possible scattering processes we have the same total cross section, \( \sigma \), i.e. the Thomson cross section \( \sigma_T \). From kinematics we know that in the electron rest frame, the energy of the scattered photon is bigger than the energy in any other frame, and this implies that the frequency of the scattered photon has to be bigger than the Doppler shifted frequency,

\[ \omega'_s = g(\theta) \omega_s \leq w_s \]  (3.6)

\(^6\)Where \( \rho_0 \) is the charge density in the rest frame.

\(^7\)[3] p.106

\(^8\)[6] p.2
Since $g(\theta)^9$ is smaller or equal to one. We now use the fact that the Thomson cross section is the same for any frequency, this implies that

$$\sigma \propto H(\omega_s - g(\theta)\omega),$$  

(3.7)

where $H$ is the Heaviside function defined as

$$H(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x < 0 \\
1/2 & \text{if } x = 0 
\end{cases}$$  

(3.8)

Thus we can rewrite equation (3.1) as

$$\frac{dN_s}{dt d\Omega d\omega_s dx dy dz dt} = \frac{d\sigma}{d\Omega} \delta(\omega_s - g(\theta)\omega) c(1 - \beta_e \cdot k/c)n_e(r_e(t), t)n_e(r_e(t), t),$$

(3.9)

where the delta function comes from the fact that the derivative of the Heaviside function is the delta function. In the expression above we see that knowledge about the differential cross section, $\frac{d\sigma}{d\Omega}$, is required in order to describe our process. We will dedicate the remainder of this chapter to find an expression for the differential cross section, $\frac{d\sigma}{d\Omega}$, and will then use the result to describe the brightness in the subsequent sections.

### 3.2 Differential cross section in electron rest frame

In this section we will follow the conventions and method presented in [6] in order to find an expression for the differential cross section, $\frac{d\sigma}{d\Omega}$, we will introduce the relevant coordinate systems and the relevant parameters that will be used throughout reminder of the report. Since Thomson scattering is an electromagnetic process we need to obtain expression for the electric fields and relate them to radiated power. We will start by deriving an expression for the differential cross section in the electron rest frame since the expression has a nice form in this frame, we will then in subsequent subsections derive the transformation of the differential cross section from the rest frame to the electron laboratory frame.

**Cross section**

In order to find the differential cross section we must first find an expression for

---

$^9g(\theta)$, is the Doppler shift factor
the Electric field arising from a photon scattering of an electron. In the electron rest frame we can now using the so called Liénard-Wiechert potentials \(^{10}\) write the electric field for a charged particle as\(^{11}\)

\[
E = \frac{q}{4\pi\varepsilon_0} \left( \hat{n} \times \left[ \hat{n} \times \dot{\beta} \right] \right) \frac{r}{c},
\]

(3.10)

where the \(r\) is distance from the electron to the observation point, \(|x - x'|\), \(\hat{n}\) is the unit vector from the electron to the point we are looking at, and the \(\dot{\beta}\) arises from the fact that the electron is accelerating when it is hit with a photon. The differential cross section is defined as

\[
\frac{d\sigma}{d\Omega} = \frac{dP}{d\Omega} < S > t,
\]

(3.11)

where \(dP/d\Omega\) is the radiated power and \(< S > t\) is the time averaged Poynting vector of the incident radiation\(^{12}\). The Poynting vector, \(S\), can be seen as the energy density of the radiation, and is defined as

\[
S = \varepsilon_0 E \times B = |E|^2 \varepsilon_0 \hat{r}. \]

(3.12)

So in order to obtain the differential cross section we have to calculate the radiated power and then divide by the average emitted energy. To obtain the radiated power we first have to find the absolute value squared of the electric field,

\[
\left\{ \begin{array}{l}
|E|^2 = \left( \frac{q}{4\pi\varepsilon_0} \right)^2 \left| \hat{n} \right| \left| \hat{n} \times \left[ \hat{n} \times \dot{\beta} \right] \right| \frac{r}{c} \\
\left| \hat{n} \times \left[ \hat{n} \times \dot{\beta} \right] \right|^2 = [2] = \left| \hat{n} (\hat{n} \cdot \dot{\beta}) - \dot{\beta} \right|^2 \\
\rightarrow |E|^2 = \left( \frac{q}{4\pi\varepsilon_0 r} \right)^2 \left| \hat{n} (\hat{n} \cdot \dot{\beta}) - \dot{\beta} \right|^2.
\end{array} \right.
\]

(3.13)

Because the electric field is a vector, it must be described with both a magnitude and a direction, the direction of the electric field is called its polarization vector and is defined as

\[
\epsilon = \frac{E}{|E|}.
\]

(3.14)

If we would like to know how much of the radiated power is radiated with respect to a specific polarization we can take the inner product of a specific polarization vector, \(\epsilon\), inside \(\left| \hat{n} (\hat{n} \cdot \dot{\beta}) - \dot{\beta} \right|\) before squaring\(^{13}\), and use the fact

\(^{10}\)The Liénard-Wiechert potentials describe the electric and magnetic field for a particle in an arbitrarily motion.

\(^{11}\)See [1] p.664-665

\(^{12}\)So in effect we are looking for the ratio of the emitted power to the incident power.

\(^{13}\)See [1] Jackson p.665
that \( \hat{\beta} \) must be perpendicular to \( \hat{n} \), to obtain

\[
|E|^2 = \left( \frac{q}{4\pi\varepsilon_0 r} \right)^2 |\epsilon^* \cdot \hat{\beta}|^2.
\] (3.15)

The radiated power can be obtained from the Poynting vector,

\[
\frac{dP}{d\Omega} = S \cdot \hat{r} r^2 = \left( \frac{q}{4\pi\varepsilon_0} \right)^2 \frac{c\varepsilon_0}{|\epsilon^* \cdot \hat{\beta}|^2},
\] (3.16)

Now if the incident radiation has a wave vector \( \mathbf{k}_0 \) the electric field is given by\(^{14}\)

\[
\mathbf{E}(\mathbf{r}, t) = \varepsilon_0 E_0 e^{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega t)},
\] (3.17)

then using the Lorentz force we obtain

\[
\dot{\beta} = \frac{ce}{m} \varepsilon_0 e^{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega t)},
\] (3.18)

Using the time average\(^{15}\) and the definition of the differential cross section,\(^{16}\)

\[
\frac{d\sigma}{d\Omega} = \frac{dp}{d\Omega},
\]

we can write the differential cross section in the electron restframe as

\[
\frac{d\sigma}{d\Omega'} = (r_0)^2 |\epsilon^* \cdot \epsilon_0|^2,
\] (3.19)

where \( r_0 \) is the classical electron radius \( r_0 = \frac{e^2}{4\pi\varepsilon_0 m_c c^2} \), and the ' denotes that we performed this calculation in the electron rest frame and the obtained expression is not valid in any other frame.

**Differential cross section in terms of incoming polarization vector**

If we now consider the problem setup in fig.1 we see that in the electron rest system it is possible to decompose the scattered photon polarization vector onto the unit vectors \( \hat{\phi}', \hat{\theta}' \) since they span a plane perpendicular to the photon propagation vector.\(^{17}\)

---

\(^{14}\)This is an approximation that ignore diverging and phase effects of the laser pulse.

\(^{15}\)See [2] p.22

\(^{16}\)Since the time average of the incident poynting vector is \( \frac{d\sigma}{d\Omega} = \frac{\varepsilon_0 E_0^2}{2} \)

\(^{17}\)The photon propagation is perpendicular to the electric and magnetic fields.
\[ \epsilon_1 = \cos \theta' (\hat{x}' \cos \phi'_e + \hat{y}' \sin \phi'_e) - \hat{z}' \sin \theta'_e, \]
\[ \epsilon_2 = -\hat{x}' \sin \phi'_e + \hat{y}' \cos \phi'_e. \]

It can be seen that \( \epsilon_2 \) is the \( \phi'_e \) unit vector and \( \epsilon_1 \) is the \( \theta'_e \) unit vector. If we assume that the incoming radiation has an electric field \( \mathbf{E}' = E'_0 (\alpha'_x \hat{x}' + \alpha'_y \hat{y}' + \alpha'_z \hat{z}') \), we can then sum over all polarization states in order to obtain the differential cross section, \( \frac{d\sigma}{d\Omega'} \), in terms of the incoming photon polarization vector

\[
\frac{d\sigma}{d\Omega'} \frac{1}{r_0^2} = \sum_{i=1,2} |\epsilon^*_i \cdot \mathbf{\alpha}'|^2 \\
= \alpha'_x^2 (1 - \cos^2 \phi'_e \sin^2 \theta'_e) \\
+ \alpha'_y^2 (1 - \sin^2 \phi'_e \sin^2 \theta'_e) \\
+ \alpha'_z^2 (1 - \cos^2 \theta'_e) \\
- 2\alpha'_x \alpha'_y \cos \theta'_e \sin \theta'_e \cos \phi'_e \\
- 2\alpha'_x \alpha'_z \sin \theta'_e \cos \phi'_e \\
- 2\alpha'_y \alpha'_z \cos \theta'_e \sin \theta'_e \sin \phi'_e. \tag{3.20}
\]
We have now obtained exactly what we were looking for, that is an expression for the differential cross section in the electron rest frame, the next task is to transform this expression to the electron laboratory frame.

3.3 Transformation between coordinate systems

In the previous subsection we obtained an expression for the differential cross section, \( \frac{d\sigma}{d\Omega'} \), but the formula is only valid in the electron rest frame. Since we will perform the scattering in the laboratory frame and not the electron rest frame, it is required to express the differential cross section, \( \frac{d\sigma}{d\Omega'} \), in the laboratory frame. It is thus required to obtain transformation laws from the electron rest system to the electron laboratory system.

A lot of natural coordinate systems exists to choose from in order to study our process, and there are some advantages to all of them. For instance when calculating the differential cross section, \( \frac{d\sigma}{d\Omega'} \), it is favourable to do it in the rest frame of the electron, while it is very favourable to work in the so called laser system where we define the z-axis to be anti parallel to the incoming photon, when studying the incoming radiation. When we are performing our experiment we are working in the stationary lab frame where the electron is moving towards the laser beam, so it is of great importance to obtain transformation laws from one coordinate system to another.

Stationary coordinate system

For our problem three particular useful coordinate systems exists. We can define a stationary frame in which the electron is traveling along the positive z-axis and the incoming radiation is at an angle \( \theta_0 \) relative to the z-axis, see fig. 2.
Figure 2: Stationary frame- This is the frame in which we are performing the real experiment

Laser frame
The laser system is defined such that the z-axis is directed in the opposite direction to the incoming photons i.e the incident photon is anti-parallel to the z-axis in this frame. The big advantage of this frame is that since the z-axis is anti-parallel to the photon propagation vector the electric and magnetic fields will lie entirely in the x-y plane and it is possible to describe their direction with a single angle. We also include three dimensional effects by introducing the angles $\xi_x$ and $\xi_y$ that represent an adjustment to the incoming photon direction. All angles and quantities in the laser system will be denoted with an l subscript, see fig. 3.
Electron lab frame
The electron lab frame is the frame we are mainly interested in. It is defined such that the z-axis is parallel to the incoming electron, and the incident photon hits the electron at the origin. The transformation to this system from the stationary coordinate system is specified by two rotations, one around the y-axis with an angle $\xi_{ye}$, and another rotation around the $x_e$ axis with an angle $\xi_{xe}$, see fig.4.
To better understand the meaning of the angles $\xi_{ye}, \xi_{xe}, \xi_{yl}, \xi_{xl}$ consider the perfect case, when all photons move along the same path and all electrons are incident at the same path. We will then have exactly the situation depicted in fig. 2. However in reality it will be impossible to have all photons focused perfectly, and the electron bunches will not all be incident on the same angle, the angles account for these effects and we will later see how to connect them to the concepts of emittance and laser focus.

**Rotation matrices**

To transform between the three coordinate systems described above we are going to need to perform rotations, and as such we need to use rotations matrices in order to transform correctly.\(^\text{18}\) The rotation from the stationary frame to the laser frame is specified by two rotations. The rotation around the z-axis can be written in matrix form as

$$R_{xl} = \begin{pmatrix}
\cos(\xi_{xl} + \theta_0) & 0 & \sin(\xi_{xl} + \theta_0) \\
0 & 1 & 0 \\
-\sin(\xi_{xl} + \theta_0) & 0 & \cos(\xi_{xl} + \theta_0)
\end{pmatrix}. \quad (3.21)$$

\(^\text{18}\)See [4] chapter 4.9
Similarly a rotation around the $x_l$ axis can be written in matrix form as

$$R_{yl} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \zeta_{yl} & -\sin \zeta_{yl} \\ 0 & \sin \zeta_{yl} & \cos \zeta_{yl} \end{pmatrix}. \quad (3.22)$$

The transformation from the stationary frame to the electron lab frame will be represented by two rotations. The first one is specified by a rotation about the $y$-axis,

$$R_{xe} = \begin{pmatrix} \cos \zeta_{xe} & 0 & \sin \zeta_{xe} \\ 0 & 1 & 0 \\ -\sin \zeta_{xe} & 0 & \cos \zeta_{xe} \end{pmatrix}, \quad (3.23)$$

and a rotation around the $x_e$-axis,

$$R_{ye} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \zeta_{ye} & -\sin \zeta_{ye} \\ 0 & \sin \zeta_{ye} & \cos \zeta_{ye} \end{pmatrix}. \quad (3.24)$$

To transform from the laser frame to the electron frame we first have to transform from the laser frame to the stationary frame\(^{19}\) and then transform from the stationary frame to the electron frame\(^{20}\). Putting all this together, the full transformation matrix becomes

$$R = R_{ye} R_{xe} R_{lx}^{-1} R_{ly}^{-1}$$

$$= \begin{pmatrix} \cos(\theta_x - \zeta_{xe}) & -\sin \zeta_{yl} \sin(\theta_x - \zeta_{xe}) & \cos \zeta_{yl} \sin(\theta_x - \zeta_{xe}) \\ \sin \zeta_{ye} \sin(\theta_x - \zeta_{xe}) & \sin \zeta_{yl} \sin \zeta_{ye} \cos(\theta_x - \zeta_{xe}) & -\cos \zeta_{yl} \sin \zeta_{ye} \cos(\theta_x - \zeta_{xe}) \\ -\cos \zeta_{ye} \sin(\theta_x - \zeta_{xe}) & -\sin \zeta_{yl} \cos \zeta_{ye} \cos(\theta_x - \zeta_{xe}) & \cos \zeta_{yl} \cos \zeta_{ye} \cos(\theta_x - \zeta_{xe}) \end{pmatrix} + \sin \zeta_{yl} \sin \zeta_{ye}$$

$$= \begin{pmatrix} \cos(\theta_x - \zeta_{xe}) & -\sin \zeta_{yl} \sin(\theta_x - \zeta_{xe}) & \cos \zeta_{yl} \sin(\theta_x - \zeta_{xe}) \\ \sin \zeta_{ye} \sin(\theta_x - \zeta_{xe}) & \sin \zeta_{yl} \sin \zeta_{ye} \cos(\theta_x - \zeta_{xe}) & -\cos \zeta_{yl} \sin \zeta_{ye} \cos(\theta_x - \zeta_{xe}) \\ -\cos \zeta_{ye} \sin(\theta_x - \zeta_{xe}) & -\sin \zeta_{yl} \cos \zeta_{ye} \cos(\theta_x - \zeta_{xe}) & \cos \zeta_{yl} \cos \zeta_{ye} \cos(\theta_x - \zeta_{xe}) \end{pmatrix} + \sin \zeta_{yl} \sin \zeta_{ye}$$

where we have defined $\theta_x = \theta_0 + \zeta_{xl}$.

\(^{19}\) Specified by the inverse of the laser rotation matrices.

\(^{20}\) Specified by by the product of two matrices.
**Lorentz boost**

Once we know how to transform between the laser frame and the electron lab frame, we need to know the transformation from the electron lab frame to the electron rest frame. This transformation is governed by a boost in the $\pm z_e$ direction depending on to which frame we are transforming to. We are only interested in how the electric and magnetic fields transform since it is those quantities that we want to express in the different frames. The transformation of the electromagnetic fields during a boost is given by\(^\text{21}\)

\[
E' = \gamma (E + c\beta \times B) - c^2 \frac{\gamma^2}{\gamma + 1} \beta (\beta \cdot E),
\]

\[
B' = \gamma (B - \frac{1}{c} \beta \times E) - c^2 \frac{\gamma^2}{\gamma + 1} \beta (\beta \cdot B).
\]

Thus the transformation of the Electric field from the electron lab frame to its rest frame is given by

\[
E'_x = \gamma (E_x - c\beta B_y),
\]

\[
E'_y = \gamma (E_y + c\beta B_x),
\]

\[
E'_z = E_z.
\]

To transform from the electron rest frame to the electron lab frame we simply let $\beta \rightarrow -\beta$.

**Transformation of angles**

We now turn our attention to how angles transform from the electron rest frame to the electron lab frame. Recall the transformation of the wave vector with respect to the Lorentz boost (z-axis)\(^\text{22}\).

\[
k'_s\parallel = \gamma (k_s\parallel - \beta \frac{\omega_s}{c})
\]

\[
k'_s\perp = k_{s\perp}
\]

Since $k = |k|\hat{r}$, $k_s = |k_s|\hat{r}$ and $|k| = \frac{\omega}{c}$, using the relativistic Doppler shift\(^\text{23}\) we can write

\[
\frac{\omega'_s}{\omega_s} = \gamma (1 - \beta \cdot k_s).
\]

\(^\text{21}\)See [1]Jackson p.558
\(^\text{22}\)See [1]Jackson p.526
\(^\text{23}\)See [1]Jackson p.526
We finally obtain

\[ \hat{r} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta \quad \Rightarrow \quad \omega' = \frac{\omega_s}{\gamma(1 - \beta \cos \theta)}, \]

[Equation (3.31)] \Rightarrow

\begin{align*}
\cos \theta_e' &= \frac{\cos \theta_e - \beta}{1 - \beta \cos \theta_e}, \\
\sin \theta_e' \cos \phi_e' &= \frac{\sin \theta_e \cos \phi_e}{\gamma(1 - \beta \cos \theta_e)}, \\
\sin \theta_e' \sin \phi_e' &= \frac{\sin \theta_e \sin \phi_e}{\gamma(1 - \beta \cos \theta_e)}. 
\end{align*} \tag{3.34}

Finally the transformation from the electron rest frame to the stationary frame can be computed with the help of

\[ \mathbf{r}_e = R_{ey} R_{ex} \mathbf{r}. \tag{3.37} \]

Since rotation preserve the length of the radial vector we obtain the relations

\begin{align*}
\cos \phi_e \sin \theta_e &= \cos \phi \sin \theta \cos \xi_{xe} - \sin \phi \sin \theta \sin \xi_{xe} \\
\sin \phi_e \sin \theta_e &= -\cos \phi \sin \theta \sin \xi_{xe} \sin \xi_{ye} + \sin \phi \sin \theta \cos \phi \sin \xi_{ye} - \cos \theta \cos \xi_{xe} \sin \xi_{ye} \\
\cos \theta_e &= \cos \phi \sin \xi_{xe} \cos \xi_{ye} + \sin \phi \sin \theta \sin \xi_{xe} + \cos \theta \cos \xi_{ye} \cos \xi_{xe} \tag{3.39}
\end{align*}

We now got all the transformation relations we need for the angles, and using the result above we can express an angle in the electron rest frame in terms of the stationary frames angles.

