Airborne Sound Insulation of Single and Double Plate Constructions

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Preface

This thesis is the result of my research work at the Division of Building Technology, Department of Building sciences at the Royal Institute of Technology, Stockholm.

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This thesis comprises of an introduction and the following papers:


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Paper A-E
1. General Introduction

The sound insulation demands for dwellings and public building has increased over the years as the numbers of sound sources has grown. From the outside our homes are exposed to noise from cars, trains, airplanes, etc. Noise intrudes from our neighbours and their television and stereo equipments. Also noise from spaces for mechanical services systems tends to become more important due to increasing energy saving demands.

This thesis was prompted by the need for more precise prediction models for the sound reduction index of building walls and partitions. With good prediction models, fewer mistakes are made and the walls and partitions might be designed with better precision. Hence, time and money can be saved.

The models presented in this thesis deal with the airborne sound reduction index of single and double plate systems. They can be used for walls, floors, windows etc.

It should be noted that not only the sound reduction index of a wall or partition determines the sound level in a room. This level will also depend on the flanking transmission from connecting parts, a topic which is not touched upon in this work.

2. Objectives

The aim of the work is to develop more precise prediction models for the sound reduction index of single and double plate systems. In particular, improved descriptions of the area dependency of the sound reduction index is desirable.
3. The Sound Insulation of a Single Plate

In paper A, an analytical model for predicting the sound reduction index of a single plate is presented. The model is developed for a thin plate excited by a diffuse sound field. The resulting model is verified both numerically and by measurements. The velocity distribution of the plate is derived from the Kirchhoff plate equation in the frequency domain. In the resulting plate velocity, the different wave forms included stand out. It is seen how the velocity is a function of forced waves and free waves; i.e. the exponential near fields and the propagating waves. The solution for the plate velocity is verified numerically. As the plate is finite, there will also be a resonant field in the plate. Waves will be reflected at the boundaries and travel back into the excited part of the plate. The sound transmission due to the resonant field is calculated and added to the forced and free transmission. The analytical model is valid both below, at and above the critical frequency.

The results show that the common asymptotic expression for the forced transmission at low frequencies, the ‘mass law’, underestimates the reduction index for small plates. The results also show that the resonant transmission at low frequencies is strongly dependent on details of the plate field.

Special interest is paid to the area dependency of the sound reduction index. Calculated results are presented for a 0.5 m², 1.7 m² and 4.4 m² plate of 16 mm chipboard. These results, which agree well with measured values, show that the sound reduction index decreases with increasing area in the whole frequency range. The sound insulation of small plates might thence be considerably underestimated when prediction models based on theory for infinite plates are used.

In paper B, the analytical model for predicting the sound reduction index of a single plate presented in paper A is compared with prediction models by Ljunggren[1], Josse and Lamure[2], Sewell[3] and Leppington et al.[4]. In particular, the area dependency of the sound reduction index of these different prediction models is studied.

In Figure 1 the measured results for a 4.4 m² chipboard plate is compared with calculated results of the prediction model in paper A, the models by Ljunggren and the prediction model recommended in the European Standard[5]. This is
basically the model by Josse and Lamure, though the area-factor of the transmission at low frequencies has been modified, and the radiation factors of Maidanik\cite{6} are used.

\textbf{Figure 1.} The sound reduction index of a 16 mm, 4.4 m$^2$ chipboard. The Young's modulus is $E = 1.5$ GPa and $\rho = 626$kg/m$^3$, the loss factors $\eta = 4\%$.

(--x--) Measured results. (---) Calculated according to paper A. (-----) Calculated according to Ljunggren\cite{1}, see paper B. (---) Calculated according to EN 12354-1:2000.
4. The Sound Insulation of a Double Plate System

In this thesis, two different prediction models for floating floor, or double plate systems in general, are presented. The main differences of these two models are presented here.

In paper C, an analytical model is presented for the sound reduction index of finite size floating floors. The model is valid for two linear elastic plates with a resilient layer in between where the top plate and the resilient layer compose the floating floor. The bottom plate, the load-bearing slab, is assumed to be excited with a diffuse airborne sound field. In this prediction model, which will be referred to as “the floating-floor model”, it must be assumed that the load bearing plate is much heavier than the floating slab.

The main idea is that once one of the plates reaches the critical frequency, \( f_c \), the sound transmission at the coincidence angle will totally dominate the result. Therefore the solution for the sound reduction index is solved for frequencies below, between and above the critical frequencies of the plates. Both plates exhibit resonant and forced fields. The sound transmission factors for all four combinations are derived separately, and the transmission factors are summed up in the end. Above the critical frequency of the load-bearing plate, but below that of the floating slab, the main coupling between the plates will occur at the coincidence angle of the load-bearing plate. For low frequencies, below the critical frequency \( f_c \) of both plates, the sound reduction index is derived using the same concept of resonant and forced velocity fields of a plate as that used by Ljunggren[1]. It is also assumed that the intermediate layer is fairly resilient so that the force acting on plate 1 can be taken as the displacement of plate 3 times the stiffness of the spring. Knowing the excitation of plate 1, the sound transmission of the total construction can be derived. Above the critical frequency of both plates, transmission will occur at the angle of coincidence of each plate. As the plates will interact, the sound insulation improvement will to some extent depend on the properties of the load-bearing slab.

In paper D and E a more precise prediction model is presented. This will be referred to as the “double plate model”. The velocity distributions of the plates are
derived from the Kirchoff thin plate theory. In paper D, the responses of the two plates are derived using a two-dimensional Fourier transform technique and evaluated by means of contour integration in the complex wave number plane. It is shown that the response of the plates can be expressed as the sum of the amplitudes of a number of propagating and evanescent plate waves. The general properties of each term are discussed.

In paper E, the Fourier transform of the plate velocity for the excited part of the plate is used for calculating the non-resonant sound transmission. The expression for the plate velocity outside the excited area is used for calculating the resonant part of the sound transmission.

In both the floating-floor model and the double plate model, it is assumed that the intermediate layer can be described by the simple spring model, which limits the model to be valid for $d < \lambda/4$, where $d$ is the thickness of the resilient layer and $\lambda$ is the wavelength in air. Both models is also limited to thin plates, i.e. $h < \lambda_B / 6$, where $h$ is the thickness of the plate and $\lambda_B$ is the bending wavelength in the plate. The floating-floor model is only valid for frequencies well above the mass-spring-mass frequency $f_B$. However, the double plate model is not based on an asymptotic solution and hence it is continuous and valid through both the mass-spring resonance and the coincidence region. The main difference in applicability though, is that in the floating-floor model, the load bearing plates must be much heavier than the floating slab. In the double plate model, the plates are allowed be identical.

In Figure 2 the sound reduction indices are calculated, both with the floating-floor model and with the double plate model. In this case the plate thicknesses are strictly speaking not different enough for the floating floor model to give a good agreement for high frequencies. On the other hand, because of the cavity with of 30 mm, this limits the validity of both models to 2800 Hz.
Figure 2.1. The sound reduction index of a double plate system (——) Measured results, (- - -) Floating-floor model, (-----) Double plate model.

The measurements and calculations in Figure 1 are made for a double plate system of two chipboards, 16 mm and 12 mm thick, with a 35 mm layer of soft mineral wool in between. The excited area is 4.4 m². The mass-spring-mass resonance is 149 Hz and the critical frequencies are 2508 Hz and 3191 Hz respectively. The measurement accuracy is further discussed in paper E. Note, the calculated values are only valid to 2800 Hz, since the spring model is only valid up to d<λ/4.

4.1. A Comparison with Gudmundsson’s model

Gudmundsson[8] solved the problem using spatial Fourier Transforms, starting with Heckl’s solution for single infinite plates (see Cremer et al[7]). In the case of a diffuse field excitation, assuming that the same force is acting on the floating slab as on the structural slab, the sound insulation improvement when adding a locally reacting floating floors, is presented as
\[ \Delta R = 10 \log \left[ \int \frac{\left| N_0 + N_3 \right|^2 (1 - \tilde{\xi}^2) \xi \, d\tilde{\xi}}{\int \left| N_1 + N_3 + N_3 N_1 / N_2 \right|^2 (1 - \tilde{\xi}^2) \xi \, d\tilde{\xi}} \right], \] (1)

where

\[ N_0 = \frac{j \omega \rho c}{\sqrt{1 - \tilde{\xi}^2}}, \] (2)

\[ N_1 = \left[ B_1 (k\tilde{\xi})^4 - \omega^2 m_1 \right] + j \left[ \frac{\omega \rho c}{\sqrt{1 - \tilde{\xi}^2}} + B_1 \eta_1 (k\tilde{\xi})^4 \right], \] (3)

\[ N_2 = s(1 + j \eta_2). \] (4)

Here \( N_3 \) is defined analogously to \( N_1 \) and \( \tilde{\xi} = k_r / k \), where \( k \) is the exciting wave number. For each diffuse field component there is a wave number along the surface \( k_r = k \sin(\theta) \), where \( \theta \) is the angle of incidence. Gudmundsson now introduces the resonant waves by a correction factor as

\[ \Delta R_{\text{res}} = -10 \log(1 + Y_{SF} \sigma_{SF}), \] (5)

where \( \sigma_{SF} \) is set to 0,1 at \( f_c/2 \) and equals 1 at \( f_c \), but is 0 for all frequencies below \( f_c/2 \). \( Y_{SF} \) is the ratio between resonant and forced waves. In the case of a simply supported plate with a free slab, the ratio is

\[ Y_{SF} = \frac{U}{\beta \eta_1 S} \left( \frac{k_{g3}}{k_{g1}} \right)^7, \] (6)

The total sound reduction index is calculated as

\[ R = R_{\text{single}} + \Delta R + \Delta R_{\text{res}}, \] (7)

where

\[ R_{\text{single}} = 10 \log \left[ 8 (\rho c \omega)^2 \int \frac{\tilde{\xi}}{\left| N_0 + N_3 \right|^2 (1 - \tilde{\xi}^2)} \, d\tilde{\xi} \right]. \] (8)
Figure 3. The sound reduction index of a double plate partition.
(---) Measured results, (----) Double plate model according to paper E,
(-----) Calculations according to Guðmundsson.

The measurements and calculations in Figure 3 are made for the same double plate system of chipboards as in Figure 2.

4.2. A Comparison with the Modified Sharp Model
In 1978 Sharp\textsuperscript{[9]} presented a simple prediction model for double plate systems which is still often used, see for example Ballagh\textsuperscript{[10]}. It is divided into three different frequency ranges as follows

\begin{align}
R &= 20\log(f (m_1 + m_2)) - 47 \quad f < f_0 \\
R &= R_1 + R_2 + 20\log(fd) - 29 \quad f_0 < f < f_1
\end{align}

(9)

(10)
where \( f_i = 55/d \) and \( f_0 \) is the mass-spring-mass resonance. \( R_i \) and \( R_3 \) are the sound reduction indices for each plate. For frequencies below the critical frequency the sound reduction index for the single plate is calculated according to the well-known mass-law

\[
R_i = 20 \log(f m_i) - 47.
\]

At and above the critical frequency the modified mass law according to Cremer\(^{[11]}\) is used

\[
R_i = 20 \log(f m_i) - 10 \log \left( \frac{2 \eta \omega}{\pi \omega_c} \right) - 47.
\]

To improve the accuracy at low frequencies, Sewell\(^{[3]}\) derived a correction factor that will consider the influence of plate size as

\[
\Delta R = -10 \log \left[ \ln(k \sqrt{A}) \right] + 20 \log \left[ 1 - \left( \frac{\omega}{\omega_c} \right)^2 \right].
\]

According to Ballagh\(^{[10]}\) this is to be used for \( f < 200 \) Hz for a normal test settings of 10-12 m\(^2\). In the calculated example below, Figure 4, with an area of 4.4 m\(^2\), the correction factor was less than 0.2 dB and hence disregarded. The expressions according to Sharp do not take into account the properties of the material in the cavity. In Fahy\(^{[12]}\) an alternative expression for high frequencies are presented,

\[
R = R_i + R_3 + 8.6 \alpha \beta + 20 \log(\beta/k),
\]

where the propagation coefficient \( \gamma = \alpha + i \beta \) can, according to Delany and Bazley\(^{[13]}\), be taken as

\[
\gamma = \frac{\omega}{c_0} 0.189 \left( \frac{\rho_o f}{\sigma} \right)^{-0.595} + \frac{i \omega}{c_0} \left[ 1 + 0.0978 \left( \frac{\rho_o f}{\sigma} \right)^{-0.705} \right],
\]

where \( \sigma \) is the flow resistivity. When the example presented in Figure 4 was calculating using Fahy’s expression the last two terms in equation 15 gave a contribution varying from 7.8 to 9.4, i.e. slightly higher than the 6 dB in Sharp’s model.
Figure 4. The sound reduction index of a double plate partition.

(———) Measured results, (−−−−) Double plate model according to paper E.
(−−−−) Calculations according to Sharp.

The measurements and calculations in Figure 4 are made for the same double plate system as in Figure 2. It is seen that both prediction models are fairly good below $f_c$. 
5. The Benefits of a Double Plate System

When a double plate construction is used for additional sound insulation it is likely that one will gain a large improvement in sound insulation for a large frequency range at frequencies above the mass-spring-mass resonance, but the sound reduction may also decrease at low frequencies. This deterioration will occur in a frequency range which already has a poor sound insulation and the loss in sound insulation may well be perceived as more importance than the improvement in high frequencies. Kropp\textsuperscript{[14]} has investigated the possibilities to optimising the sound insulation of double wall construction at low frequencies by altering the design. With plates of identical or close critical frequencies being the worst case, the article discusses just how asymmetric the design should be.

In order to motivate the use of double plate constructions a comparison between the sound insulation of a single plate construction and that of a double plate construction might be in place. In Figure 5 three plates of the same surface weight are compared. The sound reduction index for the single plate is calculated according to the prediction model presented in paper A. The sound reduction index of the double plate constructions are calculated according to the prediction model presented in paper E. A concrete plate of 8 cm is compared with a 3 + 5 cm construction and a 4 + 4 cm construction, both with an cavity of 1 cm. Below the mass-spring resonance $f_0$, the calculated curves are almost the same, since the double plates will behave as a single plate at low frequencies. At the mass-spring resonance of each double plate constructions the sound reduction index of the double plate constructions is somewhat lower than that for the single plate. The sound reduction index at this frequency is heavily influenced by the loss factor. In this case the loss factor 4\% is used.

Most important though, for frequencies above $f_0$, the sound reduction index of the double plate system is considerably higher than in the single plate case. In the double plate case the sound reduction index increases with approximately 25 dB/octave, while the increase is about 8 dB/octave in the single plate case. Since each plate in the double plate construction is much thinner than in the single plate case, the critical frequency will be higher and the negative effect of $f_c$ will be less and start higher up in frequency. It is seen that the coincidence effect is much worse in the case with two equally thick plates, than in the case of 3 + 5 cm concrete plates where the plates will reach coincidence at different frequencies.
Figure 5. A solid plate compared with a double plate system with same surface weight. L = 6 m. (-----) Single plate of 8 cm concrete, $f_c = 230$ Hz.
(— — —) Double plate construction of 4+4 cm concrete and 1 cm cavity, $f_{cl} = 462$ Hz, $f_0 = 86$ Hz.
(-----) Double plate construction of 3+5 cm concrete and 1 cm cavity, $f_{cl} = 369$ Hz, $f_{cl2} = 616$ Hz, $f_0 = 88$ Hz.

If the total thickness is crucial, the comparison between two plates of the same total thickness is more fair. In figure 6 the sound reduction index of a plate of 8 cm concrete is compared with that of a construction of 3+4 cm concrete with a 1 cm cavity in between. Just as in Figure 5, the sound reduction index of the double plate construction is still considerably better.
Figure 6. A solid plate compared with a double plate system with same total height. \( L = 6 \text{ m} \). (---) Single plate of 8 cm concrete, \( f_c = 230 \text{ Hz} \).
(---) Double plate construction of 3 + 4 cm concrete and 1 cm cavity, \( f_{c1} = 462 \text{ Hz}, f_{c2} = 616 \text{ Hz}, f_{0} = 92 \text{ Hz} \).
6. Conclusion

A new theoretical model for the airborne sound insulation of single plates is presented. The calculated results show a good agreement with measured results. Measured and calculated results show that the sound reduction index decreases with increasing area.

Two different theoretical models for the airborne sound insulation of a double plate system are also presented. The calculated results show a good agreement with measured results. Again, measured and calculated results show that the sound reduction index decreases with increasing area.

Since the prediction models of paper A for single plates, and especially that of paper D for double plate systems, might be a bit complicated and needs some computational time, some simpler models are evaluated. In Figure 1, chapter 3, measured sound insulation of the single plate is compared with calculated results from the prediction model in paper A, the thin plate model by Ljunggren\textsuperscript{11} and the prediction model recommended in the European Standard\textsuperscript{6}. It is seen that the prediction model of paper A gives the best precision around \(f_c\), but at the frequency regions above and beneath, the simpler models may just as well be used.

In the double plate case it could be concluded that the floating floor model works well, for the cases where it is applicable. For low frequencies, below \(f_c\), the prediction model by Sharp\textsuperscript{9} with the corrections by Sewel\textsuperscript{3} and Fahy\textsuperscript{12} (see chapter 4.2) works well, though it could not be recommended for higher frequencies.
7. Abstract of the Appended Papers

Paper A: Airborne Sound Insulation of a Thin Plate of Finite Dimensions

An analytical model for predicting the sound reduction index of a single plate is presented. The plate is assumed to be excited by a diffuse sound field and the velocity distribution of the plate is derived from the Kirchoff plate equation in the frequency domain. The resulting Fourier transform is evaluated using residue calculus and the solution is verified numerically. The analytical model is valid for a wide frequency range, both below, above and at the critical frequency. Special interest is paid to the area dependency of the sound reduction index. Calculated results are presented for a 0.5 m², 1.7 m² and 4.4 m² plate of 16 mm chipboard. These results, which agree well with measured values, show that the sound reduction index decreases with increasing area in the whole frequency range.

Paper B: Sound Insulation of a Single Plate: A Discussion on the Area Dependency of the Sound Reduction Index (conference paper)

This paper presents a theoretical and experimental analysis of the air-borne sound transmission through a single plate. An analytical model for the sound reduction index of a single plate is presented. The plate is assumed to be excited by a diffuse sound field and the velocity distribution of the plate is derived from the Kirchoff plate equation in the frequency domain. The resulting Fourier transform is evaluated using residue calculus. The analytical model is valid for a wide frequency range, both below, above and at the critical frequency. Special interest is paid to the area dependency of the sound reduction index. Calculated results are presented for a 0.5 m², 1.7 m² and 4.4 m² plate of 16 mm chipboard. These results, which agree well with measured results, show that the sound reduction index decreases with increasing area at and below the critical frequency.
Paper C: Airborne Sound Insulation of Floating Floors

An analytical model is presented for the sound reduction index of finite size floating floors. The model is valid for two elastic plates with a resilient layer in between. The bottom plate, the load-bearing slab, is assumed to be excited with a diffuse airborne sound field. The top plate and the resilient layer compose the floating floor. The problem is solved for frequencies below, between and above the critical frequencies of the plates. Above the critical frequency of the load-bearing plate, but below that of the floating slab, the main coupling between the plates will occur at the coincidence angle of the load-bearing plate. Above the critical frequency of both plates, transmission will occur at the angle of coincidence of each plate. As the plates will interact, the sound insulation improvement will to some extent depend on the properties of the load-bearing slab. In the article it is shown how the sound reduction index depends on the physical parameters and the geometry of the plates.


