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Nonlinear dynamics of large amplitude modes in a magnetized plasma

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We derive two equations describing the coupling between electromagnetic and electrostatic oscillations in one-dimensional geometry in a magnetized cold and non-relativistic plasma. The nonlinear interaction between the wave modes is studied numerically. The effects of the external magnetic field strength and the initial electromagnetic polarization are of particular interest here. New results can, thus, be identified. © 2014 AIP Publishing LLC.

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I. INTRODUCTION

Large amplitude electron plasma oscillations are in most cases treated by means of perturbation methods. However, there are a few particular cases for which exact analytical solutions can be found, e.g., Ref. 1–5. In such schemes, one first makes an Ansatz, often by trial and error, on the spatial behavior of the physical variables, such that the partial differential equations (PDE:s) reduce to a system of coupled nonlinear ordinary differential equations for the temporal evolutions of the wave amplitudes. Such systems (see also Refs. 6–8) can be very useful, in particular, in comparisons with more general, although approximate, PDE:s derived by other techniques.

Recently, we considered wave propagation in a cold plasma.⁹ In that case, we had, due to mathematical difficulties, to assume that the plasma was unmagnetized in its equilibrium state. In the present paper, we have however been able to consider wave propagation in a magnetized plasma for the case, where the waves propagate in the direction of a constant magnetic field $B_0\hat{z}$. In this way, it has been possible for us to derive a generalized system of coupled ordinary differential equations. The presence of the external magnetic field leads to a richer dynamics of the coupled system, as is shown numerically.

II. BASIC EQUATIONS AND DERIVATIONS

Let us start from the basic equations for a cold non-relativistic electron plasma. We then have

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0, \quad (1)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\frac{e}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (3)$$

$$\nabla \times \mathbf{B} = -e\mu_0 n\mathbf{v} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}, \quad (4)$$

and

$$\nabla \cdot \mathbf{E} = -\frac{e(n - n_0)}{\epsilon_0}. \quad (5)$$

Here, n is the electron number density, n_0 is the constant ion number density, \mathbf{v} is the electron fluid velocity, \mathbf{E} is the electric field, \mathbf{B} is the magnetic field, $-e/m$ is the electron charge to mass ratio, μ_0 is the magnetic vacuum permeability, and c is the speed of light. Next, we consider one-dimensional spatial variations in the z -direction, i.e., $\nabla \rightarrow \hat{z}\partial/\partial z$. Furthermore, we make the Ansatz $n = n(t)$, $\mathbf{B} = B_x(t)\hat{x} + B_y(t)\hat{y} + B_0\hat{z}$, $\mathbf{v} = (u_x(t)\hat{x} + u_y(t)\hat{y} + u_z(t)\hat{z})z$, and $\mathbf{E} = (\epsilon_x(t)\hat{x} + \epsilon_y(t)\hat{y} + \epsilon_z(t)\hat{z})z$. Substituting this into Eqs. (1)–(4), we obtain

$$u_z = -\frac{1}{n} \frac{\partial n}{\partial t}, \quad (6)$$

$$\frac{\partial u_z}{\partial t} + u_z^2 = -\frac{e}{m} (\epsilon_z + u_x B_y - u_y B_x), \quad (7)$$

$$\frac{\partial u_x}{\partial t} + u_z u_x = -\frac{e}{m} (\epsilon_x + u_y B_0 - u_z B_y), \quad (8)$$

$$\frac{\partial u_y}{\partial t} + u_z u_y = -\frac{e}{m} (\epsilon_y - u_x B_0 + u_z B_x), \quad (9)$$

$$\epsilon_y = \frac{\partial B_x}{\partial t}, \quad (10)$$

$$\epsilon_x = -\frac{\partial B_y}{\partial t}, \quad (11)$$

$$u_x = \frac{\epsilon_0}{en} \frac{\partial \epsilon_x}{\partial t} = -\frac{\epsilon_0}{en} \frac{\partial^2 B_y}{\partial t^2}, \quad (12)$$

$$u_y = \frac{\epsilon_0}{en} \frac{\partial \epsilon_y}{\partial t} = \frac{\epsilon_0}{en} \frac{\partial^2 B_x}{\partial t^2}, \quad (13)$$

and

$$\epsilon_z = -\frac{e(n - n_0)}{\epsilon_0}. \quad (14)$$

Equations (6), (12), and (13) together with (8) and (9) yield

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$$\begin{aligned} & \frac{\partial}{\partial t} \left(-\frac{\varepsilon_0}{en} \frac{\partial^2 B_y}{\partial t^2} \right) + \left(-\frac{1}{n} \frac{\partial n}{\partial t} \right) \left(-\frac{\varepsilon_0}{en} \frac{\partial^2 B_y}{\partial t^2} \right) \\ &= -\frac{e}{m} \left(-\frac{\partial B_y}{\partial t} + \frac{\varepsilon_0}{en} \frac{\partial^2 B_x}{\partial t^2} B_0 + \frac{1}{n} \frac{\partial n}{\partial t} B_y \right) \end{aligned} \quad (15)$$