**Invariant quantities**

We now turn our attention to a specially useful Lorentz invariant quantity, namely the normalized vector potential,

\[ a_0 = \frac{e}{mc} \sqrt{\left| A_\mu A^\mu \right|}. \tag{3.41} \]

We restrict ourselves to the case of a charged particle in an electromagnetic field, where \((A^\mu) = (0, \mathbf{A})\). Furthermore since \( \mathbf{E} = -\partial_t \mathbf{A} \), it can be seen that \( \mathbf{A} = \frac{E}{\omega} \), plugging this into equation (3.41) we obtain

\[ a_0 = \frac{e}{mc} \sqrt{\frac{E^2}{\omega^2}} = \frac{eE}{mc \omega}. \tag{3.42} \]

\(^{24}\text{See[1] p.239}\)

---

20
Since $A_\mu A^\mu$\textsuperscript{25} is a Lorentz invariant quantity we conclude that $a_0$ is also a Lorentz invariant quantity. This statement is really powerful since we can now find a relation between the electric fields and the angular frequencies in the electron lab and rest system.

$$a'_0 = \frac{eE'_0}{mc\omega'_0} = \frac{eE_0}{mc\omega_0} = a_0 \quad (3.43)$$

$$\Rightarrow \frac{E'_0}{E_0} = \frac{\omega'_0}{\omega_0} \quad (3.44)$$

As a final step we can now use the Doppler shift of the angular frequency to obtain the relation:

$$\frac{E'_0}{E_0} = \gamma(1 - \frac{c}{\omega_0} \beta \cdot k_0) \equiv g'(\theta_x, \xi_{yl}, \xi_{xe}, \xi_{ye}). \quad (3.45)$$

The quantity $g'$ may be written out in terms of the angles of our different systems if we use the rotation matrix in equation (3.25), and the definition of the laser system in which the photon is incident through the negative $z_l$ axis,

$$k_{0l} = -\frac{\omega_0}{c} \hat{z}_l. \quad (3.46)$$

Similarly in the electron lab system this vector can be expressed as

$$(k_0)z = R_{z\hat{z}}k_{0l}i = -\frac{\omega_0}{c} (\cos \xi_{yl} \cos \xi_{ye} \cos(\theta_x - \xi_{xe}) + \sin \xi_{yl} \sin \xi_{ye}). \quad (3.47)$$

Then from equation (3.45) we obtain

$$g'(\theta_x, \xi_{yl}, \xi_{xe}, \xi_{ye}) = \gamma(1 + \beta [\cos \xi_{yl} \cos \xi_{ye} \cos(\theta_x - \xi_{xe}) + \sin \xi_{yl} \sin \xi_{ye}]). \quad (3.48)$$

### 3.4 Differential cross section in the stationary frame

Now that we have obtained all the transformation relations between our different coordinate systems we may write down the differential cross section. We will start by defining the incident electric and magnetic fields in the laser system, then transform these fields to the electron rest system using equation (3.25), equation (3.28), and then transform back to the stationary system.

\textsuperscript{25}See [3]p.108
Electric and Magnetic fields

In the laser system the incident photon wave vector is anti-parallel to the $z_l$ axis, and we may thus describe the fields as

\[ E_{xl} = E_0 \cos \phi_p, \quad E_{yl} = E_0 \sin \phi_p, \quad (3.49) \]
\[ cB_{xl} = E_0 \sin \phi_p, \quad cB_{xl} = -E_0 \cos \phi_p, \quad (3.50) \]

where the angle $\phi_p$ defines the polarization vector, $\alpha$. The electric and magnetic fields may then be transformed to the electron rest system

\[ E_e = R E_l, \quad (3.51) \]
\[ B_e = R B_l. \quad (3.52) \]

The $E_e$ field can then be boosted to the electron rest frame using equation (3.28) and if we use the definition of the polarization vector in the electron rest frame

\[ \alpha'_x = \frac{E'_x}{E'_0} = \gamma \left[ \cos \phi_p \left( (\cos(\theta_x - \xi_{xe}) + \beta (\cos \xi_{yl} \cos \xi_{ye} + \sin \xi_{yl} \sin \xi_{ye} \cos(\theta_x - \xi_{xe}))) \right) - \sin \phi_p \left( \sin \xi_{yl} \sin(\theta_x - \xi_{xe}) + \beta (\sin \xi_{ye} \sin(\theta_x - \xi_{xe}))) \right) \right] \quad (3.53) \]
\[ \alpha'_y = \frac{\gamma}{\gamma'} \left[ \cos \phi_p \left( \sin(\theta_x - \xi_{xe}) (\sin \xi_{ye} + \beta \sin \xi_{yl}) \right) + \sin \phi_p \left( \cos(\theta_x - \xi_{xe}) (\beta + \sin \xi_{yl} \sin \xi_{ye}) + \cos \xi_{yl} \cos \xi_{ye} \right) \right] \quad (3.54) \]
\[ \alpha'_z = -\frac{\gamma}{\gamma'} \left[ \cos \phi_p \left( \sin(\theta_x - \xi_{xe}) \cos \xi_{ye} \right) + \sin \phi_p \left( \sin \xi_{yl} \cos \xi_{ye} \cos(\theta_x - \xi_{xe}) - \cos \xi_{yl} \sin \xi_{ye} \right) \right] \quad (3.55) \]

Now let us consider the transformation of the differential cross section from the electron rest frame to the lab frame. Using the chain rule we obtain

\[ \frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega'} \frac{d\Omega'}{d\Omega} = \frac{d\sigma}{d\Omega'} \frac{d\cos \theta_e}{d\Omega'} \frac{d\cos \theta_e}{d\cos \theta_e} \]
\[ = \frac{d\sigma}{d\Omega'} \frac{1 - \beta^2}{(1 - \beta \cos \theta_e)^2} \quad (3.56) \]

where we have used the fact that $d\Omega = -d(\cos \theta) d\phi$ and equation (3.38). Finally using equation (3.38), equation (3.19), we obtain the expression for the differential
cross section in the electron rest frame, expressed in the coordinates of the stationary frame

\[
\frac{1}{r_0^2} \frac{d\sigma}{d\Omega'} = \alpha_x'^2 \left(1 - \frac{\cos^2 \phi_e \sin^2 \theta_e}{\gamma^2 (1 - \beta \cos \theta_e)^2}\right) \\
+ \alpha_y'^2 \left(1 - \frac{\sin^2 \phi_e \sin^2 \theta_e}{\gamma^2 (1 - \beta \cos \theta_e)^2}\right) \\
+ \alpha_z'^2 \left(1 - \frac{(\cos \theta_e - \beta)^2}{(1 - \beta \cos \theta_e)^2}\right) \\
- 2\alpha_x'\alpha_y' \left(\frac{(\cos \phi_e \sin \theta_e)(\sin \phi_e \sin \theta_e)}{\gamma^2 (1 - \beta \cos \theta_e)^2}\right) \\
- 2\alpha_x'\alpha_z' \left(\frac{(\cos \phi_e - \beta)(\cos \phi_e \sin \theta_e)}{\gamma(1 - \beta \cos \theta_e)^2}\right) \\
- 2\alpha_y'\alpha_z' \left(\frac{(\cos \phi_e - \beta)(\sin \phi_e \sin \theta_e)}{\gamma(1 - \beta \cos \theta_e)^2}\right)
\]

where the components \( \alpha_i' \) is given by equation (3.53), and the angles may be written in terms of the angles of the stationary frame using equation (3.38). To obtain the differential cross section in the electron lab frame, we got to use equation 3.56. Where the components \( \alpha_i' \) is given by equation (3.53), and the angles may be written in terms of the angles of the stationary frame according to equation (3.38). We now have exactly found what we were looking for, an expression for the differential cross section, \( \frac{d\sigma}{d\Omega'} \), in the stationary frame. To see some of the features of the differential cross section let us plot the angular dependence, \( \theta \), for some different electron energies. The energy and \( \theta \) dependence of the normalized differential cross section \( \frac{1}{r_0^2} \frac{d\sigma}{d\Omega} \) is depicted in fig.5-8 for the case \( \phi = \phi_p = 0 \).
Fig. 5-6 shows that as the energy of the electron is increased the differential cross section, $\frac{d\sigma}{d\Omega}$, gets a bigger peak at $\theta = 0$. Fig. reffigureangdep3 shows the dependence of the differential cross section, $\frac{d\sigma}{d\Omega}$, in the $\theta = 0$ direction, with respect to the electron focus angles $\xi_{xe}, \xi_{ye}$. And from Fig. 7 we see that the differential cross section, $\frac{d\sigma}{d\Omega}$, rapidly approach zero as these angles grow. Fig. 8 illustrates that when the electron energy is increased the differential cross section rapidly increases as well.
4 Scattering rate

In this section we are going to study the scattering rate, \( \frac{dN_s}{dt} \), and explore what parameters it depends on. Even though the scattering rate is not the main quantity of interest in this report, it is still valuable to know how many scattered photons that is actually created in the process. We will also investigate the form of the electron density, \( n_e \), and the photon density \( n_\gamma \), and on what parameters they depend. Once the scattering rate scattering rate is known, it is easy to obtain the total number of scattered photons, \( N_s \), by just integrating the scattering rate over time.

4.1 Scattering density

From equation (3.4) we have

\[
\frac{dN_s}{d^4x} = c\sigma(1 - \beta_e \cdot \frac{k}{\omega})n_e(r,t)n_\gamma(r,t). \tag{4.1}
\]

We are going to consider the case when both the electron and the photon beams are cylindrical, and in order to simplify the calculations it is useful to express equation (3.4) in cylindrical coordinates,

\[
\frac{dN_s}{d^4x} = \frac{d^4N_s}{rdrd\phi dzdt}
\]

[angular symmetry] \Rightarrow \frac{dN_s}{drdzdt} = 2\pi r\sigma c(1 - \beta_e \cdot \frac{k}{\omega})n_e(r,z,t)n_\gamma(r,z,t). \tag{4.2}

The formula above is valid for all collisions of two cylindrical beams, but we now restrict ourselves to the case of a head on collision along the z-axis. In this case the incoming photon wave vector is anti-parallel to the incoming electrons, and we obtain

\[
\frac{dN_s}{drdzdt} = \{ |k| = \frac{\omega}{c}, \beta_e = \beta_e \hat{z} \} = 2\pi r\sigma c(1 + \beta_e)n_e(r,z,t)n_\gamma(r,z,t) \tag{4.3}
\]

Now since we are interested in measuring the scattering rate, \( \frac{dN_s}{dt} \), we have to consider where we do the measurement. This is because information does not travel instantaneously and we have to take into account the time it takes to travel from the scattering position to the detector where we measure the scattered photon\(^{26}\). To this end let us assume that we have placed an imaginary detector at a position \( z_d \), which means that we make the assumption that the emission took

\(^{26}\)I.e the retarded time.
place on the z-axis the time it takes for the photon to travel from a position $z$ to the detector is $\frac{z_d - z}{c}$. Thus if the emission took place at a time $t$, the time we measure the radiation at our detector is

$$t_d = t + \frac{z_d - z}{c}. \tag{4.4}$$

Since the position of our imaginary detector is arbitrary we might as well set it to 0. Thus we obtain the expression

$$\frac{dN_s}{drdzdt_d} = 2\pi r \sigma c (1 + \beta_e) n_e(r, z, t_d + \frac{z}{c}) n_\gamma(r, z, t_d + \frac{z}{c}) \tag{4.5}$$

In order to integrate away the $r, z$ dependence and obtain the scattering rate, $\frac{dN_s}{dt}$, we have to know the form of the electron density, $n_e$, and the photon density, $n_\lambda$.

### 4.2 Electron and Photon densities

Up until now we have not actually specified how the electron and photons densities actually look, in this subsection we will show how it is possible to model the densities and on what parameters they depend.

**Photon density**

To model the photon density we are going to use a Gaussian beam approximation, meaning that we model the photon density as a Gaussian distribution.

![Gaussian beam parameters](image)

*Figure 9: Gaussian beam parameters Obtained from [9]*

To model the beam as a Gaussian we have to know the parameters shown in fig.9, where $w_0$ is the beam waist, $z_R = \frac{\pi w_0^2}{\lambda}$ is the distance along the beam to

---

27This approximation is valid for small collision angles, $\theta_0$, for big collision angles a more general formalism has to be used, See section 6.4
where the cross sectional area is doubled. The radius of the beam at a position $z$ is then $w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2}$. To model the temporal part of the photon density we assume that the photon pulse has a duration $\Delta t$. Since the total number of photons in the pulse is given by $N_\gamma = \frac{W}{\hbar \omega_0}$, where $W$ is the energy of the pulse and $\omega_0$ is the photon wavelength we can model the photon distribution as

$$n_\lambda(r, z, t_d) = \frac{N_\lambda}{\sqrt{\frac{3}{2} \Delta t c w_0^2 (1 + \left(\frac{z}{z_R}\right)^2)}} 
\times \exp\left(-2\left(\frac{t_d + \frac{z}{\Delta t}}{\Delta t}\right)^2\right) 
- \frac{2}{w_0^2 (1 + \left(\frac{z}{z_R}\right)^2)^2}, \quad (4.6)$$

The pre factors in front of the exponential comes from the choice normalization of the photon density, $n_\lambda$, i.e we have choose a normalization such that

$$\int n_\lambda(r, z, t_d) dt dtdz = N_\lambda. \quad (4.7)$$

### Electron density

It is possible to model the electron beam in analogy to the laser pulse, where the radius of the beam is given by $w(z) = r_b \sqrt{1 + (k_f z)^2}$, where $r_b$ is the focal radius and $k_f$ is the inverse beta function, given in terms of the normalized emittance, $\epsilon$, as $k_f = \frac{\epsilon}{r_b^2 \gamma_c}$. The electron density distribution is thus given as

$$n_e(r, z, t_d) = \frac{N_e}{\sqrt{\frac{3}{2} \Delta t c r_b^2 (1 + (k_f z)^2)}} 
\times \exp\left(-\left(\frac{t_d + \frac{(1-\beta_0)z}{\Delta t}}{\Delta t}\right)^2\right) 
- \frac{r^2}{r_b^2 (1 + (k_f z)^2)^2}, \quad (4.8)$$

where the $-\beta_0$ term comes from the fact that the electron is only moving at $\beta_0$ percent of the speed of light, and is moving in the opposite way compared to the incident photons.

---

28Where we approximate the frequency as the center frequency.
29See [7]
4.3 Scattering rate

Now that we have obtained expressions for the electron and photon densities, we can obtain the scattering rate by using

$$\frac{dN_s}{dt_d} = \int \frac{dN_s}{dr dz dt_d} dz dr,$$

(4.9)

where the integral is performed from $-\infty$ to $+\infty$ for the $z$ integral, and the radial integral is performed from 0 to $+\infty$. In terms of the electron and photon density, $n_e, n_\lambda$, we can now express the scattering density, $dN_s/dt_d$, in terms of the normalized axial position $z = 2\sqrt{2z/c\Delta t}$, normalized beta function $\eta = k_f c \Delta t / 2\sqrt{2}$, and the normalized Rayleigh length $\mu = c \Delta t / 2\sqrt{2} z R$.

In terms of these parameters we have that $(k_f z)^2 = \eta \bar{z}$, $\frac{\bar{z}}{\bar{r}} = \mu \bar{z}$. The scattering density thus becomes

$$\frac{dN_s}{dz dt_d} = 2 \pi r \sigma (1 + \beta_0) \frac{N_e N_\lambda}{\pi^3 r_b^2 w_0^2 \Delta \tau} \frac{1}{(1 + (\mu \bar{z})^2)(1 + \eta \bar{z})^2)}$$

$$\times \exp(-\left\{ \frac{t_d}{\Delta \tau} + \bar{z} \left(1 - \beta_0\right) \Delta t \right\}^2)
- \left\{ \sqrt{2} \frac{t_d}{\Delta \tau} + \bar{z} \right\}^2
- \frac{2r^2}{w_0^2 (1 + (\mu \bar{z})^2)} - \frac{r^2}{r_b^2 (1 + (\eta \bar{z})^2)}).$$

(4.10)

We can perform the radial part of this integral with the help of equation (C.1) in the appendix to obtain

$$\frac{dN_s}{dz dt_d} = \sigma (1 + \beta_0) \frac{r_b^2}{w_0^2} N_e N_\lambda \frac{1}{\Delta \tau}$$

$$\times \frac{1}{1 + \mu^2 \bar{z}^2 + 2 \frac{t_d}{\Delta \tau} (1 + \eta^2 \bar{z}^2)}$$

$$\times \exp(-\left\{ \frac{t_d}{\Delta \tau} + \bar{z} \left(1 - \beta_0\right) \Delta t \right\}^2)
- \left\{ \sqrt{2} \frac{t_d}{\Delta \tau} + \bar{z} \right\}^2).$$

(4.11)

Since we are working with highly relativistic electrons, we see that the term $1 - \beta_0$ is going to be very close to 0, and thus we can as an approximation

---

30See [5]
31See [5] p.6
ignore all terms proportional to $1 - \beta_0$. The $\bar{z}$ integration can be carried out with the help of equation (C.3) in the appendix, and we obtain

$$\frac{dN_s}{dt_d} = \sigma \frac{r_b^2}{w_0^2} (1 + \beta_0) \frac{N_e N_L}{\Delta \tau} \exp\left\{ -\left( \frac{t_d}{\Delta \tau} \right)^2 \right\} \times \frac{1 - \text{Erf}\left\{ \sqrt{\eta + \chi\mu} / (\eta \mu + \chi \eta^2) \right\}}{\sqrt{\left( \mu^2 + \chi \mu \eta \right) (1 + \chi \mu^2)}}. \tag{4.12}$$

Where we have introduced $\chi = 2 \frac{r_b^2 \eta}{w_0^2 r}$, and Erf is the error function defined as

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx \tag{4.13}$$

If we now introduce the overlap function $\mathcal{F}$ defined as

$$\mathcal{F}(\chi, \eta, \mu) = \frac{1 - \text{erf}\left\{ \sqrt{\eta + \chi \mu} / (\eta \mu + \chi \eta^2) \right\}}{\sqrt{\left( \mu^2 + \chi \mu \eta \right) (1 + \chi \mu^2)}}, \tag{4.14}$$

we can write the scattering rate as:

$$\frac{dN_s}{dt_d} = \sigma (1 + \beta_0) \frac{r_b^2}{w_0^2} \frac{N_e N_L}{\Delta \tau} \mathcal{F} e^{-\left( \frac{t_d}{\Delta \tau} \right)} \tag{4.15}.$$  

It is possible to view this as a Gaussian distribution centered around$^{32} t_d = 0$,

$$\frac{dN_s}{dt_d} = N_s e^{-\left( \frac{t_d}{\Delta \tau} \right)}, \tag{4.16}$$

where the total number of scattered photons are given by

$$N_s = \sigma (1 + \beta_0) \frac{r_b^2}{w_0^2} \frac{N_e N_L}{\Delta \tau} \mathcal{F}. \tag{4.17}$$

Now that we got an expression for the total number of scattered photons, $N_s$, we can identify some interesting features, for instance we see that $N_s$ is only directly proportional to the electron bunch duration, $\Delta \tau$, while the laser pulse duration, $\Delta t$, only appears implicitly inside the overlap function, $\mathcal{F}$.

