Inflation Mechanics of Hyperelastic Membranes

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Abstract

The applications of inflatable membrane structures are increasing rapidly in the various fields of engineering and science. The geometric, material, force and contact non-linearities complicate this subject further, which in turn increases the demand for computationally efficient methods and interpretations of counter-intuitive behaviors noted by the scientific community. To understand the complex behavior of membranes in biological and medical engineering contexts, it is necessary to understand the mechanical behavior of a membrane from a physics point of view.

The first part of the present work studies the pre-stretched circular membrane in contact with a soft linear substrate. Adhesive and frictionless contact conditions are considered during inflation, while only adhesive contact conditions are considered during deflation. The peeling of membrane during deflation is studied, and a numerical formulation of the energy release rate is proposed. It is observed that the pre-stretch is having a considerable effect on the variation of the energy release rate.

In the second part, free and constrained inflation of a cylindrical membrane is investigated. Adhesive and frictionless contact conditions are considered between the membrane and substrate. It is observed that the continuity of principal stretches and stresses depend on contact conditions and the inflation/deflation phase. The adhesive traction developed during inflation and deflation arrests the axial movement of material points, while an adhesive line force created at the contact boundary is responsible for a jump in stretches and stresses at the contact boundary. The pre-stretch produces a softening effect in free and constrained inflation of cylindrical membranes.

The third part of the thesis discusses the instabilities observed for fluid containing cylindrical membranes. Both limit points and bifurcation points are observed on equilibrium branches. The secondary branches emerge from bifurcation points, with their directions determined by an eigen-mode injection method. The occurrence of critical points and the stability of equilibrium branches are determined by perturbation techniques. The relationship between eigenvalue analysis and symmetry is highlighted in this part of the thesis.

Descriptors: Membranes, Hyperelastic models, Constrained inflation, Energy release rate, Adhesive contact condition, Stability, Critical points

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Sammanfattning

Uppblåsbara membranstrukturer används i ökande grad inom olika tillämpningsområden inom teknik och naturvetenskap. Analysen av dessa är komplicerad genom att de påverkas av geometriska och materiella ickelinjariteter, men också ickelinjariteter i belastningar och kontaktvillkor. Detta ökar behoven av effektiva beräknings- och tolkningshjälpmedel för de ofta icke-intuitiva responsaspekterna. För att förstå det komplexa beteendet hos membran i biologiska och medicintekniska sammanhang är det nödvändigt att kunna analysera det mekaniska och fysiska beteendet av membran.

Den första delen av det aktuella arbetet studerar ett för-sträckt cirkulärt memb ran i kontakt med ett flexibelt linjärt skikt. Adhesiva och friktions-lösa kontaktvillkor jämförs under uppblåsning, medan endast adhesiva kontakter beaktas under avlastning. Avskalningen av membranet från skiktet under avlastning studeras. En numerisk beskrivning av energi-frigörandet föreslås för detta fenomen. Studien visar att för-sträckningen av membranet har en betydelesefull effekt på den energifrigöring som sker.

I den andra delen av studien studeras fri och villkorad uppblåsning av ett cylindriskt membran. Adhesiva och friktionslösa förhållanden mellan membranet och begränsningsytan beaktas. Studien visar att kontinuiteten i huvudtöjningar och -spänningar beror på de antaganden som görs för kontakten, men också på i vilken fas analysen sker. De adhesiva kontaktspänningar som uppstår under uppblåsning och avlastning bromsar materiella punkters axiella rörelser. En adhesiv linjekraft i kontaktgränsen leder däremot till en diskontinuitet i töjning och spänning. Försträckningen leder till ett mjuknande i responsen under fri och villkorad uppblåsning av de cylindriska membranen.

Den tredje delen av avhandlingen diskuterar de instabiliteter som observerats för vätskefyllda cylindriska membran. Både gräns- och förgreningspunkter existerar på jämviktskurvorna. Sekundära lösningskurvor utgår från förgreningspunkterna, med riktningar som kan bestämmas via en metod som innebär en injektering av egenmoderna i förgreningen. Förekomsten av kritiska jämviktspunkter bestäms med hjälp av perturbationstekniker. Relationen mellan egenvärdesanalyser och strukturens symmetriegenskaper är en aspekt som diskuteras i avhandlingen.

Nyckelord: Membran, Hyper-elastiska modeller, Begränsad inflation, Energifrigöring, Adhesiv kontakt, Stabilitet, Kritiska punkter

Preface

This thesis discusses inflation mechanics of inflated hyperelastic membranes of different geometries. Semi-analytical and numerical formulations are proposed in this thesis. It thereby gives the basis for qualitative verifications of results obtained by other methods and experiments. A brief introduction to the basic concepts and methods is presented in the first part. The second part contains three articles. The papers are adjusted to conform with the present thesis format for consistency, but their contents have not been changed as compared with their original counterparts.