and

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\frac{\varepsilon_0}{en} \frac{\partial^2 B_x}{\partial t^2} \right) + \left(-\frac{1}{n} \frac{\partial n}{\partial t} \right) \left(\frac{\varepsilon_0}{en} \frac{\partial^2 B_x}{\partial t^2} \right) \\ &= -\frac{e}{m} \left(\frac{\partial B_x}{\partial t} + \frac{\varepsilon_0}{en} \frac{\partial^2 B_y}{\partial t^2} B_0 - \frac{1}{n} \frac{\partial n}{\partial t} B_x \right). \end{aligned} \quad (16)$$

Similarly, substituting (6), (12)–(14) into (7) gives

$$\begin{aligned} & -\frac{\partial}{\partial t} \left(\frac{1}{n} \frac{\partial n}{\partial t} \right) + \left(\frac{1}{n} \frac{\partial n}{\partial t} \right)^2 \\ &= \frac{e^2}{m} \left[\frac{(n - n_0)}{\varepsilon_0} + \frac{\varepsilon_0}{ne^2} \left(B_y \frac{\partial^2 B_y}{\partial t^2} + B_x \frac{\partial^2 B_x}{\partial t^2} \right) \right], \end{aligned} \quad (17)$$

where n_0 is the constant background density.

Next, we introduce normalized variables according to $t \rightarrow \omega_p t$, $n \rightarrow N = n/n_0$, and $B_{x,y} \rightarrow eB_{x,y}/m\omega_p$, where $\omega_p = (n_0 e^2 / \varepsilon_0 m)^{1/2}$. Moreover, we introduce a complex magnetic field defined by $B_+ = B_x + iB_y$. The three Eqs. (15)–(17) can then be rewritten as two coupled equations

$$\begin{aligned} & -\frac{\partial}{\partial t} \left(\frac{1}{N} \frac{\partial N}{\partial t} \right) + \left(\frac{1}{N} \frac{\partial N}{\partial t} \right)^2 \\ &= (N - 1) + \frac{\omega_c}{N} \left(\text{Im} B_+ \frac{\partial^2 (\text{Im} B_+)}{\partial t^2} + \text{Re} B_+ \frac{\partial^2 (\text{Re} B_+)}{\partial t^2} \right) \end{aligned} \quad (18)$$

and

$$\frac{\partial}{\partial t} \left(\frac{1}{N^2} \frac{\partial^2 B_+}{\partial t^2} \right) + i \frac{\omega_c}{N^2} \frac{\partial^2 B_+}{\partial t^2} = -\frac{\partial}{\partial t} \left(\frac{B_+}{N} \right), \quad (19)$$

where $B_+ = \text{Re} B_+ + i \text{Im} B_+$ and the dimensionless quantity ω_c is given by $\omega_c = eB_0/m\omega_{p0}$. Equation (18) describes electrostatic density oscillations driven by electromagnetic (EM) modes, and Eq. (19) describes the influence of density perturbations on the left- and right-hand polarized modes. In order to illustrate the electromagnetic polarizations, we let $N \rightarrow 1$ and consider the linearized version of (19). Integrating the resulting equation once and taking the integration constant as zero, we obtain

$$\left(\frac{\partial^2}{\partial t^2} + i\omega_c \frac{\partial}{\partial t} + 1 \right) B_+ = \left(\frac{\partial}{\partial t} + i\omega_L \right) \left(\frac{\partial}{\partial t} - i\omega_R \right) B_+, \quad (20)$$

where we have introduced the frequency of the left hand mode $\omega_L \equiv \sqrt{1 + \omega_c^2/4} + \omega_c/2$ and of the right hand mode $\omega_R \equiv \sqrt{1 + \omega_c^2/4} - \omega_c/2$. In the absence of electromagnetic fields, $B_+ = 0$, Eq. (18) has a nonlinear solution for N