A new model for the acoustic properties of a finite-size double plate system is presented. The plates are modelled using the Kirchoff plate equation while the intermediate layer is modeled assuming a local reaction of spring character. One of the plates is excited by a diffuse airborne sound field over a finite area. The responses of the two plates are derived using a two-dimensional Fourier transform technique and evaluated by means of contour integration in the complex wave number plane. Some general properties of the solution are discussed in the present paper; expressions for the sound reduction index are presented and discussed in a companion article. It is shown that the response of the plates can be expressed as the sum of the amplitudes of a number of propagating and evanescent plate waves. These waves are of the same type for the two plates. One of these waves describes the forced motion; two others describe free plate waves propagating away from the two boundaries of the excited area, the wave number is here the same as for the uncoupled plate; another two describe the corresponding bending wave near-fields excited at the boundaries; the four remaining expressions describe similar propagating waves and near fields but with a wave number which describes a coupled motion of the two plates.

An analytical model for predicting the sound reduction index of a finite size double plate system is presented. The plates are assumed to be excited by a diffuse sound field and the velocity distributions of the plates are derived from the Kirchhoff thin plate theory. The intermediate layer is modelled as a locally reacting spring. The model is valid and continuous through both the mass-spring-mass resonance and the coincidence region. The results from the analytical model show good agreement with measured results for chipboard and plasterboard partitions. Calculated results, as well as measured ones, show that the sound reduction index decreases with increasing area in the whole frequency range.
References

Airborne Sound Insulation of a Thin Plate of Finite Dimensions

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Abstract
An analytical model for predicting the sound reduction index of a single plate is presented. The plate is assumed to be excited by a diffuse sound field and the velocity distribution of the plate is derived from the Kirchoff plate equation in the frequency domain. The resulting Fourier transform is evaluated using residue calculus and the solution is verified numerically. The analytical model is valid for a wide frequency range, both below, above and at the critical frequency. Special interest is paid to the area dependency of the sound reduction index. Calculated results are presented for a 0.5 m², 1.7 m² and 4.4 m² plate of 16 mm chipboard. These results, which agree well with measured values, show that the sound reduction index decreases with increasing area in the whole frequency range.

1 Introduction
The present work is prompted by the need to predict the sound reduction index of a single wall over a wide frequency range and in particular how this index is influenced by the geometry of the wall. Several authors has studied the present problem but with different boundary conditions and range of frequencies. Cremer[1] was among the first who studied the sound transmission through infinite plates. Ljunggren presented in two articles[2,3] the sound transmission of both thin and thick walls of finite dimensions. Osipov et al.[5] compared three simplified models for predicting the sound insulation for low frequencies: the infinite plate model, the baffled plate model and the room-plate-room model. Laboratory measurements showed a strong modal behavior of the low frequency sound transmission. Here the sound insulation depended both on the properties of the plate and by the room dimensions and geometry.

In Sewell[6] the area dependency of a single plate system is investigated. He found that the resonant transmission decreases with increasing area in opposite to the behavior of forced transmission which increases with increasing area, i.e.
the total sound reduction index depends on two functions varying with the plate area in different ways. Generally, but not always, the forced transmission will dominate the low frequency region and resonant transmission will dominate higher frequencies (still below the critical frequency). If this is the case, Sewell[6] maintained that the total sound reduction index will decrease with increasing area at low frequencies but increase with increasing area at higher frequencies up to the critical frequency above which the sound transmission does not depend on the area. In M. Villot et al.[7] the influence of the area was investigated by using a spatial windowing technique. A part of an infinite plate was assumed to be excited. The model does not include resonant transmission which gives an observable deviation between measurements and theory for the coincidence frequency region. A. Tadeu and J. António et al. have published a number of articles on single and double walls [8-11]. The full scattering phenomena was modeled, with propagation of all wave types within the solid layer and guided waves traveling along the panel, though the plates were assumed to be infinite and thus the resonant waves are not included. The measurements, presented in [11], of single plates of glass and steel for areas from 0.6x0.6 m² to 1.8x1.8 m² show a decrease in the sound reduction index for an increase of the area.

The work presented in this paper deals with prediction models for the insulation of homogenous single plates with respect to diffuse airborne sound field. An analytical model for predicting the sound reduction index of a single plate is presented. Furthermore, a theoretical and experimental analysis of the air-borne sound transmission through a single plate is presented. Flanking transmission is not included. The plate is assumed to be excited by a diffuse sound field and the velocity distribution of the plate is derived from the Kirchoffs plate equation in the frequency domain. The resulting Fourier transform is evaluated using residue calculus and the solution is verified numerically.

In Ljunggren[4] a model with similar starting conditions is presented. As in M. Villot et al.[7] the spatial windowing technique was used, but since Ljunggren[4] uses a finite plate the reverberant field was also regarded. The main difference from the present work is that Ljunggren used asymptotic expressions for the plate field when evaluating the transmission while the complete expressions are used in the present case. The advantage of using the complete expressions is that the sound reduction index curve will be continuous since it is valid both below, at and above the critical frequency.
2 The sound reduction index

2.1 Forced and free waves

Consider a plate partly excited over the area $-L/2 < x < L/2$ by an incident, plane sound wave. The plate is initially considered as two dimensional and infinite. The incident and reflected wave act on the plate with a pressure $p(x)$ as

$$p(x) = \hat{p} e^{-\beta x} e^{i\omega t}. \quad (1)$$

Here $x$ is the coordinate according to Figure 1, $\omega$ is the angular frequency, $t$ is the time and $k$ is the trace wave number of the exciting pressure along the plate surface,

$$k = k_a \sin \theta, \quad (2)$$

where $\theta$ is the angle of incidence according to Figure 1 and $k_a$ is the wave number of sound in air.

![Diagram of baffled plate](image)

*Figure 1. The baffled plate. Region 1 is the excited part of the plate while region 2 and 3 are the non-excited parts of the plate shielded by baffles, in the positive and negative $x$-direction, respectively.*
The Kirchhoff plate equations for plate of uniform thickness, made of an isotropic, homogeneous linear elastic material, is written as follows

\[ B \frac{\partial^4 v}{\partial x^4} - \omega^2 m v = j \omega p(x), \]

(3)

where \( v \) is the plate velocity, \( m \) is the surface mass and \( B \) is the complex bending stiffness as

\[ B = \frac{Eh^3}{12(1-\mu^2)} (1 + j \eta). \]

(4)

For the evaluations presented later on, the loss factor is taken as

\[ \eta = \eta_m + \frac{m}{\sqrt{f \cdot 485}}, \]

(5)

where \( f \) is the frequency and \( \eta_m \) is the internal loss factor, see EN ISO 140-1[12]. Introducing the wave number of the free bending waves in plates \( k_b \) as

\[ k_b = \sqrt{\frac{\omega^2 m}{B}}, \]

(6)

the plate equation now becomes

\[ k_s^4 v(k_s) - k_b^4 v(k_s) = \frac{j \omega}{B} p(k_s). \]

(7)

\( k_s \) is the spatial Fourier transform of the wave number. The Fourier transform of the exciting pressure is calculated as

\[ p(k_s) = \int_{L/2}^{L/2} p(x) e^{-j k_s x} dx, \]

(8)

i.e.

\[ p(k_s) = \hat{p} \frac{j e^{-j(k+k_s)\frac{L}{2}} - j e^{j(k+k_s)\frac{L}{2}}}{(k+k_s)}. \]

(9)

which gives
\[ v(k_x) = \frac{j \omega p(k_x)}{B \left(k_x^2 - k_y^2\right)} = -\omega \hat{p} e^{-j(k_x L/2)} - e^{j(k_x L/2)} \frac{k_x}{k_x^2 - k_y^2} \] (10)

The sound transmission factor of an element is defined as
\[ \tau(\theta) = \frac{P(\theta)}{P_m(\theta)} \] (11)

where \( P(\theta) \) is the radiated power and \( P_m(\theta) \) is the input power, which can be calculated as
\[ P_m(\theta) = \hat{p}^2 L \frac{\cos(\theta)}{8 \rho c} \] (12)

The radiated power from the excited part of a large plate can be calculated as (Cremer et al.\textsuperscript{[1]} p. 528)
\[ P(\theta) = \frac{\rho c k_x}{4\pi} \text{Re} \left[ \int_{k_x}^{k_x} \frac{v(k_x) v^*(k_x)}{\sqrt{k_x^2 - k_y^2}} dk_x \right] = \frac{\rho c k_x}{4\pi} \int_{k_x}^{k_x} \frac{|v(k_x)|^2}{\sqrt{k_x^2 - k_y^2}} dk_x \] (13)

With the Fourier transformed velocity \( v(k_x) \) according to Equation (16), the radiated power becomes
\[ P(\theta) = \frac{\rho c k_x}{4\pi} \left| \frac{-\omega \hat{p}}{B} \right|^2 \int_{k_x}^{k_x} \frac{e^{j(k_x L/2)} + e^{-j(k_x L/2)} |k_x^2 - k_y^2| \left(k_x^2 - k_y^2\right) \left(k_x^2 - k_y^2\right)}{\sqrt{k_x^2 - k_y^2}} dk_x \] (14)

It is now suitable to rewrite the two-dimensional problem into three dimensions. This is here achieved by changing the length \( L \) for the mean free path according to Kosten\textsuperscript{[12]} as
\[ L_m = \frac{\pi S}{U} \] (15)

where \( S \) is the area of the excited part of the plate and \( U \) is the perimeter of the area \( S \). In the work by Ljunggren\textsuperscript{[2]} a mean projected trace length is derived as \( L_m = (2S/\pi)^{1/2} \) as the mean value of \( \sqrt{L^2} \). This was used instead of Kosten's mean free path for frequencies above \( f_c \) where the transmission was proportional to \( L_2 \). For square plates the two expressions are equal, and in practice the
difference between length and width of a plate needs to be 1:14 before the difference is 3 dB, see Figure 2.

![Figure 2](image)

Figure 2. The mean free path according to Kosten versus the mean projected trace length according to Ljunggren. The plate dimensions are (----) \( b \cdot b \), (- - -) \( b \cdot 4b \), (-----) \( b \cdot 14b \). Lines without markers are calculated according to Kosten \(^{12} \), lines with markers (*) according to Ljunggren \(^{23} \).

In the following, all calculations in this article are based on “\( L \)” as \( L_m \) according to Equation (15). The reason for this is that the transmission factor derived in this work is rather proportional to \( L \) than \( L^2 \). To achieve an excitation corresponding to a diffuse sound field the sound transmission is integrated over the angle of incidence by use of Paris’ formula

\[
\bar{\tau} = 2 \int_0^{\pi/2} \tau(\theta) \sin(\theta) \cos(\theta) \, d\theta .
\] (16)

Note, the full integration range from 0 to \( \pi/2 \) is, and should be, used. A common “trick” to reduce the difference between theoretical results and results from measurements in laboratory is, otherwise, to limit the integration angle, e.g. Cremer\(^{11} \) used 78°, (see also discussion in Villot\(^{71} \)).
To elucidate the different wavetypes included in \( \tau(\theta) \) the plate velocity of Equation (10) is solved. The inverse Fourier transform of the velocity of the plate can be taken as

\[
\nu(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \nu(k_x) e^{jk_x x} dk_x,
\]

(17)

which gives

\[
\nu(x) = \frac{-\omega \hat{p}}{B2\pi} \int_{-\infty}^{\infty} \left( e^{-j(k+x_{L}/2)} - e^{j(k+x_{L}/2)} \right) e^{jk_x x} \frac{k_x}{(k_x^a - k_B)} dk_x.
\]

(18)

This expression can be evaluated by means of the calculus of residues. In order to get the solution for region 1, i.e. \(-L/2 < x < L/2\), the integration needs to be rewritten as

\[
\nu(x) = \frac{-\omega \hat{p}}{B2\pi} \int_{C_1} e^{-j(k-x_{L}/2+jk_B)x} \frac{k_x}{(k_x^a - k_B)} dk_x - \int_{C_2} e^{j(k-x_{L}/2-jk_B)x} \frac{k_x}{(k_x^a - k_B)} dk_x,
\]

(19)

where the integration contours presented in Figure 3 are suitable as the contribution from the arcs of the integration contours are then negligible.

*Figure 3. The poles of the integrand, Equation 18, and the integration contours \( \Gamma_1 \) and \( \Gamma_2 \).*
As the pole of \( k_f \) is situated on the real axis the principle value is taken. The evaluated result becomes

\[
\nu_1(x) = \frac{je^\Phi}{B} \left[ \frac{e^{-jk_x}}{k^4 - k_B^4} + \frac{e^{j(k-k_B)^2/2j} + jk_xe^{-jk_x}}{4k_B^2(k_B - k)} + \frac{e^{-jk_x}}{4k_B^2(k_B + k)} + \frac{e^{jk_B}}{4k_B^2(k_B - jk)} + \frac{e^{-jk_x}}{4k_B^2(k_B + jk)} \right].
\] (20)

Presented in this way the different parts of the velocity are easily interpreted. The first term of the solution represents the forced wave propagating with the wave number \( k \). The second and third term denoted “free standing waves” are the waves propagating in each direction from the boundary \( x = \pm L/2 \) with the bending wave number \( \pm jk_B \). The last two terms represent the nearfield are generated at the boundary \( x = \pm L/2 \) and has an exponential decay of \( \pm jk_B \) away from the point of generation. In order to verify the Equation (20), the integral in Equation (18) was evaluated numerically. This is presented in the appendix.

### 2.2 Reverberant field

The reverberant field is calculated by using classical SEA expressions, and hence valid under the conditions for this theory (see for example Craik[16]). The plate is now considered finite. The excited area is denoted \( S \), and the area of the whole plate is denoted \( S_{tot} \) (i.e. \( S_{tot} \geq S \)). The waves propagating away from the perimeter of the excited area will create a reverberant field with a mean square velocity given by

\[
\langle \nu^2 \rangle = \frac{P_R}{\omega m S_{tot} \eta},
\] (21)

where \( \eta \) is the loss factor. In the case of plane wave excitation the power traveling away in one direction from the line \( x = L/2 \) can be written as (compare Cremer et. al.[1], p. 109, Ljunggren[2])

\[
P_R = 2c_B m U \langle v_x^2 \rangle \langle \cos(\psi) \rangle,
\] (22)

where \( c_B \) is the velocity of the bending wave and \( U \) is the length of the perimeter. \( \langle v^2_x \rangle \) is the mean squared velocity of the plate in region 2 (see Figure 1) and \( \psi \) is the direction of the free waves, see Figure 4. This is solved from the transform of the velocity, \( v(k_x) \) according to Equation 10. By use of the integration contour \( \Gamma_2 \) the velocity of region 2, i.e. \( x \geq L/2 \) can be derived as
\[ v_2(x) = \frac{\hat{p}k_g}{2m\omega} \left[ \frac{\sin[(k - k_g)L/2]e^{-jk_gx}}{(k - k_g)} + \frac{\sin[(k + jk_g)L/2]e^{-jk_gx}x}}{(jk - k_g)} \right], \quad (23) \]

and at \( x = L/2 \), \( v_2 \) becomes

\[ v_2 = \frac{\hat{p}k_g}{2m\omega} \left[ \frac{\sin[(k - k_g)L/2]e^{-jk_gL/2}}{(k - k_g)} + \frac{\sin[(k + jk_g)L/2]e^{-jk_gL/2}}{(jk - k_g)} \right]. \quad (24) \]

So far the solution is valid for the case of an excitation perpendicular to the plate. This will later be integrated over the angle of incidence \( \theta \), as in the case with non reverberant transmission in order to correspond to an excitation of a diffuse sound field. Further, in case of oblique incidence the free waves emerge from the boundary at an angle that may differ from that of the exciting wave. The direction of the free waves is obtained from the trace wave number of the forcing wave along the boundary together with the free bending wave number, as

\[ k_g \sin \psi = k \sin \phi, \quad (25) \]

where \( \psi \) is the angle between the direction of the free waves and the boundary normal, and \( \Phi \) is the corresponding angle of the forcing wave, see Figure 4. The mean value of \( \cos(\psi) \) in Equation (24) can now be taken as (Ljunggren\textsuperscript{[23]})

\[ \langle \cos \psi \rangle = \frac{2}{\pi} \int_{\phi}^{\Phi} \frac{e^{\frac{\pi}{2}}} {\left[ 1 - (k \sin \phi / k_g)^2 \right]^2} d\phi. \quad (26) \]
Figure 4. $\psi$ is the angle between the direction of the free waves and the boundary normal, and $\Phi$ is the corresponding angle of the forcing wave. $\theta$ is the angle of incidence.

The transmission due to the reverberant field is defined as the relation between the radiated power and the input power as

$$\tau = \frac{P_{\text{rad}}}{P_{\text{in}}},$$  \hspace{1cm} (27)

where $P_{\text{in}}$ is calculated according to Equation (12) above. $P_{\text{rad}}$ is defined as

$$P_{\text{rad}} = \langle v_r^2 \rangle S \rho c \sigma,$$  \hspace{1cm} (28)

where $S$ is the area of excited part of the plate and $\sigma$ is the radiation factor. The sound transmission due to resonant waves becomes

$$\tau_{\text{res}} = \frac{32}{k_\eta L S_{\text{tot}}} \left( \frac{\rho c}{\dot{p}} \right)^2 \langle \cos \psi \rangle \sigma \int_0^{\pi} \langle v_r^2 \rangle^2 \sin(\theta) d\theta.$$  \hspace{1cm} (29)

### 2.3 Prediction formula

The sound reduction index is defined as

$$R = -10 \log \tau.$$  \hspace{1cm} (30)
The sound reduction index with respect to the forced and free fields can be evaluated directly from the following expressions

$$R_f = -10 \lg \left[ \frac{2 \rho c \omega^2}{B} \frac{k_x}{\pi L} \int_{0}^{\pi} \int_{-\infty}^{\infty} \frac{e^{i(k_x-k_x')/2} + e^{-i(k_x+k_x')/2}}{\sqrt{k_x^2 - k_x'^2}} \sin(\theta) d\theta d\theta \right]$$

[31]

with \( L \) according to Equation (15). The sound reduction index for resonant waves becomes

$$R_{res} = -10 \lg \left[ \frac{8k_b}{\eta LS_m} \frac{U_s}{\rho c} \sigma \langle \cos(\psi) \rangle \right]$$

[32]

where \( \langle \cos(\psi) \rangle \) according to Equation (26) and the radiation factors \( \sigma \) presented by Leppington[15] are used.

$$\sigma_f = \frac{U_c}{4\pi^2 \sqrt{f/f_c} \sqrt{S/\mu^2 - 1}} \left( \ln \left( \frac{\mu + 1}{\mu - 1} \right) + \frac{2\mu}{\mu^2 - 1} \right), \quad f < f_c$$

[33]

where \( \mu = \sqrt{f_c/f} \)

$$\sigma_{\mu} = \sqrt{\frac{2\pi f}{c}} l_s \left( 0.5 - 0.15 \frac{f_c}{f} \right), \quad f = f_c$$

[34]

$$\sigma_{\mu} = \frac{1}{\sqrt{1 - \frac{f_c}{f}}}, \quad f > f_c$$

[35]

where \( l_s \) and \( l_q \) are the dimensions of the excited part of the plate.
In Figure 5 and 6 the non-resonant and the resonant part of the sound reduction index are presented separately. This shows that for both wave types there is a strong area dependency for low frequencies. The sound reduction index decreases with increasing area. However, above the critical frequency only the resonant part of the transmission is affected by the area size. The local maxima in the \( R_{\text{res}} \) curve below \( f_c \) is caused by local minima and maxima in the first and second term of the velocity \( v_2 \), Equation (24). The calculations are performed for a 16 mm chipboard since this is used for measurements described later on.

**Figure 5.** Area dependency of sound reduction index due to forced and free waves (——) 0.5 m²; (- - - - -) 1.7 m² and (---) 4.4 m². All calculations are made for a 16 mm chipboard with \( E = 1.5 \) GPa, \( \eta_{\text{int}} = 4 \% \), \( \rho = 62 6 \text{kg/m}^3 \).
Figure 6. The resonant part of the sound reduction index, $R_{res}$ (-----) 0.5 m$^2$, (- - -) 1.7 m$^2$ and (-----) 4.4 m$^2$. Same material properties as in Figure 5.

