Practice to inspire: Mathematics teaching in one Hungarian grade one classroom

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Abstract

In this paper I introduce and categorise the concept of foundational number sense (FONS). Broadly described as the number related competences expected of grade one children, which research has shown to be necessary for the later study of mathematics, FONS is operationalised as an eight dimensional framework for analysing the number-related opportunities teachers present to their students. Drawing on data from a case study of exemplary teaching in grade one classrooms, I analyse one teacher's, Klara's, practice against the framework to show not only that she provides some profound opportunities for her students to learn but does so in ways that reflect the long-standing Hungarian tradition of mathematics as a problem solving discipline taught in collaborative and socially dynamic ways.

Introduction

This paper, in its celebration of Hungarian mathematics teaching, is also a celebration of and tribute to the life and work of Julianna Szendrei. I made my first visit to Budapest more than twenty years ago. At that time I was working at the Manchester Metropolitan University and became involved in an Erasmus exchange scheme with colleagues at ELTE. Adopting the typical imperialist mentality of the English, my colleague, Gillian Hatch, and I arrived thinking that we would be showing our Hungarian hosts how mathematics should be taught. In fact, within minutes of the start of our first lesson, we realised not only how arrogant our perspectives had been but, importantly, how our views on mathematics teaching were being changed forever.

So impressed were we by the quality of the mathematics being learned and the manner in which it was being taught, that once the Erasmus project ended we sought funding to allow us to continue our exploration of Hungarian mathematics classrooms. This led to a further collaboration with colleagues at ELTE, funded by the British Council and the Hungarian Ministry of Culture. After this, I directed an EU-funded video study of the teaching of mathematics in grades 5-8 in England, Finland, Flanders, Hungary and Spain. Throughout these projects and across the years I have written extensively on my interpretation of a long and very successful educational tradition and, with each piece of writing, I have had Julianna in mind. I know from conversations we have had over the years that editors of international journals have frequently asked her to review my papers. This has not only highlighted the very high regard in which Julianna has been held internationally but acted as an academic barometer for me; what would Julianna make of this article? Would she recognise what I am trying to say? Would she approve of my interpretations of the data?

The focus of thisparticular paper is a new field to me. It is one I believe would have been of interest to Julianna and demonstrates well how, even with the youngest of children, Hungarian teachers encourage high levels of mathematical thinking in their students. It concerns the teaching and learning of foundational number sense, which I describe later. But first, I consider number sense more generally to situate the specific foundational number sense in which we are interested.

Number sense in general

Described as a "traditional emphasis in early childhood classrooms" (Casey et al. 2004: 169), number sense is acknowledged as a major objective of many early years' mathematics curricula

(Howell and Kemp 2005; Yang and Li, 2008). Evidence from all round the world indicates that number sense is a predictor of later mathematical success, both in the short (Aubrey and Godfrey, 2003; Aunio and Niemivirta, 2010; Passolunghi and Lanfranchi, 2012) and the longer term (Aubrey et al. 2006; Aunola et al., 2004). In particular, and perhaps unsurprisingly, elementary counting and enumeration skills have been found to be predictive of later arithmetical competence in England, Finland, Flanders, USA, Canada and Taiwan respectively (Aubrey and Godfrey, 2003; Aunola et al., 2004; Desoete et al, 2009; Jordan et al., 2007; LeFevre et al., 2006; Yang and Li, 2008).In other words, there is an international consensus that poorly developed number sense underlies later mathematical failures (Jordan et al, 2009; Gersten et al., 2005; Malofeeva et al., 2004). However, while it is important to understand the consequences of poorly or inappropriately developed number sense, evidence indicates that it is not a well-defined construct. Indeed, Griffin (2004: 173), posed and answered her own, rhetorical question. “What is number sense? We all know number sense when we see it but, if asked to define what it is and what it consists of, most of us, including the teachers among us, would have a much more difficult time”.