Paper 1. Amit Patil, Anirvan DasGupta & Anders Eriksson, 2014
Contact Mechanics of a Circular Membrane Inflated Against a Soft Adhesive Substrate. *Submitted to International Journal of Solids and Structures.*

Paper 2. Amit Patil, Arne Nordmark & Anders Eriksson, 2014
Free and Constrained Inflation of a Pre-stretched Cylindrical Membrane. *Proceedings of Royal Society A,* **470***, 20140282.*

Paper 3. Amit Patil, Arne Nordmark & Anders Eriksson, 2015
Instability Investigation on Fluid-loaded Pre-stretched Cylindrical Membranes. *Submitted to Proceedings of Royal Society A.*

The following publications are not included in the thesis, but related to the presented work.

Before joining KTH

1. Amit Patil & Anirvan DasGupta, 2013
Finite Inflation of an Initially Stretched Hyperelastic Circular Membrane. *European Journal of Mechanics- A/Solids,* **41***, 28–36.*

2. Amit Patil & Anirvan DasGupta, 2015
Constrained Inflation of a Stretched Hyperelastic Membrane inside an Elastic Cone. *Meccanica, In Press.*

After joining KTH

1. Amit Patil, Arne Nordmark & Anders Eriksson, 2014
Finite Inflation of Fluid-filled Pre-stretched Cylindrical Membranes. *27th Nordic Seminar on Computational Mechanics,* **27***, 224–227.*

2. Anders Eriksson, Arne Nordmark, Amit Patil & Yang Zhou, 2014
Parametric Stability Investigations for Hydro-statically Loaded Membranes. *Submitted to Computers and Structures.*

Division of work among authors

This project was first introduced by Professor Anders Erikson (AE) who is the main supervisor of this work and Arne Nordmark (AN) who has functioned as assistant supervisor. Professor Anirvan DasGupta (AD) has contributed in one research paper. Amit Patil (AP) has discussed the results and progress with AE, AN and AD for the respective manuscripts.

Paper 1

AP built up the numerical model in Mathematica and performed the simulations with feedback from AD and AE. The paper was written by AP with help from AD and AE.

Paper 2

AP built up the numerical model in Mathematica and performed the simulations with feedback from AN and AE. AN verified some results in Comsol Multiphysics. The paper was written by AP with help from AN and AE.

Paper 3

AP built up the numerical model in Mathematica and performed the simulations with feedback from AN and AE. AN verified some results in Comsol Multiphysics and AE in an independent FEM model in Matlab. The paper was written by AP with help from AN and AE.

Contents

Part I

Overview

CHAPTER 1

Introduction

1.1. Background

1.1.1. *Definitions*

The definition of a membrane varies from application field to field. A few of these definitions will be included here.

From a mechanics point of view, a membrane is a two-dimensional elastic continuum that does not resist or transmit bending moments (Hagedorn & DasGupta 2007). In general, a membrane can be viewed as a two-dimensional plane stress problem with in-plane and out-of-plane loadings.

From a micro-technology perspective, a membrane is a special type of thin film with very small thickness compared to the other dimensions (Schomburg 2011). Their thickness is in the range of 0.5-500 μm, while the lateral dimensions are typically in the range of 100 μm to 10 mm.

From a biology and chemical perspective, membranes are thin selective barriers who permit certain components and retain other certain components of mixture (Cheryan 1998).

A membrane can be further classified according to a) material: silicon, oxides, polymeric and metals; b) structure: porous and non-porous; c) application: biological, micro-technological and structural; d) mechanism of action: absorption, diffusion, ion exchange and inert, et cetera.

For this research, a polymeric, non-porous, inert membrane for structural application was considered, with applicability to models with relevance for the bio-engineering field. The membranes made of polymeric material are called elastomers, and can be described as hyperelastic material models. Well known materials for structural membranes are natural rubber, Hevea rubber, Polyethylene, Ethylene-propylene rubber, etc. For a computational point of view, the hyperelastic membrane are modeled by neo-Hookean, Mooney-Rivlin, Ogden, Arruda-Boyce models and many more (Selvadurai 2006).

1.1.2. *Membrane structures applications*

Due to numerous advantages, membrane structures are widely used in space engineering, medical sciences, bio-engineering, civil structures and mechanical engineering with many applications to be found in literature (Fung 1990; Jenkins 2001; Wan 2001). Only some of the applications are discussed in this section.

