Modeling the dynamics of toroidal Alfvén eigenmodes

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Abstract. A model describing nonlinear dynamics of a single Alfvén eigenmode excited by an inverted energy distribution of energetic ions is presented, suitable for drift orbit averaged Monte Carlo codes. The nonlinear dynamics of the wave mode is modeled with a complex wave amplitude, and is characterized by the formation of coherent structures in phase space, caused by wave-particle interaction. The transition to a quasilinear regime is modeled with a phenomenological decorrelation of the wave-particle phase. As the decorrelation is increased the coherent phase-space structures diminishes, and frequency chirping events in the marginal stability region is limited. The strength of the decorrelation modifies the saturation level and saturation time of the eigenmode amplitude.

1. Introduction

Populations of energetic ions appear in present day thermonuclear fusion experiments based on magnetic confinement, e.g. by fusion reactions, ion cyclotron resonant heating (ICRH) and heating by neutral beam injection (NBI). Shear Alfvén waves, such as the toroidal Alfvén eigenmodes (TAEs), can be excited [1, 2] by resonant ion populations. This may result in large amounts of resonant ions being ejected from the core of the plasma, which can cause a significant reduction of the heating efficiency [3–5]. Theoretical understanding of the nonlinear dynamics of such waves is of importance for the development of viable fusion power plants.

Models successfully describing many aspects of the dynamics of TAEs exist in the literature [6–13]. Major contributions to the field of TAE dynamics are based on the model by Berk and Breizman [6–8]. Numerical studies based on their model have shown that systems with a marginal linear instability of the wave growth rate give a variety of wave amplitude dynamics, including regimes of stable saturation, periodic and chaotic modulation, and explosive amplitude evolution [14–17]. In particular, it was found that when the dynamical friction (drag) is much larger than the velocity space diffusion of the particles the mode may grow explosively [16, 17]. Other important results include the creation of phase space hole and clump pairs around the wave resonance near marginal
Modeling the dynamics of TAEs

stability [17–19], which can be regarded as long lived Bernstein-Greene-Kruskal (BGK) modes [20].

In this paper, a one dimensional Monte-Carlo model for the nonlinear interaction between an oscillating wave field and an ensemble of charged particles is developed. It is a refined version of the model presented in Ref. [21]. The presented model attempts to mimic the scenario of a fusion plasma confined in a tokamak, in which a set of energetic ions interacts with a discrete Alfvén eigenmode, e.g. a TAE. The model includes basic dynamics necessary for the nonlinear excitation of a wave mode, describing phenomena such as the growth and saturation of a wave mode, and the generation of coherent structures in the distribution function. One of the major issues to be treated in this paper is the transition from the nonlinear regime to the quasilinear regime by studying effects of a partial phase decorrelation.

2. The model

The presented model is based on the formalism described in Ref. [15]. It assumes that particle orbits are integrable and periodic in the absence of perturbations, which makes the use of action-angle variables applicable [22]. The Hamiltonian of the system is described in a localized region of phase space around a point on the resonant surface of the eigenmode. Coordinates are chosen in such a way that the lowest order perturbation of the Hamiltonian in mode amplitude only depends on the angle $\xi$ and time $t$, and the unperturbed Hamiltonian only depends on the action $I$, conjugate to $\xi$. Then the Hamiltonian is described as

$$H = H_0(I) + 2 \text{Re} \ C(t) V(I_r) e^{i(\xi - \omega_0 t)} ,$$

(1)

where $\omega_0$ is the unperturbed eigenmode frequency, $C(t)$ is the complex mode amplitude (varying on time scales $\ll \omega_0^{-1}$), and $V(I_r)$ is a proportionality factor between $dI/dt$ and $2 \text{Re} iC(t) V(I_r) e^{i(\xi - \omega_0 t)}$ at the eigenmode resonance $I = I_r$, which is determined by the unperturbed particle orbits and the structure of the eigenmode. From the Hamiltonian in eq. (1) the kinetic equation for particles close to resonance can be derived accordingly:

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial I} \frac{\partial f}{\partial \xi} - \frac{\partial H}{\partial \xi} \frac{\partial f}{\partial I} \approx \frac{\partial f}{\partial t} \bigg|_{\text{coll}} ,$$