$^{32}$Temporal Gaussian distribution.
5 Head-on Brightness

In the previous sections we have developed the tools that is required to calculate the brightness, in this section we are going to use these tools directly in order to calculate the brightness. In this section we will be concerned with the Brightness for a head-on collision, even though in most cases the collision is not head-on. The reason that we are taking this approach before going to the general case, is that the head-on collision is very illustrative, and for small deviations from a perfect head-on collision we can still use the head-on collision results as an approximation. We will restrict ourselves to the scattering in the $\theta = \phi = 0$ direction, and assume that the scattering is "perfect". More specifically we will study the head-on collision brightness for a cylindrical beam, and we will model our beam as a Gaussian distribution using the tools developed in section 4. We start from the definition of the brightness and take into account important effects such as the finite bandwidth of the laser pulse, and then move on to take into account the energy spread within the electron bunches.

5.1 Laser pulse frequency spread

To take into account the frequency spread within the laser pulse when calculating the brightness, $B$, we start from equation (3.19), we see that for the case of an incident radiation beam with polarization in the x-direction ($\phi_p = 0, \xi_l y = \xi_l x = 0 = \xi_e x = \xi e y$), and if we are only considering the on-axis brightness, i.e $\theta = \phi = 0$, the cross section in the stationary frame takes the form

$$\frac{d\sigma}{d\Omega} = r_0^2 \left( \frac{\gamma + v}{\gamma - v'} \right)$$ (5.1)

where we for convince have introduced the normalized beta factor $v = \beta \gamma$. From equation (3.4) we obtain

$$\frac{dN_s}{d^4 x d\Omega d\omega_s} = r_0 2 \frac{\gamma + v}{\gamma - v} (1 + \beta_0) n_e n_p \delta(\omega_s - g(\theta) \omega),$$ (5.2)

where the Doppler up shift factor, $g(\theta)$, is given by

$$\frac{g'}{\gamma (1 - \beta \cos(\theta_e))}$$ (5.3)

$\xi_l l = \xi y l = \xi x e = \xi y e = 0$

$\xi e x = \xi e y$ See [5]p 5-8
and $g'$ is defined in equation (3.48). To take into account the finite bandwidth of the photon beam, we model the frequency spread as a Gaussian distribution and integrate over all frequencies to obtain

$$S_\omega = \frac{r_0^2 \gamma + v}{\gamma - v} (1 + \beta_0) n_e n_\lambda \int_{-\infty}^{+\infty} \delta(\omega_s - g(\theta) \omega) \times \exp \left[ - \left( \frac{\omega - \omega_0}{\Delta \omega} \right)^2 \right] d\omega.$$  

(5.4)

Now using equation (C.6) and equation (C.7) in the appendix, we obtain

$$S_\omega = \frac{r_0^2 \gamma + v}{\gamma - v} (1 + \beta_0) n_e n_\lambda \frac{1}{\sqrt{\pi \Delta \omega}} \int_{-\infty}^{+\infty} \delta(\omega_s - g(\theta) \omega) \times \exp \left[ - \left( \frac{\omega - \omega_0}{\Delta \omega} \right)^2 \right] d\omega.$$  

(5.5)

where $\Delta \omega$ is given by the relation $\Delta \omega \Delta t = \sqrt{2}$.

### 5.2 Energy spread

Now that the frequency spread within the laser pulse has been taken into account, we need to consider how an energy spread within the electron bunches will affect the brightness, $B$. Remember that we are interested in the brightness and the reason we consider all these effects is to obtain as valid value of the brightness, $B$, as possible. We model the energy spread as a Gaussian and integrate over the local on-axis spectral brightness

$$S_\gamma = \int_{1}^{+\infty} S_\omega \exp \left[ \left( \frac{\gamma - \gamma_0}{\Delta \gamma} \right)^2 \right].$$  

(5.6)

The integration ranges for the energy spread distribution are from 0 to $+\infty$, but if we restrict ourselves to the cases where $\gamma_0 > 1$ and $\Delta \gamma \ll 1$, the non-zero part of the integral will be within a small distance from $\gamma_0$. We can then as a good approximation let the lower integration range go to $-\infty$ and use equation (C.2) from the appendix, to obtain

$$S_\gamma = \int_{-\infty}^{+\infty} \frac{\gamma + v (1 + \beta_0)}{\gamma - v} g(\theta) \exp \left[ - \left( \frac{\omega}{g(\theta)} - \omega_0 \right)^2 \right] - \left( \frac{\gamma - \gamma_0}{\Delta \gamma} \right)^2.$$  

(5.8)
For a head-on on-axis collision, \( \theta_0 = \theta = \phi = 0 \), this results in a Doppler upshift factor

\[
|g(\theta)|_{\theta=0} = \frac{\gamma + v_z}{\gamma - v_z}.
\] (5.9)

Since we are working with highly relativistic electron bunches, we can use the fact that \( \gamma \) is large, this implies

\[
\gamma^{-2} << 1.
\] (5.10)

From the definition of the normalized beta factor, \( v = \beta \gamma \), we can write \( \gamma^2 \) as

\[
\gamma^2 = 1 + v^2.
\] (5.11)

In a perfect world all the electrons would move in a straight line along the \( z \)-axis, but because of electric fields in radial direction the electrons will have a small velocity in the radial directions. This can be visualized by dividing the normalized beta factor, \( v \), into a \( z \)-component, \( v_z = \beta_z \gamma \), and a component in the plane perpendicular to the \( z \)-axis, \( v_\perp = \beta_\perp \gamma \). To make the evaluation of the \( S_\gamma \) integral easier it is favourable to rewrite the integrand somewhat before performing the integral. The \( \gamma^2 \) factor can be rewritten as

\[
\gamma^2 = 1 + v_z^2 + v_\perp^2.
\] (5.12)

Since \( \beta_0 \) do not have any perpendicular component we can identify it with \( \beta_z \), and rewrite, \( \frac{(1 + \beta_0)}{|g(\theta)|} \), as

\[
\frac{(1 + \beta_0)}{|g(\theta)|} = \frac{\gamma - v_z \gamma + v_z}{\gamma + v_z \gamma} = \left( \frac{\gamma - v_z}{\gamma} \right).
\] (5.13)

Furthermore we can simplify \( \frac{\gamma + v}{\gamma - v} \left( \frac{\gamma - v_z}{\gamma} \right) \) by using the fact that \( 1 + v_\perp^2 = \gamma^2 - v_z^2 \), to obtain

\[
\frac{\gamma + v}{\gamma - v} \left( \frac{\gamma - v_z}{\gamma} \right) = \frac{(\gamma + v)^2}{\gamma(\gamma + v_z)}(1 + v_\perp^2),
\] (5.14)

where we used that

\[
\frac{\gamma + v}{\gamma - v} = \frac{(\gamma + v)^2}{\gamma^2 - v^2} = (\gamma + v)^2.
\] (5.15)
If we now consider that $v_\perp$ is very small (true for most cases) we can simplify further by using the approximation

$$\frac{(\gamma + v)^2}{\gamma(\gamma + v_\perp)} \approx 2,$$

(5.16)

to obtain

$$\frac{\gamma + v}{\gamma - v} \left( \frac{\gamma - v_\perp}{\gamma} \right) \approx 2(1 + v_\perp^2).$$

(5.17)

A final simplification can be made if we introduce $\delta = \gamma - \gamma_0$, and use the Taylor-expansion

$$\frac{1}{\gamma^2} \approx \frac{1}{(\gamma_0 + \delta)^2} \approx \frac{1}{\gamma_0^2} \left( 1 - 2\frac{\delta}{\gamma_0} \right).$$

(5.18)

Remember that the purpose of all these simplifications was perform the integration $S_\gamma$, and using the approximations above and the fact that $d\delta = d\gamma$ we can calculate $S_\gamma$ with the help of equation (C.2) in the appendix, and after the smoke clears we obtain

$$S_\gamma = \frac{2r_0^2 n_e n_s}{\sqrt{\pi} \omega_0} \frac{1 + v_\perp^2}{\sqrt{\delta \omega^2 + \delta \gamma^2 \chi^2(1 + v_\perp^2)^2}} \times \exp \left[ -\frac{(\chi(1 + v_\perp^2) - 1)^2}{\delta \omega^2 + \delta \gamma^2 \chi^2(1 + v_\perp^2)^2} \right],$$

(5.19)

where we have introduced the relative energy spread $\delta \gamma = \frac{\Delta \gamma}{\gamma_0}$, normalized Doppler up-shift $\chi = \frac{\omega_s}{4 \gamma_0^2 \omega_0}$ and the relative spectral width $\delta \omega = \frac{\Delta \omega}{\omega_0}$.

### 5.3 Emittance contribution

Emittance is a measure of the spread of the phase space coordinates of electrons, or in other words it measures the spread of the electron velocity. For instance a small emittance means that we have a beam where the electrons are confined to a small volume and have a similar momentum. For our case the relevant parameter is the normalized beta parameter, $v$. In particular we want to study the behaviour of the beam perpendicular to the collision axis, i.e perpendicular to the $z$-axis. Thus the parameter of interest is the transverse component of the normalized beta function $v_\perp$. For a perfect beam $v_\perp$ would be constant equal to zero, but for a real beam we can model the behaviour of $v_\perp$ by assuming an
Gaussian dependence entered around 0. Now since the directions perpendicular to the z-axis span a plane, and since we are considering a cylindrical beam, the transverse velocity plane area element is $v_\perp dv_\perp d\phi$. Since we are modelling our beam as a perfect cylinder we have an angular symmetry and may integrate away the $\phi$ dependence to obtain $2\pi$. All this results that we get a more exact value for the brightness, $B$, and taking emittance into account we define $S_\epsilon$ as

$$S_\epsilon = \frac{1}{\pi\Delta v_\perp} \int_0^{+\infty} S_\gamma(v_\perp) \exp \left( - \left\{ \frac{v_\perp}{\Delta v_\perp} \right\}^2 \right) 2\pi v_\perp dv_\perp.$$  \hfill (5.20)

The meaning of $S_\epsilon$ is that the smaller the standard deviation, $\Delta v_\perp$, from 0, or in other words the more electrons that got a small radial velocity the higher brightness, $B$, we obtain.

### 5.4 Peak on-axis brightness

In this section we are going to calculate the peak brightness, in other words the maximum brightness that we obtain. The reason for this is that once we have obtained the peak brightness there is an easy way to get the average brightness\textsuperscript{35}. We have now taken into account both the energy spread and the emittance contributions to the brightness, all that is left now is to integrate the electron and photon density along the z-axis. Since we are assuming a Gaussian distribution for both the electron and photon distributions, they are given by equation(4.8) and equation(4.6)

$$n_\lambda(r, z, t_d) = \frac{N_\lambda}{\sqrt{\pi} \Delta tcw_0^2 \sqrt{1 + \left( \frac{z}{z_R} \right)^2}}$$

$$\times \exp \left( - \frac{t_d + 2\beta e_z}{\Delta t} \right)^2$$

$$- 2\frac{r^2}{w_0^2 \left( 1 + \left( \frac{z}{z_R} \right)^2 \right)^2},$$  \hfill (5.21)

$$n_e(r, z, t_d) = \frac{N_e}{\sqrt{\pi} \Delta \tau cr_b^2 \sqrt{1 + (kfz)^2}}$$

$$\times \exp \left( - \left( \frac{t_d + (1-\beta_0)z}{\Delta \tau} \right)^2 \right)$$

$$- \frac{r^2}{r_b^2 \left( 1 + (kfz)^2 \right)^2}. \hfill (5.22)$$

\textsuperscript{35}See section 8
The integration of $n_\lambda \cdot n_e$ over $z$ is in general very complicated, but if we assume that the beams perfectly overlap at $t_d = 0$, in order words that there do not exist ant temporal offset, we see from the behavior of $n_\lambda n_e$ that they will obtain their maximum when $r=0$. Thus we have to perform the integral

$$I = \int_{-\infty}^{+\infty} n_e(0, z, 0)n_\lambda(0, z, 0)dz$$

$$= \left(\frac{\pi}{\sqrt{2}}\right)^3 \frac{N_e N_\lambda}{c^2 r_b^2 w_0^2 \Delta \tau \Delta t} \int_{-\infty}^{+\infty} \frac{dz}{1 + (k_f z)^2}[1 + \left(\frac{z}{\bar{z}}\right)^2]$$

$$\times \exp \left[-\frac{z^2}{2} \left(\frac{1 - \beta_0}{\Delta \tau^2} + \frac{8}{\Delta t^2}\right)\right]$$

(5.23)

To perform this integral we will assume that $\frac{[1-\beta_0]}{\Delta \tau^2} << \frac{8}{\Delta t^2}$, and following the steps of section 4.3, we introduce the normalized beta function $\eta = \frac{k_f \Delta t}{c \sqrt{2}}$ and the normalized Rayleigh length $\mu = \frac{c \Delta t}{2 \sqrt{2} z_R}$.\n
$$I = \int_{-\infty}^{+\infty} n_e(0, z, 0)n_\lambda(0, z, 0)dz$$

(5.24)

$$= \frac{N_e N_\lambda}{\pi^2 c^2 r_b^2 w_0^2 \Delta \tau \Delta t} \int_{-\infty}^{+\infty} \frac{e^{-z^2}}{(1 + (\eta z)^2)(1 + (\mu z)^2)}d\bar{z}$$

(5.25)

$$= \frac{N_e N_\lambda}{\pi^2 c^2 r_b^2 w_0^2 \Delta \tau \Delta t} \frac{\eta e^{\frac{1}{\eta^2}}[\text{Erf}(\frac{1}{\eta}) - 1] - \mu e^{\frac{1}{\mu^2}}[\text{Erf}(\frac{1}{\mu}) - 1]}{\mu^2 - \eta^2}$$

(5.26)

Using the result above and equation (5.20) we can obtain an expression for the on-axis peak brightness. The brightness is usually expressed in units of a small surface element $\delta \Sigma = 1 \text{mm}^2$, a small solid angle element $\delta \Omega = 1 \text{mrad}^2$, and 0.1 percent of the fractional bandwidth, i.e $\delta \omega_s = 10^{-3} \omega_s$. The brightness can then be expressed as

$$B = \frac{dN_s}{dx dy dt d\Omega d\omega_s} \delta \Omega \delta \Sigma \delta \omega_s$$

$$= \frac{10^{-15} \omega_s N_e N_\lambda}{\pi^2 c^2 r_b^2 w_0^2 \Delta \tau} \frac{\eta e^{\frac{1}{\eta^2}}[\text{Erf}(\frac{1}{\eta}) - 1] - \mu e^{\frac{1}{\mu^2}}[\text{Erf}(\frac{1}{\mu}) - 1]}{\mu^2 - \eta^2}$$

$$\times \frac{1}{\pi \Delta v_\perp} \int_{0}^{+\infty} S_\gamma(v_\perp) \exp \left(-\left\{\frac{v_\perp}{\Delta v_\perp}\right\}^2\right) 2\pi v_\perp dv_\perp$$

(5.27)

\[36\] This assumption is valid for most highly relativistic cases
The integral over $S_\gamma$ can also be performed with the help of equation (C.4) in the appendix, and we obtain

$$
B = \frac{4 \times 10^{-15} \gamma_0^2 N_e N_\lambda r_0^2}{\pi^2 \Delta v_\perp r_b^2 \Delta \tau w_0^2} \exp \left[ \frac{\chi - 1}{2\chi \Delta v_\perp^2} \left( 2 + \frac{\delta \omega^2 + \delta \gamma^2 \chi^2}{2\chi(\chi - 1)\delta v_\perp^2} \right) \right] \\
\times \left[ 1 - \text{erf} \left( \frac{\chi - 1}{\sqrt{\delta \omega^2 + \delta \gamma^2 \delta \chi^2}} \left( 1 + \frac{\delta \omega^2 + \delta \gamma^2 \chi^2}{2\chi \Delta v_\perp^2} \right) \right) \right] \\
\times \frac{\eta \omega^{\frac{1}{\mu}} [\text{Erf} \left( \frac{1}{\eta} \right) - 1] - \mu \omega^{\frac{1}{\mu}} [\text{Erf} \left( \frac{1}{\mu} \right) - 1]}{\mu^2 - \eta^2}.
$$

(5.28)

We now obtained exactly what we were looking for, an expression for the on-axis brightness for a head-on collision. This expression is of great use since no integrations need to be performed and it is a "nice" analytic expression. It is important to remember that the above brightness expression is only valid for a head-on collision of a very relativistic electron beam, and for non-head collision additional effects must be accounted for. Even though the expression for the peak on-axis brightness we have obtained in this section is useful, we still need an expression for the brightness in a non head-on collision, to that end we will in the next section use the same methods and steps developed in this section but apply them on a non head-on collision.
6 Non Head-on collision

In the last section we only considered the case where the electron bunches and photon pulses collide head-on. Even though we obtained an expression for the brightness, $B$, we limited the analysis to the case of a head on collision. In this section we are going to consider collisions in an arbitrary geometry, in doing so we will use the techniques and methods developed in section 5.

![Figure 10: Scattering geometry for a non head-on collision](image)

We are first going to consider the case with no laser or electron focusing effects, just like in section 5. The reason for this is that for the general case that involves the general cross section formula (equation 3.2) it is impossible to qualitatively study the resulting brightness due to the complexity of the problem. So even though the idealized case is just an approximation it still provides valuable insight in the behavior of the brightness. The general case is studied at the end of this section.

6.1 Nearly Head-on collision

To derive an expression for the peak brightness of a nearly head-on collision, we start from the same starting point as we did for the head-on collision in

---

37 $\phi_p = 0, \xi_{lx} = \xi_{ex} = \xi_{ey}$

38 i.e. collisions that have a small angle, $\theta_0$, between the electrons and the photons

---

37

38
where $\theta_0$ is the angle between the electron bunches and the photon pulses. From the general differential cross section formula, equation (3.2), for the case when $\theta = \phi = \xi_{yl} = \xi_{xl} = \xi_{el} = \xi_{el} = \phi_p$, we observe the behavior depicted in fig.11.

\begin{align*}
\frac{dN_s}{d\Delta x d\Omega d\omega_s} &= r_0^2 \frac{d\sigma}{d\Omega}(\theta_0)(1 + \beta_0 \cos(\theta_0)) n_e n_p \delta(\omega_s - g(\theta) \omega), \quad (6.1)
\end{align*}

It can be seen from fig.3.19 that for small collision angles the cross section is nearly equal the the head-on cross section in the energy range we are working with. Thus as a good approximation we can approximate the small angle non head-on collision cross section as the head-on collision cross section. This means that we consider the cross section independent on the collision angle, $\theta_0$, and it is important to remember that this is only valid for small collision angles, $\theta_0$, and this approximation will cease to be valid as the collision angle increases. We can also as a very good approximation use that for small angles $\cos \theta_0 \approx 1$. Now following the steps of section 5 we obtain the relation

\begin{align*}
\frac{dN_s}{dxdydzdt_d\Omega d\omega_s} = \frac{n_e n_p}{\pi \Delta v_\perp} \int_0^{+\infty} S_\gamma(v_\perp) \exp \left( - \frac{v_\perp}{\Delta v_\perp} \right)^2 2\pi v_\perp dv_\perp. \quad (6.2)
\end{align*}
This expression looks similar to the expression that we obtained for the head-on collision, but this expression is only correct for small deviations from a head-on collision and a big difference is hidden within the photon density, $n_\lambda$. We can without loss of generality assume that the electron bunches still move along the positive $z$-axis, and we can use the electron density given in equation (4.8). Since the photon pulses are incoming with an angle, we have to modify both the time and the spatial part of the photon density distribution in order to correctly describe the brightness in a non head-on collision.