1.1.2a. *Space applications.* Deployable membrane structures have much lower weight, higher packaging efficiency and lower cost than conventional space structures, so the membrane structures are becoming preferable for space applications. The applications of inflatable technology in space engineering are numerous, including solar

(a) (b)

Figure 1.1: a)Thermal shields by James Webb Space Telescope NASA,USA (http:// www. jwst. nasa. gov/). b) Solar sails by Japan Space Agency, Japan (http://www. isas. jaxa. jp/)

sails, scientific balloons, thermal shields, pressurized habitats in space, solar arrays, and telescope mirrors. The thermal shield developed by NASA and solar sails developed by the Japan space agency are the examples of growing applicability of deployable structures in space.

1.1.2b. *Architecture and civil structures.* The inflatable structures can be classified into two types. The first type of structures, known as air supported structures consists of membranes restrained by cables on the ground and pressurized by air from below with several civil structure applications. The second type of membrane structures are closed and inflated by pressurized air, e.g. airbags in automobiles. One of the reasons for using membrane structures in architecture and civil engineering is the possibility to easily reshape them under different load conditions to adapt to environmental changes. The tensegrity structures along with membrane structures are used as stadium roofs (Fig. 1.2).

Figure 1.2: Inflated membrane structure, Allianz Arena stadium, Munich, Germany (https: //www. allianz-arena.de)

1.1.2c. *Mechanical engineering.* The closed membrane structures inflated by some kind of gas are used as load carrying or impact absorbing appliances. The fluid inflated rubber tubes for tractors is a common example, while air inflated membrane structures can be used as airbags in automobiles to absorb impact load during accidents (Fig. 1.3).

Figure 1.3: Inflated membrane as airbags (http://www. carsp. ca)

1.1.2d. *Bio-engineering and medical engineering.* The contact of cells within themselves or with a soft substrate, the function of vesicles and the contact mechanics of a balloon with stent and blood vessels can be better understood by studying inflation mechanics of membranes. The application of an inflated balloon for an expanding stent in balloon angioplasty is explained in Fig. 1.4.

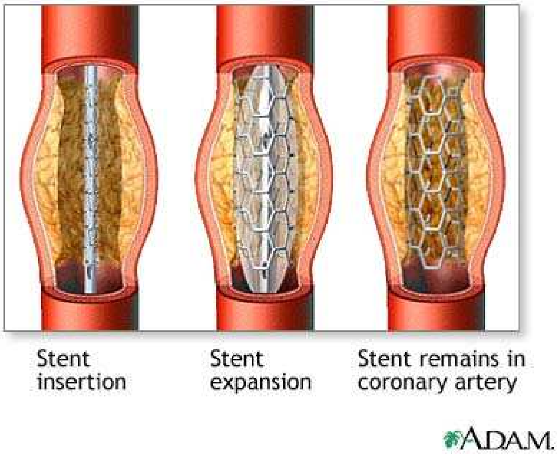

Figure 1.4: Stent expansion in Balloon Angioplasty by inflated membrane (http://www.adam.com/)

1.2. Aim and outline of the thesis

The main objective of this thesis is to study free and constrained inflation of hyperelastic membranes with different parameters like pre-stretch, loading medium, constraining properties and contact conditions. The numerical analysis of membranes encounters numerous non-linearities like geometric non-linearity due to large displacements, and material non-linearity due to large strains, force non-linearity due to displacement dependent loading, and also contact non-linearity. These non-linearities make computations complex, and poses many mathematical challenges.

They will also frequently give results which show counter-intuitive behavior. This underlines the necessity to investigate membrane structures with different parameters, geometries and material models. The objective of the study can be summarized into the following parts

1. To study contact mechanics of a homogeneous, isotropic, incompressible initially flat, pre-stretched circular membrane against a substrate under axisymmetric conditions with frictionless and adhesive contact conditions.
2. To study free and constrained inflation of a homogeneous, isotropic, incompressible, pre-stretched cylindrical membrane under axisymmetric conditions with frictionless and adhesive contact conditions.
3. To investigate instability phenomena associated with bifurcations, limit points, and wrinkling for fluid-loaded pre-stretched cylindrical membranes.

The problem of contact between an inflated membrane and an elastic surface appears in various contexts in different scientific applications. The contact of elastomers is best modeled using dry adhesive friction theory as noted by Savkoor (1987). For adhesive contact problems, continuum theories considering elastic deformation of the bodies involved have been presented by Johnson, Kendall and Roberts (JKR) (Johnson 1997) and Derjaguin, Muller and Toporov (DMT) (Derjaguin *et al.* 1975). Researchers, like Long *et al.* (2010); Long & Hui (2012); Srivastava & Hui (2013*a*,*b*) have studied delamination and energy release rates for circular and rectangular membranes. The first part of this thesis focuses on the contact mechanics of circular membrane with a soft substrate with adhesive and frictionless contact conditions. The delamination of the membrane during deflation is considered.