(2)

where $\Omega(I) = dH_0/dI$ is the associated particle frequency determining the resonance condition $\Omega(I_r) = \omega_0$, and $\partial f/\partial t|_{\text{coll}}$ is the collision operator for particles in $I, \xi$ space. Locally around the resonance $\Omega(I) \approx \omega_0 + (I - I_r)d\Omega(I_r)/dI$. The quantity $d\Omega(I_r)/dI$ depends on the specifics of the adiabatic invariant motion along the given projection. As given in Ref. [18], which uses a Hamiltonian on the same form as in eq. (1), the mode amplitude equation is given by

$$\frac{dC}{dt} = -\frac{i\omega_0}{2\epsilon} \int d\Gamma \ V^*(I_r) e^{-(\xi - \omega_0 t)} f(\xi, \Omega, t) - \gamma_d C ,$$

(3)
Modeling the dynamics of TAEs

where \(d\Gamma\) is a volume element in phase space, and \(\epsilon\) determines the wave mode energy according to

\[
W_w = 2\epsilon|C|^2. \tag{4}
\]

In this paper, the following parameters are used

\[
v = \frac{nI}{mR_0}, \quad \phi = \frac{\xi - \omega_0 t}{n}, \quad E = 2iC^*,
\]

\[
\frac{d\omega}{dv} = \frac{mR_0}{n^2} \frac{d\Omega}{dI}\Big|_{I_t}, \quad q = \frac{n}{R_0} V(I_t),
\]

\[
\int d\phi dv v = \frac{\omega_0}{nm} \left( \frac{d\Omega}{dI}\Big|_{I_t} \right)^{-1} \int d\Gamma,
\]

where \(v\) is the ion velocity, \(m\) is the ion mass, \(q\) is the ion charge, \(n\) is the toroidal mode number, \(R_0\) is the major radius of the torus, \(\phi\) is the toroidal angle, and \(\omega(v) \equiv d\phi/dt\).

Using these variables, the equations (2) – (4) can be rewritten as

\[
\frac{\partial f}{\partial t} + \frac{d\omega}{dv}(v - v_t) \frac{\partial f}{\partial \phi} + \frac{q}{m} \text{Re}(Ee^{-in\phi}) \frac{\partial f}{\partial v} = \frac{\partial f}{\partial t} \bigg|_{\text{coll}}, \tag{5}
\]

\[
\frac{dE}{dt} = -\frac{q}{\epsilon} \int d\phi dv f(\phi, v, t)ve^{in\phi} - \gamma_d E, \tag{6}
\]

\[
W_w = \frac{\epsilon|E|^2}{2}. \tag{7}
\]

Starting from the collisionless case, where the right hand side of eq. (5) vanishes, and with an initial distribution function \(f(\phi, v, t = 0) = f_0(v)\), the wave mode amplitude starts to evolve exponentially when \(t \gg (nv_t d\omega/dv)^{-1}\) according to \(|E(t)| = E_0 e^{(\gamma_L - \gamma_d)t}\), where

\[
\gamma_L = \frac{\pi^2 q^2 v_t}{nm|\epsilon|} \left( \frac{d\omega}{dv} \right)^{-1} \frac{df_0}{dv}\bigg|_{v_t}. \tag{8}
\]

This quantity, which is the collisionless growth rate of the wave perturbation, will here be referred to as the linear growth rate. Another useful quantity is the bounce frequency of particles deeply trapped by the wave field, which is

\[
\omega_B = \sqrt{n|q||E| \frac{d\omega}{dv}}. \tag{9}
\]

In the limit of a constant wave amplitude, particles being trapped by the wave field satisfy the condition \(|v - v_t| \leq v_s\), where \(v_s\) is the separatrix width, given by

\[
v_s = 2\sqrt{\frac{|q||E|}{nm} \left( \frac{d\omega}{dv} \right)^{-1}}. \tag{10}
\]
In the presented Monte Carlo model, the energetic tail distribution function is represented by a set of discrete particles (markers) of a single ion species. In the collisionless case, their individual equations of motion are

$$\frac{dv_k}{dt} = \frac{q}{m} \text{Re} \left( E e^{-im\phi_k} \right),$$  \hspace{1cm} (11)

$$\frac{d\phi_k}{dt} = \frac{d\omega}{dv} (v_k - v_r).$$  \hspace{1cm} (12)

The total energy of the system is simply expressed as

$$W_{\text{tot}} = \sum_{k=1}^{N_0} \frac{m_k v_k^2}{2} + \frac{\epsilon |E|^2}{2} \approx \sum_{k=1}^{N_p} s_k \frac{m_k v_k^2}{2} + \frac{\epsilon |E|^2}{2}. \hspace{1cm} (13)$$