My colleagues’ and my interpretation of the literature is that number sense can be categorised in three related ways. The first, preverbal number sense (Butterworth, 2005; Ivrendi, 2011; Lipton and Spelke, 2005), reflects those elements of number sense that are innate to all humans and comprises an understanding of small quantities in ways that allow for comparison. For example, children as young as six months can discriminate between numerosities in a 1:2 ratio, while adults can discriminate ratios as close as 7:8 (Feigenson et al., 2004). In a related way, children at ages 3 and 4 can estimate accurately the numerosity of sets containing up to five items (Gelman and Tucker, 1975). This numerical discrimination, which underpins the acquisition of verbal counting skills (Gallistel and Gelman, 2000) and arithmetic (Zur and Gelman, 2004), is independent of formal instruction, developing as an innate consequence of human, and other species’ evolution (Dehaene, 2001; Feigenson et al., 2004).

Of course, the distinction between those elements of number sense that are innate and those that are not is not always transparent. For example, by age four or five children have normally begun to acquire counting skills and an awareness of quantity that allows them to respond to questions concerning ‘more’ and ‘less’, while by the time they start school they have typically acquired a sense of a mental number line (Aunio et al., 2006; Griffin, 2004). However, such number-related understandings are frequently dependent on individual family circumstances (Zur and Gelman, 2004), indicating that instruction, whether implicit or explicit, may be necessary for their development. This uncertainty with respect to instruction leads us to the second perspective, which is the explicit focus of this paper. Foundational number sense (FONS), which builds on children’s preverbal number sense, comprises those number-related understandings that require instruction and which typically occur during the first years of school (Ivrendi, 2011; Jordan and Levine, 2009). It is a “construct that children acquire or attain, rather than simply possess” (Robinson et al., 2002: 85) and reflects, inter alia, elementary conceptions of number as a representation of quantity or a fixed point in the counting sequence (Griffin, 2004).

However, before discussing FONS I present the third perspective, which has been labelled applied number sense. For some writers, applied number senserefers to a set of core number-related understandings that permeate all mathematical learning (Faulkner, 2009, Faulkner and Cain, 2013; National Council of Teachers of Mathematics, 1989). But more typically it refers to the “basic number sense which is required by all adults regardless of their occupation and whose acquisition by all students should be a major goal of compulsory education" (McIntosh et al., 1992: 3). Understanding issues pertaining to applied number sense, though important, is not the objective of this paper. My interest lies in understanding how teachers induct their young students into foundational number sense.

Most number sense-related studies, whether preverbal, foundational or applied, have examined children's competence (See, for example, Aunio and Niemivirta, 2010; Chard et al., 2005; Clarke and Shinn, 2004; Desoete et al., 2009, 2012; Geary et al., 2009; Holloway and Ansari, 2009). Few have analysed the number sense-related opportunities that teachers create for their children. In this paper we summarise the development and implementation in a Hungarian context of a simple to operationalise framework for analysing FONS-related opportunities in different cultural contexts. Such a tool, which offers an appropriately operationalised definition of number sense, represents a unique contribution to the mathematics education literature with the potential to inform teacher education, facilitate classroom evaluations and provide a warranted tool for use in cross-cultural studies of early years' mathematics teaching. But first, the literature on FONS is examined in order to synthesis a definition appropriate for our task.

Foundational number sense in particular

It has been argued that FONS is to the development of mathematical competence what phonic awareness is to reading (Gersten and Chard, 1999), in that early deficits tend to lead to later difficulties (Jordan et al., 2007; Mazzocco and Thompson, 2005). Significantly, it has been shown to be a more robust predictor of later mathematical success than almost any other factor (Aunio and Niemivirta, 2010; Byrnes and Wasik, 2009). So, what are the characteristics of FONS? Broadly speaking it has been described as the ability to operate flexibly with number and quantity (Aunio et al., 2006; Clarke and Shinn, 2004; Gersten and Chard 1999) and can be expressed in terms of rather vague attributes like "awareness, intuition, recognition, knowledge, skill, ability, desire, feel, expectation, process, conceptual structure, or mental number line" (Berch, 2005: 333).