The second part of the thesis is devoted to free and constrained inflation of a pre-stretched cylindrical membranes. The constrained inflation of cylindrical membranes are of practical importance for problems like blow moulding of plastic bottles, balloon angioplasty, and thermoforming. Recently, Patil & DasGupta (2013*b*) demonstrated a stretch-induced softening/stiffening phenomenon during free inflation of unstretched and prestretched hyperelastic circular flat membranes. The same phenomenon needs to be investigated for cylindrical membranes. Frictionless and adhesive contact conditions are considered during inflation, while adhesive contact conditions are considered during deflation.

Due to the highly non-linear nature of membrane structures, they are prone to become unstable under displacement dependent loads like gas or fluid pressures. This aspect is studied in the third section of the thesis. The stability investigation of inflated membranes is concentrated on three main topics: limit point instability, bifurcation instability and wrinkling. A previous experimental investigation (Pamplona *et al.* 2001) has shown a bifurcation of the membrane prior to the limit point with fluid volume as the control variable. Axisymmetric and asymmetric equilibrium solutions are detected on the primary and secondary equilibrium branches, and connected at a set of critical points. Stability analysis of equilibrium solutions are carried out on the primary and secondary branches.

CHAPTER 2

Inflation of a circular membrane against a soft substrate

Contact and interaction of membranes with other structures is unavoidable in many applications of inflatable structures. The treatment of such problems is complex in nature. Contact of cells with other cells or surroundings, adhesive contact of thin films, contact of a balloon with stent or artery, and contact of a toroidal rubber tube on a steel rim are a few examples. The contact mechanics of bi-layer membranes can be useful in order to estimate mechanical properties (Evans 2009). In the past, contact mechanics of membranes have been well studied with rigid substrates, [Feng & Yang (1973); Kumar & DasGupta (2013); Long *et al.* (2010)]. The constrained inflation problem with a soft substrate, however, remains to be explored. The flexibility of the surface brings further complexity into the mechanics, as the substrate surface deforms with inflation and relaxes with deflation, giving interesting behaviors. The continuity of stretches and stress resultants depend on the continuity of field variables which in turn depend on contact conditions during the inflation and deflation phases. The constraining surface can be modeled in different ways. In many applications, the constraining surface possesses geometric as well as material non-linearities. For our analysis, we have considered a linear spring foundation, which is stiff in the radial direction compared to the axial direction. This assumption restricts the generality of the model, but is useful for many applications where the membrane is compliant compared to the substrate, which is a common case. The constrained problem can be solved by numerical techniques like finite element analysis (Khayat & Derdouri 1994*a*,*b*). A semi-analytical technique proposed in this thesis is believed to be more convenient and can be used as a verification tool for finite element simulations.

The contact between a membrane and a substrate, or between two membranes is of complex nature. Chernyak& Leonov (1986) have mentioned that the Amontons-Coulomb law is not applicable to elastomeric friction on a smooth solid surface, which is adhesive in nature. The elastomeric friction can be best modeled by an adhesive friction theory, as advocated by Savkoor (1987). We have considered contact between membrane and substrate at a macroscopic level. A popular technique to measure the adhesion between a soft and a hard solid is the Johnson Kendall Roberts (JKR) test (Johnson *et al.* 1971). In the JKR test, a soft hemisphere is brought into contact with a flat surface by a compressive force. Since the contact area increases with adhesion for the same applied force, a measurement of the contact area versus force can be used to find out the work of adhesion, of the surfaces in contact. Important work on adhesion measurement and contact mechanics has been reported by Shanahan (2000) and Wan (2001). Recently, Long *et al.* (2010) have developed expressions for the energy release rate in terms of contact angle, stretches and stresses at the contact boundary. They used a fraction mechanics approach to deduce energy release rate. Researchers like Srivastava & Hui (2013*a*,*b*) have extended this approach and used it also for rectangular membranes. To understand the lamination and delamination of a membrane from the substrate, it is better to view the gap between the membrane and the substrate as an external crack and apply fracture mechanics approach. During inflation, the crack recedes and adhesion energy is released to the system. During deflation, the crack advances and elastic energy is released by the system to create new surfaces. The energy release rate is defined as the energy released by the elastic system per unit contact area created. The condition for making or breaking contact is when the energy release rate equals to the effective work of adhesion of the interfaces.