In Figure 7 the total sound reduction index is presented, i.e. the forced part according to Equation (31) together with the resonant part according to Equation (32).
Figure 7. The calculated sound reduction indexes. $R_{\text{calculated}}$ (---) 0.5m$^2$; (-x-x-) 1.7m$^2$ and (□□□) 4.4m$^2$ according to Equation (31) and (32). Same material properties as in Figure 5.

In Figure 8 and 9 the influence of varying loss factors on the sound reduction index is illustrated. It is found that, in the non-resonant part of the sound reduction, an increase of the loss factor will increase the sound reduction at and above the critical frequency. In the resonant part of the transmission though, an increase of the loss factor will increase the sound reduction index for all frequencies.
Figure 8. The influence of varying loss factors on the non-resonant part of the sound reduction index. The sound reduction is calculated for a 4.4 m², 16 mm thick chipboard with $E = 1.5$ GPa and $\rho = 626$ kg/m³, with the internal loss factors as: (solid) $\eta_{\text{int}} = 1\%$, (dashed) $4\%$, (dash-dotted) $8\%$, (dotted) $32\%$. 
3 Measurements

Measurements were made in order to show the area dependency of the sound reduction index. A 16 mm chipboard was used as a test plate. The modulus of elasticity and the loss factor was measured from a beam of the plate to 1.5 GPa and 4% respectively. To achieve a simply supported mounting the plate was hanging in rubber bands from the ceiling in the aperture between two reverberation chambers. The total plate area Stot was 2.44 x 2.42 m². The edges of the plate were covered by baffles, and the excited part of the plate S was varied in size from 0.55 m² to 4.4 m² by altering the sizes of the baffles. The baffles were made of 400 mm mineral wool with 26 mm gypsum board on each side. Measurements were performed according to ISO standard\textsuperscript{17}. The measurement setup is presented in Figure 10.
Figure 10. The measurement setup.

The measurements were performed in 1/3-octave bands for the frequency range 50 – 5000 Hz. The computations are performed in the frequency range 1 – 5612 Hz, with an increment of 1 Hz, and then averaged into 1/3-octave bands to match the measurements.

Figure 11. The measured sound reduction indexes. $R_{\text{measured}}(-\quad--\quad-)0.5 \text{ m}^2$; $(-\times\quad-x-\times-\times-)$1.7 m$^2$ and $(-\quad-\quad-\quad-\quad-)4.4 \text{ m}^2$. 
Figure 12. Comparison between measured and calculated sound reduction indexes. \( R_{\text{measured}} \) (---) and \( R_{\text{calculated}} \) (-----) according to Equation (30) and (31). For calculations the Young’s modulus \( E = 1.5 \) GPa, \( \rho = 626 \) kg/m\(^3\), and the loss factor \( \eta_{\text{int}} = 4\% \) are used. The area is \( S = 0.5 \) m\(^2\).

In Figure 11 the measured result for a 16 mm chipboard wall is presented. The excited area is varied from 0.5 m\(^2\) to 4.4 m\(^2\). In Figure 12 the measured results are compared with the calculated sound reduction index. The calculated and measured sound reduction indexes match fairly well. It should be observed that the modal density of the plate is low at low frequencies, which may account for the discrepancy between measured and calculated results at low frequencies. In both the calculated results and in the measured results the area dependency of the sound reduction index appears. The sound reduction will increase for decreasing area both for resonant and non-resonant case for frequencies at and below the critical frequency. Above the critical frequency only the resonant transmission is area dependant, and here as well the sound reduction will increase with decreasing area.

The calculated result are also compared with a measured result for a plasterboard partition, see Figure 13. This result was published by R. A. Novak\(^{[20]}\). In this case a partition that consists of a 2.99 x 2.50 m\(^2\) double layer
plasterboard, attached with studs c/c 600 mm, was tested. The studs are not taken into account in the calculations; still the agreement is not too bad.

![Sound Reduction Index Graph](image.png)

*Figure 13. Comparison between measured and calculated sound reduction indexes for a 2 x 13 mm plasterboard partition. For the calculations, the Young's modulus E is taken as 2.5 GPa, surface mass $m = 9$ kg/m$^2$ and the loss factor $n_{\text{int}} = 6\%$. The area is 2.99 x 2.50 m$^2$. (— — ) $R_{\text{measured}}$ (— — — ) $R_{\text{calculated}}$."

4 Conclusions

This work presents an analytical model for predicting the sound reduction index of a single plate. Flanking transmission is not included. The plate is assumed to be excited by a diffuse sound field and the velocity distribution of the plate is derived from the Kirchoff plate equation in the frequency domain. In the resulting plate velocity, the different wave forms included stands out. It is seen how the velocity is a function of forced waves and free waves; i.e. the bending nearfields and the propagating waves. The solution of the plate velocity is verified numerically. As the plate is finite, there will also be a resonant field in the plate. Waves will reflect at the boundaries and travel back into the excited part of the plate. The sound transmission due to the resonant
field is calculated and added to the forced and free transmission. The analytical model is valid both below, at and above the critical frequency. The analytical model is confirmed by measurement results.

In both the calculated and in the measured results the area dependency of the sound reduction index appears. The sound reduction will increase for decreasing area. Small plates might thence be considerably underestimated when prediction models based on theory for infinite plates are used.

5. Acknowledgements

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6. References


Appendix

In order to verify the Equation (20) and (23), the integral in Equation (18) was evaluated numerically. This integral, which is in the form of infinite integral, is not easy to evaluate since its integrand shows an oscillating nature with a very slow convergence, which makes conventional methods inapplicable. Ir [18,19] Sidi studied such types of integrals and demonstrated that a fast convergence can be obtained once a W-transformation is applied; note that the W-transformation is based on extrapolation of the sequences of partial sums of the original integral. It can be shown that as $k_x \to \infty$, the integrand has Poincaré-type asymptotic expansions with view of the condition stated in [18,19]. Accordingly, the integral may be partitioned as

$$
\int_{-\infty}^{\infty} f(k_x, x)dk_x = I_1 + I_2, f(k_x, x) = \frac{(e^{-i(k+k_x)L/2} - e^{i(k+k_x)L/2})e^{ik_x}}{(k + k_x)(k_x^+ - k_y^+)} \tag{A1}
$$

$$
I_1(x) = \int_{0}^{\infty} f(k_x, x)dk_x = \int_{0}^{k_{x1}} f(k_x, x)dk_x + \int_{k_{x1}}^{k_{x2}} f(k_x, x)dk_x + \int_{k_{x2}}^{k_{x3}} f(k_x, x)dk_x + \ldots \tag{A2}
$$

$$
I_2(x) = \int_{-\infty}^{0} f(k_x, x)dk_x \tag{A3}
$$

where $k_{x1}, k_{x2}, k_{x3}, \ldots$ are the zeros of the function $f(k_x, x)$.

$I_2(x)$ is in turn broken down following the same procedure as above. These integrands have a number of poles and, to shift the resulting singularities from the real axis, a small portion of damping, $\eta_p$, is introduced to warrant the numerical stability as (see Cremer[11])

$$
k_y \approx k_y(1 - \frac{i}{2} \eta_p) \tag{A4}
$$

The calculations were carried out using $\eta_p = 4\%$. Note that this choice of damping loss factor is made for convenience only. Experiments showed that this value had no influence on the result, provided that $\eta_p$ is chosen within reasonably limits. Each term on the right side of Equation (A2) is evaluated numerically using the high order recursive adaptive quadrature method, which is useful for functions that change rapidly over an interval. Then, the W-algorithm of Sidi[18,19] is applied. The results are shown in Figure A1 and
Figure A2. As can be shown the agreement is reasonable and the validity of the analytical expressions, Equation (A2) and Equation (A3) is verified. It is thought, that the discrepancy between the two evaluations is related to the extrapolation method of Sidi, which is an approximate method.

\[ v_1(x) \]

\[ v_2(x) \]

\[ x (m) \]

Figure A1. The analytical expressions, Equation (20) and (23) versus the numerical evaluation of the velocity integral, Equation (18). (-----) the real part of the analytical expression; (■) the real part of the numerical evaluation of the integral. The plate is made of chipboard, length of plate is 5 m, thickness 0.016 m, frequency 0.5 kHz, angle of incidence is $\pi/3$ radian.
Figure A2. The analytical expressions, Equation (20) and (23) versus the numerical evaluation of the velocity integral, Equation (18). (- - -) the imaginary part of the analytical expression; (●) the imaginary part of the numerical evaluation of the integral. The plate is made of chipboard, length of plate is 5 m, thickness 0.016 m, frequency 0.5 kHz, angle of incidence is π/3 radian.
Sound Insulation of a Single plate:

A Discussion on the Area Dependency of the Sound Reduction Index

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Abstract [203] This paper presents a theoretical and experimental analysis of the air-borne sound transmission through a single plate. An analytical model for the sound reduction index of a single plate is presented. The plate is assumed to be excited by a diffuse sound field and the velocity distribution of the plate is derived from the Kirchhoff plate equation in the frequency domain. The resulting Fourier transform is evaluated using residue calculus. The analytical model is valid for a wide frequency range, both below, above and at the critical frequency. Special interest is paid to the area dependency of the sound reduction index. Calculated results are presented for a 0.5 m², 1.7 m² and 4.4 m² plate of 16 mm chipboard. These results, which agree well with measured results, show that the sound reduction index decreases with increasing area at and below the critical frequency.

1 INTRODUCTION

The sound transmission of single plates is one of the fundamental problems in building acoustics and prediction model of sound insulation is particularly important at the critical frequency. This work was initially prompted by the need to predict the sound reduction index of a single wall for a wide range of frequencies; moreover, from the practical point of view it is of interest to investigate how this index can be influenced by the geometry of the wall.

The work presented in this paper deals with prediction models for the insulation of homogenous single plate with respect to diffuse airborne sound field. Flanking transmission is not included. An analytical model for predicting the sound reduction index of a single plate is presented and compared with measurements and models by other authors [1-4]. The full derivation of the model is presented in [5]. In [6] Ljunggren presented a model with similar starting conditions. However, his model is only valid at low frequencies, below the critical frequency. Moreover, Ljunggren did not evaluate his model numerically nor by measurements.

2 FORCED AND FREE WAVES

The plate is assumed to be excited by a diffuse sound field and the velocity distribution of the plate is derived from the Kirchoffs plate equation in the frequency domain. Here it is assumed that the longitudinal waves can be neglected as it can be shown that the displacement due to the longitudinal waves are very small compared to that of the displacement induced by the free bending wave. The
plate is assumed to be excited over the length \(-L/2 < x < L/2\). Under the present prerequisites, the low-frequency Lamb modes can also be neglected. The Fourier transform of the plate velocity becomes \(^{[5]}\)

\[
\nu(k_x) = \frac{j \omega}{B} \frac{p(k_x)}{k_x^2 - k_y^2} = -\frac{\omega \hat{P}}{B} \frac{e^{-j(k_x+x)L/2} - e^{i(k_x-x)L/2}}{(k_x + k_x)^2 - (k_x - k_y)^2}.
\]

(1)

Here \(\omega\) is the angular frequency, \(k\) is the trace wave number of the exciting pressure along the plate surface, \(k = k_x \sin \theta\), where \(\theta\) is the angle of incidence and \(k_x\) is the wave number in air. The wave number of the free bending waves is denoted \(k_y\). \(B\) is the bending stiffness and \(p(k_x)\) is the Fourier transform of the pressure due to the incident and reflected wave which act on the plate. The radiated power from excited part of an large plate, calculated according to Cremer et al.\(^{[7]}\) (p. 528, eq. 61) using the Fourier transformed velocity \(\nu_j(k_x)\) above, can be written as

\[
P(\theta) = \frac{\rho c k_x}{4\pi} \left| -\frac{\omega \hat{P}}{B} \int_{-k_x}^{k_x} \frac{e^{-j(k_x-x)L/2} + e^{-j(k_x+x)L/2}}{(k_x^2 - k_y^2)(k_x^2 - k_y^2)} \right|^2 dk_x,
\]

(2)

where \(c\) is the velocity of sound in air and \(\rho\) is density of air. It may now be suitable to rewrite the two-dimensional problem into three dimensions. For low frequencies below the critical frequency \(f_c\) this can be achieve by changing the length \(L\) for the mean free path according to Kosten\(^{[8]}\) as \(L_m = \pi S/2U\) where \(S\) is the area of the excited part of the plate and \(U\) is the perimeter of the area \(S\). For high frequencies \((f > f_c)\), the mean projected trace length derived in Ljunggren\(^{[1]}\) \(L_m = (2S/\pi)^{1/2}\) can be used. To achieve an excitation corresponding to a diffuse sound field the sound transmission factor is integrated over the angle of incidence by use of Paris’ formula. The sound transmission is defined as the ratio between the radiated power and the incident power as

\[
\bar{T} = 2 \int_0^\frac{\pi}{2} \frac{P(\theta) \sin(\theta) \cos(\theta)}{P_m(\theta)} d\theta = \frac{16 \rho c}{p^2 L} \int_0^\frac{\pi}{2} P(\theta) \sin(\theta) d\theta.
\]

(3)

Note that the full integration range from 0 to \(\pi/2\) is, and should be, used. A common “trick” to reduce the difference between theoretical results and results from measurements in laboratory is, otherwise, to limit the integration angle, e.g. Cremer\(^{[9]}\) used 78°, see also discussion in Villot\(^{[10]}\). Hence the sound reduction index for the non reverberant part becomes

\[
R_f = -10 \log \left[ \frac{2 \rho c \omega}{B} \frac{k_x}{\pi L} \int_0^\frac{\pi}{2} \int_{-k_x}^{k_x} \frac{e^{-j(k_x-x)L/2} + e^{-j(k_x+x)L/2}}{(k_x^2 - k_y^2)(k_x^2 - k_y^2)} \sin(\theta) d\theta \right]
\]

(4)

(with \(L\) according to Eq. 14). The inverse Fourier transform of the velocity of the plate is solved by means of the calculus of residues.

\[
\nu_i(x) = \frac{j \omega \hat{P}}{B} \left[ \frac{e^{-jx}}{k_x^2 - k_y^2} + \frac{j k_x}{4k_x^2(k_x - k)} + \frac{j k_x}{4k_x^2(k_x + k)} + \frac{e^{-jx}}{4k_x^2(k_x + jk)} + \frac{e^{-jx}}{4k_x^2(k_x - jk)} \right]
\]

(5)
and the velocity of region 2, i.e. \( x \geq L/2 \) can be derived as

\[
v_2(x) = \frac{\rho k_B}{2m_0} \left[ \frac{\sin[(k - k_B)L/2]e^{-jkBx}}{(k - k_B)} + \frac{\sin[(k + jk_B)L/2]e^{jkBx}}{(jk - k_B)} \right]. \tag{6}
\]

These expressions are evaluated numerically in [5].

3 REVERBERANT FIELD

The plates is now considered finite. The excited area is denoted \( S_e \) and the area of the whole plate is denoted \( S_{tot} \). As the plate is finite, there will also be a resonant field in the plate. Waves will reflect at the boundaries and travel back into the excited part of the plate. From the power traveling away, in each direction of \( x = \pm L/2 \), the sound transmission due to resonant waves can be calculated as (see Kernen\(^{[3]}\))

\[
\tau_{res} = \int_0^\pi \frac{U S}{L S_{tot}} \frac{(\rho c)^2}{\rho k_B \eta} \left< v_2^2 \right> \sigma \sin(\theta) d\theta, \tag{7}
\]

where \( \left< v_2^2 \right> \) is the mean square velocity of the plate at \( x=L/2 \) according to equation 6 and \( \eta \) is the loss factor. The radiation factor according to Leppington\(^{[11]}\) may be used.

4 COMPARISON WITH OTHER MODELS

4.1 Ljunggren

The sound reduction index according to Ljunggren\(^{[1]}\) is divided in a forced and a resonant part, which is to be added logarithmically:

\[
R_{\text{forced}} = 20 \lg \frac{\omega m}{2 \rho c} - 10 \lg S_d, \quad f < f_c \tag{8}
\]

\[
R_{\text{resonant}} = 20 \lg \frac{\omega m}{2 \rho c} + 10 \lg \frac{2 \eta}{\pi} + 10 \lg \frac{f}{f_c} - 10 \lg \frac{S_g S_s}{S_{tot}}, \quad f < f_c \tag{9}
\]

\[
R_{\text{forced}} = 20 \lg \frac{\omega m}{2 \rho c} + 10 \lg \frac{2 \eta_{\text{eq}}}{\pi} + 10 \lg \frac{f}{f_c} + 10 \lg \left( 1 - \frac{f_c}{f} \right), \quad f > f_c \tag{10}
\]

\[
R_{\text{resonant}} = 20 \lg \frac{\omega m}{2 \rho c} + 10 \lg \frac{2 \eta}{\pi} + 10 \lg \frac{f}{f_c} + 10 \lg \left( 1 - \frac{f_c}{f} \right) - 10 \lg \frac{S_g S_s}{S_{tot}}, \quad f > f_c \tag{11}
\]

The radiation factor \( s_d \) is calculated according to

\[
s_d = \frac{\pi}{2} s_x(\theta) \cdot \sin(\theta) d\theta \tag{12}
\]

where

\[
s_x(\theta) = \frac{k_0}{(L/2) \pi} \int_{k_x} \frac{\sin^2((L/2)(k_x - k))}{(k_x - k)^2 \sqrt{k_x^2 - k^2}} dk_x. \tag{13}
\]
4.2 Josse and Lamure

The sound reduction index according to Josse and Lamure\cite{2} can be calculated as

\[
R = 20 \lg \frac{\omega m}{2 \rho c} - 10 \lg \left[ 2 + \frac{4 c^2}{\pi^2 \eta f_c \sqrt{f f_c}} \left( \frac{a^2 + b^2}{a^2 b^2} \left( 1 + \frac{2 f}{f_c} + \frac{3 f^2}{f_c^2} \right) \right) \right],
\]

\[
R = 20 \lg \frac{\omega m}{2 \rho c} + 10 \lg \frac{2 \eta}{\pi} + 10 \lg \frac{f}{\Delta f},
\]

\[
R = 20 \lg \frac{\omega m}{2 \rho c} + 10 \lg \frac{2 \eta}{\pi} + 10 \lg \frac{f}{f_c} \left( 1 - \frac{f_c}{f} \right),
\]

where \( \eta \) corresponds to the loss factor of internal and radiational losses and \( a \) and \( b \) referring to the plate dimensions.

When comparing the model presented in this work with that of Ljunggren, the main difference between the two models is that calculations according to this work are valid continuously over the whole frequency range, while the model of Ljunggren\cite{1} is valid only well below and well above the critical frequency.

![Figure 1. The sound reduction index according to eq. 4 and 7 (Kerner\cite{3}): line without markers, Ljunggren\cite{1}: line with dots, Josse and Lamure\cite{2}: line with crosses. Solid line is calculated for 4.4m² plate, dotted line for plate 0.5m². Both resonant and forced waves are included.](image)
In the model by Ljunggren as well as the model presented in this work, the sound reduction index will decrease with increasing plate area for frequencies below coincidence. In Josse, on the other hand, the area dependency is the opposite. When using the model of Josse an increase of area will also increase the sound reduction. These sound reduction indexes are compared in Figure 1. Both forced and resonant waves are including since the area dependency of Josse is connected to the resonant part. The model by Josse is not area dependent above the critical frequency, unlike the others which has a certain area dependency in the resonant part of the transmission for high frequencies as well.