To identify and make operational an appropriate categorisation of FONS a constant comparison analysis was undertaken (Glaser and Strauss, 1967; Corbin and Strauss, 1990) of the research literature focused on those elements of number sense typically taught to grade one (plus or minus one year) students; that is, students in their first year of formal instruction. In so doing we identified refereed articles and book chapters located in the research domains of mathematics education, psychology, special educational needs and generic education. This collection of papers was then randomly assigned an ordering. The first article was then read and an initial categorisation made. This approach, for example, would have placed *rote counting to five* and *rote counting to ten*, two narrow categories discussed by Howell and Kemp (2006), within the same broad category of systematic counting. The second article in the random list was then for evidence confirming or refining an initial category and then for evidence of categorisations not present in the first. Any new categorisations would then be tested against articles read earlier. This process was repeated for the whole set of articles and yielded eight categorisations of foundational number sense, which are summarised briefly below.

Number recognition

> FONS involves being able to recognise number symbols and know associated vocabulary and meaning (Malofeeva et al., 2004). It entails being able to both identify a particular number symbol from a collection of number symbols and name a number when shown that

symbol (Clarke and Shinn, 2004; Gersten et al., 2005; Van de Rijt et al., 1999; Yang and Li, 2008).

Systematic counting

FONS incorporates systematic counting (Berch, 2005; Clarke and Shinn, 2004;Gersten et al., 2005; Griffin, 2004; Van de Rijt et al., 1999) and includes notions of ordinality and cardinality (Ivrendi, 2011; Jordan et al., 2006; LeFevre et al., 2006; Malofeeva et al. 2004; Van Luit and Schopman, 2000). Children can count to twenty and back or count upwards and backwards from an arbitrary starting point (Jordan and Levine, 2009; Lipton and Spelke, 2005). They know that each number occupies a fixed position in the sequence of all numbers (Griffin et al., 2004).

Awareness of the relationship between number and quantity

FONS includes an awareness of the relationship between number and quantity. Children understand not only the one-to-one correspondence between a number's name and the quantity it represents but also that the last number in a count represents the total number of objects (Jordan and Levine, 2009; Malofeeva et al, 2004; Van Luit and Schopman, 2000).

Quantity discrimination

FONS includes awareness of magnitude and of comparisons between different magnitudes (Clarke and Shinn, 2004; Griffin, 2004; Ivrendi, 2011; Jordan et al., 2006; Jordan and Levine 2009; Yang and Li, 2008) and deploys language like 'bigger than' or smaller than' (Gersten et al., 2005). Children understand that eight represents a quantity that is bigger than six but smaller than ten (Baroody and Wilkins, 1999; Lembke and Foegen, 2009).

An understanding of different representations of number

FONS incorporates an understanding that numbers can be represented differently (Ivrendi, 2011; Jordan et al., 2007; Yang and Li, 2008). This includes understanding the number line (Siegler and Booth 2004; Booth and Siegler, 2006, 2008), simple partitions as representations of a number (Hunting, 2003; Thomas et al., 2002), fingers as representations(Fayol et al., 1998; Gracia-Bafalluy and Noël, 2008;Jordan et al., 1992; Noël, 2005), and manipulatives, particularly linking cubes (Van Nes and Van Eerde, 2010).

Estimation

FONS aware children are able to estimate, whether it be the size of a set (Berch, 2005; Jordan et al., 2006, 2007; Kalchman et al., 2001; Malofeeva et al 2004; Van de Rijt et al., 1999) or an object (Ivrendi, 2011). Estimation involves moving between representations - sometimes the same, sometimes different - of number, for example, placing a number on an empty number line (Booth and Siegler, 2006).

Simple arithmetic competence

A FONS aware child will perform simple arithmetical operations (Ivrendi, 2011; Jordan and Levine 2009; Malofeeva et al., 2004; Yang and Li, 2008). Such skills,which Jordan and Levine (2009) describe as the transformation of small sets through addition and subtraction, underpin later arithmetical and mathematical fluency (Berch, 2005; Dehaene, 2001; Jordan et al., 2007).