CHAPTER 3

Constrained inflation of a cylindrical membrane

Constrained inflation of cylindrical membranes finds application in manufacturing processes like blow moulding of plastic bottles (Khayat & Derdouri 1994*a*,*b*) and in medical engineering like balloon angioplasty. Here, the focus more is on the balloon angioplasty applications. Many possibilities exist for analysis in order to optimize the stenting procedure based on patient-specific needs. The balloon angioplasty is well studied in recent years with the help of finite element analysis and experimental methods with elaboration on the stenting procedure (Martin & Boyle 2013; Morgan & Walser 2010). Gasser & Holzapfel (2007) studied balloon angioplasty by considering an elastic-nonelastic behavior for the artery in simulating balloon-artery interaction with a point-to-surface strategy. However, Holzapfel *et al.* (2002) developed a model of angioplasty which is based on the magnetic resonance imaging and mechanical testing. As cylindrical membranes are used for widening narrowed or obstructed arteries in angioplasty, an extensive study is a demanding task. With some assumptions on the constraining surface, contact conditions and model scaling, we aimed to develop a simple model for balloon angioplasty.

The free inflation of a cylindrical membrane has been studied by researchers such as Pamplona *et al.* (2001, 2006*a*), whereas confined inflation of a cylindrical membrane is studied by Khayat & Derdouri (1994*a*,*b*). Several research works in the past have considered the membrane contact problem with rigid frictionless surfaces. However, contacts with no-slip and adhesion on soft surfaces have been scarcely discussed. Adhesion on soft surfaces is more complex than contact problems involving hard surfaces, because the surface conforms (relaxes) with inflation (deflation). In the case of an inflated membrane interacting with a soft adhesive substrate, changes in contact conditions with inflation and deflation have a further level of complexity owing to material and geometrical nonlinearities.

Mooney-Rivlin and neo-Hookean material models for membrane and linear elastic foundation for substrate were considered for the numerical analysis, while a principle of stationary potential energy was used to obtain equilibrium equations along with boundary conditions. A finite difference approach was used to discretize the resulting differential equations into a set of algebraic equations. For most application of non-linear mechanics, Newton-like methods are most popular for obtaining equilibrium states, due to their straight-forward numerical implementation. However, this type of algorithms is sensitive to the initial guess to the solution, and becomes expensive in the computations of the Jacobian matrix and its inverse. So, a fictitious time integration method (FTIM) proposed by Chein-Shan & Atluri (2008) was used for this study to obtain an algorithm where the inverse of the Jacobian matrix is not required for solution.

When adhesive inflation is considered, the contact area depends on the history of contact and has to be determined incrementally by imposing a kinematic constraint on all material points in the contact zone. For adhesive deflation on the other hand, the contact area attained at the end of the inflation phase is assumed to remain constant with different deflation pressures.

CHAPTER 4

Stability analysis of a fluid loaded cylindrical membrane

Inflatable membrane structures are very sensitive to stability phenomena, due to inherent material, geometric and force non-linearities. Although fluid pressure is a displacement-dependent follower force, a potential for this force can in most cases be written, so the system remains conservative. The principle of potential energy was here used to obtain the needed governing equations. Even if the concept of stability is dynamic in nature, time can be taken out and stability criteria strictly become static for conservative system. For the inflated membranes, quasi-static analysis were carried out, where thermal and dynamic (inertia, damping) effects were neglected. The tangent stiffness matrix was obtained from double differentiation of potential energy with respect to displacement field, which determined the stability of conservative system, according to the following criteria.

- If the tangent stiffness matrix is positive definite; then all eigenvalues are positive, and the potential energy is strictly minimum and all variations demands added energy. In result, the equilibrium solution is stable.
- If the tangent stiffness matrix is positive semi-definite; then one or more eigenvalues are zero. In result, the equilibrium solution is probably a critical point.
- If the tangent stiffness matrix is indefinite; then one or more eigenvalues are negative, and the potential energy has a saddle point. In result, the equilibrium solution is unstable, and certain incremental displacements will occur with release in energy.

The stability of a system is a local property, and the stability of an equilibrium solution can be evaluated without consideration of the path needed to reach this state. For conservative systems, the stability of the system changes only at critical points on the equilibrium path, so one solution between each pair of critical points is sufficient to predict whether a path is stable or not. The equilibrium path can be classified as primary or secondary. On primary equilibrium paths, solution of cylindrical membrane maintains symmetry, while the equilibrium solution breaks symmetry on secondary equilibrium branch. At a bifurcation point, one or more secondary equilibrium branches emerge from the primary path.