**Photon density**

The photon density distribution, $n_\lambda$, for a head on collision is given by equation (4.6), this distribution will still hold true if we work in the photon laser frame, but when we calculate our brightness we would like to express the photon distribution in terms of the stationary coordinate system. If we assume that non head-on and the head-on distributions are both defined relative to the same $x$-axis, then we can obtain the coordinates of the non head-on collision distribution via a rotation around the $x$-axis of the head-on collision distribution. This means that we are using the fact that we know the form of the photon density, $n_\lambda$, for the case when the photons move along the positive $z$-axis and we can transform this expression into the new desired form by performing a rotation in the $y,z$ plane. Using equation (3.3) for a rotation in the $y,z$ plane, one finds

$$
\begin{align*}
  x' &= x, \\
  y' &= y \cos \theta_0 + z \sin \theta_0, \\
  z' &= -y \sin \theta_0 + z \cos \theta_0.
\end{align*}
$$

With these relations we can express the photon density in the stationary frame as

$$
\begin{align*}
n_\lambda(r, z, t_d) &= \frac{N_\lambda}{\sqrt{\pi} 2 \Delta tcw_0^2 (1 + \left(\frac{-y \sin \theta_0 + z \cos \theta_0}{z_R}\right)^2)} \\
&\times \exp \left[-2\left(\frac{t_d + 2\frac{-y \sin \theta_0 + z \cos \theta_0}{c}}{\Delta t}\right)^2 - 2\frac{x^2 + \left(y \cos \theta_0 + z \sin \theta_0\right)^2}{w_0^2 (1 + \left(\frac{-y \sin \theta_0 + z \cos \theta_0}{z_R}\right)^2)^2}\right].
\end{align*}
$$

To perform the integral over the electron and photon distributions, we will use the fact that for small collision angles, $\theta_0$, we expect that the peak-brightness still occurs at the point $t_d=x=y=0$, and that it is still valid to integrate over the $z$-coordinate, this approximation will of course break down as the collision angle increases.

---

<sup>39</sup>We can always choose our coordinate axis so that this is the case.
θ₀, gets bigger. Thus we can then write the integral over the electron and photon densities for the peak-brightness as

\[
I = \int_{-\infty}^{+\infty} n_e(0, z, 0)n_\lambda(0, z, 0)dz
\]

\[
= \frac{N_e N_\lambda}{\left(\frac{\pi}{\sqrt{2}}\right)^3 c^2 r_b^2 w_0^2 \Delta t \Delta t} \int_{-\infty}^{+\infty} \frac{dz}{\left[1 + (k_f z)^2\right]\left[1 + \left(\frac{z \cos \theta_0}{z_R}\right)^2\right]}
\]

× \exp \left[-\frac{z^2}{c^2} \left(\frac{[1 - \beta_0]^2}{\Delta \tau^2} + \frac{2(1 + \cos^2 \theta_0)}{\Delta t^2}\right) - 2 \frac{z^2 \sin^2 \theta_0}{w_0^2 (1 + \frac{z^2 \cos^2 \theta_0}{z_R^2})}\right]. \quad (6.7)

This expression can be simplified if we introduce the non-head-on normalized beta function \( \eta = \frac{k_f c \Delta t}{2 \sqrt{2} \sqrt{1 + \cos \theta_0}} \), the normalized inverse Rayleigh length \( \mu = \frac{c \Delta t}{2 \sqrt{2} z_R} \) and the non-head-on normalized axial position \( \bar{z} = \frac{2 \sqrt{2} \sqrt{1 + \cos \theta_0} z}{c \Delta t} \). We also make the approximation that \( \frac{[1 - \beta_0]^2 + 2(1 + \cos^2 \theta_0)}{\Delta \tau^2} \approx \frac{2(1 + \cos^2 \theta_0)}{\Delta t^2} \). \( ^{40} \) Thus the integration over the electron and photon density yields

\[
I = \int_{-\infty}^{+\infty} n_e(0, z, 0)n_\lambda(0, z, 0)dz
\]

\[
= \frac{N_e N_\lambda \sqrt{1 + \cos \theta_0}}{\pi^3 c^2 r_b^2 w_0^2 \Delta t} \int_{-\infty}^{+\infty} \frac{dz}{\left[1 + \left(z \eta\right)^2\right]\left[1 + \left(z \mu\right)^2\right]}
\]

× \exp \left[-\bar{z}^2 - 2 \frac{z^2 \Delta t^2 \sin^2 \theta_0}{w_0^2 (1 + \left(z \mu\right)^2)}\right] \quad (6.8)

For all practical purposes this integral will be evaluated numerically, but for completeness and in analogy with section 5, we can derive an analytic expression for this integral if we use the Taylor expansion of the exponential and only keep the first two terms, this approximation is valid if the collision angle, \( \theta_0 \), is small.

\( ^{40} \)This approximation is still valid for most relativistic cases if the collision angle, \( \theta_0 \), is sufficiently small.
From the form of the integral $I$ we see that the brightness scales as $\frac{N_e N_\lambda}{\Delta \tau}$. Thus an easy way to increase the brightness is, as expected, to increase the number of particles in the electron bunches, photon pulses, and to decrease the bunch duration. These observation only hold as long as the approximations we made in this section is valid, and increasing the number of particles will eventually encounter a difficulty since it will effect the radius of focus, $r_b$, and the focal radius, $w_0$, and in turn the expression for the brightness. We can also see from the form of equation (6.9) that it approaches the same expression as in section 5 when the collision angle, $\theta_0$, approaches zero. With the help of the distribution integral in equation (6.9) and equation (5.20), we can express the non head-on collision peak brightness as

$$B = \frac{dN_s}{dx dy dt d\Omega d\omega_s} \delta \Omega \delta \Sigma \delta \omega_s \left( 10^{-15} \omega_s \int_{-\infty}^{+\infty} n_e(0,z,0)n_\lambda(0,z,0) dz \right)$$

$$\times \int_{0}^{+\infty} S_\gamma(v_\perp) \exp \left( - \left( \frac{v_\perp}{\Delta v_\perp} \right)^2 \right) 2\pi v_\perp dv_\perp,$$  

(6.10)

where $\Delta v_\perp$ is given in terms of the normalized emittance, $\epsilon$, according to $\Delta v_\perp = \frac{\epsilon \beta}{r_b}$. We have now obtained exactly what we were looking for, an expression for

---

41 This integral was performed with Wolfram Mathematica 9
the peak on-axis brightness, B, for the case of a non head-on collision. This is a very general result and can with great success describe the brightness, B, however to take into account laser and electron focus effects it is necessary to start from the result in section 3.

6.2 Laser and electron focusing effects

In this subsection we will analyse the brightness, B, obtained from equation (3.19). We will in this subsection explore how to include laser and electron focus effects when calculating the brightness, B. The reason that it may be of interest to include these effects is that it gives a more accurate description of the brightness. We will begin by finding an expression for the peak on-axis brightness in analogy with section 5. We will then proceed by taking into account laser and electron focus effects, and then finally develop the case of general collision angles, θ₀.

On-axis peak spectral brightness

In section 5 we began by assuming the ideal case of no photon or electron focusing effects. In this section we will not put any constraints on these parameters and in this case we do not get an easy analytic expression for the differential cross section, \( \frac{d\sigma}{d\Omega} \), instead we have to use equation (3.19). From equation (3.4) we obtain

\[
\frac{dN_s}{d^4xd\Omega d\omega_s} = \frac{d\sigma}{d\Omega} \frac{g'}{\gamma_0} n_e n_p \delta(\omega_s - g(\theta, \xi_{ly}, \xi_{lx}, \xi_{ex}, \xi_{ey})\omega),
\]

where we have used \( g' = g'(\theta_0, \xi_{ly}, \xi_{lx}, \xi_{ex}, \xi_{ey}) \), defined in equation (3.48). We are still only interested in the radiation in the forward direction i.e \( \theta = \phi = 0 \), but we do not put any constraints on the collision angle, \( \theta_0 \). To take into account the finite bandwidth within the photon pulse we assume that the frequency spread follows a Gaussian distribution, and integrate over all possible frequencies to obtain

\[
S_{\omega} = \frac{d\sigma}{d\Omega} \frac{g'}{\gamma_0} n_e n_p \sqrt{\pi \Delta \omega} \int_{-\infty}^{+\infty} \delta(\omega_s - g(\omega)) \exp \left[ -\left( \frac{\omega - \omega_0}{\Delta \omega} \right)^2 \right] d\omega.
\]

This integral can then be performed with equation (C.7) in the appendix, to obtain

\[
S_{\omega} = \frac{d\sigma}{d\Omega} \frac{g'}{\gamma_0} n_e n_p \lambda \sqrt{\pi \Delta \omega} |g| \exp \left[ -\left( \frac{\omega_s - \omega_0}{\Delta \omega} \right)^2 \right].
\]

\[42\] \( \xi_{ly} = \xi_{lx} = \xi_{ex} = \xi_{ey} = 0 \)

\[43\] With standard deviation \( \Delta \omega \)
To account for the energy spread we follow the steps of section 5.2, to obtain

$$S_\gamma = \frac{g'}{\gamma_0} \frac{n_e n_\lambda}{\pi \Delta \omega \Delta \gamma} \int_{-\infty}^{+\infty} \frac{d\sigma}{d\Omega} \frac{1}{|g|} \exp \left[ -\left( \frac{\omega - \omega_0}{\Delta \omega} \right)^2 \right].$$

(6.14)

**Laser focusing effects**

Now that we have an expression for \( S_\gamma \) we take the next step by taking into account laser focusing effects. Recall from equation (4.6) that the Rayleigh length, \( z_R \) is given by

$$z_R = \frac{\pi w_0^2}{\lambda_0}.$$  

(6.15)

However this expression is only true if the spatial part of the laser is distributed as a perfect Gaussian distribution. In practice this is not the case, and one usually introduce the so called \( M^2 \) parameter to represent how close the laser is to being a perfect Gaussian distribution. The Rayleigh length is then simply modified to

$$z_R = \frac{\pi w_0^2}{\lambda_0 M^2}.$$  

(6.16)

It is then possible to express an effective emittance for the laser beam \(^{44}\) as

$$\epsilon_L = \frac{M^2 \lambda_0}{\pi} = \frac{w_0 w'_0}{w_0},$$

(6.17)

where \( w'_0 \) represents the width of the angular divergence. The width of the angular divergence is typically given in both the \( x, y \) directions, but if we for simplicity assume that these are equal we obtain the relation

$$\Delta \xi_{yl} = \Delta \xi_{l x} = \frac{\epsilon_L}{w_0}.$$  

To model the laser focus throughout the beam we have to introduce some distribution \( f(\xi_{ly}, \xi_{lx}, \Delta \xi_{ly}, \Delta \xi_{lx}) \). We place no particular constraint on this distribution other than it should be normalized such that integrating over all angles, \( \xi_{lx}, \xi_{ly} \) gives unity. We will make the choice of a Gaussian distribution, but the final expression for the brightness is valid for an arbitrary distribution, \( f \), as long as the above criteria are met. Our choice of laser angle distribution, \( f \), is that of a Gaussian distribution,

$$f(\xi_{ly}, \xi_{lx}, \Delta \xi_{ly}, \Delta \xi_{lx}) = \frac{2}{\pi \Delta \xi_{ly} \Delta \xi_{lx}} \exp \left[ -2 \left( \frac{\xi_{lx}}{\Delta \xi_{lx}} \right)^2 \right] \exp \left[ -2 \left( \frac{\xi_{ly}}{\Delta \xi_{ly}} \right)^2 \right].$$

(6.18)

\(^{44}\)See [6]
Electron beam focus effects

To account for electron beam focus effects we will proceed completely analogous to the laser focus effects. The divergence of the electrons at the beam waist is given by

$$\Delta \xi_{ye} = \Delta \xi_{xe} = \frac{\xi}{r_b},$$  \hspace{1cm} (6.19)

where we have assumed that the spot size, $r_b$, is the same in both the $x$, $y$ directions. To model the electron beam focus effects we will assume that the the angles are spread out according to some distribution, $g(\xi_{ye}, \xi_{xe}, \Delta \xi_{ye}, \Delta \xi_{xe})$, this distribution, is as for the case of the laser distribution also arbitrary, but it has to be normalized such as the integration over all angles gives one. We will assume a Gaussian distribution of the electron angles, but the expression for the brightness is equally valid for any choice of distribution that obeys the above conditions. Our choice of distribution, $g$, is

$$g(\xi_{ye}, \xi_{xe}, \Delta \xi_{ye}, \Delta \xi_{xe}) = \frac{2}{\pi \Delta \xi_{ye} \Delta \xi_{xe}} \exp \left[ -2 \left( \frac{\xi_{ye}}{\Delta \xi_{ye}} \right)^2 \right] \exp \left[ -2 \left( \frac{\xi_{xe}}{\Delta \xi_{xe}} \right)^2 \right].$$

We can then obtain an expression for the on-axis brightness

$$\frac{dN_s}{dxdydt} = \int S_T g(\xi_{ye}, \xi_{xe}, \Delta \xi_{ye}, \Delta \xi_{xe}) f(\xi_{ly}, \xi_{lx}, \Delta \xi_{ly}, \Delta \xi_{lx}) d\xi_{xl} d\xi_{yl} d\xi_{xe} d\xi_{ye}.$$  \hspace{1cm} (6.20)

The peak brightness can then be obtained by integrating over the spatial coordinates and multiplying with the appropriate factors as described in equation (5.27). The expression for the brightness described above can not be solved analytically, so we must resort to purely numerical simulations. However due to the dimensionality of the integral that have to performed, converges slowly for common numerical integration schemes, we should use a Monte Carlo integration technique to perform these integrations. We have obtained a completely general expression for the peak on-axis brightness, $B$, taking into account energy spread, finite bandwidth and electron, photon focusing effects.

6.3 Scattered frequency spread

We know from equation (3.48) that the energy of the scattered radiation depends on a variety of factors, in particular it depends on the energy of the incident

\[^{45}\text{See [6]}\]
electron beam, the frequency of the incident radiation, as well as laser and electron focusing effects. And thus a spread in these parameters will result in a spread of the scattered radiation. If we only look in the direction specified by \( \theta = 0, \phi = 0 \) and consider a head on collision the relative frequency spread of the scattered radiation can be approximated as

\[
\frac{\Delta \omega_s}{\omega_s} \approx \sqrt{\frac{\gamma_0^4 (\Delta \xi_{xe}^2 + \Delta \xi_{ye}^2)}{4} + \frac{\Delta \gamma^4}{\gamma_0^2} + \frac{(\Delta \xi_{xl}^2 + \Delta \xi_{yl}^2 + \Delta \xi_{ye}^2 + \Delta \xi_{xe}^2)}{8}} + \frac{\Delta \omega^2}{\omega_0^2}.
\]

(6.22)

6.4 General collision angle

We have up until now only considered the case of small collision angles, \( \theta_0 \), and the difficulty arising from a general collision angle comes mainly from the difficulty to determine over what path to integrate the electron bunch and laser pulse density function. There exists a couple of ways to deal with this problem, one of them is to bypass the problem entirely and just integrate over whole space-time to obtain, the total number of scattered photons, \( N_s \), and then use the relation

\[
B \approx \frac{f N_s}{4 \pi^2 \epsilon^2},
\]

(6.23)

where \( f \) is the repetition rate. This is a rough approximation, but it can still provide a way to determine the approximate value of the brightness.

To describe the situation of a general collision angle without any major approximation it is necessary to first recall the definition of the detector time is

\[
t_d = \frac{z_d - z}{c}.
\]

(6.24)

In the definition of the detector time, \( t_d \), in equation (6.24) it is assumed that the scattering takes place on the z-axis, however if we want to describe the general case we have to allow the emission of the x-rays to take place at a general position. So we will assume that the emission takes place at a point \((x, y, z)\) and redefine the detector times as

\[
t_d = \frac{\sqrt{x^2 + y^2 + z^2}}{c},
\]

(6.25)

\(^{46}\text{See [6]p.12}\)
where we have put the detector position, $z_d$, to 0 since the detector position is arbitrarily. In section 5 and section 6 we had to calculate the z integral over the photon, $n_\lambda$, electron density, $n_e$,

$$I = \int_{-\infty}^{+\infty} n_e(0, z, 0) n_\lambda (0, z, 0) d,$$  \hspace{1cm} (6.26)

in order to obtain the brightness. For a general angle we assume that the x,y point is distributed according to some probability distribution\(^{47}\) and instead of only integrating over z, we integrate over all volume to obtain

$$I \rightarrow I = \frac{1}{2\pi \sigma_x \sigma_y} \int_{\mathbb{R}^3} n_e(x, y, z, t_d) n_\lambda (x, y, z, t_d)$$

$$\times \exp \left[- \left( \frac{x^2}{2\sigma_x^2} \right) \right] \exp \left[- \left( \frac{y^2}{2\sigma_y^2} \right) \right] dV. \hspace{1cm} (6.27)$$

Note that the unit of the integral in equation (6.27) is the same as in section 5\(^{48}\) just as in the small angle case. From the form of the electron and photon density\(^{49}\) it is clear that the peak on-axis brightness occur at detector time $t_d = 0$, just as in the small angle case. Note that a reasonable choice of the x,y standard-deviation is

$$\sigma_x \sim w_0, \quad \sigma_y \sim w_0.$$  

This technique works equally well for small scattering angles, but then it is required to calculate a volume integral instead of a line integral, and this will cause the computational time to increase, it is thus favorable to use the small angle approximations when it is applicable in order to decrease reduce numerical calculation time.

\(^{47}\) In our case we assume a Gaussian distribution.

\(^{48}\) The standard deviation for the x,y Gaussian distribution\(( \sigma_x, \sigma_y )\) got the unit of length.

\(^{49}\) See 4.8 and equation (6.6).
7 More general laser pulse

Up until now we have worked with very symmetrical laser pulses\textsuperscript{50}, meaning that we have not considered that the central frequency may vary across the laser pulse, or that the spatial distribution does not have to be cylindrical. In this section we are going to consider the chirped lasers and flattened lasers, and study how these types of laser pulses effect the brightness, $B$.

7.1 Flattened laser pulse

Recall from section 6 that the photon density, $n_{\lambda}$, for a general collision angle can be written as

$$n_{\lambda}(r, z, t_d) = \frac{N_{\lambda}}{\sqrt{\frac{\pi}{2}}} \Delta t c w_{0}^{2} \left(1 + \left(\frac{-y \sin \theta_0 + z \cos \theta_0}{z_R}\right)^2\right)^{\frac{3}{2}} \sqrt{\pi^2} \left[\exp \left(-\frac{2}{w_0^2} \left(\frac{t_d}{\Delta t} + \frac{2}{c} \frac{-y \sin \theta_0 + z \cos \theta_0}{\Delta t}\right)^2 - 2 \left(x^2 + \left(y \cos \theta_0 + z \sin \theta_0\right)^2\right)\right]$$

(7.1)

A flattened beam means that we let one of the radial components\textsuperscript{51} to be modified to $\frac{x}{w_0} \rightarrow \left(\frac{x}{w_0}\right)^p$, where $p$ is the flattening parameter\textsuperscript{52}.