Awareness of number patterns

> FONS includes awareness of number patterns and, in particular, being able to identify a missing number (Berch, 2005; Clarke and Shinn, 2004; Gersten et al., 2005; Jordan et al., 2006, 2007; Van Luit and Schopman 2000).

In sum, our systematic analysis of the literature identified eight distinct but not unrelated characteristics of FONS. The fact that they are not unrelated is important as number sense

> "relies on many links among mathematical relationships, mathematical principles..., and mathematical procedures. The linkages serve as essential tools for helping students to think about mathematical problems and to develop higher order insights when working on mathematical problems" (Gersten et al., 2005: 297).

In other words, without the encouragement of such links there is always the risk that children may be able to count but not know, for example, that four is bigger than two (Okamoto and Case, 1996). The eight categories of FONS are summarised below in table 1.

FONS category	Teachers encourage their children to
Number recognition	identify a particular number symbol from a collection of number symbols and name a number when shown its symbol
Systematic counting	count systematically, both forwards and backwards and from arbitrary starting points
Relating number to quantity	understand the one-to-one correspondence between a number's name and the quantity it represents
Quantity discrimination	compare magnitudes and deploy language like 'bigger than' or 'smaller than'
Different representations	recognise, work with and make connections between different representations of number
Estimation	estimate, whether it be the size of a set or an object
Simple arithmetic	perform simple addition and subtraction operations
Number patterns	recognise and extend number patterns and, in particular, identify a missing number

Table 1: Operational definitions of the FONS categories

Applying the FONS framework to Hungarian lessons: A case study

In the following we apply the FONS framework to excerpts drawn from the starts of three different lessons taught by the same grade one teacher, whom we call Klara. These derived from a study of exemplary Hungarian primary mathematics teaching undertaken by my colleague Jenni Back. Klara works in a provincial Hungarian city and is regarded against various local criteria as an effective teacher of mathematics. Lessons were video-recorded in ways that would optimise the capture of the teacher's actions and utterances. Klara was filmed over several lessons spread over several weeks to minimise the likelihood of show-piece lessons. The lessons were accompanied by a contemporary translation provided by a home-based English-speaking colleague. This translation was augmented by me, who has many months' experience of analysing both live and recorded Hungarian mathematics classrooms. Finally, each lesson was viewed simultaneously and repeatedly by three researchers, Andrews, Back and Judy Sayers.

This led to our agreeing which components of FoNS were addressed, both implicitly and explicitly, during any particular episode.The three episodes chosen for presentation here were thought to be illustrative of Klara's practice and, due to their falling at the start of lessons, in need of less explanatory detail that other episodes would have required. The lessons were taught mid-way through the students' first year of formal schooling and were focused on the domain of natural numbers to 20. There was no attempt to extend the set of numbers within which Klara operated.

Episode 1

Excerpt: The lesson began with Klara presenting the configuration, prepared before the start of the lesson, shown in figure 1.

–	–	–	–	–	–	–	–	–
3	7	6	10	_	_	_	_	_

Figure 1: Klara's initial configuration

Having been invited to do so, the class read out the numbers in unison as Klara pointed to each in turn. Next, moving from left to right, she invited volunteers to explain how each number could be derived from the one preceding it. Students volunteered that the first jump was add four, then subtract one and add four. With each offering Klara wrote the operation underneath, as shown in figure 2.

–	–	–	–	–	–	–	–	–
3	7	6	10	_	_	_	_	_
	+4	-1	+4					

Figure 2: Initial configuration with operations added

This led to Klara inviting predictions as to what operation would be expected next, followed by the outcome of that operation. Consequently, following a student suggestion, Klara wrote -1 below and to the right of the 10, before adding 9 to the list of numbers.Eventually, this process of inviting the operation in order to identify the next number in the sequence led to the results shown in figure 3.