The stability investigation of inflated membranes is concentrated on three main topics: limit point instability, bifurcation instability and wrinkling. The limit and bifurcation points are critical points on the equilibrium paths. At a limit point, the tangent to the equilibrium path is transversal to the loading axis, while at a bifurcation point, two or more equilibrium branches emerges. Detailed investigations of limit point instabilities, with some experimental results, are discussed by Alexander (1971*a*), Alexander (1971*b*), Eriksson & Nordmark (2012), Patil & DasGupta (2013*b*), Verron *et al.* (1999), and Tamadapu *et al.* (2013). The results on stability and bifurcation of membranes are discussed by Alexander (1971*b*), Haughton & Ogden (1979), Haughton & Ogden (1978), Eriksson (2014), Chaudhuri & DasGupta (2014),and Biscari & Napoli (2009), Khayat *et al.* (1992), and Chen (1997). At a bifurcation point, several methods have been proposed for calculation of the secondary branches [Kouhia & Mikkola (1989, 1998); Riks (1979); Shi (1995); Shi & Moita (1996); Wagner & Wriggers (1988)], but also in relation to generalised path-following [Eriksson (1994); Kouhia & Mikkola (1999)]. Wrinkling corresponding to more or less localized regions of compressive stresses, often occur for inflated membrane structures, and can be seen as related to geometric effects. Wrinkling is normally not directly observed from the tangent stiffness matrix of the system. Important contributions on this topic are given by Evans (2009), Li & Steigmann (1994*a*), Li & Steigmann (1994*b*), Pipkin (1986), and Steigmann (1990).

CHAPTER 5

Summary of papers

Paper 1

Contact Mechanics of a Circular Membrane Inflated against a Soft Adhesive Substrate

In this paper, finite inflation of a hyperelastic flat circular membrane against a soft adhesive substrate, and peeling upon deflation is studied. The membrane is modeled by either a homogeneous, isotropic and incompressible Mooney-Rivlin or a neo-Hookean model. The soft substrate is assumed to be a distributed linear stiffness in the direction normal to the undeformed surface. It is assumed that the substrate is more stiff in the radial direction than in the transverse direction. The adhesive contact is considered to be perfectly sticking with no tangential slip between the dry surfaces of the membrane and the substrate. During adhesive inflation and deflation, the area of contact depends upon the history of contact.

The principle of stationary potential energy is used to obtain the governing equations and boundary conditions, which are transformed to a non-linear two-point boundary value problem by a careful choice of field variables for efficient computation. A Runge-Kutta method is used to obtain solutions to the two-point boundary value problem by assuming some initial conditions for certain dependent variables. These initial conditions are then corrected using a two-dimensional bisection method.

It is found that, the continuity of stretches and stresses depend on the contact conditions as well as the inflation/deflation phase. Interestingly, stretch locking in an adhesive contact is found to result in a higher indentation on the substrate than in a frictionless contact.

Peeling at the contact junction is studied from a fracture mechanics point of view. The energy release rate is deduced as a change in potential energy per unit new area delaminated. A numerical formulation for the energy release rate calculation is proposed in this paper. It is observed that pre-stretch has a more significant effect on the energy release rate than what is discussed in literature, when a membrane is pulled from a rigid substrate.

Paper 2

Free and Constrained Inflation of a Pre-stretched Cylindrical Membrane

The aim of this paper is to study free and constrained inflation of a pre-stretched hyperelastic cylindrical membrane and a subsequent constrained deflation. The membrane material is assumed as a homogeneous and isotropic Mooney-Rivlin solid. The constraining soft cylindrical substrate is assumed to be a distributed linear stiffness normal to the undeformed surface. Frictionless and adhesive contact are modelled during the inflation as an interaction between the dry surfaces of the membrane and the substrate. An adhesive contact is modelled during deflation.

The principle of stationary potential energy is used to obtain governing equations and boundary conditions, which are discretized by finite differences into a set of algebraic equations. To make the solution process less dependent on the initial guess, and to avoid computation of the inverse of the Jacobian, a fictitious time integration method (FTIM) is used.

Continuity in the principal stretches and stresses at the contact boundary is dependent on the contact conditions and the inflation/deflation phase. The pre-stretch reduces circumferential stretch and stress, which gives a softening effect on the free and constrained inflation. It is observed that the limit point pressures vary with pre-stretch. Interesting trends are observed in the stretch and stress distributions after the interaction of the membrane with the soft substrate, which underlines the significant effects on the response from material parameters, as well as from pre-stretch and constraining properties.

Paper 3

Instability Investigation on Fluid-loaded Pre-stretched Cylindrical Membranes

This paper discusses the evaluation of instabilities on the quasi-static equilibrium path of fluid-loaded pre-stretched cylindrical membranes, and the switching to a secondary branch at a bifurcation point. A simple one parameter neo-Hookean material model is used to describe the hyperelastic behaviour of the membrane.

The free inflation problem yields governing equations and boundary conditions, which are discretized by finite differences and solved by a Newton-Raphson method for different fluid levels. Commonly, limit points are encountered on the equilibrium path, which makes solution difficult with a Newton-Raphson method without proper initial guesses. An incremental arclength-cubic extrapolation method is used to get proper initial guesses and facilitate an automatic tracking of the equilibrium path.