### 4.3 Sewell and Leppington

The sound reduction index for forced transmission for frequencies below the critical frequency can according to Sewell[3] be calculated as

\[
R = 20 \log_{10} \left( \frac{m \omega}{2 \rho c} \right) + 20 \log_{10} \left( 1 - \frac{f^2}{f_c^2} \right) - 10 \log_{10} \left( \ln(k \sqrt{S}) + 0.160 - U(\Lambda) + \frac{1}{4 \pi k_c^2 S} \right)
\]  

(17)

where \( S \) is the area of the test object, which is assumed to be a plate simply supported in an infinite baffle, \( S = ab \). \( U(\Lambda) \) is a form factor which can be calculated as

\[
U(\Lambda) = -0.804 - \left( \frac{1}{2} + \frac{\Lambda}{\pi} \right) \ln \Lambda + \frac{5 \Lambda}{2 \pi} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \Lambda^{2n+1}}{2 \pi n (n+1)(2n+1)^2}
\]  

(18)

for \( \Lambda \leq 1 \), where \( \Lambda \) is the relation between the height and width of the testing area, \( A = a/b \). \( \Omega \) is calculated as

\[
\Omega = \frac{m \omega}{2 \rho c} \left( 1 - \frac{\omega^2}{\omega_c^2} \right) = \frac{m \omega}{2 \rho c} \left( 1 - \frac{f^2}{f_c^2} \right).
\]  

(19)

Sewell’s expression for forced transmission is valid under the condition that \( m > 10 \text{ kg/m}^2 \). In the case of a concrete plate, this is fulfilled at a plate thickness of 5 mm.

Leppington et al.[4] published in 1987 a slightly modified version of Sewell’s expression for the sound reduction index. The expression for the forced transmission factor valid for frequencies well below the coincidence frequency was presented as

\[
\tau = \left( \frac{\rho_c c_o}{\pi f m (1 - f^2 / f_c^2)} \right)^2 \left( \ln \left( \frac{2 \pi f \sqrt{S}}{c_o} \right) + 0.160 - U(a/b) + \frac{1}{4 \mu^2} \left[ 4 \mu^2 - 1 \right] \left[ (\mu^2 - 1) \ln(\mu^2 - 1) + (2 \mu^2 + 1) \ln(\mu^2 + 1) - 4 \mu^2 - 8 \mu^2 \ln \mu \right] \right)
\]  

(20)

where \( \mu = (f_c f)^{1/2} \) and \( U_{lep}(a/b) \) is the form factor according to

\[
U_{lep}(x) = U_{lep}(1/x) = \frac{1}{2 \pi} \left( x + \frac{1}{x} \right) \ln(1 + x^2) - \left( \frac{1}{2} + \frac{x}{\pi} \right) \ln x - \frac{2 x}{\pi} \int_0^x \left( \frac{\pi}{t} \right)^{1/2} \text{arctan}\left( \frac{\pi}{t} \right) \, dt.
\]  

(21)
In Figure 4 the non-resonant parts of the sound transmission below the coincidence region are considered. Results from the models of Sewell and Leppington are here compared with that presented in this work, according to eq. 4 and 7. For mid- and high frequencies (still below \( f_c \)) though the agreement is good. It should be noted though, that both Sewell and Leppington has a limit in frequency downwards, due to the term \( \ln(k_0 S^{1/2}) \). This implies that these models are not suitable for small plates.

![Figure 2. The non-resonant parts of the sound transmission below the coincidence region. Leppington 0.5m\(^2\)(—) and 4.4 m\(^2\) (-----); Sewell 0.5m\(^2\)(—.--) and 4.4 m\(^2\)(—.--); Kernen 0.5m\(^2\)( ) and 4.4 m\(^2\)( — ).](image)

5 MEASUREMENTS

Measurements are made in order to show the area dependency of the sound reduction index. A 16 mm chipboard is used as a test plate. To achieve a simply supported mounting the plate is hanging in rubber bands from the ceiling in the aperture between two reverberation chambers. The total plate area \( S_{tot} \) is 2.44 x 2.42 m\(^2\). The edges of the plate is covered by baffles, and the excited part of the plate \( S \) is varied in size from 0.55 m\(^2\) to 4.4 m\(^2\) by altering the sizes of the baffles. The baffles are made of 400mm mineral wool with 26mm gypsum board on each side. An analyzer Brüel & Kjaer 2260 with a microphone Brüel & Kjaer 4189 was used together with a loudspeaker "Mackie Industrial Art. 300A". Measurements are performed according to ISO standard.

![Figure 3. The measurement setup.](image)
The measurements are performed in 1/3-octave bands for the frequency range 50 – 5000 Hz. The computations are performed in the frequency range 1 – 5612 Hz, with an increment of 1 Hz, and then averaged into 1/3-octave bands to match the measurements. In the resonant part of the transmission the radiation factors presented by Leppington\textsuperscript{[12]} are used.

![Graph showing sound reduction index vs. frequency](image)

Figure 4. Comparison between measured and calculated sound reduction indexes. $R_{\text{measured}}$ (---) 0.5 m\textsuperscript{2}; ( - - - - - ) 1.7 m\textsuperscript{2} and (- - - - - ) 4.4 m\textsuperscript{2}; $R_{\text{Kernen}}$ (-----) 0.5 m\textsuperscript{2}; ( - - - - ) 1.7 m\textsuperscript{2} and (- - - - ) 4.4 m\textsuperscript{2} according to eq. 4 and 7. The modulus of elasticity $E$ is 1.5 GPa, the loss factor $\eta$ about 8%.

The calculated and measured sound reduction indexes matches well. In both cases it is seen that the sound reduction indexes are decreasing with increasing area for the region at and below the critical frequency.

6 CONCLUSIONS

The calculation model is both compared with the model presented by Ljunggren\textsuperscript{[2]} and confirmed by measurements. The solution presented in this paper does not take into account the radiation loss of the air. This is however, a reasonable approximation as the loss factor of the air is much smaller than the loss factor of the plate.

In both the theoretical model and in the measurements the area dependency of the sound reduction index appears. The sound reduction will increase for decreasing area both for resonant and non-resonant case for frequencies at and below the critical frequency. Above the critical frequency only the resonant transmission is area dependent, and here as well the sound reduction will increase with decreasing area.

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8 REFERENCES

Airborne Sound Insulation of Floating Floors

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Summary

An analytical model is presented for the sound reduction index of finite size floating floors. The model consists of two elastic plates with a resilient layer in between. The bottom plate, the load-bearing slab, is assumed to be excited with a diffuse airborne sound field. The top plate and the resilient layer compose the floating floor. The problem is solved for frequencies below, between and above the critical frequencies of the plates. Above the critical frequency of the load-bearing plate, but below that of the floating slab, the main coupling between the plates will occur at the coincidence angle of the load-bearing plate. Above the critical frequency of both plates, transmission will occur at the angle of coincidence of each plate. As the plates will interact, the sound insulation improvement will to some extent depend on the properties of the load-bearing slab. In the article it is shown how the sound reduction index depends on the physical parameters and the geometry of the plates.

1. Introduction

1.1 Historical review

Probably the most famous work in this field was published by Cremer[1] in 1952, when he presented his well-known equation

\[ \Delta L = 40 \lg \left( \frac{f}{f_0} \right), \tag{1} \]

where \( f \) is the frequency and \( f_0 \) is the mass-spring-mass resonance frequency. Cremer's model treats two homogeneous infinite plates with an intermediate resilient layer that acts as a spring. In many cases, however, this equation was
found to overestimate the sound insulation improvement. In 1954 Cremer\textsuperscript{[2]} explained this as caused by sound bridges, which has been a popular explanation ever since.

In 1956, Gösele\textsuperscript{[3]} points out that equation (1) shows good agreement with measurement results when the resilient layer has a relatively high flow resistance. To explain the overestimated sound insulation improvement observed in certain cases, he suggested that wave propagation in the resilient layer could have a deteriorating effect.

Shortly after, Foti\textsuperscript{[4]} published a new theory where the plates are no longer infinite. Both plates are considered to be simply supported and their vibrations are described by the sum of eigenfunctions. Still, his theory leads to the same frequency dependence as Cremer’s equation, i.e. an increase by 40 dB/decade.

In 1967 Cremer et al.\textsuperscript{[5]} examined the case where the structural slab is primarily excited. They found that the sound reduction improvement could be written as

\[
\Delta L \approx 40 \log \left( \frac{f}{f_0} \right) + 10 \log \left( \frac{f_{c3}}{f_{c1}} \right)^2 - 1 \right) \right)^2 ,
\]

where \(f_{c1}\) and \(f_{c3}\) are the coincidence frequencies of the plates. If \(f_{c1} \times f_{c3}\) this expression turns into equation (1).

A different type of resonantly reacting floating floors was examined by Vér\textsuperscript{[6]}.\textsuperscript{[7],[8]}. In his model the floating slab is not resting on a continuous resilient layer but on load-bearing resilient unit-mounts. The space between these mounts is filled with flow-resistive material in order to prevent horizontal propagation of sound waves. A number of assumptions are made, such as that there is no correlation of the slab motion at the different mounts. The unit mounts are modelled as pure springs, and all sound transmission from plate 1 to plate 2 is assumed to be transmitted through these mounts. By use of SEA (Statistical Energy Analysis) Vér derived the sound reduction improvement of floating floors as

\[
\Delta L = 10 \log \left( \frac{2.3c_{1}I_{1}I_{2} \eta_{0} n_{1} f_{s}^{3}}{2 \rho_{0} f_{s}^{4}} \right) ,
\]
where \( n' \) is the number of resilient mounts per unit area, \( \eta_i \) and \( t_i \) are the loss factor and the thickness of plate 1, and \( c_{L1} \) is the longitudinal wave speed in plate 1.

In 1973 Nilsson\(^9\) presented a model for finite size floating floors. The load-bearing slab, plate 3, is assumed to be simply supported and excited by a bending moment along one of the boundaries. The floating slab has free ends and rests on a continuous elastic interlayer. The coupling between the plates is still considered to be local.

If the floating slab is highly damped Nilsson’s expression agrees exactly with equation (2) by Cremer. If plate 1 is lightly damped the sound insulation improvement is derived as

\[
\Delta L = 10 \log \left( \frac{f^{N2} \eta_2 2\pi (f_{c3}^3 - f_{c1}^3) \sqrt{ab}}{ef_0^2 f_{c3} f_{c1} (a+b)} \right) + 10 \log \left( \frac{\sigma_3}{\sigma_1} \right),
\]

(4)

in the case of \( k_{B1} \ll k_{B3} \), and

\[
\Delta L = 10 \log \left( \frac{f^{N2} \eta_2 2\pi (f_{c3}^3 - f_{c1}^3) \sqrt{ab}}{ef_0^2 f_{c3} f_{c1} (a+b)} \right) + 10 \log \left( \frac{\sigma_3}{\sigma_1} \right),
\]

(5)

if \( k_{B1} \gg k_{B3} \). The first part of the equations represents the velocity level difference. By addition of the second term the full expression for the sound insulation improvement is obtained. Note here that \( \eta_2 \) is a loss factor assigned to the intermediate layer, \( a \) and \( b \) are the dimensions of the plates and \( \sigma_1 \) and \( \sigma_2 \) the radiation factors of the plates. In this case \( \Delta L \) will increase by 25 dB/decade, assuming that \( \eta_2, \sigma_1 \) and \( \sigma_2 \) are independent of the frequency.

In 1979 Ljunggren \([10]\) published measurements that showed how different parameters influence the sound reduction improvement of floating floors. For frequencies above the critical frequency of the floating slab, plate 1, the loss factor of plate 1 is shown to increase the sound insulation by approximately \( 10 \log(\eta) \) and increased thickness of plate 1 will increase the insulation by \( 30 \log(t) \).

In 1982 Widén \([11]\) presented measurements where he compared concrete tiles with a continuous concrete plate. By this subdivision of the plate, each tile behaves as a lumped mass on a spring, and the sound insulation of the floating floor is increased by approximately 6 – 10 dB.
Gudmundsson\textsuperscript{[12]} solved the problem using spatial Fourier Transforms, starting with Heckl's solution for single infinite plates (see Cremer et al.\textsuperscript{[5]}). In the case of a point force excitation, assuming that the same force is acting on the floating slab as on the structural slab, the insulation improvement for locally reacting floating floors is presented as

\begin{equation}
\Delta L = 10 \log \left( \frac{\int_0^\infty \left[ N_3 + N_0 \right]^2 \xi^2 d\xi}{\int_0^\infty \left[ N_1 + N_3 + N_2 N_1 / N_2 \right]^2 \xi^2 d\xi} \right),
\end{equation}

where

\begin{equation}
N_0 = \frac{\eta_{\rho c}}{\sqrt{1-\xi^2}},
\end{equation}

\begin{equation}
N_1 = B_1 (k \xi)^4 - \omega^2 m_1 + j \frac{\eta_{\rho c}}{\sqrt{1-\xi^2}} + B_1 \eta_1 (k \xi)^4,
\end{equation}

\begin{equation}
N_2 = s(1 + j \eta_2).
\end{equation}

Here $\rho$ is the density of the air, $\omega$ is the angular frequency, $m$ is the mass per square meter and $B$ is the bending stiffness. $N_3$ is defined analogously to $N_1$ and $\xi = k_r / k$, where $k$ is the exciting wave number and $k_r$ is a transform wave number.

Gudmundsson\textsuperscript{[12]} solved this equation numerically and when compared, it agreed well with equation (1) in the case of a heavy structural slab and with equation (2) if the structural slab was light. In the case of airborne excitation, he deduced an equation similar to equation (6), with the square roots $\sqrt{1-\xi^2}$ replaced with $(1-\xi^2)$.

In the case of resonantly reacting floating floors, Gudmundsson pointed out that the sound isolation improvement should be corrected with a term

\begin{equation}
\Delta \Delta L_{\text{res}} = -10 \log \left( 1 + Y \frac{\sigma_{l,\text{res}}}{\sigma_{l,\text{forced}}} \right),
\end{equation}

where $\sigma_{l,\text{res}}$ and $\sigma_{l,\text{forced}}$ are the radiation factors of the resonant and the forced field of the floating slab. For $f < f_c$ $\sigma_{l,\text{res}}$ is approximated to 0, for $f > f_c$, $\sigma_{l,\text{res}}$
\( \approx l \) and \( \sigma_{\text{forced}} \approx l \). In the case of a free floating slab and a simply supported load-bearing slab the ratio factor, defined as \( Y = \frac{v^2_{1,\text{res}}}{v^2_{1,\text{forced}}} \) where \( v_{1,\text{res}} \) and \( v_{1,\text{forced}} \) the velocity of the resonant and forced waves, is presented as

\[
Y_{SF} = \frac{U^2}{3\pi \eta_i S} \left( \frac{k_{B3}}{k_{B1}} \right)^7.
\]  

(11)

If the load-bearing slab instead is clamped, Gudmundsson gives the ratio factor as

\[
Y_{CF} = \frac{I_{CF} U^2}{4\pi \eta_i S} \left( \frac{k_{B3}}{k_{B1}} \right),
\]  

(12)

where \( I_{CF} \) is the squared equivalent width of the coupling strip \( (w_{CF}) \), averaged over the angles of incidence, \( S \) is the plate area and \( U \) is the length of the plate boundary. This expression has to be evaluated numerically.

### 1.2 An outline of the present work

In the present article, an analytical model for predicting the sound reduction index for airborne sound of finite floating floors is presented. In the following, the upper floating slab is named plate 1, the load bearing plate is named plate 3. Both the plates have a finite area. In between the plates is a resilient layer, layer 2. Plate 3 is excited by a diffuse airborne sound field, which is composed of plane waves propagating at an angle \( \theta \), see Figure 1.

The problem is treated somewhat differently for different frequency regions. The low frequency range is defined as the range below the critical frequency, \( f_c \), of both plates. The mid-frequency range is between the critical frequencies of the plates. The high frequency range is above \( f_c \) for both plates.

![Figure 1. The floating floor model.](image)
For low frequencies, the sound reduction index is derived using the same concept of resonant and forced velocity fields of a plate as that used by Ljunggren\textsuperscript{[13]}. Plate 3 is assumed to be excited by an incident propagating plane sound wave with the trace wave number $k$. It is assumed that the influence of the intermediate layer can be described by the simple spring model used by e.g. Cremer et al.\textsuperscript{[5]}. It is also assumed that the intermediate layer is fairly resilient so that the force acting on plate 1 can be taken as the displacement of plate 3 times the stiffness of the spring. Knowing the excitation of plate 1, the sound transmission of the total construction can be derived.

Both plates exhibit resonant and forced fields. The sound transmission factors for all four combinations are derived separately, and the transmission factors are summed up in the end.

For midrange frequencies, i.e. $f_{c3} < f < f_{c1}$, the coupling of the plates behaves differently. In order to describe the coupling the concept of “plate admittance” is now introduced. It is defined as $Y = v/p$, where $v$ is the forced velocity response of the plate and $p$ is the pressure acting on the plate. This admittance is a complex quantity and a function of the frequency as well as the forcing wave number. The main behaviour of the normalised admittance of the two plates is illustrated in Figure 2 as a function of the angle of incidence. $Y_0$ is here defined as $Y_0 = v_0/p_0$ where $v_0$ is $10^9$ m/s and $p_0$ is $2 \cdot 10^{-5}$ Pa.

If a diffuse airborne sound field excites plate 3, Figure 2 shows that a predominant part of the response of this plate is due to the coupling at the coincidence angle $\theta_{c3}$. As the response of plate 1 does not vary very much with the angle of incidence and as the sound reduction index of plate 3 is determined by the coupling with the airborne sound field at this angle, it is thought that all important coupling between the plates occurs at $\theta_{c3}$.
Figure 2. Plate admittance at 180 Hz, the mid frequency range. (—) Plate 1, 0.05 m thick, concrete, $f_{c1}=360$ Hz. (—) Plate 3, 0.2 m thick, concrete, $f_{c3}=90$ Hz.

Figure 3. Plate admittance at 540 Hz, the high frequency range. (—) Plate 1, 0.05 m thick, concrete, $f_{c1}=360$ Hz. (—) Plate 3, 0.2 m thick, concrete, $f_{c3}=90$ Hz.
The examples have been calculated assuming a density of 2400 kg/m$^3$, a Young’s modulus of 26 GPa and a loss factor of 2%.

In the case of high frequencies, it can be assumed that plate 3 will couple to plate 1 at two angles, $\theta_{c3}$ and $\theta_{c1}$. The normalised admittance with respect to the forced response is illustrated in Figure 3. For the coupling that occurs at $\theta_{c3}$ the conditions are the same as for midrange frequencies. The contributions from the two peaks of the plate admittance are assumed to be uncorrelated. The sound transmission factor for the whole construction can thus be written as the sum of the transmission at $\theta_{c3}$ and the transmission at $\theta_{c1}$.

The aim of this article is to show the influence of the area, the critical frequency and the damping of the floating slab on the sound insulation.

1.3 Main limitations and assumptions of the model

The plates are assumed to be of uniform thickness and made from a homogeneous, isotropic and linear elastic material. It is also assumed that plate 3 is much heavier than plate 1. The model is only valid for frequencies well above the mass-spring-mass frequency $f_0$. No account is taken of the specific boundary conditions at the edges of the plates.

As mentioned earlier, it is assumed that the intermediate layer can be described by the simple spring model (see Cremer et al.\textsuperscript{[5]}) and that the frequency is high enough so that the force acting on plate 1 can be taken as the displacement of plate 3 times the spring constant $s$.

Further, it is assumed that the area of the load-bearing slab, $S_3$, and the floating slab, $S_1$, is larger than the excited part of the area, $S_5$. It is also assumed that the excited area of the load-bearing slab is the same size as the radiating area of the floating slab and that they are situated directly above each other (see Figure 4).