–	–	–	–	–	–	–	–	–
3	7	6	10	9	13	12	16	15
	+4	-1	+4	-1	+4	-1	+4	-1

Figure 3: First complete configuration

Commentary: In this introductory episode, which lasted exactly two minutes, each question elicited a response from a different student. Each answer was written in yellow, to contrast it with the white of the original four numbers. Interestingly, while we would argue the task was mathematically sophisticated the students' responses indicated that these were not unfamiliar activities, with all participants understanding the nature of the tasks and their roles in its completion. The task, despite being contextualised to relatively small numbers, seemed to demand high levels of mathematical thinking and reasoning. The sequence itself was complicated by the additional fact of its being alternating.

During this period it seems to us that Klara had engaged in encouraging several foundation number sense characteristics. Firstly, the opening in which children were invited to recognise and read the numbers on the board was clearly focused on the recognition of number symbols and their vocabulary. It could, as the evidence of the lesson suggested, also be inferred that she was encouraging them attribute meaning to both symbols and words. Secondly, in the ways in which successive numbers were identified, Klara was encouraging explicitly an engagement with simple arithmetical operations. Thirdly, the episode was clearly located in an activity focused on an understanding of number patterns and the ways in which knowledge of the embedded relationships can identify missing numbers.

Excerpt: On completion of the first task, Klara showed the class a set of cards, each of which had a letter written on it. She announced that they were going to play a game, which involved her asking questions to which the answer would be one of the numbers in the sequence. Each correct answer would yield a letter, as a reward, that will eventually spell out a word that would tell the class where it would be going in the story of this particular day. The following reflects the first minute of next five minutes of discourse.

Klara So my first statement is… please look only at the numbers on the board… I am thinking of the largest one-digit number. Balasz?

Balasz	Six (Pupils protest)
Klara	Look at the sequence again, and please, correct yourself.
Balasz	Seven
Klara	Look at the number line… Ferenc?
Ference	Nine
Klara	That’s right. So I will give you a reward for the nine (Klara attached a card, showing the letter Í, to the board above the number nine). When you know the solution, please keep it secret… The next number I am thinking of… You mustn’t look behind you (Referring to a picture on the back wall) is the value of the black stick in our collection. Perszi?
Perszi	Eight (Pupils protest)
Klara	(to Perszi) Look around, the others don’t agree with you… Mara?
Mara	Seven
Klara	Let’s see who’s correct. (They all look at the back wall, where they can see the members of the Cuisenaire rod collection and their values)
All	(In chorus) Mara was correct (at which point Klara places a card with the letter Á above the seven).

Commentary: In this excerpt could be seen evidence of Klara encouraging the development of several number sense characteristics. For example, in inviting her students to consider the largest one-digit number she appeared to present an explicit focus not only on the recognition of number symbols, their vocabulary and meaning but also awareness and comparisons of magnitude. The second problem, focused on the relationship between a Cuisenaire rod and its numerical value appeared to address not only an understanding of different representations of number but also an awareness of the relationship between numbers and quantities. Thus, in under a minute Klara was observed to privilege four of the six categories of number sense. Also, embedded in this episode, it seems to us, was an expectation that students would not only begin to acquire the skills of mathematical problem solving but also assume responsibility for the correcting of mistakes. For example, in relation to Balazs’s error, Klara appeared not to examine the nature of his error but, through her instruction to *look at the sequence again, and please correct yourself* and *look at the number line*, to highlight her intention of not correcting him herself but encouraging him to resolve his problems for himself.

Excerpt: The lesson continued with Klara asking a different form of question for each of the numbers in the sequence. These included statements like, a number two smaller than nine, the largest two digit number, the smallest one digit number, a number whose digits add up to 4, and so on. In each case at least one child was involved in publicly responding to the questions posed and with each correct response a new letter was attached to the board. Eventually, as shown in figure 4, the following emerged with only the number ten left without its corresponding letter.

B		Á		B				Í		N		H		Á		Z
3		7		6		10		9		13		12		16		15
	+4		-1		+4		-1		+4		-1		+4		-1	

Figure 4: Almost complete configuration

Commentary: During this phase of the episode several number sense categories were addressed. Every statement could be construed as focusing on different representations of number in the sense, for example, that a number two smaller than nine is a different representation of seven. Simple arithmetical operations were addressed by the same statement as well as thirteen being presented as a number whose digits sum to four. Awareness and comparisons of magnitude were addressed in statements like the largest two digit or smallest one digit number and so on.