Limit points and bifurcation points are observed on the equilibrium path when fluid level is seen as the controlling parameter. With perturbation techniques, it is established that the pre-limit primary branch is stable, while the post-limit primary branch is unstable with respect to fluid level. At a bifurcation point, an eigen-mode injection method is employed to switch to a secondary equilibrium branch. This secondary branch is unstable with respect to fluid level, and the obtained bifurcation point classified as super-critical.

A limit point with respect to fluid level is observed for a partially-filled membrane when the aspect ratio (length/radius) is high, whereas for smaller aspect ratios, the limit point with respect to fluid level is observed when the cylinder is over-filled. Pre-stretch is observed to have a stiffening effect in the pre-limit zone and a softening effect in the post-limit zone.

CHAPTER 6

Conclusions and future work

6.1. Conclusions

The inflation mechanics of hyperelastic membranes with different geometries and parameters like pre-stretch, loading medium, and constraining properties are studied in this thesis.

The analysis of hyperelastic membranes is highly non-linear and requires good mathematical modeling with robust numerical solution techniques. In literature, numerous hyperelastic models like neo-Hookean, Mooney-Rivlin, Ogden, Arruda-Boyce, Yeoh, Blatz-Ko and many more are proposed (Selvadurai 2006). Each model has its own advantage in numerous fields. We have chosen neo-Hookean and Mooney-Rivlin models for our study. Neo-Hookean is simplest model from computational point of view and has good agreement with experimental results at moderate strains. A Mooney-Rivlin model is an extension of neo-Hookean model with some hardening characteristics. The substrate geometry, stiffness and contact conditions are also critical parameters which complicate the solution procedure (Long & Hui 2012) in constrained inflation. For this study, a linear substrate with either adhesive or frictionless contact conditions are considered. The adhesion is modeled as a sticky contact between the membrane and substrate.

Different numerical procedures are used to obtain the solution of an inflated membrane depending on membrane and substrate geometry, contact conditions and loading medium. The numerical and semi-analytical methods are common in this field. The numerical methods comprise finite element method, finite difference method, RayleighRitz method, modified Galerkin method, while semi-analytical method are typically based on conversion of set of higher order differential equations into set of first order differential equations and employ simple numerical methods like Runge-Kutta for solution. For this study, the principle of minimum potential energy is used to obtain equilibrium equations. For an air inflated circular membrane, the differential equations are converted into a two point boundary value problem by a careful choice of variables. A Runge-Kutta method is employed to obtain the solutions. For an air inflated cylindrical membrane, a finite difference approach is used to discretized equilibrium equations into algebraic equations, and fictitious time integration method is then used to obtain solution of algebraic equations. For a fluid-loaded cylindrical membrane a finite difference approach along with Newton-Raphson method is used to get solution. A perturbation technique is used to ascertain stability of equilibrium solutions.

The orders of continuity of the principal stretches and stresses of an inflated/deflated membrane are found to be dependent on the contact conditions. It is also observed that, during free inflation, the stretches becomes anisotropic as pressure increases and for frictionless constrained inflation, the stretches in the contacting region readjust and tend to remain isotropic. For air inflated circular membranes, the pre-stretch gives a stiffening effect before limit point and a softening effect after the limit point, while for air inflated cylindrical membrane the pre-stretch introduces softening for neo-Hookean as well as Mooney-Rivlin membranes in the pre-limit and post-limit zones. For fluid loaded cylindrical membranes, pre-stretch produces a stiffening effect in the pre-limit zone and a softening effect in the post-limit zone.

For a circular membrane, the energy release rate in the case of peeling during deflation has been determined. For lower deflation pressures, a pull-off instability will occur, while for higher deflation pressures there will be no pull-off instability, and the energy release rate becomes zero at a certain contact radius.

The stability of inflated membranes is dependent on the geometry, pre-stretch, loading medium and control parameter. For air inflated circular and cylindrical membranes, the limit point pressure depends upon combinations of material parameters and pre-stretch for free inflation. At a certain value of pre-stretch, a cusp point is observed for Mooney-Rivlin membrane. Above this, a pre-stretched membrane does not exhibit a limit point behavior. For fluid-containing cylindrical membranes, limit and bifurcation points are discovered.

In general for pressurized membrane, it is necessary to qualify the occurrence of critical points with respect to the chosen control parameters, as the stability conclusions are highly dependent on the precise control. The numerical investigation in this thesis is carried out by considering fluid level as the primary controlling parameter, whether this is inside or above the membrane. Limit points with respect to fluid level and with respect to fluid volume are detected. In general it is noted that the volume calculation is not always unique, as it depends on an assumption on a tank geometry above the membrane, but we have taken only volume inside membrane as control parameter. Bifurcation points with respect to fluid level are detected for the pre-stretched cylindrical membrane. As principal stress resultants do not become negative, wrinkling does not occur for fluid loaded pre-stretched cylindrical membrane in the considered pressure ranges and parameters.