\textsuperscript{50}Symmetrical in terms of temporal and spatial distributions.
\textsuperscript{51} $x$ or $y$ for $\theta_0 = 0$
\textsuperscript{52}See fig. 12
Figure 12: Flattening for a Gaussian distribution, $\sim \exp \left[ - (x^2)^p \right]$

We will consider that the it is the x-coordinate that is flattened, all the steps in this section may be equally well performed for the y-coordinate, the problem is that it gets more messy when $\theta_0 \neq 0$. It is pretty straight forward to modify the laser pulse density, the one thing that is important is to get the right normalization factors. The reason for this is that when $p \neq 1$ we are not dealing with a Gaussian anymore, and the normalization factors do not have an easy expression. Let's begin by only writing out the normalization factors that is coming from the temporal and the $y, z$ part of the density,

$$\frac{1}{\sqrt{\frac{2}{\pi}} \Delta t c w_0} \sqrt{\frac{1}{1 + \left( \frac{-y \sin \theta_0 + z \cos \theta_0}{z_R} \right)^2}}.$$  (7.2)
The normalization factor that is arising from the x-part of the density is then the integral over the exponential factor,

\[
\int_{-\infty}^{+\infty} \exp \left[ -2 \left( \frac{x^2}{w_0^2} \right)^p \frac{1}{1 + \left( \frac{1}{-y \sin \theta_0 + z \cos \theta_0 z_R} \right)^2} \right] dx.
\] (7.3)

The normalization factor above looks complicated but we can use the fact that the x part is first raised to the power of two and then raised with the broadening parameter, \(p\), this makes the integral even, and we can rewrite the normalization factor in terms of the gamma function, \(\Gamma\), as

\[
\int_{-\infty}^{+\infty} \exp \left[ -ax^b \right] dx = [\text{even integrand}]
= 2 \int_0^{+\infty} \exp \left[ -ax^b \right] dx = \left[ \frac{t}{dt} = ax^b \right] = 2 \int_0^{+\infty} t^{\frac{1-b}{b}} a^{-\frac{1}{b}} e^{-t} dt
= 2a^{-\frac{1}{b}} \frac{1}{b} \Gamma \left( \frac{1}{b} \right),
\] (7.4)

where \(\Gamma\) is the gamma function. For our case \(b = 2p\), and \(a = \frac{2}{w_0 p (1 + (\frac{2}{z_R} y \sin \theta_0 + z \cos \theta_0)^2)}\).

Now using all of this together we can write the laser pulse density as

\[
n_{\lambda}(r, z, t_d) = \frac{N_{\lambda}}{\sqrt{\pi} \Delta t c w_0} \frac{1}{\sqrt{1 + \left( \frac{-y \sin \theta_0 + z \cos \theta_0}{z_R} \right)^2}} \frac{1}{\frac{2}{p^2} a^{-\frac{1}{p}} \Gamma \left( \frac{1}{2p} \right)}
\times \exp \left[ -2 \left( \frac{t_d + \frac{2}{c} y \sin \theta_0 + z \cos \theta_0}{\Delta t} \right)^2 \right] \times \exp \left[ -2a \left( \frac{x^2}{w_0^2} \right)^p \right].
\] (7.5)

It should now be obvious why considering \(y\) as the flattening variable is a mess, when \(\theta_0 \neq 0\) the procedure of determining the normalization factor gets very complicated.

### 7.2 Chirped laser pulse

A chirped laser pulse is a laser pulse where the central frequency varies within the laser pulse. A chirped laser will give a lower peak-brightness, but it may also shorten the scattering time, thus it is possible that the average brightness may

\[\text{Effectively we let } x \to |x| \text{ in the exponent.}\]
increase even though the peak-brightness decrease. Luckily this new phenomena only comes into play in frequency exponential factor of equation \((7.6)\), as

\[
S_\gamma = \frac{g'}{\gamma_0} \frac{n_e n_\lambda}{\pi \Delta \omega \Delta \gamma} \int_{-\infty}^{+\infty} \frac{d\sigma}{d\Omega} |g| \exp \left[ -\left( \frac{w_s - \omega_0}{\Delta \omega} \right)^2 \right]. \tag{7.6}
\]

The nature of chirped beams means that \(\omega_0 = \omega_0(z)\), this simple fact implies that all our calculations for the simplified version of brightness are unusable, since they assumed that \(\omega_0 = \text{const} \). However, the machinery we developed for the general formula\(^{54}\) still holds since we have not made any assumptions about the collision angle, \(\omega_0\). Let us rewrite the exponential factor in \(S_\gamma\) as

\[
\exp \left[ -\left( \frac{w_s - \omega_0(z)}{\Delta \omega} \right)^2 \right] = \exp \left[ -\omega_0(0)^2 \left( \frac{w_s - \omega_0}{\Delta \omega} \right)^2 \right] = \exp \left[ -\omega_0(0)^2 \left( 1 - \frac{\delta \omega_0(z)}{\omega_0(0)} \right)^2 \right] = \exp \left[ -\left( \frac{\delta \omega_0(z)}{\omega_0(0)} \right)^2 \right], \tag{7.7}
\]

where we have used \(\frac{w_s}{\omega_0(0)} = 1^{55}\), and defined \(\delta \omega_0(z) \equiv \omega_0(z) - \omega_0(0)\). Now depending on the specific chirpness of the beam we can choose \(\frac{\delta \omega_0(z)}{\omega_0(0)}\) freely. Using this we can continue with the analysis performed in section 6, just keeping in mind that \(S_\gamma\) is a function of \(z\).

\(^{54}\)See section 6
\(^{55}\)See equation 6.12 in the appendix
8 Result

We have through sections 3-6 analyzed the behaviour of the total number of scattered photons, \( N_s \), and the peak on-axis brightness, \( B \). In this section we are going to use the results obtained from these sections.\(^{56}\). In this report we are mainly interested in scattered photon wavelengths in the range 3.3-4nm\(^{57}\), this is because wavelengths in this range do not get absorbed by biological tissue and are thus very useful for biologists and chemists.

8.1 Experimental parameters

Our experimental parameters that we study is given in fig.13a-14.

<table>
<thead>
<tr>
<th>IR Laser</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Wavelength</td>
<td>nm</td>
<td>1028</td>
</tr>
<tr>
<td>Pulse Duration</td>
<td>ps</td>
<td>2-3</td>
</tr>
<tr>
<td>Msquare factor</td>
<td></td>
<td>1.3</td>
</tr>
<tr>
<td>Energy</td>
<td>( \mu J )</td>
<td>( \frac{2}{3} )</td>
</tr>
<tr>
<td>Repetition rate</td>
<td>MHz</td>
<td>176</td>
</tr>
</tbody>
</table>

(a) IR laser parameters

<table>
<thead>
<tr>
<th>Optical enhancement cavity</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Enhancement factor</td>
<td></td>
<td>1500</td>
</tr>
<tr>
<td>Spot size</td>
<td>( \mu m )</td>
<td>50</td>
</tr>
</tbody>
</table>

(b) Optical enhancement cavity parameters

The enhancement factor determines how many times the IR laser pulse energy gets multiplied, and in this case the enhancement factor is 1500, so we end up with an energy of 1 mJ per laser pulse.

\(^{56}\)See D in the appendix for the code used to create the result.

\(^{57}\)Water window
Since we are mainly interested in the water window, we will only consider electron energies of (3.48) 14.56-16 MeV, but the electrons have a maximal energy of 15 MeV, so we only need to consider electrons with an energy of 14.56-15 MeV. We will use an electron beam that got a charge of 100 pC, energy of 14.7 MeV, and a bunch duration, $\Delta \tau$ of 2 ps. We will also use a laser pulse duration of 2 ps, and work with a collision angle, $\theta_0$, of 18°. We will now use the brightness formula derived in section 6 to analyze its behavior. The results are summarized in fig.15-20.
8.2 Simulation result

Figure 15: Peak brightness, $B$, versus the collision angle, $\theta_0$. All experimental parameters, except the collision angle, are set to the values described in subsection 8.1.

Figure 16: Peak brightness, $B$, versus energy spread, $\Delta \gamma$. All experimental parameters, except the energy spread, are set to the values described in subsection 8.1.

Figure 17: Peak brightness, $B$, versus the electron bunch duration, $\Delta \tau$. All experimental parameters, except the electron bunch duration, are set to the values described in subsection 8.1.

Figure 18: Peak brightness, $B$, versus the laser pulse duration, $\Delta t$. All experimental parameters, except the laser pulse duration, are set to the values described in subsection 8.1.
Figure 19: Peak brightness, $B$, focal radius $w_0$. All experimental parameters, except the focal radius, are set to the values described in subsection 8.1.

Figure 20: Peak brightness, $B$, versus the radius at focus, $r_b$. All experimental parameters, except the radius at focus, are set to the values described in subsection 8.1.

Figure 21: Scattering rate, $\frac{dN_s}{dt_d}$, versus detector time $t_d$, for different values of the flattening parameter $p$. All experimental parameters, except the broadening, are set to the values described in subsection 8.1.

Figure 22: Scattering rate, $\frac{dN_s}{dt_d}$, versus detector time $t_d$, for different values of the flattening parameter $p$. All experimental parameters, except the broadening, are set to the values described in subsection 8.1.
Figure 23: Scattering rate, $\frac{dN_s}{dt_d}$ versus detector time $t_d$, for different values of the broadening parameter $p$. The broadened beam profile is normalized the same regardless of $p$, resulting in a higher(lower) number of photons in the laser pulse, $N_\lambda$, when $p$ is bigger(smaller) than 1.

### 8.3 Optimization

The optimization of performed over the simplified model described in section 6, the reason that we do not use the general model is that for optimization in the genetic algorithm the brightness, $B$, needs to be evaluate thousands of time, and due to limited computing power we do not have time\(^{58}\) to use the general formula\(^{59}\). Instead we will use the simplified formula for the brightness and make the assumption that the general formula should follow the same trend, this method will of course only yield an approximate optimization. We will then in the end use the more general formula to calculate the optimal value given the

\(^{58}\) In this report
\(^{59}\) See equation (5.27)
The genetic algorithm was performed with the help of MATLAB, and the result is summarized in figure 24.

<table>
<thead>
<tr>
<th>Optimal parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Bunch duration, $\text{ps}$</td>
<td>0.5</td>
</tr>
<tr>
<td>Laser pulse duration, $\text{ps}$</td>
<td>2</td>
</tr>
<tr>
<td>Focal radius, $w_0$, $\mu$m</td>
<td>20</td>
</tr>
<tr>
<td>Radius at focus, $r_b$, $\mu$m</td>
<td>20</td>
</tr>
<tr>
<td>Gamma spread, $\Delta \gamma$</td>
<td>0.2</td>
</tr>
<tr>
<td>Emittance, $\epsilon$, mm mrad</td>
<td>1.5</td>
</tr>
<tr>
<td>Broadening parameter, $p$</td>
<td>1</td>
</tr>
</tbody>
</table>

*Figure 24: Optimization parameter, all the parameters were allowed to vary about their designated values described in section 8.1, by 50% of their original value.*

And the optimal peak on-axis brightness obtained was $1.88 \times 10^{23}$ Photons $\text{mrad}^{-2}\text{mm}^{-2}\text{BW}^{-0.1}\%$.

The validity and usefulness of this result is discussed in section 9.

---

60 For the case of an collision angle of 18°
9 Discussion

For the entirety of this report our goal have been to optimize the brightness, B, and in section 8 we saw the brightness dependence on the parameters of interest. From the optimization in section 8.3 we obtain that the maximum value occurs when we got as low energy spread, \( \Delta \gamma \), focal radius, \( w_0 \), radius at focus, \( r_b \), electron bunch duration, \( \Delta \tau \), laser pulse duration, \( \Delta t \) and emittance, \( \epsilon \) as possible. This result is not that surprising since they were exactly what we might have expected. However, when optimizing this process we have to consider that in the physical world everything is not perfect, and no electron bunch or laser pulse will have exactly the specified parameters, in particular they might vary for each bunch and pulse. So we also have to consider that even though increasing one of these parameters yields a bigger peak brightness, B, it might also render the average peak brightness more unpredictable since a little change to the parameters might have a big effect on the peak brightness. We will in this section discuss the constraints of the relevant parameters of section (8), and investigate how changing of one of them might produce experimental complications.

9.1 Laser pulse focal radius

From fig.19 in section 8, it is shown that the peak on-axis brightness, B, increases significantly when the focal radius, \( w_0 \), gets small enough \(^6\). So at first glance the conclusion is to minimize the focal radius in order to maximize the peak on-axis brightness. There are however some additional considerations that has to be taken into account. From fig.19 we note that as the focal radius, \( w_0 \), gets smaller the tilt of the curve also becomes steeper. From an engineering standpoint it is impossible to make exactly the same parameters for every laser pulse, electron beam overlap, and thus for small focal radii a relatively small difference between two overlaps will result in a big change in both the average and the peak on-axis brightness. It is thus important to consider how precise and consistent one can make the target focal radii before trying to minimize it; also it is important to know how big of a fluctuation of the brightness is acceptable when specifying the focal radius.

Another important aspect to consider is that throughout this report we have assumed that the collision of the laser pulse and the electron beam is perfect, meaning that we have assumed that they focus perfect onto each other. Since we are dealing with time scales of the order \( \sim \) ps, it is hard to get a perfect overlap

\(^6\) \( w_0 < 40 \mu m \)
of the laser pulse and the electron beam, a more realistic case is to consider some small displacement between the pulse and the beam. So let’s make an estimate how much a small displacement of the laser pulse relative to the electron beam may effect the brightness. The beam waist at a point \( z \) is described as

\[
w(\z) = w_0 \sqrt{1 + \left( \frac{z}{z_R} \right)^2},
\]

(9.1)

where the Rayleigh range is given by \( z_R = \frac{\pi w_0^2}{\lambda} \). The Rayleigh range is defined as the \( z \)-coordinate where the beam width, \( w \), is \( \sqrt{2} w_0 \); and thus a smaller Rayleigh range, \( z_R \), means that the laser pulse diverges faster. In practice this means that for a small focal radius, \( w_0 \), a small displacement of the laser pulse focus relative to the electron beam focus will have a bigger effect on the beam waist, \( w \), and will thus effect the brightness. So let us consider the case when the mismatch of the laser pulse relative to the electron beam is of the order of \( \Delta t \sim 1 \text{ps} \). Since the electrons are highly relativistic we can approximate their speed as the speed of light, \( c \), and this implies that in the time \( \Delta t \) the electron beams have traveled \( \Delta l = c \Delta t \sim 0.3 \text{mm} \). Thus in order for a displacement of the order \( \Delta l \) not to have big impact on the brightness we must require that \( z_R \gg \Delta l \), so let us consider a Rayleigh range of a order of magnitude bigger than the displacement, i.e \( z_R \sim 3 \text{mm} \). This corresponds to a focal radius of

\[
w_0 = \sqrt{\frac{z_R \lambda}{\pi}} \sim 30 \mu \text{m}.
\]

(9.2)

Thus we see that for any focal radii smaller than 30\( \mu \text{m} \) a small displacement will have a large impact on the brightness. So let us see what is the maximum displacement possible if we had a focal radius of \( w_0 = 10 \mu \text{m} \). This focal radii correspond to a Rayleigh length of \( z_R \sim 0.3 \text{mm} \), and thus if we assume that \( z_R \) have to be a magnitude bigger than the displacement we find that the maximal displacement time is \( \Delta t \sim 0.1 \text{ps} \). To achieve a maximal displacement of \( \Delta t \sim 0.1 \text{ps} \) is extremely problematic from an engineering standpoint, and for the FREIA experiment it is out of the question. Thus there exists constraints on the matching of the laser pulse with the electron beam for a given focal radii, and even though a small focal radius will increase the brightness for a perfect scenario, there will be a difficulty to align the pulses and the beams sufficiently well to increase the brightness consistently. More likely the brightness will in fact decrease since the effective focal radius, beam waist, will be bigger when there is a small displacement, than it would for a bigger focal radius.

\[62\]See section (4.2)
Thus when decreasing the focal radius, $w_0$, in order to increase the brightness, it is essential that the technology is available in order to consistently have the same focal radius and have a small enough temporal displacement such that the average brightness lie within the specifications. The constraints we put on the focal radius and the displacement of the laser pulse with the electron beam can equally well be put on the radius at focus, $r_\text{f}$, and the divergence of the electron beam in analogy with the focal radius, and thus when reducing the radius at focus the same considerations have to be taken into account.

### 9.2 Laser pulse duration

From fig.18 we see that another way to increase the brightness is to decrease the laser pulse duration, $\Delta t$. This result is expected since it is natural to expect that when the same amount of photons interact during a shorter time, the intensity and the brightness of the resulting Thomson scattering will increase.

It is promising that in fig.18 the steepness of the graph stays about the same throughout the whole range of pulse duration. This means that a small difference in laser pulse duration in two pulses will result in about the same difference in peak on-axis brightness regardless of the target laser pulse duration\(^\text{63}\). From an engineering standpoint there is still the possibility that it is harder to consistently get near the target laser pulse duration when the duration gets lower, and this got to be taken into account when choosing the target laser pulse duration.

When considering what laser pulse duration to choose it is important to not only consider the brightness, but also consider what frequency spread of the resulting x-rays is required. From equation (6.22) and from relation between the laser pulse duration and the laser bandwidth, $\Delta \omega_0$\(^\text{64}\), we see that a shorter laser pulse duration will result in a longer bandwidth and thus resulting in a bigger frequency spread for the produced x-rays. Thus it is important to not only consider the brightness when optimizing with respect to the laser pulse duration, but to also consider the target frequency spread of the produced x-rays. If a maximal frequency spread, $\Delta \omega_s$, is decided, it is important to maximize the brightness with respect to the laser pulse duration while taking the available technology and into account, and making sure that the choose laser pulse duration is such that the frequency spread, $\Delta \omega_s$, is within the specifications. For example for the

\(^{63}\text{In the laser pulse interval } 4\text{ps} > \Delta t > 1\text{ps.}\)

\(^{64}\Delta t \Delta \omega_0 = 2\)
FREIA experiment the target bandwidth is the water window\textsuperscript{65}, and it may then be favorable to choose the energy of the beam such that the center wavelength of the scattered x-ray photons lie in the middle of this range, and the laser pulse duration is specified such that the frequency bandwidth stay within the water window.

### 9.3 Electron bunch duration

From the behavior of the brightness in fig.\textsuperscript{17} it is possible at first glance to deduce that in order to increase the brightness just choose as low of a electron bunch duration, $\Delta \tau$ as possible, however from an engineering standpoint the steepness of the curve increase significantly as the bunch duration gets shorter, and thus a small deviation from the target bunch duration will result in a bigger deviation in the average brightness for smaller bunch duration’s relative to longer ones.

The electron bunch duration is determined from the ballistic bunching energy chopper, and from Liouville’s theorem\textsuperscript{66} we have that the electron bunch duration times the energy spread got to be constant, and a increase of the bunch duration will result in an increase in the energy spread. Also in order to optimize the energy spread it is possible to ”cut” the energy beam\textsuperscript{67}, and this decreases the charge of the beam, and since the charge is directly proportional to the number of electrons and in turn directly proportional to the brightness, this means that the brightness will decrease. So the even though the relation of the brightness with respect to the bunch duration looks straight forward in fig.\textsuperscript{17}, there exists restrictions in the technology that makes this relation more complicated. So in order to choose the optimal bunch duration a lot of thought have to be made on the accelerator specifications and how a given bunch duration effects the beam charge, energy spread, and radius at focus.