Excerpt: Having identified all the letters bar the one linked to ten Klara passed responsibility to her students and invited them to offer statements appropriate to that number. This led to the following:

Mara: It is the bigger neighbour of the number 9.

Ildikó: It is the smallest 2-digit number.

Eva: It is the smaller neighbour of the number 11.

Gabor: The sum of its digits is 1.

Judit: Even number.

Klara Have we got anything else?

Zsolt: It is the sum of the 1 and 9.

Klara: Yes, the sum of the 1 and 9… and who knows the letter in my hand?

All: Sz

Klara: Yes, and where are we going today?

All: Bábszínház. (Puppet theatre)

Commentary: In this final phase of the episode Klara's students offered further evidence that the promotion of number sense is an integral element of their experience. For example, both understanding of different representations of number and recognition of number symbols their vocabulary and meaning was, we believe, implicit in all the contributions. Simple arithmetic could be seen in Gabor's statements that the sum of ten's digits is one and Zsolt's suggestion that ten is the sum of one and nine. Awareness of number patterns was implicit in Judit's even number suggestion, while awareness of magnitude was implicit in Mara's bigger neighbour of nine and Eva's smaller neighbour of eleven. Admittedly, within these statements was little evidence of children showing either an awareness of the relationship between numbers and quantities or skills of systematic counting but it would be unrealistic to expect all categories of numbers sense to be addressed in such a short episode.

Episode 2

Excerpt: Klara began the lesson by opening the front of board to reveal five arithmetic problems, each written on a laminated sheet and affixed to the board in the arrangement of sums shown in figure 5. These were 7+6=, 3+9=, 17-8=, 13-5= and 16-9=. Children were invited to solve the problems individually. As they worked, many were seen to use their fingers. Once she was content that the class was ready, Klara invited answers, which she wrote neatly next to the sums. Each answer was provided by a different volunteer. Next she asked her students to give her the list of answers in descending order. A number of students were involved, with Klara writing their suggestions in the spaces she had prepared below the sums.

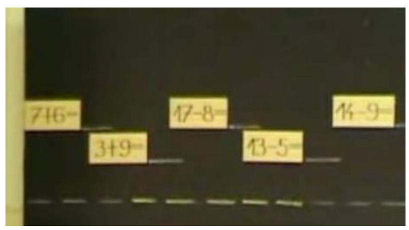

Figure 5: Klara's five problems and the board arrangement around them

Commentary: From the perspective of FONS, this excerpt found Klara engaging her class in simple arithmetic operations. It could also be argued that her children's use of fingers, which was certainly not discouraged, indicated an emphasis on different representations of number. The listing of the answers in numerical order involved quantity discrimination and number recognition, although the latter was implicit in all Klara did.

Excerpt: Klara asked students to tell her how they could get from one number to the next in the sequence she had just created. This led to different children telling her -1, -3, -1, -3. This process was then used to extend the sequence to the right; a child suggestedthat to get the next number they would have to subtract one. This led to four being added to the sequence, as shown in figure 6. Eventually, this process led to zero on the right, with Klara having prepared just enough spaces for the answers. Throughout, the emphasis was on repeating the same pattern - what would happen if the pattern repeats as we have seen it already? Next she asked them to see if they could work out how to move leftwards, using what they already know. In response to her request, there was a chanted response that 16-3 = 13. Klara wrote both 16 and -3 in the correct spaces. Eventually, a process involving several students, led to 20 at the far left of the table.

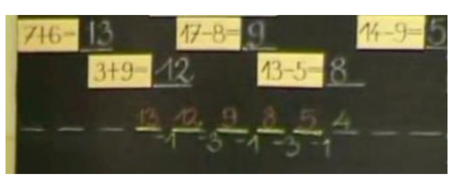

Figure 6: Extending the sequence

Commentary: This last excerpt incorporated a number of FONS-related components. Klara was clearly working with a number pattern and encouraging children to find missing numbers in it. It could also be argued that her writing the sequence from highest to lowest could be seen as supporting systemic counting. Her placing of the operations beneath the sequence indicated an expectation of simple arithmetical operations with, of course, the continuing focus on number recognition. Interestingly, the complexity of the task increased as she began moving leftwards.