2. The analytical model

The whole construction is assumed to be surrounded by air but the influence of the radiation load on the plates is neglected throughout. The model can therefore be used only for sufficiently heavy plates. Plate 3 is assumed to be
excited by an incident propagating plane sound wave. The incident and the reflected wave act on the plate with a pressure \( p(x) \) where

\[
p(x) = p_0 e^{-j \omega t} e^{j k x}.
\]  

(13)

Here \( x \) is the coordinate according to Figure 4, \( \omega \) is the angular frequency, \( t \) is the time and \( k \) is the trace wave number of the exciting pressure along the plate surface,

\[
k = k_0 \sin \theta,
\]  

(14)

where \( \theta \) is the angle of incidence according to Figure 1 and \( k_0 \) is the wave number in air. The area of plate 1 and 3 are denoted \( S_1 \) and \( S_3 \), respectively, and \( S_3 \) is the area of the excited part of plate 3.

\[\text{Figure 4. Definition of the areas } S_1, S_3 \text{ and } S_3\]

2.1 Forced and resonant fields

As the concepts of forced and resonant response are defined differently in the literature, an elucidation may be appropriate here.

Consider first an infinite plate, excited over part of the area (see Figure 5) by an airborne sound field in a room below. A bending wave field will then be excited in the plate. In the part of the plate which is inside the boundaries of the room, the bending wave field will be strongly correlated to the forcing airborne field. This bending wave field is called forced.

Due to the excitation, a bending wave field will also be generated in the plate outside the boundaries of the room. This field consists of waves propagating at the phase speed of the free bending wave. For high frequencies, the waves will
mainly emerge from the boundaries towards the unexcited part of the plate, but for lower frequencies the wave propagating in the opposite direction must also be taken into account. In reality, no plates are infinite, and there will be reflections at the boundaries of the plate. If the plate is not too large and the damping is not too high, the reflections will give rise to a field where the amplitude does not vary very much over the surface: a resonant field.

Figure 5. Illustration of the generated bending field outside the excited area. $S_3$ is the total area of plate 3, $S_S$ is the excited part of the area.

2.2 The low frequency range where $f<\frac{c_3}{k}$, $f<\frac{c_1}{k}$

This first part considers excitation with wave numbers smaller than the bending wave number of both plate 3 and plate 1.

2.2.1 Transmission due to forced/forced fields

In this first case forced fields in both plates are considered. The exciting wave number $k_a$ and hence also the trace wave number $k$ is much smaller than both the bending wave number of plate 3 and that of plate 1. The transverse velocity of the excited part of plate 3 can be taken as (Ljunggren\textsuperscript{[13]})

$$v_3(x) = \frac{P_{in}}{j\omega m_3} \left[ e^{-jk_a x} - \frac{1}{2} e^{-jk_a x} \cos(k_a x) - \frac{1}{2} e^{-k_a x} \cosh(k_a x) \right], \quad (15)$$

where $m$ is the surface weight and $P_{in}$ is the amplitude of the exciting pressure. The time factor $e^{j\omega t}$ is omitted here and in all similar cases in this paper. The
first part of the expression represents the forced wave. The other two terms
represent the free waves, one of which is the propagating wave and the other
an exponential nearfield. Since the amplitude of the free waves is smaller than
that of the forced wave and the radiation factor usually smaller, the free waves
are neglected. Hence, expression (15) is simplified to

\[ v_{3,f}(x) = \frac{P_m}{j \omega m_3} e^{-j \omega x}. \] (16)

The first index of the velocity specifies the plate number, the second index is \( f \)
as in "forced". The motion of plate 3 will cause a pressure on plate 1,

\[ p_{1}(x) = p_{1} e^{-j \omega x} = \frac{s v_{3,f}}{j \omega} e^{-j \omega x}, \] (17)

where \( s \) is the spring stiffness of the elastic intermediate layer. In the same
manner the velocity of the forced vibrations of plate 1 can be taken as

\[ v_{1,f}(x) = \frac{P_1}{j \omega m_1} e^{-j \omega x} = \frac{s p_m}{-j \omega^3 m_1 m_3} e^{-j \omega x}. \] (18)

The Fourier transform of the velocity can be written as

\[ v_{1,f}(k_x) = \int_{-L}^{L} v_{1,f}(x) e^{j k_x x} dx = \frac{s p_m}{-j \omega^3 m_1 m_3} \frac{2 \sin[(k_x - k)L]}{(k_x - k)}, \] (19)

where \( L \) is the length according to Figure 4. The radiated power can be
calculated according to Cremer et al.\(^{[5]}\) as

\[ P_{rad} = \left( \frac{\rho c k_x}{4 \pi} \right) \int_{k_x}^{k_x} \frac{|v(k_x)|^2}{\sqrt{k_x^2 - k_s^2}} dk_x = \]

\[ = \frac{4 \rho s^2 p_m^2}{4 \pi \omega^3 m_1^2 m_3} \int_{-L}^{L} \frac{\sin^2[(k_x - k)L]}{(k_x - k)^2} \frac{1}{\sqrt{k_x^2 - k_s^2}} dk_x, \] (20)

where \( \rho \) is the density of air and \( c \) is the velocity of sound in air. The radiation
efficiency \( \sigma \) is introduced as (see also Ljunggren\(^{[12]}\))

\[ \sigma = \left( \frac{k_x}{\pi L} \right) \int_{-L}^{L} \frac{\sin^2[(k_x - k)L]}{(k_x - k)^2} \frac{1}{\sqrt{k_x^2 - k_s^2}} dk_x. \] (21)
As plate 3 is considered heavy, the power incident on the plate for this two-dimensional case, can be calculated as (Ljunggren\textsuperscript{[13]})

\[ P_{in} = \frac{P_{in}^2 2L \cos \theta}{8 \rho c}. \]  \hspace{1cm} (22)

The transmission factor is taken as the ratio between the power radiated from the structure and the incident power, which gives

\[ \tau(\theta) = \frac{P_{rad}}{P_{in}} = \left( \frac{\rho cs}{\omega^3 m_1 m_3} \right)^2 \frac{4\sigma}{\cos \theta}. \]  \hspace{1cm} (23)

If the transmission factor is integrated over the angle of incidence according to Paris' formula,

\[ \bar{\tau} = 2 \int_{0}^{\pi/2} \tau(\theta) \cos \theta \sin \theta \, d\theta, \]  \hspace{1cm} (24)

the averaged transmission factor for the three-dimensional case becomes

\[ \bar{\tau}_{r, f} = \left( \frac{2 \rho cs}{\omega^3 m_1 m_3} \right)^2 2\sigma_d. \]  \hspace{1cm} (25)

The first index \( f \) stands for forced transmission in plate 3, second index \( f \) for forced transmission in plate 1, \( r \) as in resonant transmission will also be used later on. The radiation factor \( \sigma_d \) is defined as

\[ \sigma_d = \int_{0}^{\pi/2} \sigma \sin \theta \, d\theta. \]  \hspace{1cm} (26)

\( \sigma_d \) is further discussed in the conclusions.

2.2.2 Transmission due to forced/resonant fields

The possibility of a forced wave in plate 3 giving rise to a reverberant field in plate 1 will now be considered. The power radiated from the upper surface of plate 1 can be written as

\[ P_{rad} = \langle v_{zr}^2 \rangle \rho c S_z \sigma_1, \]  \hspace{1cm} (27)

or, with the help of the relation between the velocity of the reverberant and forced vibrations derived in the appendix:
\[
\frac{\langle v^2 \rangle_{l,cr}}{\langle v^2 \rangle_{l,cr}} = \frac{\pi}{4\eta_i} \frac{S_3}{S_1} \frac{f_{cl}}{f} \sigma_i, \tag{28}
\]

and equation (18) as
\[
P_{\text{rad}} = \frac{s^2 p^2_{\text{in}}}{\omega^6 m^2_i m^2_3} \frac{\pi}{8\eta_i} \frac{S_3}{S_i} \frac{f_{cl}}{f} \rho c S_3 \sigma_i^2. \tag{29}
\]

Here \(\eta_i\) is the loss factor of plate 1 and \(\sigma_i\) is the radiation factor with respect to low frequencies as
\[
\sigma_i = \frac{1}{2\pi^2} \frac{U_3^2 \rho c}{S_i} \sqrt{\frac{f}{f_{cl}}}, \tag{30}
\]

where \(U_3\) is the length of the boundary of the excited part of the plates. In this three-dimensional case the incident power must be taken as
\[
P_{\text{in}} = \frac{p^2_{\text{in}} S_3 \cos \theta}{8\rho c}, \tag{31}
\]

which gives the transmission factor as
\[
\tau_{f_{sr}}(\theta) = \frac{p_{\text{rad}}}{p_{\text{in}}} = \left( \frac{2\rho c s}{\omega^3 m^2_i m^2_3} \right)^2 \frac{\pi}{4\eta_i} \frac{S_3}{S_i} \frac{f_{cl}}{f} \sigma_i^2 \frac{1}{\cos \theta}. \tag{32}
\]

The average transmission factor, integrated according to equation (24), becomes
\[
\bar{\tau}_{f_{sr}} = \left( \frac{2\rho c s}{\omega^3 m^2_i m^2_3} \right)^2 \frac{\pi}{2\eta_i} \frac{S_3}{S_i} \frac{f_{cl}}{f} \sigma_i^2. \tag{33}
\]

### 2.2.3 Transmission due to resonant/forced fields

In this case, the possibility of resonant waves in plate 3 giving rise to a forced field in plate 1 is considered. The radiated power can in this case be written as
\[
P_{\text{rad}} = \langle v^2 \rangle_{l,cr} \rho c S_3 \sigma_3. \tag{34}
\]
Since the vibrational field in plate 1 is forced, the radiation factor of plate 1 is equal to that of plate 3, i.e. \( \sigma_2 \) is calculated in the same manner as equation (30) (using \( \lambda_{C3}, f_{C3} \) instead of \( \lambda_{C1} \) and \( f_{C1} \)). \( \langle v_{1,f}^2 \rangle \) is calculated according to the first part of equation (18) where the squared pressure exciting plate 1 in this case will be

\[
p_1^2 = \frac{s^2 \langle v_{3,r}^2 \rangle}{\omega^2}.
\]  

(35)

To calculate the velocity of the resonant field in plate 3, the relation between the velocity of the reverberant and the forced field may be used (see appendix I)

\[
\frac{\langle v_{3,r}^2 \rangle}{\langle v_{3,f}^2 \rangle} = \frac{\pi}{4\eta_3} \frac{S_3}{S} \frac{f_{C3}}{f} \sigma_3
\]  

(36)

and the radiated power becomes

\[
P_{rad} = \langle v_{3,f}^2 \rangle \frac{\pi}{4\eta_3} \frac{S_3}{S} \frac{f_{C3}}{f} \sigma_3 \frac{s^2}{\omega^2 m_1} \rho \varepsilon S_3 \sigma_3.
\]  

(37)

With \( P_{in} \) according to equation (31) and the velocity of plate 3 according to equation (16), the transmission factor becomes

\[
\tau(\theta) = \left( \frac{2\rho cs}{\omega m_1 m_3} \right)^2 \frac{\pi}{4\eta_3} \frac{S_3}{S} \frac{f_{C3}}{f} \sigma_3^3 \frac{1}{\cos \theta}.
\]  

(38)

The averaged transmission factor integrated according to equation (24) is

\[
\tau_{r,f} = \left( \frac{2\rho cs}{\omega m_1 m_3} \right)^2 \frac{\pi}{2\eta_3} \frac{S_3}{S} \frac{f_{C3}}{f} \sigma_3^2.
\]  

(39)

### 2.2.4 Transmission due to resonant/resonant fields

In the fourth and last case for the low frequency region, it is assumed that the resonant waves in plate 3 cause a resonant field in plate 1. The radiated power is taken as

\[
P_{rad} = \langle v_{r}^2 \rangle \rho \varepsilon S_3 \sigma_1.
\]  

(40)
With the velocity of the resonant field in plate 3 according to equation (36) and (16), the pressure exciting plate 1 can be calculated according to equation (35). The velocity of the reverberant field in plate 1 may now be calculated with the help of equation (28) and (18), and the radiated power becomes

\[
P_{\text{rad}} = \frac{\rho cs^2 p_{\text{in}}^2}{\omega^2 m_1 m_2^2} \frac{\pi^2 S_3^2}{4\eta_1 \eta_3} \frac{f_{\text{Cl}} f_{\text{C3}}}{f} \sigma_1^2 \sigma_3. \tag{41}
\]

With \( P_{\text{in}} \) according to equation (31) the transmission factor becomes

\[
\tau(\theta) = \frac{P_{\text{rad}}}{P_{\text{in}}} = \left( \frac{2\rho cs}{\omega^2 m_1 m_3} \right)^2 \frac{\pi^2 S_3^2 f_{\text{Cl}} f_{\text{C3}}}{2\eta_1 \eta_3 S_1 S_3} \frac{1}{f^2} \sigma_1^2 \sigma_3 \cos \theta \tag{42}
\]

and

\[
\bar{\tau}_{r,r} = \left( \frac{2\rho cs}{\omega^2 m_1 m_3} \right)^2 \frac{\pi^2 S_3^2 f_{\text{Cl}} f_{\text{C3}}}{2\eta_1 \eta_3 S_1 S_3} \sigma_1^2 \sigma_3. \tag{43}
\]

Four different transmission factors for the low frequency region are now derived. These are all summed up to a total transmission factor,

\[
\bar{\tau}_{\text{tot}} = \bar{\tau}_{r,r} + \bar{\tau}_{r,f} + \bar{\tau}_{r,s,f} + \bar{\tau}_{r,s,r}, \tag{44}
\]

\[
\bar{\tau}_{\text{tot}} = \left( \frac{2\rho cs}{\omega^2 m_1 m_3} \right)^2 \left( 2\sigma_d + \frac{\pi}{2\eta_1} \frac{S_3}{S_1} f \sigma_1^2 \right)
\]

\[
+ \frac{\pi}{2\eta_1} \frac{S_3}{S_1} f \sigma_3^2 + \frac{\pi^2}{4\eta_1 \eta_3} \frac{S_3}{S_1 S_3} f^2 \sigma_1^2 \sigma_3. \tag{45}
\]

### 2.3 The mid-frequency range

The mid-frequency range well above the critical frequency of plate 3 but below that of plate 1 will now be considered. As explained in the introduction, all important coupling between the plates will here occur at the coincidence angle \( \theta_{\text{C3}} \). This implies that plate 3 will excite plate 1 with the wave number \( k_{31} \).

If plate 3 is to be considered large in the sense that \( \eta k_{31} L/4 \gg 1 \), the plate would be expected to be dominated by forced vibrations. However, since this model is mainly aimed for floating floors in dwellings, these plates will be considered
small and not fulfill the criteria above. In Ljunggren\textsuperscript{13} it is shown that for plates of more modest dimensions, the response will be dominated by the free vibrations and if the plate is not too small, by the propagating waves alone.

Since plate 1 is still excited with small wave numbers though, both resonant and forced transmission will be of importance. First, the forced field is considered. The power radiated from plate 1 is taken as

$$P_{rad} = \langle v_{1,f}^2 \rangle \rho c S_S \sigma_S,$$  \hspace{1cm} (46)

where $\langle v_{1,f}^2 \rangle$ is calculated according to equation (18) and (35). The velocity of the reverberant field in a plate of finite dimensions can be taken as

$$\langle v_{3,r} \rangle = \frac{P_R}{\omega m_3 \eta_3 S_S},$$  \hspace{1cm} (47)

where $P_R$ is the power travelling away from the excited area,

$$P_R = 2c_g m_3 \nu_{3,R}^2 \Lambda.$$  \hspace{1cm} (48)

$\Lambda$ is the length of the boundary at $x = L$ and $\langle v_{3,R}^2 \rangle$ is the mean square value of the velocity of the free propagating bending wave $\nu_{3,R}(x)$. Using the two-dimensional model of Figure 4, Ljunggren\textsuperscript{13} has shown that $\nu_{3,R}$ can be written as

$$\nu_{3,R}(x) = \frac{p_{in} k_{33}}{2 \omega m_3} \sin\left[\frac{(k - k_{33})L}{2}\right] e^{-k_{33}x}.$$  \hspace{1cm} (49)

The velocity of the reverberant field becomes

$$\langle v_{3,r} \rangle = \frac{p_{in}^2 k_{33} \Lambda}{4 \omega^2 m_3^2 \eta_3 S_S^2} \sin^2\left[\frac{(k - k_{33})L}{2}\right],$$  \hspace{1cm} (50)

and the radiated power from plate 1

$$P_{rad} = \frac{s^2}{2 \omega^3 m_1 m_3} \frac{p_{in}^2 k_{33} \Lambda}{4 \omega^2 m_3^2 \eta_3 S_S^2} \sin^2\left[\frac{(k - k_{33})L}{2}\right]\rho c S_S \sigma_S.$$  \hspace{1cm} (51)

With $P_{in}$ according to equation (22) the transmission factor becomes

$$\tau(\theta) = \left(\frac{2 \rho c s}{\omega^3 m_1 m_3}\right)^2 \frac{S_S k_{33}}{S_3 \eta_3 L \sigma_3} \sin^2\left[\frac{(k - k_{33})L}{2}\right] \frac{1}{\cos \theta}.$$  \hspace{1cm} (52)
where $\sigma_3$ is the radiation factor of plate 3. For frequencies above the coincidence frequency, the radiation factor can be evaluated in the same way as in Ljunggren's paper [13] to

$$\sigma_3 = \frac{1}{\sqrt{1 - \frac{f_{cl}}{f}}}.$$  \hfill (53)

The average transmission factor integrated according to Paris' formula becomes

$$\bar{\tau}_{r,f} = \left( \frac{2 \rho c s^3}{\omega^3 m m_3} \right)^2 \frac{\pi}{2 \eta_3 \eta_3} \frac{f_{cl} f_{c3}}{f} \sigma_3^2.$$ \hfill (54)

Second, the case of resonant transmission in plate 1 is considered. The radiated power must in this case be taken as

$$P_{rad} = \left( \nu^2_{\nu_3} \right) \rho c S_3 \sigma_1.$$ \hfill (55)

Once again, the relation between the resonant and forced field according to equation (28) may be used and the radiated power becomes

$$P_{rad} = \frac{s^2 \Theta^2}{8 \omega^3 m_3 m_3^2} \frac{\pi}{2 \eta_3 \eta_3} \frac{k_{by} f_{cl} f_{c3}}{f} \sigma_1^2 \frac{\sin^2[(k - k_{by})L]}{(k - k_{by})^2}.$$ \hfill (56)

The corresponding transmission factor is

$$\tau(\theta) = \left( \frac{s \rho c}{\omega^3 m m_3} \right)^2 \frac{S_3^2}{S_1 S_3 L} \frac{\pi}{2 \eta_3 \eta_3} \frac{k_{by} f_{cl} f_{c3}}{f} \sigma_1^2 \frac{\sin^2[(k - k_{by})L]}{(k - k_{by})^2} \frac{1}{\cos \theta}.$$ \hfill (57)

and the averaged transmission factor for the case of resonant transmission in both plates becomes

$$\bar{\tau}_{r,r} = \left( \frac{2 \rho c s^3}{\omega^3 m m_3} \right)^2 \frac{\pi^2}{4 \eta_3 \eta_3} \frac{S_3^2}{S_1 S_3} \frac{f_{cl} f_{c3}}{f^2} \sigma_1^2 \sigma_3.$$ \hfill (58)

The total transmission factor for the mid-frequency region is given as

$$\bar{\tau} = \bar{\tau}_{r,f} + \bar{\tau}_{r,r} = \left( \frac{2 \rho c s^3}{\omega^3 m m_3} \right)^2 \frac{\pi}{2 \eta_3 \eta_3} \frac{S_3 f_{c3}}{S_3 f} \left( \sigma_3^2 + \frac{\pi}{2 \eta_3 \eta_3} \frac{S_3 f_{cl} f_{c3}}{f} \sigma_1^2 \sigma_3 \right).$$ \hfill (59)
2.4 The high frequency range

The high frequency range where \( f > f_{c3} \) and \( f \geq f_{c1} \), will now be considered. As in the mid-frequency range, all important coupling between the plates will occur at the coincidence angle, but in this case at both \( \theta_{c1} \) and \( \theta_{c3} \). Plate 3 will excite plate 1 with the wave numbers \( k = k_{B1} \) and \( k = k_{B3} \). The mean transmission factor for high frequencies is calculated as the sum of the contribution from the transmission at \( \theta_{c1} \) and \( \theta_{c3} \).