### 9.4 Energy spread

From figure \textsuperscript{16} it can be seen that lowering the energy spread, $\Delta \gamma$, is an efficient way to increase the brightness. However according to Liouville’s theorem\textsuperscript{68} we have

$$\Delta p_i \Delta q^i = \text{constant},$$

\textsuperscript{65}3.3nm-4nm.
\textsuperscript{66}See [11] p.62
\textsuperscript{67}This method is used in FREIA
\textsuperscript{68}See [11] p.62
where $p_i$ are the conjugate momentum coordinates and $q_i$ are the canonical coordinates. Since the momentum are related to the energy and in turn the gamma factor, this implies that for the electron beam $\Delta \gamma r_b^2 = \text{constant}$. Thus a decrease in the energy spread, $\Delta \gamma$, will result in an increase in the radius at focus, $r_b$. So when optimizing the brightness with respect to the energy spread it is important to take into account that a change in the energy spread will also change the radius at focus and thus it is required to weigh the gain of decreasing the energy spread with the expense of increasing the radius at focus.

### 9.5 Angular dependence

From fig.15 we see that the peak on-axis brightness is optimized when the collision angle, $\theta_0$, is as small as possible. From an engineering stand point it is relatively easy to have a very small deviation from the target collision angle regardless of what it is. The reason that we use the angle $\theta_0 = 18^\circ$ in this report and not the optimal choice $\theta_0 = 0^\circ$ is that the electron bunches have to be injected into the optical enhancement cavity in order for them to interact with the laser pulses. So in order to not send the remaining laser pulse into the electron accelerator it is necessary to introduce a collision angle $\theta_0 \neq 0^\circ$. Thus the collision angle is largely dependent on the experimental setup and will be hard to change once the experiment is constructed.

### 9.6 Flattening of laser pulse

The idea of flattening the laser pulse is to increase the interaction time of the laser pulse with the electron beam. In fig.21 and fig.22 a re-normalized flattened beam profile is used, meaning that it is normalized according to section 7.1. It is then natural to expect the brightness and total number of scattered photons to be lower since we are effectively lowering the maximum value of the Gaussian profile in order to keep the normalization. Another possible situation would be to flattening the laser distribution, but keeping the old normalization, in effect increasing the total number of photons, $N_\lambda$. In this situation we expect that the brightness and the total number of scattered photons should increase, and according to fig.23 this is indeed the case. In fig.23 it can be seen that the scattering rate increase when $p > 1$ and decrease when $p < 1$.

---

69 This means the laser pulse is normalized so that the total number of photons $N_\lambda$ is the same as the non flattened case.
10 Conclusion

In section 9 a discussion was made about the results obtained in 8, and it was seen that at first glance the optimal on axis peak-brightness was obtained when the relevant parameters defined in section 8 was as low as possible. In section 9 it was pointed out that it is not realistic to minimize every parameter since there exists various constraints on the technology available. For instance the lowering of the focal radius, $w_0$, will indeed increase the brightness for a perfect focus of the laser pulse and the electron beam, but for a smaller focal radius the laser pulse diverges faster, making the available technology for matching the electron focus with the laser focus relevant.

There exists some additional consequences of reducing the parameters like for instance the reduction of the pulse duration, $\Delta t$, will result in the increase of scattered x-ray radiation bandwidth, and for FREIA it is important to have the x-ray bandwidth lie within the 3.3-4.4 nm range$^{70}$, so a reduction of the pulse duration could in effect enlarge the x-ray bandwidth such that it gets larger than the target range. This is also true for other parameters like the energy spread, $\Delta \gamma$, since the available energy spread depends on the electron bunch duration and the radius at focus.

So in conclusion it is important to first consider the technological limitations before determining the specifications of the x-ray radiation. Changing some of the parameters will result in more of a increase in brightness than others, however thought have to be taken to ensure that the limiting engineering constraints do not prove to be insufficient, resulting in a lower x-ray quality then intended.

---

$^{70}$The water window.
A Quantum electrodynamics corrections

In the previous sections we have based all our calculations and reasoning on the classical formulas for the differential cross section, $\frac{d\sigma}{d\Omega}$. But since for high enough energies (or few enough photons) the classical formulas fail, we have to use quantum electrodynamics instead. In this section we will not study the brightness directly, but we will instead investigate how valid the Thomson cross section, $\sigma_T$, is and possible corrections coming from quantum electrodynamics.

A.1 Klein–Nishina formula

The Klein–Nishina formula named after Oskar Klein and Yoshio Nishnia is the quantum field theory lowest order contribution to compton scattering (which reduces to thompson scattering in the classical limit). This formula is derived in section B in the appendix, and if it takes the following form in terms of the polarization vectors $\hat{e}$:

$$\frac{d\sigma}{d\Omega'} = r_0^2 \left( \frac{k'}{k} \right)^2 \left[ |\vec{e}^* \cdot \vec{e}_0|^2 + \frac{(k - k')^2}{4kk'} \left( 1 + (\vec{e}^* \times \vec{e}) \cdot (\vec{e}_0 \times \vec{e}_0^*) \right) \right], \quad (A.1)$$

where $k$ is the incident photon wave-vector and $k'$ is the outgoing radiation wave vector. To evaluate the polarization vector multiplication we will use the same technique as we did for the classical derivation. Note that the first contribution, $|\vec{e}^* \cdot \vec{e}_0|^2$, is exactly the one we calculated for the classical formula, and thus we can skip the calculation of it here. In the electron rest system we can then decompose the scattered radiation polarization vectors in terms of the $\hat{e}_1$, $\hat{e}_2$, unit vectors,

$$\vec{e}_1 = \cos \theta'_e (\hat{x}' \cos \phi'_e + \hat{y}' \sin \phi'_e) - \hat{z}' \sin \theta'_e, \quad (A.2)$$

$$\vec{e}_2 = -\hat{x}' \sin \phi'_e + \hat{y}' \cos \phi'_e. \quad (A.3)$$

If we assume that the incident radiation has an electric field

$$\vec{E}' = E'_0 (\alpha'_x \hat{x}' + \alpha'_y \hat{y}' + \alpha'_z \hat{z}'), \quad (A.4)$$

the incident radiation polarization vector is given by $\vec{e}_0 = (\alpha'_x, \alpha'_y, \alpha'_z)$. Note that since the incident wave vector is real, the cross product will automatically be

---

71 See [?]p.697
72 See equation 3.2
equal to zero and we do not have to calculate any of the cross products. Using the result of equation 3.2 we then obtain

\[
\frac{d\sigma}{d\Omega'} \frac{1}{r_0^2} = \sum_{i=1,2} \left( \frac{k'}{k} \right)^2 \left[ |e^*_i \cdot e_0|^2 + \frac{(k - k')^2}{4kk'} \right]
\]

\[
= \left( \frac{k'}{k} \right)^2 \left[ \alpha'_x^2 (1 - \cos^2 \phi'_e \sin^2 \theta'_e) + \alpha'_y^2 (1 - \sin^2 \phi'_e \sin^2 \theta'_e) + \alpha'_z^2 (1 - \cos^2 \theta'_e) - 2\alpha'_x \alpha'_y (\cos \phi'_e \sin \theta'_e) (\sin \phi'_e \sin \theta'_e) - 2\alpha'_x \alpha'_z \cos \theta'_e \sin \theta'_e \cos \phi'_e - 2\alpha'_y \alpha'_z \cos \theta'_e \sin \theta'_e \sin \phi'_e - \frac{(k - k')^2}{2kk'} \right] \tag{A.5}
\]

Now that we have the differential cross section, \(d\sigma/d\Omega'\), in the electron rest frame, we now have to transform it to the stationary frame. The way to perform this transformation is exactly the same as for the classical differential cross section in section 3, and the final result for the differential cross section, \(d\sigma/d\Omega\), in the stationary frame is

\[
\frac{1}{r_0^2} \frac{d\sigma}{d\Omega'} = \left( \frac{k'}{k} \right)^2 \left[ \alpha'_x^2 \frac{(1 - \cos^2 \phi_e \sin^2 \theta_e)}{\gamma^2(1 - \beta \cos \theta_e)^2} + \alpha'_y^2 \frac{(1 - \sin^2 \phi_e \sin^2 \theta_e)}{\gamma^2(1 - \beta \cos \theta_e)^2} + \alpha'_z^2 \frac{(1 - \cos^2 \theta_e)}{(1 - \beta \cos \theta_e)^2} - 2\alpha'_x \alpha'_y \frac{(\cos \phi_e \sin \theta_e) (\sin \phi_e \sin \theta_e)}{\gamma^2(1 - \beta \cos \theta_e)^2} - 2\alpha'_x \alpha'_z \frac{(\cos \theta_e - \beta) (\cos \phi_e \sin \theta_e)}{\gamma(1 - \beta \cos \theta_e)^2} - 2\alpha'_y \alpha'_z \frac{(\cos \theta_e - \beta) (\sin \phi_e \sin \theta_e)}{\gamma(1 - \beta \cos \theta_e)^2} - \frac{(k - k')^2}{2kk'} \right] \tag{A.6}
\]

where the components \(\alpha'_\nu\) is given by equation (3.53), and the angles may be written in terms of the angles of the stationary frame according to equation (3.38). To obtain the differential cross section in the electron lab frame, we have to use
equation (3.56). Now that we have obtained an expression for the differential cross section, $\frac{d\sigma}{d\Omega}$, in the rest frame, the next step is to find a relation between the incident and the scattered wave vectors $k$ and $k'$ respectively.

A.2 Wave vector relation

The scattering situation we are considering is presented in the figure ??.

Since the sum of the 4-momentum vectors have to be the same before and after the scattering we have that

$$p_1 + p_2 = q_1 + q_2. \quad (A.8)$$

Now since we are not interested in the angle between scattered the electron and the scattered photon, we can simply rearrange the terms and square to remove that particular 4-momentum,

$$q_2^2 = p_1^2 + p_2^2 + q_1^2 + 2(p_1 p_2) - 2(q_1 p_1) - 2(q_1 p_2). \quad (A.9)$$

Since photons are massless and any 4-momentum vector satisfies the relation $p^2 = m^2c^2$, all photon 4-momentum vectors squared equals zero. Then

$$(p_1 p_2) = \hbar \frac{\omega}{c} (mc, 0) = \hbar \omega, \quad (A.10)$$

$$(p_1 q_1) = \hbar^2 \frac{\omega'}{c} (mc, 0) = \hbar^2 (kk' - \cos \theta_e kk'), \quad (A.11)$$

$$(q_1 p_2) = \hbar (\frac{\omega'}{c}, k') = \hbar \omega', \quad (A.12)$$

where $\theta_e'$ is the angle between $p_1$ and $q_1$. Plugging this into equation (A.9) and rearranging the terms we obtain

$$\frac{k'}{k} = \frac{1}{1 + \frac{\hbar \omega}{mc^2} (1 - \cos \theta_e')} \quad (A.13)$$

Using the transformation relations for the angles from the electron rest frame to the stationary frame\textsuperscript{73}, we obtain the relation for the incident and the scattered

\textsuperscript{73}See equation 3.38
wave vectors in the coordinates of the stationary frame,

\[
\frac{k'}{k} = \frac{1}{1 + \frac{\hbar \omega}{m c^2} \left( 1 - \left( \frac{\cos \beta - \beta}{1 - \beta \cos \theta} \right)^2 \right)},
\]

(A.14)

\[
\cos \theta_e = \cos \phi \sin \xi_{xe} \cos \xi_{ye} + \sin \phi \sin \theta \sin \xi_{ye} + \cos \theta \cos \xi_{ye} \cos \xi_{xe}.
\]

(A.15)

If we compare the Klein-Nishina formula with the original formula for Thompson scattering, we observe that the Klein-Nishina formula depends on the frequency of the incident radiation, whereas the Thomson scattering formula does not. It is thus reasonable from the form of the differential cross section, \(\frac{d\sigma}{d\Omega}\), to expect that the Klein-Nishina deviates from the Thomson formula as the incident photon frequency, \(\omega_0\), is increased.

\[\text{Figure 25: The Klein-Nishina formula for different incident frequencies, with polarization angle } \phi_p = \frac{\pi}{4}, \gamma = 1\]

66
As can be seen in figure 25, the behaviour is exactly what we expect, for low frequencies the Klein-Nishina formula is the same as the Thompson formula, but as the frequency increases the validity of the Thomson approximation deteriorates. In our case we use a 1 nm laser, which corresponds to a angular frequency of $\omega_0 \approx 2 \times 10^{15}$ Hz, and thus our original assumption was correct: it is valid to use the Thomson formula for the frequencies we work with, but for 100 times higher frequencies or so the Thomson formula gets corrections from the Klein-Nishina formula.

### B Klein–Nishina formula derivation

To derive the Klein-Nishina formula we are going to start from a purely quantummelectrical, QED, lagrangian, $\mathcal{L}$, and only calculate the lowest order contribution to the cross section. We are going to use the following conventions in section ??, we will also denote the mass of the electron with $m$. We will however, use a different choice of metric tensor, $g$, such that

$$ (g_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (B.1) $$

And we will use the Weyl representations of the gamma matrices. We will also use the Feynman slash convention and note that all the calculations will be performed in natural units i.e $c = \hbar = 1$

$$ \gamma^\mu p^\mu, \quad (B.2) $$

$$ a^\mu b_\mu \equiv ab. \quad (B.3) $$

The lagrangian, $\mathcal{L}$, for Spinor electrodynamics is

$$ \mathcal{L} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\Psi} (i \partial - m) \Psi + e \bar{\Psi} \gamma^\mu \Psi A^\mu, \quad (B.4) $$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, $A^\mu$ is the photon field, $\Psi$ is the electron spinor field. The gamma matrices are given by

$$ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad (B.5) $$

\[^{74}\text{The reason for this is that it makes some of the calculations a bit less messy, but it is equally valid to use the metric defined in section ??}\]

\[^{75}[10]\text{p.351}\]
where $\sigma^i$ are the Pauli matrices. The Feynman rules can then be obtained from the lagrangian, $\mathcal{L}$, and we thus obtain the Feynman rules\textsuperscript{76}

\[ p_2 \gamma \]

\[ e^- \quad p_1 \quad p_1 + p_2 \quad e^- = ie\gamma\mu \]

\[ i \quad j \quad p = -\frac{i}{p^2 + m^2 - i\epsilon} \]

For an internal Dirac spin-$\frac{1}{2}$ fermion with mass $m$

\[ e^- \quad s, p \quad j \quad e^- \quad u_j(\vec{p}, s) \quad \text{For an initial Dirac spin-$\frac{1}{2}$ fermion with mass } m \]

\[ e^- \quad s, p \quad j \quad e^- \quad \bar{u}_j(\vec{p}, s) \quad \text{For a final Dirac spin-$\frac{1}{2}$ fermion with mass } m \]

\[ \gamma \quad \lambda, p \quad \mu \quad \gamma \quad \epsilon_{\mu}(\vec{p}, \lambda) \quad \text{For an initial photon} \]

\textsuperscript{76}See [10] Chapter 9-11
For a final photon

Now that we know the Feynman rules we can calculate the scattering amplitude for Compton scattering. We will only consider the lowest order contribution to the scattering amplitude, meaning that we will only consider tree diagrams with 2 vertices. The scattering amplitude then become

\[ iT = p_1 \epsilon^*_\mu(p_1, \lambda) \epsilon^\nu(p_2, \gamma) \]

where

\[ T = \epsilon_2^* \epsilon_2^\nu \bar{u}_1^\alpha A_{\mu\nu} u_1, \quad (B.6) \]

\[ A_{\mu\nu} = \gamma_\nu \left(-\gamma_1 - \gamma_2 + m\right) \gamma_\mu - s + m^2 - t + m^2, \quad (B.7) \]

In the expression above the notation \( \epsilon_2^\nu \equiv \epsilon_2^\nu(\lambda')^v \) has been used and we have used the mandelstam variables \( s, t \) defined as

\[ s = -(p_1 + p_2)^2 = -(q_1 + q_2)^2, \quad (B.8) \]

\[ t^2 = -(p_1 - q_2)^2 = -(p_2 - q_1)^2, \quad (B.9) \]

\[ u^2 = -(p_1 - p_2)^2 = -(q_1 - q_2)^2, \quad (B.10) \]

The Mandelstam variables satisfy \( s + t + u = 2m^2 \), as can be easy check by plugging in the definitions. Since \( A_{\mu\nu} = A_{\nu\mu} \quad (B.7) \) we have \( T = e^2 \epsilon_2^\nu \epsilon_2^\gamma \bar{u}_1 A_{\rho\eta} u_1' \) \quad (B.11) \)

We then have that \( |T|^2 = \bar{T}T \). To evaluate this expression we have to write out the summation variables explicitly and move them around to form a trace:

\[ \bar{T}T = e^4 \epsilon_2^* \epsilon_2^\nu \epsilon_2^\gamma \epsilon_2'^\rho \bar{u}_1' A_{\mu\nu} A_{\nu\rho} u_1 u_1'. \quad (B.12) \]
Rearranging the terms in the expression above gives us the sought after trace:

\[
e^A e^\star^\mu e^\nu e^\nu e^\eta e^\phi u^\nu_1 u^\nu_1 (A^\mu\nu)_{\alpha\beta} u^\nu_{1\alpha} (A^\rho\eta)_{\sigma\chi} u^\nu_{1\chi}
\]

(B.13)

\[
e^A e^\star^\mu e^\nu e^\nu e^\eta e^\phi u^\nu_1 u^\nu_1 (A^\mu\nu)_{\alpha\beta} u^\nu_{1\alpha} (A^\rho\eta)_{\sigma\chi}
\]

(B.14)

\[
e^A e^\star^\mu e^\nu e^\nu e^\eta e^\phi Tr \left[ u^\nu_1 u^\nu_1 A^\mu\nu u^\nu_1 A^\rho\eta \right]
\]

(B.15)

If we now make the assumption that we can not measure the photon polarization and electron spin final state we can sum over all combinations. If we also assume that we have no knowledge of the initial photon polarization and electron spin we can take the average over all combinations. This enables us to use the sum formulas for the spinors and the polarization vector.

\[
\sum_{s=\pm} u_s \bar{u}_s = -p + m,
\]

(B.16)

\[
\sum_{\lambda=\pm} e^\eta_2 e^\star^\mu = s^\eta^\mu.
\]

(B.17)

Using these relations we can define the spin averaged scattering amplitude, \( \langle |T|^2 \rangle \), as

\[
\langle |T|^2 \rangle \equiv \frac{1}{4} \sum_{\lambda,\lambda'} \sum_{s,s'} |T|^2
\]

\[
= \alpha^2 Tr \left[ A^\mu\nu (-p_1 + m) A^\rho\eta (-q_2 + m) \right] s^\rho^\eta^\mu
\]

\[
= \alpha^2 Tr \left[ A^\mu\nu (-p_1 + m) A^\nu\mu (-q_2 + m) \right]
\]

(B.18)

where we have defined \( \alpha = \frac{e^2}{\pi} \). We can write the expression for the spin averaged cross section more compact by introducing the expressions

\[
\langle \Phi_{tt} \rangle = Tr \left[ \gamma^\mu (-p + q_2 + m) \gamma^\nu (-p_1 + m) \gamma^\nu (-p_1 + q_2 + m) \gamma^\mu (-q_2 + m) \right],
\]