Table 6: The five new problems

Excerpt: Next Klara revealed some more problems, also fixed to the board, as shown in figure 6, although the fifth was left blank. Already prepared were spaces for the answers. On the front desk Klara had placed five red trapezia, each with an answer on it. A girl was invited to locate the correct answer, 13, to the first problem, which she then placed above the first sum to create an image that looked like a house and returned to her seat with a very proud smile. The second child, a boy, was unsure and Klara supported his getting the correct answer, 5, by using her hands and fingers in ways he was intended to mimic, which he did. After the first four houses were completed Klara produced the answer for the final question, which had yet to be defined, and placeds it on the board. Students now volunteered possible sums, which Klara wrote in the prepared places, as shown in figure 7.

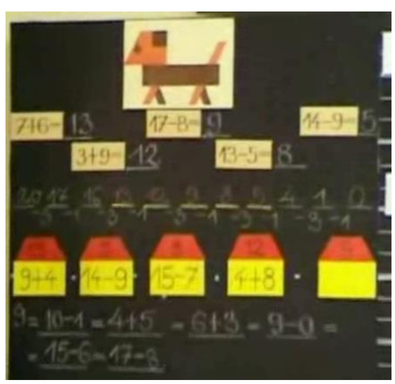

Figure 7: The completed task

Commentary: With respect to FONS components, Klara engaged her children in simple arithmetical operations. The creation of a series of equivalent sums could be construed as another example of her encouraging children to engage with different representations of number, as could her use of her hands in discussing the second problem with the boy concerned. Number recognition was implicit in all she did. Finally, on a different theme, the act of inviting children to offer such different sums could be construed as an act of problem posing, which research shows is a substantial contributor to children's learning of mathematics.

Episode 3

Excerpt: Klara revealed three pictures fixed to the board. One was a three by three square with three inner squares shaded, the second showed the five dots of one side of a standard die, and the final a picture of a closed fist with the thumb extended. Beneath this was drawn a number line with an arrow pointing left from zero and a second arrow pointing right from what would be eight. Klara asked her class what numbers are represented and, given each in turn, wrote 3, 5, 1 alongside their respective images. Once written she asked the class to repeat the numbers, which it did, chanting három, öt, egy as she pointed to each in turn. At this point Klara removed a laminated paper leaf from a collection affixed to the board.

Commentary: In this first instance was evidence of at least two components of FONS. Firstly, students were engaging with different representations of number and, secondly, they were involved in number recognition.

Excerpt: Next she invited a child to the board, handed him a stick, and asked him to count out three on the number line. He did this with confidence, although initially he counted to five. When prompted he correctedhimself, much to the amusement of all, and Klara wrote*3* at the correct point and the class did the same in their books. A girl came next and, in similar manner, tapped out five starting at zero, after which Klara wrote 5 in its allotted place. Another girl, hesitantly but correctly, tapped out one, which was also written up. Thus far all has been written in white chalk.

Commentary: In this second short excerpt, Klara involved students in another representation of number, systematic counting, particularly emphasising the role of zero, and a further number identification by writing the numbers in the correct place on the number line.

Excerpt: Another leaf was now removed and a picture began to emerge. Next Klara asked her class what is special about the numbers and was told that they are odd. She then asked what is missing between the one and the three. She was informed that it was two, and wrote it in red before repeating the process for four, which led to a discussion about odds and evens. Finally, the last of the leaves was removed to leave a picture of what looked like hen house. This led to a lengthy and excited discussion as to what it might be as there were no hens in evidence. Eventually, she placed a goose, a duck, a turkey and a cow in the picture before placing a hen in the hen house and a pig in the shed next to it.