6.2. Future work

The semi-analytical and numerical formulation presented in this thesis are intended to analyse hyperelastic membranes of different geometries with different inflating media. It is observed that the geometry of the membrane, loading medium, and pre-stretch produces interesting instability phenomena and softening/stiffness effects. So, the inflation mechanics of membranes with different geometries with different parametrization will be the future direction to pursue. As the current work focuses on two instabilities, i.e., limit point and bifurcation instability, wrinkling instability will be an interesting behavior to investigate in future.

The following studies could be interesting and achievable, based on the current work:

- To investigate the inflation of rectangular and square membranes with different pre-stretches and loading media. In the present work, we observed that eigenvalue analysis has a strong connection with symmetry of the system. Specifically, the response at bifurcations depend on the symmetry of the structural system. The present formulations can be extended to study stability analysis of rectangular and square membranes, with more emphasis on symmetry-eigenvalue relationship. There are two ways to achieve a pre-stretched rectangular membrane. The first is by uniformly stretching an initial rectangular configuration, and the second is by non-uniformly stretching square configuration. Although the final geometries before inflation are the same, these two methods have different impacts on the inflated behavior of the membrane. A combination of a finite difference approach with the Newton-Raphson method could be a good numerical technique to obtain solutions. As compared to cylindrical membranes, where one-dimensional finite difference is sufficient to get primary equilibrium solutions, the discretization process for rectangular membranes requires two-dimensional finite difference approach due to the lack of symmetry about the transverse axes. This makes a solution procedure tedious and time consuming. Applying general finite element softwares could be a good option for verification and prediction of results.

The challenges in the proposed research directions primarily lies in localized deformation and wrinkling with certain material models. When hyperelastic membranes inflated with fluid, specifically on secondary branch, a localized deformation will occur which demands non-uniform and dense mesh. Also wrinkling can occur with very high hydrostatic pressure, and after bifurcation where principal stretch directions are unknown a priori, this makes handling of wrinkling in computational models cumbersome.

- It is observed in previous literature, that boundary conditions affect the instability phenomenon. During manufacturing of membrane, various geometric defects are developed like thickness variations due to manufacturing tolerances. The fluid or gas inflated thin cylindrical membrane of non-uniform thickness can undergo wrinkling instability with specific boundary conditions. To obtain equilibrium solutions for wrinkled membranes, a combination of standard and relaxed forms of strain energy functions need to used. The relaxed form of strain energy function ensures that no compressive stresses will occur in the solution. The work on this topic is already started, and will consider different loading, boundary and geometric conditions.

  In the literature, the work on relaxed strain energy function for an isotropic material is carried out extensively. However, for an anisotropic membranes the relaxed strain energy is not straight forward and we will explore this topic in future. Even for non-axisymmetric isotropic membranes, where principal stretch directions are unknown a priori, the implementation of a relaxed strain energy function requires eigenvalues of the local strain tensor.
- Future work will aim to come closer to the membrane application in medical engineering. In balloon angioplasty, the hyperelastic balloon inflates and expands stents which are made of biocompatible materials and positioned inside blood vessels. Many finite element studies have been carried out in this regard but are computationally complex. Simple and computationally cheap models are not available in the literature. To understand this problem deeply, a quasi-static and dynamic study of cylindrical membrane constrained by a stent and artery will be done. The knowledge gained from paper 2 can be extended to yield a simple model for balloon angioplasty, by considering a hyperelastic anisotropic model for the constraining surface and the elastic-plastic model for the stents. As the real problem is dynamic in nature, a dynamic model will be proposed. For such complex problem, finite element method would be appropriate and necessary. So a finite element formulation will be proposed to analyze constrained inflation of

cylindrical membranes.

The challenges in the proposed research directions primarily lie in the inclusion of the irregular geometry and plasticity of the stents, anisotropy of blood vessel and plaque, and contact non-linearities. In real situation, the inflating medium interacts with plaque, stent, blood vessel, but at the same time opposes blood pressure, which makes this problem highly complex. Optimization of cylindrical balloons with stress and displacement constraints in angioplasty is a future task needed to take up after quasi-static and dynamic analysis.

Basically, the future work will be focused on two research strategies. The first focus is on to study computational mechanics of hyperelastic membranes with different geometries and boundary conditions. The stability investigations is a main interest of the future work. The second approach is based on application of our work toward medical engineering specifically balloon angioplasty. We want to extend the study from the second paper, and propose quasi-static/dynamic model for balloon angioplasty.

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