2.4.1 Coupling at \( \theta_{c3} \)

In the case of coupling at \( \theta_{c3} \) plate 3 is assumed to be excited with \( k = k_{B3} \) and so the resonant waves will be predominant. Plate 1 on the other hand is still excited with wave numbers smaller than \( k_{B1} \), which means that both forced and resonant transmission will be of importance. This implies that the plates will behave in the exact same manner as in the mid-frequency range and the transmission factors are given by equations (54) and (58).

2.4.2 Coupling at \( \theta_{c1} \)

The radiated power in the case of coupling at \( \theta_{c1} \) is taken as

\[
P_{rad} = \frac{1}{c_{S} \sigma_{1} \rho} \left( v^2_{1,\sigma} \right).
\]

As plate 3 is excited with a wave number larger than \( k_{B3} \), the velocity of the excited part of the plate will according to Ljunggren\textsuperscript{[13]} be

\[
v_{3}(x) = \frac{jap_{m} k_{B}}{Bk_{i}^{2}} e^{-jk_{B}x} + \frac{p_{m} k_{B}}{2\omega m_{k} k_{i}^{2}} \left[ e^{-jk_{B}x} \sin(k_{B}L - k_{B}L) + e^{jk_{B}x} \sinh(k_{B}x - jk_{B}L) \right].
\]

where the first part of the equation represents the forced wave. The second and third terms are the free waves: the propagating bending wave and the exponential nearfield. Since the amplitude of the free waves is smaller than that of the forced wave, these free waves are disregarded. With \( k = k_{B1} \), the squared pressure on plate 1 calculated according to equation (17) becomes

\[
p_{1}^{2} = \frac{s^{2} \left( v^2_{3,\sigma} \right)}{-\omega^{2}} = \frac{p_{m}^{2} s^{2} f_{c3}^{4}}{2\omega^{2} m_{k}^{2} f_{c1}^{4}}.
\]
Since plate 1 is excited at its angle of incidence, the resonant waves will dominate. Using the same technique as in the preceding case, the radiated power can be derived as

\[
P_{rad} = \left( \frac{P_{1}k_{11}}{2\omega m_{1}} \right)^{2} \frac{S_{1} \rho c \sigma_{1} \lambda}{k_{11} \eta_{1} S_{1}} \frac{\sin^{2}[(k-k_{11})L]}{(k-k_{11})^2},
\]

where \( \sigma_{1} \) is the radiation efficiency at high frequencies according to equation (53) and \( \lambda \) is the length of the boundary at \( x = L \). The transmission factor becomes

\[
\tau(\theta) = \left( \frac{2 \rho c s}{\omega^{2} m_{1} m_{s}} \right)^{2} \frac{k_{11}}{4 \eta_{1}} \frac{S_{s}}{S_{1} L \int f_{ci}^{4}} \frac{\sin^{2}[(k-k_{11})L]}{(k-k_{11})^2} \frac{1}{\cos \theta}
\]

and

\[
\bar{\tau}_{r+r} = \left( \frac{2 \rho c s}{\omega^{2} m_{1} m_{s}} \right)^{2} \frac{\pi}{2 \eta_{1}} \frac{S_{s}}{S_{1} \int f_{ci}^{4}} \frac{f_{ci}^{4}}{f} \frac{\sigma_{1}^{2}}{f}
\]

The total transmission factor for high frequencies is now calculated as the sum of the contributions at \( \theta_{ci} \) and \( \theta_{ci} \) as

\[
\bar{\tau} = \bar{\tau}_{r+r} + \bar{\tau}_{r+r} = \left( \frac{2 \rho c s}{\omega^{2} m_{1} m_{s}} \right)^{2} \left( \frac{\pi}{2 \eta_{1}} \frac{S_{s}}{S_{3}} \frac{f_{ci}^{4}}{f} \frac{\sigma_{1}^{2}}{f} + \frac{\pi}{2 \eta_{1}} \frac{S_{s}^{2}}{S_{1} S_{3}} \frac{f_{ci}^{4}}{f} \frac{\sigma_{1}^{2}}{f} \right)
\]

Inspection shows that this expression is valid not only above \( f_{ci} \) but also at \( f_{ci} \), providing that an appropriate radiation factor is used.

3. Discussion

3.1 The characteristics of the present model

The sound reduction indices given in section 3 show that the airborne sound insulation is a function of the areas of both plates. In dwellings the area of the load-bearing slab, \( S_{s} \), is often much larger than both \( S_{1} \) and \( S_{3} \). While the excited part of the plate is intended to represent the area of a room, the load-bearing plate can extend over the entire flat. This is the case if the walls within the flat are lightweight constructions, e.g. gypsum panels and steel studs. The floating slab area can represent one or more rooms. In most lab measurements
though, the areas $S_t$, $S_r$ and $S_s$ are the same and the area relation is little investigated.

In Figure 6 and 7, $S_s$ and $S_t$ are varied from 10 to 200 m$^2$, while the other areas are kept fixed. Increasing the excited and radiating area $S_s$ will decrease the sound insulation for all frequencies, but to a varying extent. Increasing the area $S_t$ will increase the sound insulation in all frequency regions although, for low and mid-frequencies, the influence of $S_t$ is less than for high frequencies (see also the expressions for the transmission factors presented in section 2).

![Graph showing sound reduction index, $R$, for varying $S_s$. $S_s=200$ m$^2$, $S_s=200$ m$^2$ $S_s=10$ m$^2$ (---), 40 m$^2$ (-----), 120 m$^2$ (- - -), 200 m$^2$ (-----).]
In these and the following calculated examples it is assumed, where nothing else is mentioned, that the plates consist of concrete with \( \rho = 2400 \text{ kg/m}^3 \), \( E = 3 \times 10^7 \text{ N/m}^2 \), the loss factor \( \eta \) of plate 1 and 3 is 0.5 \% and 4 \%, respectively, \( S_1 = 20 \text{ m}^2 \), \( S_2 = 20 \text{ m}^2 \), \( S_3 = 100 \text{ m}^2 \), \( t_1 = 0.050 \text{ m} \), \( t_3 = 0.200 \text{ m} \), \( s = 2 \times 10^6 \text{ kg/m}^2\text{s}^2 \).

From the expressions of the transmission factors in section 2 it is seen that the floating slab is preferably as heavy and highly damped as possible. Note though the initial assumption that \( m_2 \gg m_1 \). At low and mid-frequencies the sound reduction index will increase with \( 20 \log(m) \) for increasing mass of plate 1, and somewhat more at high frequencies.

From the sound transmission factors presented in section 2 it is evident that the sound insulation improvement of the floating floor construction will to some extent depend on the properties of plate 3. Figure 8 shows the variation of \( \Delta R \) with the critical frequency of plate 3. \( \Delta R \) is here defined as

\[
\Delta R = 10 \log \left( \frac{1}{\tau} - 10 \log \frac{1}{\tau_3} \right),
\]  

\((67)\)
where $\tau$ is the total sound transmission factor according to section 2 and $\tau_3$ is the sound transmission factor of plate 3 if there were no floating floor present according to Ljunggren\textsuperscript{[13]}. In the transmission factor of plate 3 for low frequencies (Equation (45)) the term which represents the forced transmission in plate 3 will be dominating. If the resonant part of the transmission is disregarded, $\Delta R$ for the low frequency region will be

$$\Delta R = 20 \log \left( \frac{\omega^2 m_i}{s} \right) - 10 \log \left( \frac{1}{2\eta_1 S_1} \frac{f}{2\sigma_1} \right).$$  \hspace{1cm} (68)$$

In the mid-frequency region $\Delta R$ becomes

$$\Delta R = 20 \log \left( \frac{\omega^2 m_i}{s} \right) - 10 \log \left( \frac{1}{2\eta_1 S_1} \frac{f}{\frac{\sigma_1^2}{\sigma_3}} \right).$$ \hspace{1cm} (69)$$

![Graph](image.png)

**Figure 8.** $\Delta R$ for varying critical frequency of plate 3. 
$f_{C3} = 80$ Hz (---), $f_{C3} = 120$ Hz (---), $f_{C3} = 200$ Hz (---).

When comparing the sound transmission at the low- and mid-frequency region, the sound insulation improvement of the floating floor at a certain frequency will be somewhat higher if the critical frequency of plate 3 is above the frequency studied, than beneath. Since the critical frequency of plate 3 will decide where the low frequency region ends and the mid-frequency region
begins, the properties of plate 3 will affect the sound insulation improvement of the floating floor for these regions.

Further, other parameters of plate 3 such as the area and the loss factor will to some extent have an influence on the sound insulation improvement of the floating floor.

### 3.2 Comparison with previous result

If $\Delta R$, defined as above, is compared to the $\Delta L$ presented by Cremer$^{[1]}$ in equation (2), it is seen (Figure 9) that the sound insulation mainly differs in the high frequency region.

![Graph showing $\Delta R$ and $\Delta L$](image)

*Figure 9. $\Delta R$ compared to Cremer's $\Delta L$ according to equation (2). $\Delta R$ (----), $\Delta L$ (——).*

The expression for the transmission factor for high frequencies according to equation (66) shows that the sound reduction improvement will increase with $10\log(\eta)$ for increasing loss factor. This is the same relation found in measurements by Ljunggren$^{[10]}$ and in theory by Vér$^{[6],[7],[8]}$ (see equation (3)). It is also seen that in the second term, which represents the resonant transmission in both plates and is the most dominating of the three terms in equation (66), the sound insulation will increase with $30\log(t_i)$ for increasing
thickness of plate 1 which again agrees with both Ljunggren's results [10] and the theory by Vér[6],[7],[8].

4. Conclusions

4.1 Prediction formulae

4.1.1 Low frequencies

The sound reduction index for low frequencies becomes

$$R = 20 \log \left( \frac{\omega^2 m_i m_3}{2 \rho c_s} \right) - 10 \log \left( 2 \sigma_2 + \frac{\pi}{2 \eta_i S_i} f c_l \sigma_1^2 + \frac{\pi}{2 \eta_3 S_3} f c_3 \sigma_3^2 + \frac{\pi^2}{4 \eta_i \eta_3 S_i S_3} f c_l f c_3 \sigma_1 \sigma_3 \right)$$  \hspace{1cm} (70)

It is felt that the expression for $\sigma_d$ presented by Ljunggren[13] is appropriate here. The radiation efficiency at resonant transmission, i.e. $\sigma_2$ and $\sigma_3$ is given by equation (30) for frequencies below $f_c$. However, it should be noted that a value twice as high must be used if the construction is surrounded by orthogonal walls.

4.1.2 Mid-frequency region

For the mid frequency range well above the critical frequency of plate 3 but below that of plate 1 the sound reduction index becomes

$$R = 20 \log \left( \frac{\omega^2 m_i m_3}{2 \rho c_s} \right) - 10 \log \left( \frac{\pi}{2 \eta_i S_i} f c_l \sigma_1^2 + \frac{\pi^2}{4 \eta_i \eta_3 S_i S_3} f c_l f c_3 \sigma_1 \sigma_3 \right)$$  \hspace{1cm} (71)

The radiation efficiency $\sigma_3$ can be calculated according to equation (53) here but with $f_{cl}$ exchanged for $f_{c3}$. The radiation efficiency for plate 1 on the other hand is still calculated according to (30) as the plate is excited below $f_{cl}$.

4.1.3 High-frequency region

In the high frequency region, well above the critical frequencies of both plates, the sound reduction index will be
\[ R = 20 \lg \left( \frac{\omega' \rho, \eta, \mu}{2 \rho \nu} \right) \]

\[ -10 \lg \left( \frac{\pi}{2 \eta_3} \frac{S_3 f_c}{f} \sigma_3^2 + \frac{\pi}{4 \eta_3} \frac{S_3}{S_1 S_3} \frac{f c_1 f_c^3}{f^2} \sigma_3^2 \sigma_3' + \frac{\pi}{2 \eta_1} \frac{S_3}{S_1} \frac{f c_1 f_c^4}{f} \sigma_3^2 \right) \] \quad (72)

For frequencies above the critical frequency of each plate the radiation efficiencies used is the same as in the mid-frequency case. In the case where plate 1 is excited at the critical frequency, a radiation factor valid at the critical frequency of plate 1 must be used for \( \sigma_1 \).

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6. References


Appendix

The velocity relation of a plate at low frequencies

According to equation (18) the transverse velocity of the forced wave in the excited part of a plate can be taken as

\[ v_r(x) = \frac{P}{j\omega m} e^{-j\alpha x}. \]  

(73)

Outside the excited area, for \( x > L \), waves will propagate away from the boundary. This response can be approximated to (Ljunggren\(^{[13]}\))

\[ v_{s,k}(x) = \frac{P}{4\omega m} e^{-j\beta x}. \]  

(74)

The exponential nearfield is here disregarded with the same motivation as for the excited part of the plate. The propagating waves will cause a reverberant field, see Figure 5, with the mean square velocity as

\[ \langle v_r^2 \rangle = \frac{P_R}{\omega m \eta S}, \]  

(75)

where \( S \) is the total area of the plate and \( P_R \) is the power travelling away from the excited area. In the case of excitation with plane airborne waves, the power emerging from one side of the boundary can be taken as (compare Cremer et al.\(^{[1]}\), p. 109)

\[ P_R = C_B m v^2 \Lambda \cos \Psi, \]  

(76)

where \( C_B \) is the group velocity, \( v^2 \) is the RMS value of the velocity of the propagating wave emerging from the boundary, and \( \Lambda \) is the length of the boundary at \( x = L \). The angle \( \Psi \) is the angle between the direction of the free wave and the boundary normal. Thus, the mean power \( P_R \) can be written as

\[ P_R = 2\left( \frac{P}{4\omega m} \right)^2 \left( \frac{\Lambda}{2} \right) C_B m U_S \langle \cos \Psi \rangle. \]  

(77)

There will be free waves emerging from both sides of the boundary which gives the first factor 2 in the expression above. According to Ljunggren\(^{[13]}\) the mean value of \( \cos \Psi \) can be taken as 1 for low frequencies. The mean square velocity of the resonant field will be
\[ \langle v_r^2 \rangle = \left( \frac{p}{4 \omega m} \right)^2 \frac{2 c_g U_s}{\omega \eta S}, \]  

(78)

where \( \eta \) is the loss factor of plate 1 and \( U_s \) is the length of the boundary of the excited part of the plates. The squared velocity ratio between resonant and forced vibrations will be

\[ \frac{\langle v_r^3 \rangle}{\langle v_f^3 \rangle} = \frac{U_s}{4 \eta S k_\alpha}. \]  

(79)

The relation between the resonant and the forced field in a plate can be rewritten as a function of the radiation factor as

\[ \frac{\langle v_r^2 \rangle}{\langle v_f^2 \rangle} = \frac{\pi S_s f_c}{4 \eta S f} \sigma, \]  

(80)

where \( \sigma \) is the radiation factor with respect to low frequencies as

\[ \sigma = \frac{1}{2 \pi^2} \frac{U_s A_c}{S_s} \frac{\sqrt{f}}{\sqrt{f_c}}. \]  

(81)
Acoustic Properties of Double Plate Systems

Part I: An Analytical Model

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Abstract
A new model for the acoustic properties of a double plate system is presented. The plates are modelled using the Kirchoff plate equation while the intermediate layer is modelled assuming a local reaction of spring character. One of the plates is excited by a diffuse airborne sound field over a finite area. The response of the two plates are derived using a two-dimensional Fourier transform technique and evaluated by means of contour integration in the complex wave number plane. Some general properties of the solution are discussed in the present paper; expressions for the sound reduction index of a finite size system are presented and discussed in a companion article[1]. It is shown that the response of the plates can be expressed as the sum of the amplitudes of a number of propagating and evanescent plate waves. These waves are of the same type for the two plates. One of these waves describes a forced motion; two others describe free plate waves propagating away from the two boundaries of the excited area, the wave number being the same as for the uncoupled plate; another two describe the corresponding bending wave near-fields excited at the boundaries; the four remaining expressions describe similar propagating waves and near-fields but with a wave number which describes a coupled motion of the two plates.

1. Introduction
Double walls or partitions are often used in the building industry. Thus, it is hardly surprising that much work has been devoted to the development of theoretical models for the airborne sound insulation. These models have been developed using several different approaches. One of the first ones is that of Beranek and Work[2] from 1949 where the interaction between different layers is modelled using an impedance approach. Beranek and Work considered only normal incidence for infinite systems, but the model was later extended for
oblique incidence by Ookura and Sato\textsuperscript{3}. An engineering approach for finite size systems was developed by Novak\textsuperscript{4} and shown to give good results.

A ray approach was used by London\textsuperscript{5} in 1950, again for infinite systems. This model can be seen as an extension of Cremer’s\textsuperscript{6} well-known model for single plates. However, calculated results from the London model do not always agree with measured results and many modifications have been proposed without giving an entirely convincing result. It is thought by the present author that the main reason why models of this kind do not work so well is the difficulty to model a finite-size system.

With another approach, Statistical Energy Analysis (SEA), used by Price and Crocker\textsuperscript{7} and by several other authors, the problem of finite sizes was avoided. However, in the building industry, double plate systems are in practice usually made in the form of plasterboard or chipboard partitions. It is well known that the forced airborne sound transmission is of predominant importance in these cases, which makes the application of SEA difficult.

In a thesis from 1984, Gudmundsson\textsuperscript{8} developed a model for the impact sound insulation of a finite-size double construction and applied the model to building constructions. The intermediate layer was assumed to be locally reacting and acting as a wave guide while the plates were modelled with the Kirchoff plate equation. The solution was obtained by expressing the motion of the plates in eigenfunctions. A model of a similar kind was previously developed by Nilsson\textsuperscript{9} and applied to constructions typical for the shipbuilding industry.

In the present work the general behaviour of a double plate system is investigated. The plates are modelled using the Kirchoff plate theory and the responses of the two plates are derived using a spatial window technique and evaluated by means of contour integration in the complex wave number plane. This general expression is valid for thin plates which have an intermediate layer that can be modelled as a locally acting spring. The main advantage of the model is that it makes possible to distinguish between the forced and free wave fields of the plates. To illuminate these expressions a simplification for the case of two identical plates is presented. In this case it is easy to see how the different wave types behave and compare them. In a companion article\textsuperscript{11}, these plate velocities are used for a new prediction model for the sound reduction index of finite-size systems.
2. Plate velocity

Consider two parallel plates with an intermediate resilient layer. The plates are initially considered as two-dimensional and infinitely large, but baffled outside the excited region. Plate 3, see Figure 1, is excited by an airborne sound field. The incident and the reflected wave act on the plate with a pressure as

\[ p(x) = \hat{p} e^{-jkx} e^{j\omega t}, \]

where \( \hat{p} \) is the magnitude of the pressure, \( x \) is the coordinate according to Figure 1, \( \omega \) is the angular frequency, \( t \) is the time and \( k \) is the trace wave number of the exciting pressure along the plate surface,

\[ k = k_o \sin \theta. \]

\( \theta \) is the angle of incidence according to Figure 1 and \( k_o \) is the wave number in air.

\[ B_1 \frac{\partial^4 v_1}{\partial x^4} - \omega^2 m_1 v_1 - s(v_3 - v_1) = 0 \]

Figure 1. The baffled 2-dimensional plate system.