(B.19)

\[
\langle \Phi_{ss} \rangle = Tr \left[ \gamma^\nu (-p_1 - p_2 + m) \gamma^\mu (-p_1 + m) \gamma^\mu (-p_1 - p_2 + m) \gamma^\nu (-q_2 + m) \right],
\]

(B.20)

\[
\langle \Phi_{ts} \rangle = Tr \left[ \gamma^\mu (-p_1 + q_2 + m) \gamma^\nu (-p_1 + m) \gamma^\nu (-p_1 + q_2 + m) \gamma^\mu (-q_2 + m) \right],
\]

(B.21)

\[
\langle \Phi_{st} \rangle = Tr \left[ \gamma^\nu (-p_1 - p_2 + m) \gamma^\mu (-p_1 + m) \gamma^\nu (-p_1 + q_2 + m) \gamma^\mu (-q_2 + m) \right].
\]

(B.22)

---

79 These assumptions are valid for most cases, and in particular valid for our case of inverse-Compton scattering.

80 See [10] p.240,p.359
We can then write the spin averaged cross section, $\langle |T|^2 \rangle$, as

$$\langle |T|^2 \rangle = \alpha^2 \left( \frac{\langle \Phi_{tt} \rangle}{(m^2 - t)^2} + \frac{\langle \Phi_{ts} \rangle + \langle \Phi_{st} \rangle}{(m^2 - s)(m^2 - t)} + \frac{\langle \Phi_{ss} \rangle}{(m^2 - s)^2} \right). \quad (B.23)$$

Even though it looks like we have to evaluate each of the four traces above it can be seen by looking at the structure of the traces that if $q_2 \leftrightarrow -p_2$ we get

$$\langle \Phi_{tt} \rangle \leftrightarrow \langle \Phi_{ss} \rangle, \quad \langle \Phi_{st} \rangle \leftrightarrow \langle \Phi_{ts} \rangle.$$ 

Thus it is enough for us to evaluate two of the traces, we will choose to evaluate $\langle \Phi_{ss} \rangle$ and $\langle \Phi_{ts} \rangle$, but the choice is arbitrary. To evaluate the traces above we need some relations for the gamma matrices\(^{81}\):

\begin{align*}
\gamma^\mu \gamma_\mu &= 2\theta, \quad (B.24) \\
\gamma^\mu \gamma_\mu &= -4, \quad (B.25) \\
\gamma^\mu \gamma_\mu &= 4(a \cdot b), \quad (B.26) \\
\gamma^\mu \gamma_\mu &= 2\theta \theta, \quad (B.27) \\
Tr[\theta \theta] &= -4(ab), \quad (B.28) \\
Tr[\theta \theta \theta \theta] &= 4[(ad)(bc) - (ac)(bd) + (ab)(cd)], \quad (B.29) \\
Tr[\text{odd number of } \gamma^\mu s] &= 0\{\gamma^\mu, \gamma^\nu\} = -2\theta^{\mu\nu}. \quad (B.30)
\end{align*}

We begin by evaluating the trace $\langle \Phi_{ss} \rangle$ as

$$\langle \Phi_{ss} \rangle = Tr \left[ \gamma^\nu (-p_1 - p_2 + m) \gamma_\mu (-p_1 + m) \gamma^\mu (-p_1 - p_2 + m) \gamma^\nu (-q_2 + m) \right]$$

$$= 4Tr \left[ (g_2 + 2m)(p_1 + p_2 - m)(p_1 + 2m)(p_1 + p_2 - m) \right]. \quad (B.31)$$

We will divide the trace above into 3 separate calculations:

$$\langle \Phi_{ss} \rangle = Tr \left[ \gamma^\nu (-p_1 - p_2 + m) \gamma_\mu (-p_1 + m) \gamma^\mu (-p_1 - p_2 + m) \gamma^\nu (-q_2 + m) \right]$$

$$= 4(A + B + C). \quad (B.32)$$

where

$$A = Tr \left[ g_2 (p_1 + p_2)(p_1 + p_2) \right], \quad (B.33)$$

$$B = m^2 Tr \left[ (p_1 + p_2)(p_1 + p_2) - p_1(p_1 + p_2) - g_2(p_1 + p_2) + \frac{1}{4} g_2 p_2 \right], \quad (B.34)$$

$$C = 16m^4. \quad (B.35)$$

\(^{81}\)See [10]p.294-297
After using equation (B.24) on the above expressions we obtain
\[
A = 8p_1^2(q_2^2 p_2) + 8(p_1 p_2)(q_2 p_2) + 4p_1^2(p_1 q_2),
\]
\[
B = 16m^2 \left( p_1(p_1 + p_2) - (p_1 + p_2)^2 + q_2(p_1 + p_2) \right) - 4m^2 p_1 q_2,
\]
\[
C = 16m^2.
\]
Using the definitions of the Mandelstam variables we obtain
\[
p_1 p_2 = \frac{1}{2}(m^2 - s), q_1 q_2 = \frac{1}{2}(m^2 - s),
\]
\[
p_1 q_1 = \frac{1}{2}(t - m^2), p_2 q_2 = \frac{1}{2}(t - m^2),
\]
\[
p_1 q_2 = \frac{1}{2}(u - 2m^2), q_1 p_2 = u.
\]
After some tedious but straightforward algebra we obtain
\[
\langle \Phi_{ss} \rangle = -8(st - m^2(3s + t) - m^4). \tag{B.42}
\]
From the above discussion we know that to obtain the trace \( \langle \Phi_{tt} \rangle \), we simply need to do the replacement \( q_2 \leftrightarrow -p_2 \) which is equivalent to \( s \leftrightarrow t \), and we obtain
\[
\langle \Phi_{tt} \rangle = -8(st - m^2(3t + s) - m^4). \tag{B.43}
\]
The next step is to evaluate the trace \( \langle \Phi_{ts} \rangle \), the evaluation of this trace is much more complicated than the last one, the first step is to write out the trace in powers of \( m \) as
\[
\langle \Phi_{ts} \rangle = \text{Tr} \left[ \gamma_\mu (-p_1 + q_2 + m) \gamma_\nu (-p_1 + m) \gamma^\mu (-p_1 - p_2 + m) \gamma^\nu (-q_2 + m) \right]
\]
\[= (A + B + C), \tag{B.44}
\]
where
\[
A = \text{Tr} \left[ \gamma_\mu (-p_1 + q_2) \gamma_\nu (-p_1) \gamma^\mu (-p_1 - p_2) \gamma^\nu (-q_2) \right], \tag{B.45}
\]
\[
B = m^2 \text{Tr} \left[ \gamma_\mu (-p_1 + q_2) \gamma_\nu (-p_1) \gamma^\mu \gamma^\nu \right],
+ m^2 \text{Tr} \left[ \gamma_\mu (-p_1 + q_2) \gamma_\nu \gamma^\mu (-p_1 - p_2) \gamma^\nu \right],
+ m^2 \text{Tr} \left[ \gamma_\mu \gamma_\nu (-p_1 + q_2) \gamma^\mu \gamma^\nu (-q_2) \right],
+ m^2 \text{Tr} \left[ \gamma_\mu \gamma_\nu (-p_1 + q_2) \gamma^\mu (-p_1 - p_2) \gamma^\nu \right],
+ m^2 \text{Tr} \left[ \gamma_\mu \gamma_\nu (-p_1 + q_2) \gamma^\mu \gamma^\nu (-q_2) \right],
+ m^2 \text{Tr} \left[ \gamma_\mu \gamma_\nu (-p_1 + q_2) \gamma^\mu (-p_1 - p_2) \gamma^\nu \right],
C = m^4 \text{Tr} \left[ \gamma_\mu \gamma_\nu \gamma^\mu \gamma^\nu \right]. \tag{B.46}
\]
\[82\text{See equation B.8, and keeping in mind that } p_2^2 = q_2^2 = -m_1^2 = 0\]
To evaluate the first of these traces, A, we are going to start by using the formula
\[ \gamma^\mu \not\! \! \not a \not\! \! \not b \not\! \! \not c \gamma^\nu = 2 \not\! \! \not c \not\! \! \not b \not\! \! \not a, \]
however from the definition of the Feynman slash, this can be rewritten as
\[ \gamma^\mu \not\! \! \not a \not\! \! \not b \not\! \! \not c \gamma^\nu \not\! \! \not a = 2 \not\! \! \not c \not\! \! \not b \not\! \! \not a \gamma^\nu \not\! \! \not b \not\! \! \not c. \quad (B.47) \]

We can now simplify A as
\[
A = Tr \left[ \gamma^\mu (-p_1 + q_2) \gamma^\nu (-p_1) \gamma^\mu (-p_1 - p_2) \gamma^\nu (-q_2) \right],
\]
\[ = 2 Tr \left[ (-p_1) \gamma^\nu (-p_1 + q_2) (-p_1 - p_2) \gamma^\nu (-q_2) \right],
\]
\[ = 8 (q_2 p_q) Tr \left[ (-p_1 + q_2) (-p_1 - p_2) \right],
\]
\[ = 8 (q_2 p_q) \left[ -p_1^2 - (p_1 p_2) + (p_1 q_2) + (p_2 q_2) \right]. \quad (B.48)\]

The next trace, B, can be expanded in a very much the same way, but we need an additional gamma matrix formula first, namely
\[ \gamma^\mu \not\! \! \not a \not\! \! \not b \not\! \! \not c \gamma^\nu \not\! \! \not a \not\! \! \not b \not\! \! \not c \gamma^\nu \not\! \! \not a = a^\rho g^\nu_\sigma g^\mu_\rho g^\nu_\sigma g^\mu_\rho = 4 a_v. \quad (B.49)\]

Using the formula above we can proceed in the same was as we did for A, and the result is
\[
B = -16 m^2 \left( 2 p_1 (p_1 - q_2) + p_2 (p_1 - q_2) + 2 p_1 (p_1 + p_2) + p_1 q_2 + q_2 (p_1 + p_2) \right) \quad (B.50)\]

We can then evaluate C in exactly the same manner as
\[
C = m^4 Tr \left[ \gamma^\mu \gamma^\nu \gamma^\mu \gamma^\nu \right] = -8 m^4. \quad (B.51)\]

It is then straight forward (but a bit tedious) to obtain that
\[ \langle \Phi_{ts} \rangle = 8 m^2 (4 m^2 - u). \quad (B.52)\]

And from our earlier discussion we know that we can obtain \( \langle \Phi_{ts} \rangle \) by swapping t for s, and thus we obtain:
\[ \langle \Phi_{ts} \rangle = 8 m^2 (4 m^2 - u) \quad (B.53) \]

---

83 See equation B.24
84 See equation B.39
Plugging all of this together we have finally obtained an expression for the spin averaged scattering amplitude:

\[
\langle |T|^2 \rangle = -8\alpha^2 \left( \frac{(st - m^2(3t + s) - m^4)}{(m^2 - t)^2} + \frac{-2m^2(4m^2 - u)}{(m^2 - s)(m^2 - t)} + \frac{(st - m^2(3s + t) - m^4)}{(m^2 - s)^2} \right). 
\]

(B.54)

Now that we have managed to calculate the scattering amplitude to the lowest order, the next step is to express the Mandelstam variables in terms of more familiar quantities, namely the angles between the scattered momenta and the frequencies of the incident and the scattered photons. In the rest frame of the initial electron we have

\[
p_1 = (m, 0), \quad p_2 = (\omega, 0, 0, \omega), \quad q_2 = (\omega', \omega' \sin \theta, 0, \omega' \cos \theta). 
\]

(B.55)

(B.56)

(B.57)

From 4-momentum conservation \( p_1 + p_2 = q_1 + q_2 \) we have

\[
q_1 = p_1 + p_2 - q_2. 
\]

(B.58)

We can then express the Mandelstam variables as

\[
s = -(p_1 + p_2)^2 = m^2 + 2m\omega, \quad t = -(p_1 - q_2)^2 = m^2 + 2m\omega', \quad u - (q_1 - q_2)^2 = 2\omega\omega' - 2\omega\omega' \cos \theta.
\]

(B.59)

(B.60)

(B.61)

Following the steps in section A, we obtain the relation between the incident and the scattered frequency as

\[
\omega' = \frac{m\omega}{m + \omega(1 - \cos \theta)}. 
\]

(B.62)

Plugging all this into equation (B.54), and introducing the fine structure constant \( \alpha_f = \frac{e^2}{4\pi} \), we can rewrite the scattering amplitude as

\[
\langle |T|^2 \rangle = 32\pi^2\alpha_f^2 \left[ -\sin^2 \theta + \frac{\omega}{\omega'} + \frac{\omega'}{\omega} \right]. 
\]

(B.63)

We also got the relations\(^{86}\)

\[
\frac{d\sigma}{dt} = \frac{1}{64m^2\omega^2} \langle |T|^2 \rangle. 
\]

(B.64)

\(^{85}\)In a frame where the incident radiation is incoming in the positive z-axis

\(^{86}\)See [10] p.79-84

74
We can then transform to the differential cross section, \( \frac{d\sigma}{d\Omega'} \), by taking the differential of \( \omega' \), and using the relation

\[
t = -(p_1 - q_2)^2 = m^2 + 2m\omega',
\]

(B.65)
to obtain

\[
dt = \frac{\omega'^2}{m}\frac{d\Omega}{d\Omega'}.
\]

(B.66)

We perform the transformation from \( t \) to \( \Omega \) to finally obtain the Klein-Nishina formula in the electron rest system

\[
\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt} \frac{dt}{d\Omega} = \frac{1}{64m^2\pi^2} \langle |T|^2 \rangle.
\]

(B.67)

By introducing the classical electron radius \( r_e = \frac{e^2}{4\pi m} \) we can write this as

\[
\frac{d\sigma}{d\Omega} = \frac{r_e^2}{2} \left( \frac{\omega'}{\omega} \right)^2 \left[ -\sin^2 \theta + \frac{\omega}{\omega'} + \frac{\omega'}{\omega} \right].
\]

(B.68)

## C  Integral table

footnoteAll integrals obtained from [13]

\[
\int_0^\infty r e^{-ar^2} dr = \int_0^\infty \frac{1}{2a} e^{-z} dz = \frac{1}{2a}.
\]

(C.1)

\[
\int_{-\infty}^\infty e^{-ax^2+bx+c} dx = \int_{-\infty}^\infty e^{-a(x-\frac{b}{2a})^2+\frac{b^2}{4a}+c} = \frac{1}{\sqrt{a}} e^{\frac{b^2}{4a}+c} \int_{-\infty}^\infty e^{-y^2} dy = \frac{1}{\sqrt{a}} e^{\frac{b^2}{4a}+c} \sqrt{\frac{\pi}{a}} \int_{-\infty}^\infty e^{-y^2-z^2} dy dz = \frac{\sqrt{\pi}}{\sqrt{a}} e^{\frac{b^2}{4a}+c}.
\]

(C.2)

\[
\int_0^\infty e^{-\frac{x^2}{a^2}} dx = \frac{\pi e^{a^2}}{2a} (1 - Erf(a)),
\]

(C.3)
\[ \int_0^{+\infty} \exp \left( -\frac{x^2}{4a} - bx \right) \, dx = (\sqrt{\pi a}) \exp(ab^2) [1 - \text{Erf}(b\sqrt{a})], \quad (C.4) \]

where \( \text{erf} \) is the error function defined as

\[ \text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} \, dx. \quad (C.5) \]

C.1 Dirac Delta function

Note that the delta function is a distribution and it does not make sense outside an integral. As such all the formulas within this section is implied to be within integrals\(^{87}\).

\[ \int_{-\infty}^{+\infty} f(x) \delta(x - x_0) \, dx = f(x_0) \quad (C.6) \]

\[ \delta(ax) = \frac{\delta(x)}{|a|} \quad (C.7) \]

\[ \delta(-x) = \delta(x) \quad (C.8) \]

\[ \delta(g(x)) = \frac{\delta(x - x_0)}{|g(x_0)|} \quad (C.9) \]

Where \( x_0 \) is the root of \( g(x) \).

D MATLAB code

```matlab
function f = Brightness(x)
%This function calculates the on axis brightness in a headon collision with collision angle 0 without approximation

%This function calculates the on-axis brightness in an arbitrary geometry aslong as it is a headon collision

%x(1)=delta_gamma
%x(2)=emmitance

\(^{87}\)Except equation C.6
```

76
%x(3)=radiusatfocus
%x(4)=pulseduration
%x(5)=bunchduration
%x(6)=focalradius
%x(7)=theta_o = incident radiation angle
%x(8)=energy - incident electron energy

bunchduration=x(5).*1e-12;
pulseduration=x(4).*1e-12;
radiusatfocus=x(3).*1e-6;
emmitance=x(2).*1e-6;
delta_gamma=x(1);
focalradius=x(6).*1e-6;
theta_0=x(7).*pi./180;
energy=x(8);
reprate=x(9).*1e3;
p=x(10);
t=x(11).*1e-12;
a=x(12);
k=x(13);

r_o=2.8179403267e-15; %classical electron radius
r_0=r_o;%Do not judge me
M=1.3;%See report
%constants
c=2.99e8;%Speed of light
charge=100e-12;%Charge of the electrons
pulseenergy=1e-3;%Energy of the laser pulse
incidentfreq=1.8275e15;%Pulse center frequency
hbar=1.05457160e-34;%Hbar
gamma_0=energy/0.5;%The definition of the gamma factor in terms of energy

%These quantities are defined to zero since we do not consider any electron
%or laser focus effects
delta_freq=sqrt(2)./pulseduration;%See report
angle_xe=0;
angle_ye=0;
polangle=0;
theta=0;
phi=0;
angle_ly = 0;
angle_lx = 0;

beta = sqrt(1 - 1./gamma_0.^2); % the velocity to the speed of light ratio
% This g corresponds to the g' in the report
angle_yl = angle_ly;
angle_xl = angle_lx;
theta_x = theta_0 + angle_xl; % help variable
g = gamma_0.*(1 + beta.*(cos(theta_x-angle_xe).*cos(angle_yl-angle_ye)...
    + sin(angle_yl).*sin(angle_ye).*i*cos(theta_x-angle_xe)));

% gscatt corresponds to g in the report
coshelp = cos(theta)*cos(angle_xe)*cos(angle_ye)+sin(phi)*sin(theta)...
    *sin(angle_ye)+cos(phi)*sin(theta)*sin(angle_xe)*cos(angle_ye);
gscatt = g./(gamma_0.*(1-beta*coshelp));
scattfreq = gscatt.*incidentfreq; % The frequency of the scattered photon
% We define zmax such that we only integrate over 5 sigma
xi = scattfreq./(4.*gamma_0.^2.*incidentfreq); % doppler upshift
zmax = 1./(((1./c).^2.*((1-beta).^2./bunchduration.^2 + 2.*(1+cos(theta_0).^2).../pulseduration.^2));

zmax = zmax*5000;
zmin = -zmax;