When $N_p$ number of markers are used in the simulations to represent $N_0$ physical particles, weight factors $s_k$ are introduced, where $N_0$ is the sum of all $s_k$. Similarly, the eigenmode amplitude equation is

$$\frac{dE}{dt} = -\frac{q}{\epsilon} \sum_{k=1}^{N_0} v_k e^{im\phi_k} - \gamma_d E \approx -\frac{q}{\epsilon} \sum_{k=1}^{N_p} s_k v_k e^{im\phi_k} - \gamma_d E. \hspace{1cm} (14)$$

A phenomenological decorrelation of the wave-particle phase is introduced to model decorrelation by collisions between particles or by interactions with other waves. It is modeled as a Brownian motion in $\phi$, which on differential form can be written as

$$d\phi_{k,\text{coll}} = \frac{\pi}{\sqrt{t_c}} dW^{(k)}_t,$$  \hspace{1cm} (15)

where $t_c$ is the characteristic time scale for complete phase decorrelation, and $W^{(k)}_t$ is a Wiener process in $t$ associated with particle $k$. Such a Brownian motion corresponds to a collision operator on the form

$$\left. \frac{df}{dt} \right|_{\text{coll}} = \partial^2 \left( \frac{\pi^2}{2t_c} f \right) = \frac{\pi^2}{2t_c} \partial^2 f.$$  \hspace{1cm} (16)

In this paper, $t_c$ is defined as a constant parameter of the system. Since $\phi_k$, $v_k$ and $E$ are coupled via their equations of motion, the complete problem is a system of $(2N_p + 2)$ SDEs. Assuming that $E$ varies on time scales much longer than the individual $v_k$ and $\phi_k$, the problem can be reduced to individual 2 dimensional subsystems of SDEs with one subsystem for each particle, which is much simpler to treat numerically. Using the order 1.5 strong Itō-Taylor scheme, as presented in Ref. [23], a finite time step $\Delta t$ gives the following changes in $\phi_k$ and $v_k$

$$\Delta \phi_k = \left[ \omega \Delta t + \frac{\pi}{\sqrt{t_c}} \Delta W_k + a \frac{d\omega}{dv} \frac{\Delta t^2}{2} \right]_{\phi_k,v_k,t}, \hspace{1cm} (17)$$

$$\Delta v_k = \left[ a \Delta t + \frac{\pi}{\sqrt{t_c}} \frac{\partial a}{\partial \phi} \Delta Z_k + \left( \omega \frac{\partial a}{\partial \phi} + \frac{\pi^2}{2t_c} \frac{\partial^2 a}{\partial \phi^2} + \frac{\partial a}{\partial t} \right) \frac{\Delta t^2}{2} \right]_{\phi_k,v_k,t}, \hspace{1cm} (18)$$
where

\[
\omega(v) = \frac{d\omega}{dv}(v - v_r),
\]

(19)

\[
a(\phi, t) = \frac{q}{m} \text{Re} \left( E(t)e^{-in\phi} \right),
\]

(20)

\[
\Delta W_k = \zeta_{1,k} \sqrt{\Delta t}, \quad \Delta Z_k = \frac{\Delta t^{3/2}}{2} \left( \zeta_{1,k} + \frac{\zeta_{2,k}}{\sqrt{3}} \right),
\]

(21)

and \( \zeta_{t,k} \) are normally distributed random variables of unit variance. The time dependence of the field amplitude is kept, but it is treated as a quantity independent of \( \phi_k \) and \( v_k \).

Due to the coupling of \( \phi_k \) and \( v_k \) in their corresponding drift terms, the addition of a Brownian motion in \( \phi_k \) gives rise to stochasticity in velocity space. When the wave field amplitude is approximately constant on relevant time scales, the added stochasticity for particles being far from trapped by the wave field (\(|v - v_r| \gg v_s|\)) can be described by a velocity diffusion coefficient on the form

\[
D_v = \frac{D_0}{1 + (v - v_r)^2/v_b^2}
\]

in the limit \( t \gg t_c \), where

\[
D_0 = \frac{q^2|E|^2 t_c}{\pi^2 n^2 m^2}, \quad v_b = \frac{\pi^2 n}{2t_c} \left( \frac{d\omega}{dv} \right)^{-1}.
\]