The behaviour of the plates is described by Kirchoff's plate equation for isotropic, homogeneous, linear elastic plates. For the coupled system, the response is governed by
\[ B_3 \frac{\partial^4 v_3}{\partial x^4} - \omega^2 m_3 v_3 - s(v_1 - v_3) = j \omega \ p(x), \]

(4)

where \( B \) is the bending stiffness, \( m \) is the surface mass and \( v \) is the velocity of the plate, see e.g. reference [10]. As is evident from these equations, the influence of the air surrounding the construction is neglected. The indices 1 and 3 refer to the different plates, see Figure 1. The elastic intermediate layer is modelled as a spring with the spring stiffness as, see reference [10],

\[ s = s_s + s_L = s_s + \frac{D_0 c^2}{d \sigma}, \]

(5)

where \( s_s \) is the stiffness of the “skeleton” of the material and \( s_L \) is the stiffness of the air in the pores of the material. \( s_L \) is determined by the density of air \( \rho_0 \), the velocity of sound in air \( c \), the thickness of the layer \( d \) and the porosity of the enclosed material \( \sigma \). Introducing the wave number of the free bending waves in plates \( k_B \) as

\[ k_B = j \sqrt{\frac{\omega^2 m}{B}}, \]

(6)

together with the Fourier transform of the plate velocity according to

\[ v(k_x) = \int_0^\infty v(x) e^{-j k_x x} dx, \]

(7)

the transformed plate equations become

\[ k_x^4 v_1(k_x) - k_{B1}^4 v_1(k_x) - \frac{s}{B_1} (v_3(k_x) - v_1(k_x)) = 0, \]

(8)

\[ k_x^4 v_3(k_x) - k_{B3}^4 v_3(k_x) - \frac{s}{B_3} (v_1(k_x) - v_3(k_x)) = j \omega \frac{B_3}{B_3} p(k_x). \]

(9)

The transformed velocities have the solution

\[ v_1(k_x) = \frac{j \omega \ p(k_x)}{B_1 B_3 \left( k_x^4 - k_{B1}^4 + \frac{s}{B_1} \left( k_x^4 - k_{B3}^4 + \frac{s}{B_3} \right) - s^2 \right)}, \]

(10)

\[ v_3(k_x) = \frac{j \omega \ p(k_x)}{B_1 B_3 \left( k_x^4 - k_{B1}^4 + \frac{s}{B_1} \left( k_x^4 - k_{B3}^4 + \frac{s}{B_3} \right) - s^2 \right)}. \]

(11)
The Fourier transform of the exciting pressure is defined as

\[ p(k_x) = \int_{-\pi/2}^{\pi/2} p(x) e^{-j(k_x x)} dx, \quad (12) \]

i.e.,

\[ p(k_x) = \hat{p} \frac{e^{-j(k_x L/2)} - je^{j(k_x L/2)}}{(k + k_x)}. \quad (13) \]

The inverse Fourier transform of the velocity is defined as

\[ v(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v(k_x) e^{jk_x x} dk_x, \quad (14) \]

which, after introduction of \( s_1 = s/B_1 \) and \( s_3 = s/B_3 \) gives the plate velocities as

\[ v_1(x) = -\frac{\alpha \hat{p}}{2\pi B_3} \int_{-\infty}^{\infty} \frac{(e^{-j(k_x L/2)} - e^{j(k_x L/2)}) e^{jk_x x}}{(k + k_x)(k_x^4 - k_{B1}^4 + s_1)(k_x^4 - k_{B3}^4 + s_3)} dk_x, \quad (15) \]

\[ v_3(x) = -\frac{\alpha \hat{p}}{2\pi B_3} \int_{-\infty}^{\infty} \frac{(e^{-j(k_x L/2)} - e^{j(k_x L/2)}) e^{jk_x x}}{(k + k_x)(k_x^4 - k_{B1}^4 + s_1)(k_x^4 - k_{B3}^4 + s_3)} dk_x. \quad (16) \]

3. General solution

If the denominator of the velocity in Equations (15) and (16) is set to zero, the poles are found as (see also Figure 2):

\[ k_1 = -k, \]

\[ k_{2,3} = \left( \frac{1}{2} \left( k_{B1}^4 - s_1 \right) + k_{B3}^4 - s_3 \right) \pm \frac{1}{4} \left( \left( k_{B1}^4 - s_1 \right) - \left( k_{B3}^4 - s_3 \right) \right)^2 + s_1 s_3 \right) \]

\[ k_{4,5} = j \left( \frac{1}{2} \left( k_{B1}^4 - s_1 \right) + k_{B3}^4 - s_3 \right) \pm \frac{1}{4} \left( \left( k_{B1}^4 - s_1 \right) - \left( k_{B3}^4 - s_3 \right) \right)^2 + s_1 s_3 \right) \]

\[ k_{6,7} = \left( \frac{1}{2} \left( k_{B1}^4 - s_1 \right) + k_{B3}^4 - s_3 \right) \pm \frac{1}{4} \left( \left( k_{B1}^4 - s_1 \right) - \left( k_{B3}^4 - s_3 \right) \right)^2 + s_1 s_3 \right) \]

\[ k_{8,9} = -j \left( \frac{1}{2} \left( k_{B1}^4 - s_1 \right) + k_{B3}^4 - s_3 \right) \pm \frac{1}{4} \left( \left( k_{B1}^4 - s_1 \right) - \left( k_{B3}^4 - s_3 \right) \right)^2 + s_1 s_3 \right) \] (17a-1)
Figure 2. The poles of the integrand, Equations (15) and (16), and the integration contours $\Gamma_1$ and $\Gamma_2$.

The velocities of the plates are evaluated by using the calculus of residue. In order to ensure that the contributions from the large arc of the integration contour is negligible, different contours must be used for the different regions in the x-direction. The expression for plate 1, region 1 becomes

$$v_{1,1}(x) = \frac{j\omega \hat{p}_s}{B_3} \left[ \frac{e^{-j\lambda x}}{f(k_1)} - \sum_{n=2}^{5} \frac{e^{-(k+n)\ell^2/2 + jk_n x}}{f(k_n)} - \sum_{n=6}^{9} \frac{e^{-(k+n)\ell^2/2 - jk_n x}}{f(k_n)} \right],$$

and for region 2 and 3

$$v_{1,2}(x) = \frac{j\omega \hat{p}_s}{B_3} \left[ \sum_{n=2}^{5} \frac{e^{-(k+n)\ell^2/2 + jk_n x} - e^{-(k+n)\ell^2/2 - jk_n x}}{f(k_n)} \right],$$

$$v_{1,3}(x) = \frac{j\omega \hat{p}_s}{B_3} \left[ \sum_{n=6}^{9} \frac{e^{-(k+n)\ell^2/2 + jk_n x} - e^{-(k+n)\ell^2/2 - jk_n x}}{f(k_n)} \right].$$

Note that the first index refers to the plate number, and the second index to the plate region, see Figure 1. The velocity of plate 3 is found in the same manner. The velocity in region 1 is given by

$$v_{3,1}(x) = \frac{j\omega \hat{p}}{B_3} \left[ \frac{(k_4^4 - k_4^2 + k_4^2 - k_4^4 + s_1) e^{-j\lambda x}}{f(k_1)} + \sum_{n=2}^{5} \frac{(k_n^4 - k_n^2 + k_n^2 - k_n^4 + s_1) e^{j(k+n)\ell^2/2 + jk_n x}}{f(k_n)} + \sum_{n=6}^{9} \frac{(k_n^4 - k_n^2 + k_n^2 - k_n^4 + s_1) e^{-j(k+n)\ell^2/2 - jk_n x}}{f(k_n)} \right],$$

(21)
and in region 2 and 3 by

\[
v_{3,2}(x) = \frac{j \omega \hat{P}}{B_3} \left[ \sum_{n=2}^{5} \frac{\left( n^4 - k_{n}^4 + s_1 \right) e^{-i(k+k_n)\sqrt{2}/2 + jk_s x} - e^{i(k+k_n)\sqrt{2}/2 + jk_s x}}{f(k_n)} \right],
\]

(22)

\[
v_{3,3}(x) = \frac{j \omega \hat{P}}{B_3} \left[ \sum_{n=6}^{9} \frac{\left( n^4 - k_{n}^4 + s_1 \right) e^{-i(k+k_n)\sqrt{2}/2 + jk_s x} - e^{i(k+k_n)\sqrt{2}/2 + jk_s x}}{f(k_n)} \right].
\]

(23)

The denominator is in all cases

\[
f(k_n) = 9k_n^8 + 8kk_n^5 - 5k_n^4 \left[ (k_{n1}^4 - s_1) + (k_{n3}^4 - s_3) \right] - 4kk_n^3 \left[ (k_{n1}^4 - s_1) + (k_{n3}^4 - s_3) \right]
+ (k_{n1}^4 - s_1)(k_{n3}^4 - s_3) - s_1 s_3.
\]

(24)

4. Two identical plates

The general expression is neither neat nor easily overviewed and needs some simplifications to be lucid. In the following some different cases are discussed such as different geometries and frequency regions.

If, for example, two identical plates are used, the poles are reduced to

\[
k_1 = -k, \quad k_2 = -k_B, \quad k_3 = -jk_B, \quad k_4 = jk_B, \quad k_5 = jk_B, \quad k_6 = k_B, \quad k_7 = \sqrt{k_B^4 - 2s_1}, \quad k_8 = -jk_B, \quad k_9 = -jk_B^2 - 2s_1
\]

(25a-i)

and the velocities of the plate 1 and 3 will be

\[
v_{1,3}(x) = \frac{j \omega \hat{P}}{B_3} \left[ \frac{s_1 e^{-jk_s x}}{k^4 - k_B^4} \left( k^4 - k_B^4 + 2s_1 \right) + \frac{e^{-i(k-k_B)\sqrt{2}/2 + jk_s x}}{8(k_B^4 - k^4) + 8(k_B^4 - 2s_1)^{3/2} \left( k - (k_B^4 - 2s_1) \right)^{1/2} + \frac{e^{-i(k-k_B)\sqrt{2}/2 - jk_s x}}{8k_B^4 (k - k_B)} + \frac{e^{-i(k-k_B)\sqrt{2}/2 + jk_s x}}{8k_B^4 (k + k_B)} + \frac{-e^{-i(k-k_B)\sqrt{2}/2 - jk_s x}}{8(k_B^4 - 2s_1)^{3/2} \left( k - (k_B^4 - 2s_1) \right)^{1/2} + \frac{e^{-i(k-k_B)\sqrt{2}/2 + jk_s x}}{8k_B^4 (k + k_B)} + \frac{-e^{-i(k-k_B)\sqrt{2}/2 - jk_s x}}{8k_B^4 (k + k_B)} + \frac{-e^{-i(k-k_B)\sqrt{2}/2 - jk_s x}}{8(k_B^4 - 2s_1)^{3/2} \left( k + (k_B^4 - 2s_1) \right)^{1/2} + \frac{e^{-i(k-k_B)\sqrt{2}/2 + jk_s x}}{8k_B^4 (k + k_B)}} \right] (26)
\]

and
\[ v_{\lambda_1}(x) = \frac{j \omega \rho}{B} \left[ \frac{(k^4 - k_B^4 + 2s_1) e^{ikx}}{8k_B^3(k - k_B)} + \frac{-e^{j(k+B)\sqrt{2} \sigma_{y}\gamma_{z}x}}{8k_B^3(k - k_B)} \right] \]

\[ + \frac{e^{-j(k+B)\sqrt{2} \sigma_{y}\gamma_{z}x}}{8k_B^3(k + k_B)} \frac{-e^{-j(k+B)\sqrt{2} \sigma_{y}\gamma_{z}x}}{8k_B^3(k + k_B)} \]

\[ + \frac{e^{-j(k+B)\sqrt{2} \sigma_{y}\gamma_{z}x}}{8k_B^3(k + k_B)} \frac{-e^{-j(k+B)\sqrt{2} \sigma_{y}\gamma_{z}x}}{8k_B^3(k + k_B)} \]

(27)

The first term in each expression represents the forced wave. In Figure 3 to 5 the normalized RMS-value of different parts of the expressions for the plate velocities,

\[ v_{\text{rms}} = \frac{1}{L} \frac{\rho c}{\rho} \sqrt{\int_{-L/2}^{L/2} \left| v_{\lambda_1}(x) \right|^2 \, dx} \]

(28)

is shown. It is seen that the forced part of the plate velocities have the same magnitude at very low frequencies and also around the mass-spring-mass resonance frequency, but at higher frequencies the velocity of the excited plate is much larger. Studying the denominator of Equations (26) and (27), it is also found that the first two terms in the parenthesis will have a pole at \( k = k_B \), which is the usual coincidence condition. Several other terms have a pole at \( k_B^4 = 2s_1 \), which is the mass-spring-mass resonance condition and which leads to the usual expression for the resonance frequency \( f_0 \). The behaviour of the first term within the parenthesis is interesting. Apart from the pole at \( k = k_B \), there are two other poles due to the term \( k^4 - k_B^4 + 2s_1 \). One of these is given by \( k_B^4 = 2s_1 + k^4 \). For normal incidence, \( k = 0 \), the pole describes the usual mass-spring-mass resonance. However, for oblique incidence the term \( k^4 \) will tend to increase the mass-spring-mass resonance frequency.

The other pole is related to the coincidence phenomenon. If the stiffness of the intermediate layer is small, the expression \( k^4 = k_B^4 - 2s_1 \) tends to \( k_B^4 = k^4 \) which leads to the usual expression for coincidence. With a finite stiffness the coincidence relation must be written as \( k^4 = k_B^4 - 2s_1 \), which shows that the stiffness tends to decrease the coincidence frequency. Note that the forced part of the solution for plate 3 also has an anti-resonance at \( f_0 / \sqrt{2} \).
Figure 3. The forced part of $v_1$ and $v_3$, i.e., the first term in Equations (26) and (27). (---) $v_3$, (——) $v_1$. The example is calculated for two identical plates of 5 cm concrete with a 2 cm cavity filled with mineral wool, where the structural stiffness $s_4$ (see Equation (5)) is disregarded. The concrete plates have a Young's modulus of 26 GPa, a density of 2400 kg/m$^3$ and a loss factor of 1%. The example is calculated for an angle of incidence $\theta = 30^\circ$. The mass-spring-mass resonance $f_0 = 54$ Hz and the coincidence frequency $f_c = 370 / \sin^2(30) = 1476$ Hz. $L = 6$ m is used.

Note that it is only the first terms in Equations (26) and (27) that differ in magnitude. Terms 2 to 9, which include both propagating waves and exponential near-fields, only differ in sign between the corresponding terms. Figure 4 shows term number 2, 4, 6 and 8 within the parenthesis of Equations (26) and (27), in Figure 5 the terms 3, 5, 7 and 9 are shown. It is seen that the odd terms, which include the effects of the stiffness of the spring, have the same magnitude below the mass-spring-mass resonance.
Figure 4. The even parts of Equation (26) and (27). (---) term number 2, (- - -) term number 6, (——) term number 4 and 8. The term number 4 and 8 are indistinguishable for all frequencies. The same material properties as in Figure 3 are used.

Figure 5. The odd parts of Equation (26) and (27). (---) term number 3, (- - -) term number 7, (——) term number 5 and 9. The term number 5 and 9 are indistinguishable for all frequencies. The same material properties as in Figure 3 are used.
In Figure 4 and 5 it is seen that the bending nearfield in each figure are indistinguishable for all frequencies. When \( k = k_B \) the propagating wave of \( v(k) \) will have a maximum since it includes the term \((k-k_B)\) in its denominator. The propagating wave of \( v(k) \) will not show this resonance since the denominator in this case has opposite sign for \( k_B \), i.e. the factor reads \((k+k_B)\).

4.1 Further simplifications

Well below \( f_0 \) the plates will behave as a single plate system with the surface mass \( m = m_1+M_\gamma \). If it is assumed that the plate system still can be regarded as thin the velocity according to Equation (15) can be reduced to

\[
v(x) = -\frac{1}{2\pi B} \int_{-\infty}^{\infty} \frac{e^{-(i(k-k_B)\nu/2) + e^{i(k-k_B)\nu/2}}}{(k+k_B)(k^2-k_B^2)} \, dk,
\]

(31)

which gives

\[
v(x) = \frac{j\omega \hat{p}}{B} \left[ e^{jkx} + e^{-(k-k_B)\nu/2} + e^{-(k-k_B)\nu/2} + e^{i(k-k_B)\nu/2} + e^{i(k-k_B)\nu/2} + e^{i(k-k_B)\nu/2} + e^{i(k-k_B)\nu/2} \right]
\]

(32)

At frequencies above \( f \), the bending wavelength \( k_B \) will be much larger than \( s/B \) hence the term \( s/B \) can be neglected. The expression for the velocity of plate 1 is then simplified to

\[
v_1(x) = -\frac{1}{2\pi B_3} \int_{-\infty}^{\infty} \frac{e^{-(i(k-k_B)\nu/2) + e^{i(k-k_B)\nu/2}}}{(k+k_B)(k^2-k_B^2)} \, dk
\]

(33)

where the integral again can be evaluated by use of the calculus of residues.

The wave numbers are given by

\[
k_1 = -k, \quad k_2 = -k_B, \quad k_3 = -k_B, \quad k_4 = jk_B, \quad k_5 = jk_B
\]

\[
k_6 = k_B, \quad k_7 = k_B, \quad k_8 = -j k_B, \quad k_9 = -j k_B
\]

(34a-i)

The velocity of plate 1 region 1 becomes

\[
v_{11}(x) = \frac{j\omega \hat{p} s}{B_1 B_3} \left[ e^{jkx} + e^{i(k-k_B)\nu/2} + e^{i(k-k_B)\nu/2} + e^{i(k-k_B)\nu/2} + e^{i(k-k_B)\nu/2} + e^{i(k-k_B)\nu/2} + e^{i(k-k_B)\nu/2} + e^{i(k-k_B)\nu/2} \right]
\]

\[
\times \left[ e^{-(k-k_B)\nu/2} + e^{-(k-k_B)\nu/2} + e^{-(k-k_B)\nu/2} + e^{-(k-k_B)\nu/2} + e^{-(k-k_B)\nu/2} + e^{-(k-k_B)\nu/2} + e^{-(k-k_B)\nu/2} + e^{-(k-k_B)\nu/2} \right]
\]

\[
+ \frac{4k_1^3(k_B^4-k_B^4)(k_B-1)}{4k_1^3(k_B^4-k_B^4)(k_B-1)} + \frac{4k_3^3(k_B^4-k_B^4)(k_B-jk)}{4k_3^3(k_B^4-k_B^4)(k_B-jk)} + \frac{4k_5^3(k_B^4-k_B^4)(k_B+1)}{4k_5^3(k_B^4-k_B^4)(k_B+1)}
\]
\[ + \frac{e^{-j(k_3+k_4)x}}{4k_{3}^3(k_{3}^4-k_4^4)(k_{3}+k)} + \frac{e^{-j(k_3+k_4)L/2+jk}}{4k_{3}^3(k_{3}^4-k_4^4)(k_{3}+j)} + \frac{e^{-j(k_3+k_4)L/2+jk}}{4k_{3}^3(k_{3}^4-k_4^4)(k_{3}-jk)} \] \]

(35)

The expression for the velocity of plate 3 can in the same way be simplified to

\[ v_3(x) = -\frac{\omega}{2\pi B_3} \int e^{-j(k_3+k_4)L/2} \frac{e^{-jkx}}{(k+k)(k^4-k^4_3)} dk \] \[ \int_{-\infty}^{\infty} \frac{e^{-j(k_3+k_4)L/2} e^{jkx}}{(k+k)(k^4-k^4_3)} dk \]

(36)

i.e. the expression as in Equation (33).