Commentary: In this third excerpt, in addition to recognition of numbers, there was evidence of students being encouraged to engage with number patterns, including missing numbers.

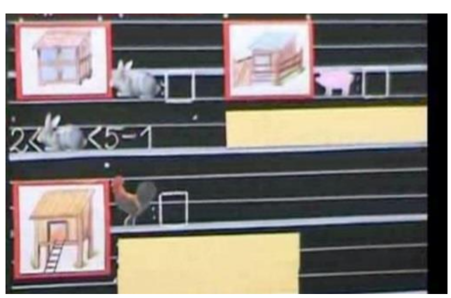

Figure 8:

Excerpt: Next Klara revealed the homes of three different animals. Each was discussed and further animals placed on the board, each one between its home and a square drawn to represent an unknown. Next, she revealed the inequality shown in figure 8, involving a rabbit at its centre. Having got a child to read the sentence, she prompted her class and soon a volunteer suggested that the right hand side, 5-1, was four, which Klara wrote above the sum. Finally, exploiting language like greater than and less than, the class alighted on the fact that the rabbit had to have a value of three, which Klara wrote in the box next to the rabbit.

Following this Klara revealed the next problem, 2+1>hen>4-3. A different child read the sentence before others offered the values 3 and 1 for the two sums, which were written above them. Finally, also invoking language of comparison, the hen wasdeclared to have a value of 2 and written in the appropriate box. The third sentence, 2+3>pig>5-4+2 was completed similarly. At this stage she revealed a picture of a sow nursing three piglets. A conversation about the number of animals in the picture led her to write beneath the picture, 4=3+1. Another child was asked if the sum could be represented differently and suggested that 4=1+3, which was also written on the board before she elicited 1+3=4 and 3+1=4, 4-1=3 and 4-3=1.

Commentary: In the above, several aspects of FONS can be inferred. Klara exploited different representations of number, she engaged children in simple arithmetical operations, she provoked discussions on quantity discrimination and she expected, in their reading of number sentences, children to recognise and engage with the vocabulary of numbers. In addition, her use of inequalities provided children with opportunities to engage with numbers at a structural level than would have been the case with equations alone.

Discussion

Over the years, drawing on various data sets, I have written about Hungarian mathematics teaching. However, this is the first time I have focused explicitly on grade one classrooms. Having done so, it is my perception that in Klara's practice can be seen all I have come to expect of the Hungarian. Indeed, whether I analyse her practice through the lens of FONS, as I have here, or compare it to other accounts of Hungarian teaching, as reflected in my earlier writing (Andrews, 2003, 2009a, 2009b, 2011; Andrews and Hatch, 2001, Andrews and Sayers, 2012, 2013) I would come to much the same conclusion; Klara's children

- learn mathematics through an engagement with meaningful mathematical problems;
- acquire a deep conceptual knowledge as the basis for procedural knowledge;
- come to see mathematics as a set of connected concepts and structures;
- engage with mathematical reasoning and are used to justifying their thinking to others
- work in a collaborative and sharing manner

Moreover, every one of the excerpts presented above showed evidence of several FONS categories, indicating an expectation that students will engage with and integrate into existing cognitive schemas several things at the same time, confirming the didactical sophistication of Hungarian teachers identified in earlier studies (Andrews, 2009a, 2009b, Andrews and Sayers, 2012, 2013). It is my view that Klara's teaching, accepting the lack of evidence of children being asked to estimate, weaves a remarkably complex but coherent web of experience. As a researcher concerned with how comparative studies of mathematics teaching can inform, in warranted ways, changes in one's own classrooms, Klara presents a model of good practice. I enjoy watching her teach as I have enjoyed watching many other Hungarian teachers over the years. Her teaching is complex, provocative and focused on generalities. She works in ways commensurate with the beliefs espoused by her Hungarian colleagues that mathematics is a problem solving activity that should not only be cognitively challenging but undertaken in a spirit of collaboration and excitement (Andrews, 2007, 2010). I believe her practice would have pleased Julianna and made her proud that such extraordinary teaching continues today.

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