3. Results

3.1. The initial distribution function

The initial distribution functions in the performed simulations are on the form

\[
f(\phi, v, t = 0) = \frac{N_0}{(2\pi)^{3/2}v_T} \exp \left( -\frac{1}{2} \left[ \frac{v - v_B}{v_T} \right]^2 \right) (1 - \sigma \cos n\phi),
\]

(23)

which models a population of energetic ions coming from a neutral beam source, with the beam velocity \( v_B = 1.2v_t \) and temperature \( v_T = 0.2v_t \). The perturbation factor \((1 - \sigma \cos n\phi)\) for \( \sigma = 0.01 \) is introduced to enforce an initial linear growth of the mode, since a system with a flat phase distribution is in unstable equilibrium. Such a system would be sensitive to statistical noise in the phase distribution of markers, which is undesirable for comparisons of similar scenarios. The markers are distributed on a grid in velocity space, with a spacing proportional to \( \exp[(v - v_B)^2/[2v_T^2]] \). For decent statistics, the particles are distributed quasi-randomly in phase \( \phi \) using a van der Corput sequence [24], with a small amplitude (pseudo-)random noise added to diminish undesirable correlations in the distribution function.
3.2. Nonlinear saturation in the presence of phase decorrelation

In Ref. [21] it was shown that there exist three characteristic regimes defined by the strength of the decorrelation in the absence of dissipative processes. These regimes also appear in the upgraded model, as illustrated in Fig. 1.a. In the weakly decorrelated regime ($4 \lesssim \gamma_L t_c < \infty$) the saturation level and time of the amplitude is much larger than in the correlated regime ($t_c = \infty$). In the intermediate regime ($1 \lesssim \gamma_L t_c \lesssim 4$) the saturation time is similar to the correlated case, but the saturation level is larger. Below the relatively sharp limit at around $\gamma_L t_c \approx 1/2$ is the strongly decorrelated regime, where the growth of the amplitude is strongly damped on time scales for relevant dynamics of more weakly decorrelated regimes. This limit corresponds to the quasilinear regime, where decorrelation time scales are shorter than any time scale relevant for the nonlinear dynamics of the system. The initial growth rate is essentially the same in the weakly and intermediately decorrelated regimes, as seen in Fig. 1.b. It is in the transition to the strongly decorrelated regime that the initial growth rate becomes lower.

As was shown in eq. (22), phase decorrelation translates into velocity space diffusion via interactions with the wave mode, which is also seen in Fig. 1.c. The induced velocity diffusion flattens a wider range of the distribution (see the distribution for $\gamma_L t_c = 16$ and $\gamma_L t_c = 1$). Since the phase decorrelation preserves the total energy of the system, given

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**Figure 1.** a – b) The initial evolution of the mode amplitude for different strengths of the phase decorrelation, with $\gamma_d = 0$, $N_p = 2 \times 10^5$ and equidistant time steps $\gamma_L \Delta t = 1/640$. (b) is the same as (a), but for $\gamma_L t \leq 5$. c) The velocity distributions for selected scenarios at the end time of the simulations ($\gamma_L t = 25$). The green dotted curve is the initial velocity distribution.
by eq. (13), the enhanced flattening of the velocity distribution allows for the higher amplitude saturation level in the weakly and intermediately decorrelated regimes. In the strongly decorrelated regime \( (\gammaLt = 1/4) \) the distribution is essentially unchanged. The absence of velocity diffusion can be explained from eq. (22). The diffusion is proportional to \( |E|^2 t_c \). Since the strong phase decorrelation shields out the dynamics necessary for the initial growth of the wave amplitude, the \( |E|^2 \) factor limits the velocity diffusion.

### 3.3. The marginally stable regime

Presented in Fig. 2.a – d is the evolution of the local time Fourier decompositions of the complex wave amplitude \( E(t) \) for different strengths of the phase decorrelation near marginal stability, with \( \gamma_d = 0.9 \gamma_L \). At each time \( t \) the frequency spectrum is evaluated for a time window of width \( t_w = 50 \gamma_L^{-1} \), centered around \( t \). The time interval is divided into an equidistant grid of \( N_w = 3.2 \times 10^4 \) points according to

\[
\tau_k(t) \equiv t + \left( \frac{k}{N_w} - \frac{1}{2} \right) t_w, \tag{24}
\]

for \( k = 0, \ldots, N_w - 1 \). Then, a discrete Fourier transform is performed on the normalized complex wave field using a Hann window:

\[
E(\omega_k, t) = \frac{n|q|}{N_w \gamma_L^2} \frac{d\omega}{d\gamma_L} \sum_{g=0}^{N_w-1} E[\tau_g(t)] \sin^2 \left( \frac{\pi g}{N_w - 1} \right) \exp \left[ -i \omega_k \tau_g(t) \right], \tag{25}
\]