5. Discussion

The relative magnitude of the terms, which describe the different waves of the solution, was discussed in conjunction with Equation (26) and (27). However, also the phase relationships are interesting. It is seen that no simple phase relationship exists between the forced waves, that is, the first term within the parenthesis, due to the factor \((k^4-k^4_3+\omega^2)\) in Equation (27). This factor depends not only on the frequency and the material parameters but also on the angle of incidence.

The behavior of the other waves are, on the other hand, very simple to describe. Thus, all the waves traveling with the wave number \(\pm k_3\) and the corresponding near-fields, are in phase. The motion of the two plates is then anti-symmetric with respect to the neutral plane of the double plate system.

It is also seen that the waves travelling with the wave number \(\pm (k_3^4-2\omega_3)^{1/4}\), and also the corresponding near-fields, are out of phase. The motion of the double plate system is then symmetric with respect to the neutral plane.

The expression for the velocity derived for region 1 is related to the non-resonant part of the sound transmission through the plate. In case of a finite plate, the waves emerging from the boundaries of region 1, i.e. at \(x = \pm L/2\), will be reflected at the plate boundaries and build up a non-forced field in the plate. If some conditions are fulfilled, this field will be reverberant and cause a reverberant transmission through the plate. The velocity of the propagating parts of \(v_{1,3}(L/2)\) and \(v_{1,3}(L/2)\) is used for calculation the resonant part of the sound reduction index as further discussed in the companion article[1].
6. Acknowledgements

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7. References


Acoustic Properties of Double Plate Systems

Part II: The Sound Reduction Index of Finite-Size Plates

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Abstract
An analytical model for predicting the sound reduction index of a finite size double plate system is presented. The plates are assumed to be excited by a diffuse sound field and the velocity distributions of the plates are derived from the Kirchhoff thin plate theory. The intermediate layer is modelled as a locally reacting spring. The model is valid and continuous through both the mass-spring-mass resonance and the coincidence region. The results from the model show good agreement with measured results for chipboard and plasterboard partitions. Calculated results, as well as measured ones, show that the sound reduction index decreases with increasing area in the whole frequency range.

1. Introduction
The development of the present model is prompted by the need to predict the sound reduction index of double plate systems. The model is derived for a mass-spring-mass system, i.e. two plates with an elastic interlayer that can be modelled as a spring. The plates are assumed to be thin, so that the Kirchhoff thin plate theory is valid, i.e. with a maximum plate thickness of $\lambda_B / 6$, where $\lambda_B$ is the wavelength of the bending wave. The complete solution of the plate velocities used are presented in a companion article[1].

2. Plate velocity
Consider two parallel plates with an intermediate elastic layer. The plates are infinitely large but baffled outside the excited region. Plate 3 is excited by an airborne sound field. The incident and the reflected wave act on the plate with a pressure
\[ p(x) = \hat{p} e^{-j k x} e^{j \omega t}, \]  
\[ (1) \]

where \( \hat{p} \) is the magnitude of the pressure, \( x \) is the coordinate according to Figure 1, \( \omega \) is the angular frequency, \( t \) is the time and \( k \) is the trace wave number of the exciting pressure along the plate surface,

\[ k = k_n \sin \theta. \]  
\[ (2) \]

\( \theta \) is the angle of incidence according to Figure 1 and \( k_n \) is the wave number of sound in air.

![Figure 1. The baffled 2-dimensional plate system.](image)

The behaviour of the plates are described by Kirchoff's plate equation in two dimensions. This two-dimensional approach is valid as long as the plates are isotropic, homogeneous and linearly elastic. The elastic intermediate layer is modelled as a locally reacting spring with the spring stiffness as, see Cremer et al.\(^{[2]}\)

\[ s = s_s + s_L = s_s + \frac{\rho_0 c^2}{d \sigma}, \]  
\[ (3) \]

where \( s_s \) is the stiffness of the “skeleton” of the material and \( s_L \) is the stiffness of the air in the pores of the material. \( s_L \) is determined by the density of air \( \rho_0 \), the velocity of sound in air \( c \), the thickness of the layer \( d \) and the porosity of the enclosed material \( \sigma \). The spring model can be assumed to be valid up to \( d < \frac{\lambda_0}{4} \). In order to cope with the finite size excitation area, a Fourier transform approach is used, see reference [1] for details, which gives the transformed velocity of plate 1 as
\[ v_i(k_x) = \frac{j \omega \rho(k_x) s}{B_1 B_3} \frac{\left( k_x^4 - k_{B1}^4 + \frac{s}{B_1} \right) \left( k_x^4 - k_{B3}^4 + \frac{s}{B_3} \right)}{s^2}, \]  

(4)

where the Fourier transform of the exciting pressure is given by

\[ p(k_x) = \hat{p} \frac{e^{-j(k+k_x)L/2} - e^{(k+k_x)L/2}}{(k + k_x)}. \]

(5)

The velocity of the plates are evaluated in reference [1] by use of contour integration. Thus, the inverse transform for the velocity of plate 1 in region 1, i.e. from -L/2 to L/2 was obtained as

\[ v_i(x) = \frac{j \omega \hat{p} \rho s_1}{B_3} \frac{\left( e^{-j(k_x)L/2} + \sum_{n=2}^{5} e^{(k_x + k_x \beta) L/2 + j \beta_x} \right)}{f(k_x)} + \sum_{n=6}^{9} \frac{e^{-(k_x + k_x \beta) L/2 - j \beta_x}}{f(k_x)}, \]

(6)

while for region 2, x > L/2, and region 3, x > -L/2, the expressions for the velocities become

\[ v_2(x) = \frac{j \omega \hat{p} s_1}{B_3} \left( \sum_{n=2}^{5} \frac{e^{-j(k_x + k_x \beta) L/2 - j \beta_x}}{f(k_x)} \right), \]

(7)

\[ v_3(x) = \frac{j \omega \hat{p} s_1}{B_3} \left( \sum_{n=6}^{9} \frac{e^{-j(k_x + k_x \beta) L/2 - j \beta_x}}{f(k_x)} \right), \]

(8)

respectively. The wave numbers of the plate waves are, see reference [1],

\[ k_1 = -k, \]

\[ k_{2,3} = \left( \frac{1}{2} \left( k_{B1}^4 - s_1 \right) + \left( k_{B3}^4 - s_3 \right) \right) \pm \frac{1}{4} \sqrt{\left( \left( k_{B1}^4 - s_1 \right) - \left( k_{B3}^4 - s_3 \right) \right)^2 + 8s_1 s_3} \right)^{\frac{1}{4}}, \]

\[ k_{4,5} = j \left( \frac{1}{2} \left( k_{B1}^4 - s_1 \right) + \left( k_{B3}^4 - s_3 \right) \right) \pm \frac{1}{4} \sqrt{\left( \left( k_{B1}^4 - s_1 \right) - \left( k_{B3}^4 - s_3 \right) \right)^2 + 8s_1 s_3} \right)^{\frac{1}{4}}, \]

\[ k_{6,7} = \left( \frac{1}{2} \left( k_{B1}^4 - s_1 \right) + \left( k_{B3}^4 - s_3 \right) \right) \pm \frac{1}{4} \sqrt{\left( \left( k_{B1}^4 - s_1 \right) - \left( k_{B3}^4 - s_3 \right) \right)^2 + 8s_1 s_3} \right)^{\frac{1}{4}}, \]

\[ k_{8,9} = -j \left( \frac{1}{2} \left( k_{B1}^4 - s_1 \right) + \left( k_{B3}^4 - s_3 \right) \right) \pm \frac{1}{4} \sqrt{\left( \left( k_{B1}^4 - s_1 \right) - \left( k_{B3}^4 - s_3 \right) \right)^2 + 8s_1 s_3} \right)^{\frac{1}{4}}, \] (9a-i)
where \( s_1 = s/B_1, s_3 = s/B_3 \). The common denominator is

\[
f(k_n) = 9k_n^8 + 8kk_n^3 - 5k_n^4 \left[ (k_{B_1}^4 - s_1) + (k_{B_3}^4 - s_3) \right] - 4kk_n^3 \left[ (k_{B_1}^4 - s_1) + (k_{B_3}^4 - s_3) \right] \\
+ \left( k_{B_1}^4 - s_1 \right) \left( k_{B_3}^4 - s_3 \right) s_1s_3.
\] (10)

### 3. The sound reduction index

The sound reduction index of the construction consists of a forced part and a reverberant part. The forced part is directly caused by the excitation and has the same \( x \)-dependence as the forcing pressure. The waves propagating away in both directions from the perimeter of the excited area will create a reverberant field. If the plate is no longer considered infinite, these waves are reflected at the plate boarders and cause a reverberant field in the excited part of the plate (region 1) as well.

#### 3.1 Forced transmission

The sound reduction index of a construction is defined as

\[
R = 10 \log_{10} \frac{1}{\tau},
\] (11)

where \( \tau \) is the sound transmission factor given by

\[
\tau(\theta) = \frac{P(\theta)}{P_{in}(\theta)},
\] (12)

and where \( P(\theta) \) is the radiated power. \( P_{in}(\theta) \) is the input power which in this case can be written as

\[
P_{in}(\theta) = p^2 L \cos(\theta) \frac{\cos(\theta)}{8 \rho c}.
\] (13)

The radiated power can be calculated directly from the transform velocity as (Cremer et al. p. 528, eq. 61\[2] )

\[
P(\theta) = \frac{\rho c k_1}{4\pi} \operatorname{Re} \left\{ \int_{-\infty}^{\infty} \frac{v(k_z) v^*(k_z)}{\sqrt{k_1^2 - k_z^2}} dk_z \right\} = \frac{\rho c k_1}{4\pi} \int_{-\infty}^{\infty} \frac{|v(k_z)|^2}{\sqrt{k_1^2 - k_z^2}} dk_z.
\] (14)
Using the expression for \( v_j(k_j) \) according to Equation (6), the radiated power becomes

\[
P(\theta) = \frac{\rho c k_x}{4\pi} \left| \frac{j_0 s_1 p}{B_3} \left[ \int k_x \frac{1}{\sqrt{k_x^2 - k_y^2}} \right] \right|^2 \frac{-j e^{i(k_x + k_y)/2} + j e^{-i(k_x + k_y)/2}}{\left( (k_x^2 - k_y^2) + s_1 \right)} dk_x \], \tag{15}

The term forced response is here used for convenience. As is evident from Equation (6), the expression for the velocity in region 1 consists not only of the forced response (the wave with wave number \(-k\)) but also of free waves emanating from the boundaries of the excited area. Two of these waves are propagating in the positive \( x \)-direction but with different wave numbers. There is also a corresponding pair of exponential near-fields. These two pairs are excited at the boundary \( x = -L/2 \). In addition, there are two pairs of waves in the negative \( x \)-direction. These are, of course, excited at the boundary \( x = +L/2 \). Similar phenomena occur in the case of a single plate, excited over a part of its area. For that case, it has been shown (Ljunggren\cite{4}) that the contribution of the waves excited at the edges is of little importance for the radiated sound. It has not yet been checked, but it is thought that the same conclusion can be drawn for the waves under discussion here because of their small radiation factors compared with that of the forced wave. Thus, the contribution to the transmission factor derived in this way is mainly determined by the forced response.

It is now suitable to rewrite the two-dimensional problem into three dimensions. This is here achieved by changing the length \( L \) for the mean free path according to Kost\[5\] as

\[
L_m = \frac{\pi S}{U} \tag{16}
\]

where \( S \) is the area of the excited part of the plate and \( U \) is the perimeter of the area. In the work by Ljunggren\cite{5} a mean projected trace length was derived as \( L_m = (2S/\pi)^{1/3} \) as the mean value of \( \sqrt{L^2} \). This was used instead of Kost's mean free path for frequencies above \( f_c \) where the transmission was proportional to \( L^2 \). In the following, all calculations in this article is based on "\( L \)" as \( L_m \) according to Eq. (16). The reason for this that the transmission factor presented in this work is proportional to \( L \) rather than \( L^2 \), see the discussion in the companion article\cite{1}.
To simulate an excitation corresponding to a diffuse sound field, the sound transmission is the integrated over the angle of incidence by use of Paris' formula

$$\tau_f = 2 \int_0^\pi \hat{\tau}(\theta) \sin(\theta) \cos(\theta) d\theta.$$  \hspace{1cm} (17)

### 3.2 Resonant transmission

The excited area is still denoted $S$, while the area of the whole plate is denoted $S_{tot}$. The waves propagating away from the perimeter of the excited area will be reflected at the boundaries of the plate. For simplicity, it is assumed that the coupling between the parts of all wave pairs (one in plate 1 and the other in plate 3) is broken up at the boundaries. Thus, it follows that the reverberant field created by the reflected waves from the boundaries travel with a wave number $k_B$ and that the mean square velocity is given by

$$\langle v_z^2 \rangle = \frac{P_B}{\cos S_{tot}}.$$  \hspace{1cm} (18)

In the case of plane wave excitation the power travelling away in one direction from the line $x = L/2$ can be written as (compare Cremer et. al\textsuperscript{[3]}, p. 109, Ljunggren\textsuperscript{[4]})

$$P_B = 2c_B m \left< v_z^2 \right> U \langle \cos(\psi) \rangle,$$  \hspace{1cm} (19)

where $U$ is the length of the perimeter and $v_z(L/2)$ is calculated according to expression (7). Only the propagating parts of the velocity should be included, of course. For the derivation of Equation (19) it has been assumed that the group velocity is $2c_B$. This is a simplification which should work well at high frequencies as $k_w$ then tend to $\pm k_B$, see reference [1]. The accuracy of the assumptions at lower frequencies is not known, due to the complications of the group velocities from Equation (9).

So far the solution is valid for the case of an excitation perpendicular to the plate. This will later be integrated over the angle of incidence $\theta$, as in the case with non-reverberant transmission in order to correspond to an excitation of a diffuse sound field. Further, in case of oblique incidence the free waves emerge from the boundary at an angle that may differ from that of the exciting wave. The direction of the free waves is obtained from the trace wave number of the forcing wave along the boundary together with the free bending wave number,
\[ k_y \sin \psi = k \sin \phi, \quad (20) \]

where \( \psi \) is the angle between the direction of the free waves and the boundary normal, and \( \Phi \) is the corresponding angle of the forcing wave, see Figure 4.

The mean value of \( \cos(\psi) \) can now be taken as (Ljunggren\(^{41}\))

\[
\langle \cos(\psi) \rangle = \frac{2}{\pi} \int_0^\pi \left[ I - \left( k \sin \phi / k_y \right)^2 \right] \frac{1}{\Phi} d\phi. \tag{21}
\]

![Figure 4](image.png)

**Figure 4.** \( \psi \) is the angle between the direction of the free waves and the boundary normal, and \( \Phi \) is the corresponding angle of the forcing wave. \( \theta \) is the angle of incidence.

The transmission due to the reverberant field is defined as the ratio between the radiated power and the input power,

\[
\tau = \frac{P_{rad}}{P_{in}}, \tag{22}
\]

where \( P_{in} \) is calculated according to Equation (13) above. \( P_{rad} \) is taken as

\[
P_{rad} = \langle v^2 \rangle S \rho c \sigma, \tag{23}
\]

where \( S \) is the area of radiating part of the plate and \( \sigma \) is the radiation factor. For the examples shown later, the radiation factor was calculated according to Leppington et al.\(^{5}\). Thus, the sound transmission factor due to the resonant field becomes

\[
\tau_{res} = \frac{2}{\pi} \int_0^\pi \frac{US}{\rho c} \frac{\langle v^2 \rangle}{k_y} \frac{\sigma \langle \cos(\psi) \rangle \sin(\theta)}{d\theta}. \tag{24}
\]

The total sound reduction index is then calculated as

\[
R_{tot} = -10 \log(\tau_{res} + \tau_f). \tag{25}
\]
4. Comparison between measured and calculated results

Measurements were made in order to show the area dependency of the sound reduction index. A test wall consisting of two chipboards, 16 mm and 12 mm thick, with a 35 mm layer of soft mineral wool in between. The density of the chipboard plates is 626 kg/m³. The area of the 16 mm plate is 2.42 m by 2.44 m, the area of the 12 mm plate is 2.1 m by 2.1 m. To achieve a simply supported mounting the plates were hanging in rubber bands from the ceiling. The edges of the plates are covered by baffles. The baffles are made of 400 mm mineral wool with two 12 mm plasterboard on each side. The plates are not in contact with the baffles, and the gaps between chipboard and plasterboard are sealed with tape.

![Diagram of the chipboard wall](image)

Figure 5. The chipboard wall.

The test wall was mounted between two reverberation chambers. The measurements were performed according to ISO standard[^6].

The measured results show that in the mid frequency range the sound reduction index improves approximately 1 dB for an aperture of 1.7 m² compared to 4.4 m², whereas for higher frequencies the sound reduction index gains up to 4 dB.

The loss factor and Young’s modulus was measured on a small sample of the chipboard and found to be approximately 4% and 1.5GPa respectively. In the present case the total plate area is larger than the exposed area. Since the vibrations will be distributed over the whole plate, the measured loss factor is corrected with the area factor $S_{tot}/S$. Hence the effective loss factor will be
\[ \eta_{\text{effective}} = \eta_{\text{measured}} \frac{S_{\text{tot}}}{S}. \] (26)

Still, the loss factor must not exceed the limit \( \eta_{\text{eq}} \), see Ljunggren\(^4\).

![Graph showing sound reduction index R vs frequency f (Hz)](image)

**Figure 6.** The chipboard partition. Measured results compared with calculated results.

- (---) Measured 1.7 m², (-----) Measured 4.4 m²,
- (-----) Calculated 1.7 m², (-----) Calculated 4.4 m².

*Note, the calculated values are only valid to 2800 Hz, since the spring model is only valid up to \( d < \lambda/4 \).*

Calculated results are also compared with measurement results for a plasterboard partition, see Figure 8. These results were published by M.M. Stani et al.\(^7\). The test wall consists of two 3.70 x 2.80 m² single layer plasterboards, attached to studs c/c 625 mm on each side of a 115 mm cavity, see Figure 7. The cavity is filled with 100 mm rock wool with a density of 29.9 kg/m³. As the cavity is rather wide the prediction model is only valid up to 740 Hz.
The Young's modulus of the plasterboards is estimated to 2.5 GPa and the surface weight to 9 kg/m². In Novak[8] the material data for the Gyproc plasterboard "normal" is given as 2.7 GPa lengthwise and 2.3 GPa crosswise, hence the mean value of 2.5 GPa was used.

In the plasterboard case, the loss factor was not measured. The sound reduction measurements followed standard laboratory setup, where the plates are more or less rigidly mounted to the boundaries, and $S = S_{nr}$. Hence, for the calculated result of M.M. Stani et. al.[7] the loss factor recommended in ISO standard[6] is used, i.e.
\[ \eta = \eta_{int} + \frac{m}{485\sqrt{f}} \] 

(27)

where \( \eta_{int} \) is set to 1%.

![Graph showing sound reduction index vs frequency]

*Figure 9. The plasterboard wall. (-----)Calculated total sound reduction index, (----) Forced part of the sound reduction index, (----) Resonant part of the sound reduction index.*

In Figure 9 the resonant and non-resonant parts of the sound transmission are shown separately.

### 5. Discussion

In the case of plasterboard partition, Figure 8, the calculated and measured results show an excellent agreement for frequencies above \( f_0 \) up to the limiting frequency above which the model is no longer valid. However, at and around \( f_0 \), the results differ. The analytical model assumes mounting of the plate without studs whereas the measured plates are mounted with studs. It is thought that the influence of the studs may well cause the discrepancy between the calculated and measured curve.

In Figure 6 as well, the results differ around \( f_0 \) where the prediction model tends to underestimate the sound reduction index. However, the discrepancy is
much smaller than in the previous case. It is interesting to note that the plates were not mounted with studs in the experiments. The discrepancy at higher frequencies is thought to be caused by flanking transmission.

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7. References