% The total number of electrons and photons
Np = pulseenergy./(incidentfreq.*hbar);
e = 1.602e-19; % Fundamental charge
Ne = charge/e;
density = @(z) generalelectrophoton(z, gamma_0, theta_0, incidentfreq...
pulseenergy, pulseduration, focalradius, radiusatfocus, bunchduration...
charge, emmitance, M, p, t);
% integration limits
w = linspace(zmin, zmax, 10000); zmin = w(find(density(w) > 0.01, 1, 'first')); % finds the proper z limits
zmax = -zmin;
delta_beta = emmitance./radiusatfocus; % Definition of delta_beta
xmin = 0;
xmax=delta_beta*5; % We only integrate over 5 sigma

test=@(z,betatransverse) chirpedintegrand(z,betatransverse,scattfreq,...
    ,gamma_0,delta_gamma,theta_0,incidentfreq,pulseduration,focalradius...
    ,radiusatfocus,bunchduration,emmitance,M,p,k,zmax,t,a);

if (isempty(zmin)) % If the integration limits are 0 we define the integral as 0
    y1=0;
else
    y1=integral2(test,zmin,zmax,xmin,xmax);
end

y3=(r_0.^2.*Ne.*Np.*c).*y1;
y3=y3./(sqrt(pi));

f=reprate.*2e-15.*y3.*scattfreq./incidentfreq;

end

function f = chirpedintegrand(z,betatransverse,scattfreq,...
    ,gamma_0,delta_gamma,theta_0,incidentfreq,pulseduration,focalradius,radiusatfocus,...
    ,bunchduration,emmitance,M,p,k,zmax,t,a);

% This function calculates the integrals over the normalized beta function
% and the electron, photon densities, for a flattened (p parameter) and
% chirped beam (a, k chirpness parameters).

hbar=1.05457160e-34;
c=2.99e8;
incidentfreq_0=incidentfreq;% Incident freq 0 is the center frequency
b=k/zmax-a.*zmax;

incidentfreq=incidentfreq.*(1+a.*z.^2+b.*z); % We assume a quadratic shape

delta_freq=sqrt(2)./pulseduration;% Def of delta_freq

relgamma=2.*delta_gamma./gamma_0;% Relative energy distribution

xi=scattfreq./(4.*gamma_0.^2.*incidentfreq); % Doppler upshift
relfreq=delta_freq./incidentfreq; %relative frequency spread

delta_beta=emmitance./radiusatfocus;%Def of delta_beta

beta=sqrt(1-1./gamma_0.^2);
incidentwavelength=2.*pi.*c./(incidentfreq);
k_f=emmitance./(gamma_0.*radiusatfocus.^2);%Radius at focus
zR=pi.*focalradius.^2./(incidentwavelength.*M);%Ray length

%--------------------Normalization of gaussian
a=2./(focalradius.^(2*p).*((1+((z.*cos(theta_0))./zR).^2));
b=2*p;
y2=2.*a.^(-1./b).*gamma(1./b).
b;
y2=1./y2;
y1=1./((pi^(5./2)).*2.*c.*2.*pulseduration.*focalradius.*...}
bunchduration.*radiusatfocus.^2);
y1=y1./sqrt((1+(cos(theta_0).*z./zR).^2));
y1=y1./((1+(k_f.*z).^2));
x2=(z./c).^2.*((t+(1-beta).*z./c).^2./pulseduration.*c.^2...}
+2.*(1+2.*cos(theta_0))./pulseduration.^2);
x2=x2+2.*(t+z./c.*cos(theta_0))./pulseduration.^2;
x3=2.*z.^2.*sin(theta_0).^2./((1+(cos(theta_0).*z./zR).^2);
x3=x3./focalradius.^2;
f1=y1.*y2.*exp(-x2-x3);

x1=2.*betatransverse.*(1+betatransverse.^2);
x2=relfreq.^2+relgamma.^2.*xi.^2.*(1+betatransverse.^2).^2;
x3=xi.*(1+betatransverse.^2)-1;
x3=x3.^2;
x4=betatransverse.^2./delta_beta.^2;
f=1./(delta_beta.^2).*f1.*x1./sqrt(x2).*exp(-x3.^2./x2 -x4);
end

%%%%%%%%%%%%%%%%%%

%This function calculates the scattering rate for different flattening parameters p
angle_xe=0;
angle_ye=0;
angle_xl=0;
angle_yl=0;
gamma_0=14.5/0.5;
polangle=0;
theta_0=0;
cross=@(theta,phi)
    Differentialcross(gamma_0,theta_0,polangle,angle_xe,angle_ye,
    ,angle_xl,angle_yl,theta,phi).*sin(theta);

thetamax=pi;
thetamin=0;
phimax=2*pi;
phimin=0;
y2=integral2(cross,thetamin,thetamax,phimin,phimax);

t=linspace(-7,7,50);
h1=linspace(0,0,length(t));
h2=linspace(0,0,length(t));
h3=linspace(0,0,length(t));
h4=linspace(0,0,length(t));
h5=linspace(0,0,length(t));
x=[0.56 2 50 5 3 50 18 14.5 176e3];

for i=1:length(t)
    h1(i)=Scatteringrate(x,t(i),y2,1);
    i
end

for i=1:length(t)
    h2(i)=Scatteringrate(x,t(i),y2,2);
    i
end
for i=1:length(t)
    h3(i)=Scatteringrate(x,t(i),y2,3);
    i
end

for i=1:length(t)
    h4(i)=Scatteringrate(x,t(i),y2,0.7);
    i
end

for i=1:length(t)

81
h5(i)=Scatteringrate(x,t(i),y2,0.4);

i

function f = Differentialcross(gamma,theta_0,polangle...
   ,angle_xe,angle_ye,angle_xl,angle_yl,theta,phi)
%This function generates the differential cross section for the input
%values
%polangle is the incoming radiations polarization angle in the xl,yl plane
%theta ,phi is the scattering angles in the stationary frame
%The angle variables specify the rotation parameters from the different
%frames, see pdf (they are called Xi there)
%freq = frequency of incoming radiation
%gamma=gamma factor
%theta_0 - angle between electron and radiation +180 degress.
r_o=2.8179403267e-15;  %classical electron radius
theta_x=theta_0+angle_xl;%help variable

beta=sqrt(1-1./gamma.^2);%the velocity to the speed of light ratio
%conversion from electron frame to the stationary frame:
A=[0 0 0];
Angletransformation;
%the Doppler shift:
g=gamma.*((1+beta.)*(cos(theta_x-angle_xe)).*cos(angle_yl-angle_ye)...
   +sin(angle_yl).*sin(angle_ye).*((1-cos(theta_x-angle_xe))));
Polarizationvector;%calls the subroutine that generates the Polarizationvectors
%The differential cross section is given by(see report)
f=(1-beta.^2)./((1-beta.*C).^2).*(alpha1.^2 .*(1-A.^2./
(gamma.^2.*(1-beta.*C).^2))...
+alpha2.^2.*(1-B.^2./(gamma.^2.*(1-beta.*C).^2))...
+alpha3.^2.*(1-((C-beta).^2./(1-beta.*C).^2))...
-2.*alpha1.*alpha2.*(C.*B./(gamma.*(1-beta.*C).^2))...
-2.*alpha1.*alpha3.*((C-beta).*A./(gamma.*(1-beta.*C).^2))...
-2.*alpha2.*alpha3.*((A-beta).*B./(gamma.*(1-beta.*C).^2)));
end

function f = Scatteringrate( x ,t_d,y2,p)
%This function calculates the on axis brightness in a headon collision with
%collision angle 0 without approximations
%This function calculates the on-axis brightness in an arbitrary geometry as long as it's a headon collision (non-head on collision is on the to-do list)

% x(1)=delta_gamma
% x(2)=emmitance
% x(3)=radiusatfoucs
% x(4)=pulseduration
% x(5)=bunchduration
% x(6)=focalradius
% x(7)=theta_o = incident radiation angle
% x(8)=energy - incident electron energy

bunchduration=x(5).*1e-12;
pulseduration=x(4).*1e-12;
radiusatfocus=x(3).*1e-6;
emmitance=x(2).*1e-6;
delta_gamma=x(1);
focalradius=x(6).*1e-6;
theta_0=x(7).*pi./180;
energy=x(8);
reprate=x(9).*1e3;
r_o=2.8179403267e-15;  \%classical electron radius
r_0=r_o;

\%constants
c=2.99e8;
charge=100e-12;
pulseenergy=1e-3;
incidentfreq=1.8275e15;
hbar=1.05457160e-34;
gamma_0=energy/0.5;
\%scattfreq=incidentfreq.*4.*gamma_0.^2;

beta=sqrt(1-1./gamma_0.^2); \%the velocity to the speed of light ratio
\%This g correpsons to the g’ in the report
% The total number of electrons and photons

Np = pulseenergy ./ (incidentfreq .* hbar);
e = 1.602e-19;
Ne = charge / e;

% Help variables yi)
t_d = t_d .* 1e-12;

%%%%% Density Integral %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

test = @(x, y, z) broadelectronphoton(x, y, z, gamma_0, theta_0, incidentfreq,
   pulseenergy, pulseduration, focalradius, radiusatfocus, bunchduration,
   charge, emmitance, t_d, 1, p);

delta_x = 1;
delta_y = delta_x;

%%%%% Integration limits/ chosen such that it covers the exponential

zmin = -0.5;
zmax = -zmin;
xmax = zmax;
xmin = -xmax;
ymax = zmax;
ymin = -ymax;

% Monte Carlo Integration

V = (xmax - xmin) .* (ymax - ymin) .* (zmax - zmin);
temp = 0;
m = 100;
n = 100000;
for i=1:m
    x=(xmax-xmin).*rand(1,n)+xmin;
    y=(ymax-ymin).*rand(1,n)+ymin;
    z=(zmax-zmin).*rand(1,n)+zmin;
    temp2=test(x,y,z);
    temp=temp+sum(temp2,'double');
end
y1=temp*V/(n*m);

%Plug everything together according to the report
f=y1.*Ne.*Np.*(1+beta).*y2.*r_o.^2;
%f=0.1.*peakbrightness.*rep.rate.*pulseduration;

end

function f = broadelectronphoton(x,y,z, gamma_0,theta_0,incidentfreq...
,pulseenergy,pulseduration,focalradius,radiusatfocus,bunchduration...
,charge,emmitance,t_d,M,p)

    hbar=1.05457160e-34;
    c=2.99e8;
    n=1000;
    x=x./n;
    y=y./n;
    z=z./n;
    beta=sqrt(1-1./gamma_0.^2);
    incidentwavelength=2.*pi.*c./(incidentfreq);
    k_f=emmitance./(gamma_0.*radiusatfocus.^2);
    zR=pi.*focalradius.^2./(incidentwavelength.*M);

    y1=1./(pi^((5./2).*2.*pulseduration.*focalradius.*bunchduration... ...
        .*radiusatfocus.^2));
    y1=y1./sqrt(1+((z.*cos(theta_0)-y.*sin(theta_0))./zR).^2);
    y1=y1./((1+(k_f.*z).^2));
    x1=(t_d+(1-beta).*z./c).^2./bunchduration.^2;
    x2=(x.^2+y.^2)./(radiusatfocus.^2.*(1+(k_f.*z).^2));
    x3=2.*(t_d+z./c+(z.*cos(theta_0)-y.*sin(theta_0))./c).^2./pulseduration.^2;

85
x4=2.*((y.*cos(theta_0)+z.*sin(theta_0)).^2./focalradius.^2.*... 
(1+((z.*cos(theta_0)-y.*sin(theta_0))./zR).^2))); 
x5=2*((abs(x./focalradius)).^(2*p)./(1+((z.*cos(theta_0)-y.... *sin(theta_0)))./zR).^2)); 

%we are considering the case where the detector time is 0; 
k=1; 
a=2./(focalradius.^(2*k).*(1+((z.*cos(theta_0)-y.*sin(theta_0))./zR).^2)); 
b=2*k; 
y2=2.*a.^(-1./b).*gamma(1./b)./b; 
y2=1./y2; 

%%%%%%%%%%%%%%%%% 
f=y1.*y2.*exp(-x1-x2-x3-x4-x5)./n^3; 
end

function f = GeneralDifferentialbrightness(x) 
%%%%%%%%%%%%%%%%%%%%%%%%%%%% 
%This code wasent used in the result section! The reason for this is that 
%the computational time was too big, instead a few points were calculated 
%to ensure that the brightness followed the same dependence on the 
%paramters as the less exact model and simulations. 
%This function calculates the on-axis brightness in an arbitrary gemetry 
%aslong as it's a headon collision (non-head on collision is on the to-do 
%list) 

%x(1)=delta_gamma 
x(2)=emmitance 
x(3)=radiusatfoucs 
x(4)=pulseduration 
x(5)=bunchduration 
x(6)=focalradius 
x(7)=theta_o =incident radiation angle 
x(8)=electron beam energy 

bunchduration=x(5).*1e-12; 
pulseduration=x(4).*1e-12; 
radiusatfocus=x(3).*1e-6;
emittance=x(2).*1e-6;
delta_gamma=x(1);
focalradius=x(6).*1e-6;
theta_0=x(7).*pi./180;
energy=x(8);%In MeV
reprate=x(9).*1e3;

r_o=2.8179403267e-15; %classical electron radius

%constants
Msquare=1.3;
c=2.99e8;
charge=100e-12;
pulseenergy=1e-3;
incidentfreq=1.8275e15;
hbar=1.05457160e-34;
gamma_0=energy/0.5; %we divide by 0.5 aka the electron rest mass in MeV
%scattfreq=incidentfreq.*4.*gamma_0.^2;
delta_freq=sqrt(2)./pulseduration;
incidentwavelength=2.*pi.*c./(incidentfreq);
polangle=0;
theta=0;
phi=0;

%Electronfocus
angle_xe=0;
angle_ye=0;
delta_angle_xe=emmitance.*2./radiusatfocus;
delta_angle_ye=delta_angle_xe;
%We assume that both the x,y components got the same distribution

%Laser focus
epsilon_l=Msquare.*incidentwavelength./pi;
delta_angle_lx=epsilon_l./focalradius;
delta_angle_ly=delta_angle_lx;
%We assume that the x,y components got the same angular distribution
%gamma_0 is the average electron lorentz factor, gamma is the lorentz factor that we integrate over

beta=sqrt(1-1./gamma_0.^2); %the velocity to the speed of light ratio
% This g corresponds to the $g'$ in the report
angle_ly=0;
angle_lx=0;
angle_yl=angle_ly;
angle_xl=angle_lx;
theta_x=theta_0+angle_xl;% help variable

\% g_scatt corresponds to $g$ in the report

g=gamma_0.*(1+beta.*(cos(theta_x-angle_xe).*cos(angle_yl-angle_ye)...
  +sin(angle_yl).*sin(angle_ye).*(1-cos(theta_x-angle_xe))));
g_scatt=g./(gamma_0.*(1-beta*coshelp));
gscatt=gscatt.*incidentfreq;

% function
spectralintegral=@(angle_ex,angle_ey,angle_lx,angle_ly,gamma)...
  Gammaintegraltest (angle_ex,angle_ey,angle_lx,angle_ly,gamma,gamma_0...,
  ,delta_gamma,theta_0,polangle,theta,phi,incidentfreq,scattfreq,...
  delta_freq,delta_angle_ly,delta_angle_lx,delta_angle_xe,delta_angle_ye);

xmax=5*delta_angle_lx;
xmin=-xmax;
ymax=5*delta_angle_ly;
ymin=-ymax;

% Electron limits
amax=5*delta_angle_xe;
amin=-amax;
bmax=5*delta_angle_ye;
bmin=-bmax;

% Integration limits for gamma, we approximate the integral from - inf to
% inf even though gamma is $\geq 1$, the error of this approximation is small
zmax=delta_gamma*20+gamma_0;
zmin=-delta_gamma*20+gamma_0;

\% MONTE-CARLO integration
V=(xmax-xmin).*(ymax-ymin).*(zmax-zmin).*(amax-amin).*(bmax-bmin);
temp2=0;
m=10000;
n=100000;
for i=1:m
a=(amax-amin).*rand(n,1)+amin;
b=(bmax-bmin).*rand(n,1)+bmin;
x=(xmax-xmin).*rand(n,1)+xmin;
y=(ymax-ymin).*rand(n,1)+ymin;
z=(zmax-zmin).*rand(n,1)+zmin;
temp=spectralintegral(a,b,x,y,z);
temp2=sum(temp,'double')+temp2;
end
y1=temp2*V/(n*m);

y2=(sqrt(pi).*delta_gamma);
f1=y1./y2;%this is the first part of the result
%Now for the integral over the electron,photon density

Np=pulseenergy./(incidentfreq.*hbar);
e=1.602e-19;
%Ne is the total number of electrons
Ne=charge/e;

density=@(z)electronphotondensity(z,gamma_0,theta_0,incidentfreq...,
pulseenergy,pulseduration,focalradius,radiusatfocus,bunchduration,...
charge,emmitance,Msquare);
%Choose the boundarys such that we cover the most of the gaussian
zmax=3000.*1./((1./c).^2.*((1-beta.^2)/bunchduration.^2 +8.*cos(theta_0)....
.^2/pulseduration.^2));
zmin=-zmax;
t=linspace(zmin,zmax,10000);
zmin=t(find(density(t)>0.1,1,'first'));
zmax=-zmin;

%Calculates the integral over the photon and electron density
f2=r_o.^2.*c.*Ne.*Np.*integral(density,zmin,zmax);
f3=scattfreq./delta_freq;
f=f1.*f2.*f3.*1e-15.*reprate;
end
Acknowledgments

This work was performed under the supervision of Vitaliy Goryashko, FREIA, Department of physics and astronomy, Uppsala University.

E References

References

[1] Classical Electrodynamics:
    Jackson, John David
    John Wiley and Sons, Inc
    3rd edition
    @1998

[2] Thidé, Bo Electromagnetic Field Theory:
    Thidé, Bo
    Uppsala, Sweden,
    2nd edition
    @1997-2011

[3] Introduction to Special Relativity:
    Rindler, Wolfgang
    Clarendon Press, Oxford
    2nd Edition
    @1991

[4] Elementary Linear Algebra:
    Rorres, Chris and Anton, Howard
    John Wiley and Sons, Inc
    10th Edition
    @2011

    Hartemann F.v, Brown W.J, Gibson D.J, Anderson S.G, Tremaine A.M,
    Springer P.T, Wootton A.J, Hartouni E.P, Barty C.P.
    Lawrence Livermore National Laboratory, Livermore, California 94550,
    USA
    2004
[6] Three-dimensional time and frequency-domain theory of femtosecond x-ray pulse generation through Thomson scattering:
Brown, Winthrop J and Hartemann, Frederic V
Lawrence Livermore National Laboratory, Livermore, California 94550, USA
11 June 2004

Oxford University press
2003

[8] Introduction to Genetic Algorithms
Sivandam S.I, Deepa S.N
Springer-Verlag Berlin Heidelberg
2008

[9] Gaussian Beam Waist picture
http://commons.wikimedia.org/wiki/File:GaussianBeamWaist.svg
Monday 26th January, 2015

[10] Quantum field theory
Srednicki, Mark, University of California, Santa Barbara
Cambridge University Press
1st edition
2012

Martin Reiser, University of Maryland
Wiley-VCH
1st edition
1994

[12] A swedish compact linac-based THz/X-ray source at FREIA
V.A Goryaskho, Uppsala University, Uppsala, Sweden
A. OPanansenko, NSC/KIPT, Kharkiv, Ukraine
V.Zhaunerychyk, University of Gothenburg, Gothenburg, Sweden

[13] Table of Integrals, Series and Products
I.S. Gradshteyn, I.M. Ryzhik
Academic Press, San Diego
4th edition
2000