**Figure 2.** a – d) Time localized Fourier decompositions of the wave amplitude \( E(t) \) near marginal stability (\( \gamma_d = 0.9 \gamma_L \)) for different strengths of the phase decorrelation, with \( N_p = 4 \times 10^6 \) and equidistant time steps \( \gamma_L \Delta t = 1/640 \). a) The white curve is the least square fit of a \( \sqrt{t} \) function to the first upward chirping event. e – f) The difference in the distribution functions \( \delta f(\phi, v, t) \equiv f(\phi, v, t) - f(\phi, v, 0) \) at the end time of the simulations (\( \gamma_L t = 200 \)) for the simulations (a) and (b).
where $\omega_k \equiv 2\pi k/t_w$ for an integer $k$.

The frequency chirping, as seen in Fig. 2.a – c, are related to the spontaneous formation of hole-clump pairs in the distribution function [17–19], where an upward chirping event is associated with a hole and vice versa. As explained in Ref. [18], by viewing the hole-clump pair as a superposition of two BGK modes, the initial evolution of the frequency shift $\omega_s$ is

$$\frac{\omega_s(t) - \omega_0}{\gamma_L} = \frac{16\sqrt{2}}{3\sqrt{3}\pi^2}\sqrt{\gamma_d t},$$

in the limit $d\omega_B/dt \ll d\omega_s/dt \ll \omega_B^2$. In Fig. 2.a the least square fit of the function $\alpha\sqrt{\gamma_d(t - t_0)}$ to the first upward chirping event is shown. The coefficient $\alpha$ was found to be 0.397, which is 9.9% less than the theoretically predicted one in eq. (26).

From Fig. 2.e and f it is seen that the hole-clump structures extend at scales $n\Delta\phi \lesssim \pi$. If the process of generating such structures happen on time scales $\gtrsim t_c$ phase decorrelation are expected to smear out these structures. From Fig. 2.f it can be noticed that hole-clump structures in the distribution are partly diminished already at $t = 200\gamma_L^{-1} = t_c/2$. The asymmetry between the evolutions of holes and clumps is within the range of statistical errors.

As seen for $\gamma_L t_c = 100$ in Fig. 2.c, frequency chirping events are still visible, but they are not moving particularly far from the resonance on the investigated time scales. Rather, the chirping events eventually seem to fade out. The process behind the diminishing of frequency chirping is presumably the phase decorrelation. For $\gamma_L t_c = 25$ no traces of frequency chirping events can be detected (see Fig. 2.d). This indicates that hole-clump structures form on time scales $\tau$ satisfying $25 \lesssim \tau \gamma_L \lesssim 100$.

4. Conclusions

The presented Monte Carlo model successfully describes several aspects of the dynamics of toroidal Alfvén eigenmodes, such as the nonlinear growth and saturation of a wave mode in the absence of wave damping, and the formation of hole-clump structures in the distribution function near marginal stability. An important aspect of the model is the inclusion of a phenomenological decorrelation of the wave-particle phase, which is shown to have a strong impact on the dynamics of the system.

In the absence of wave damping, three characteristic regimes defined by the strength of the phase decorrelation have been observed in addition to the correlated regime: the weakly, the intermediately and the strongly decorrelated regime. The weakly and intermediately decorrelated regimes show a higher saturation level of the mode amplitude than the correlated regime, but the weak regime has a longer saturation time, whereas the saturation times in the correlated and the intermediately decorrelated regimes are similar. In the strongly decorrelated regime, which corresponds to the
quasilinear regime, the wave amplitude is heavily damped on time scales relevant for more weakly decorrelated regimes.

The hole-clump structures formed near marginal stability are much similar to those described in Ref. [17–19]. When including phase decorrelation, the frequency chirping events in the local time Fourier decomposition of the mode, resulting from the hole-clump formations, are weakened and eventually vanish on time scales similar to the characteristic decorrelation time. In the regime of partial phase decorrelation the frequency chirping events are also slowed down compared to the correlated scenario.

References