$$N(t) = \frac{(1 + \Delta)}{1 + \Delta - \Delta \cos(t)}, \quad (21)$$

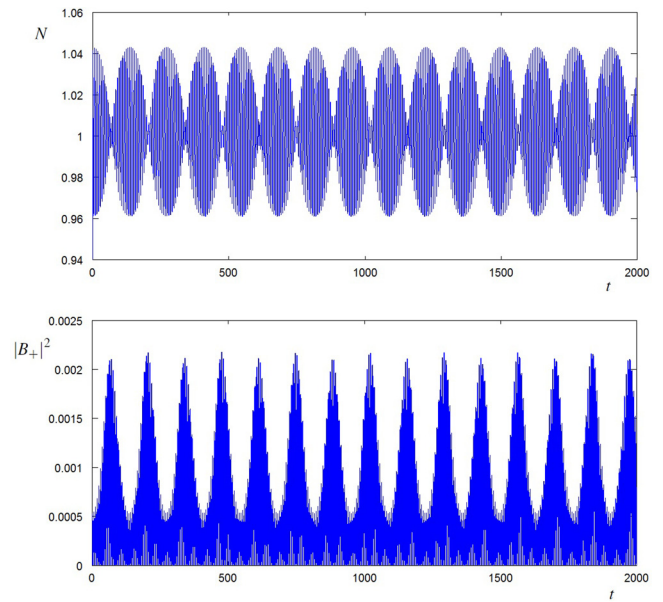


FIG. 1. The density N (upper panel) and the (wave field) magnetic energy density $|B_+|^2$ (lower panel) plotted as a function of time for $\omega_c = 0$. The initial conditions are $N = 1.04$, $dN/dt = 0$, $B_+ = 0$, $dB_+/dt = 0.02$, $d^2 B_+/dt^2 = 0$ resulting in linearly polarized electromagnetic fields.

where Δ is a parameter describing the initial electron density perturbation, or $\Delta = N(0) - 1$.⁹ Furthermore, linearizing Eq. (18) immediately gives the normalized eigenfrequency $\omega = 1$ for the electrostatic mode. The rest of the manuscript is devoted to a numerical study of the full system (18) and (19).

III. NUMERICAL RESULTS

Even for fairly modest initial amplitudes $N - 1 \sim 0.04$, the nonlinear behavior becomes apparent when the long time

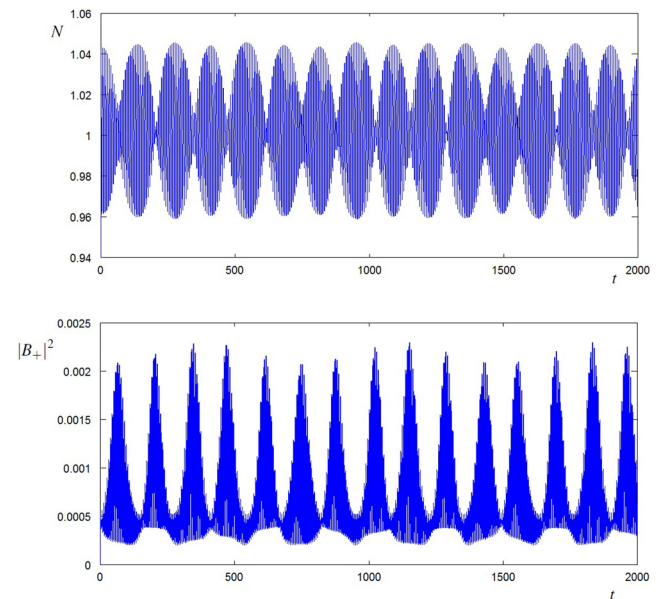


FIG. 2. The density N (upper panel) and the (wave field) magnetic energy density $|B_+|^2$ (lower panel) plotted as a function of time for $\omega_c = 0$. The initial conditions are $N = 1.04$, $dN/dt = 0$, $B_+ = 0$, $dB_+/dt = 0.02$, $d^2 B_+/dt^2 = 0.02i$ resulting in a time dependent electromagnetic polarization.

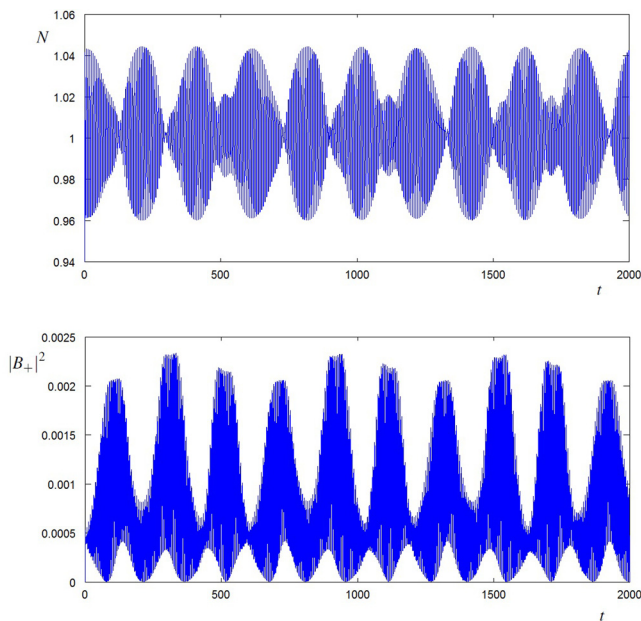


FIG. 3. The density N and the (wave field) magnetic energy density $|B_+|^2$ plotted as a function of time for $\omega_c = 0.05$. The initial conditions are $N = 1.04$, $dN/dt = 0$, $B_+ = 0$, $dB_+/dt = 0.02$, and $d^2B_+/dt^2 = 0.02i$.

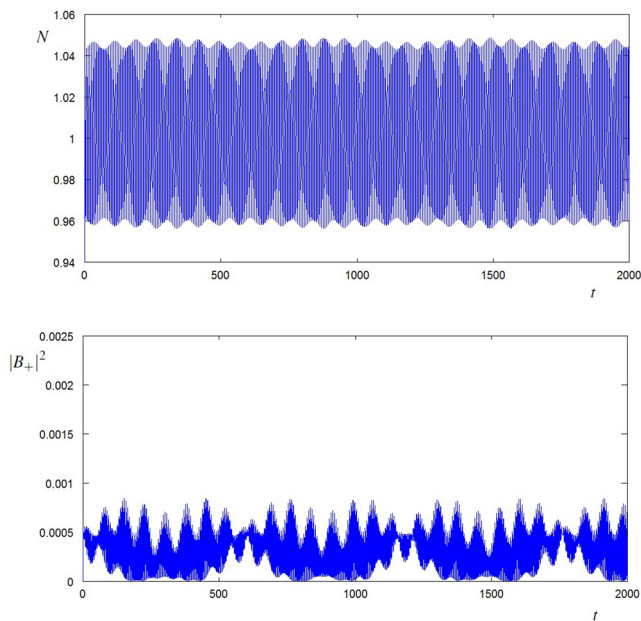


FIG. 4. The density N and the (wave field) magnetic energy density $|B_+|^2$ plotted as a function of time for $\omega_c = 0.2$. The initial conditions are $N = 1.04$, $dN/dt = 0$, $B_+ = 0$, $dB_+/dt = 0.02$, and $d^2B_+/dt^2 = 0.02i$.

evolution is studied. Starting with the simplest case of no external magnetic field ($\omega_c = 0$) and initial conditions that produce only linearly polarized magnetic fields (i.e., $B_+ - B_+^* = 0$ for all times), we find that the energy oscillates regularly between electrostatic and electromagnetic degrees of freedom, see Fig. 1. This relative simplicity is dependent on the absence of an external magnetic field that allows linearly polarized EM-modes, but also on initial conditions that keep the polarization fixed. Modifying the initial conditions slightly such that the EM-polarization is allowed to vary as a result of the nonlinear interaction, the amplitude oscillations

become significantly more complicated, see Fig. 2. While a quasi-periodic oscillation between electrostatic and electromagnetic modes still can be seen, it is clear that there is much additional structure present. For example, the peaks in the electromagnetic energy are complemented by minima that gradually change shape. The situation is further complicated when the external magnetic field is added. The evolution for the same initial conditions as in Fig. 2 is plotted in Fig. 3, with the difference that the external magnetic field is nonzero, i.e., $\omega_c = 0.05$. As can be seen, this adds several distinct features to the evolution. For example, the local maxima of the density oscillations now vary in a highly irregular manner. Further increase of the external magnetic field makes the variations in the density oscillations less pronounced (Fig. 4), but the magnetic field evolution is still highly complex.

IV. SUMMARY AND CONCLUSION

We have here derived Eqs. (18) and (19) that describe the nonlinear interaction between arbitrarily polarized electromagnetic radiation and electrostatic oscillations in a cold magnetized plasma. The ions are considered as immobile, and the electron velocity is non-relativistic, but no other amplitude restrictions have been applied in the derivation. In the absence of an external magnetic field, it is shown that the initial polarization plays an important role for the evolution of the system, and complexity is added since the electrostatic mode can give energy to one electromagnetic polarization at the same time as it gains energy from the other polarization (see Fig. 2). When adding a non-zero external magnetic field into the plasma, the left and right hand polarized modes get different frequencies, which introduces further complexities. This is seen in the structure of the density oscillations (see Fig. 3) and in the magnetic field dynamics (see Fig. 4). A more thorough study of Eqs. (18) and (19) can be a project for further research. Generalizations to include relativistic and/or thermal effects could also be of future interest. Thus, as suggested by the referee, further alternative analytical work may consider a circularly polarized wave magnetic field with constant amplitude and constant density. The resulting nonlinear dispersion relation indicates, then, no nonlinear frequency shift in our nonrelativistic case. However, including relativistic effects (e.g., Refs. 10 and 11) such an approach could lead to interesting effects.

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