Fusion Systems On Finite Groups and Alperin’s Theorem

Author:
Eric Ahlqvist, ericahl@kth.se

Supervisor:
Tilman Bauer

SA104X - Degree Project in Engineering Physics, First Level
Department of Mathematics
Royal Institute of Technology (KTH)

May 21, 2014
Abstract

Let $G$ be a group and $P$ a Sylow $p$-subgroup of $G$. A fusion system of $G$ on $P$, denoted by $\mathcal{F}_P(G)$, is the category with objects; subgroups of $P$, and morphisms induced by conjugation in $G$. This thesis gives a brief introduction to the theory fusion systems.

Two classical theorems of Burnside and Frobenius are stated and proved. These theorems may be seen as a starting point of the theory of fusion systems, even though the axiomatic foundation is due to Puig in the early 1990’s.

An abstract fusion system $\mathcal{F}$ on a $p$-group $P$ is defined and the notion of a saturated fusion system is discussed. It turns out that the fusion system of any finite group is saturated, but the converse; that a saturated fusion system is realizable on a finite group, is not always true.

Two versions of Alperin’s fusion theorem are stated and proved. The first one is the classical formulation of Alperin and the second one, due to Puig, a version stated in the language of fusion systems. The differences between these two are investigated.

The fusion system $\mathcal{F}$ of $\text{GL}_2(3)$ on the Sylow 2-subgroup isomorphic to $SD_{16}$ is determined and the subgroups generating $\mathcal{F}$ are found.

Section 1-5 is written together with Karl Amundsson and Oliver Gäfvert.

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1 Introduction

The study of fusion systems is an active field of mathematics with applications in many areas including topology, representation theory and finite group theory. The axiomatic foundations of fusion systems started out with L. Puig in the early 1990’s, but the starting point of the theory reach back to classical theorems of Burnside and Frobenius with arguments of fusion of p-elements in finite groups [5]. In [11] Puig writes that one can view the Frobenius normal p-complement criterion as a "conceptual origin" of the notion of the fusion system. It is currently being investigated if the theory of fusion systems can be used to simplify the classification of finite simple groups [2].

A fusion system on a finite group $G$ is a category on a Sylow $p$-group $P$ for some prime $p$ dividing $|G|$. The objects of the category are the subgroups of $P$ and the morphisms are all injective maps between subgroups of $P$ which are induced by conjugation with elements in $G$. By looking at the local structure of the fusion system of a group one can get information about the group $G$.

Another way of looking at fusion systems is considering abstract fusion systems that are defined on some $p$-group $P$ without requiring a larger group $G$. With this definition it is possible to look at what properties of the fusion system determine if there exists a finite group $G$ containing the fusion system. It is still being investigated what properties of the fusion system determine properties of the group $G$.

A special case of fusion systems are the saturated fusion systems and one can show that all finite groups have saturated fusion systems but not all saturated fusion systems belong to some finite group. This makes the study of saturated fusion systems very interesting since it might simplify classification of finite simple groups.

Two books were released on the subject in 2011. One by David Craven [5] and one by Michael Aschbacher, Radha Kessar and Bob Oliver [4]. These have been crucial for this thesis.

We begin with some preliminary definitions.

**Definition 1.1.** Let $p$ be a prime. A $p$-group is a group whose order is a power of $p$.

**Definition 1.2.** Let $G$ be a group of order $p^nm$, where $p$ is a prime and $p \nmid m$. Then we say that a subgroup of $G$ is a Sylow $p$-subgroup if its order is $p^k$.

**Theorem 1.3 (Sylow’s theorem).** Let $G$ be a group of order $p^nm$, where $p$ is a prime and $p \nmid m$. Then

1. $G$ has subgroups of order $1, p, p^2, \ldots, p^n$.
2. All Sylow $p$-subgroups of $G$ are conjugate.
3. A subgroup of order $p^k$, $0 \leq k \leq n$, is contained in some Sylow $p$-subgroup.
4. The number of subgroups of order $p^k$, $0 \leq k \leq n$, is congruent to 1 modulo $p$.
5. The number of Sylow $p$-subgroups equals $|G : N_G(P)|$, where $N_G(P)$ is the normalizer of $P \in \text{Syl}_p(G)$. In particular, $|\text{Syl}_p(G)|$ divides $m$.

A proof can be found in [6].
2 Fusion systems

This section aims to give an introduction to fusion systems and hopefully also serve as a motivation as to why they are interesting to study. We begin with a definition of what is meant by fusion and for elements to be fused in some finite group.

Note that if $x, y$ and $g$ are elements in some group $G$, we write $x^g = xgx^{-1}$ and $g^x = x^{-1}gx$ and the map $c_y : G \to G$ denotes conjugation by the element $y$, i.e., $c_y(g) = yg$.

**Definition 2.1.** Let $G$ be a finite group and let $H \leq K \leq G$ be subgroups of $G$.

1. Let $g, h \in H$ and suppose that $g$ and $h$ are not conjugate in $H$. If $g$ and $h$ are conjugate by an element in $K$, then $g$ and $h$ are said to be fused in $K$. Similarly, two subgroups are said to be fused if they are conjugate by an element in $K$.

2. The subgroup $K$ is said to control weak fusion in $H$ with respect to $G$ if, whenever $g, h \in H$ are fused in $G$, they are fused in $K$.

3. The subgroup $K$ is said to control $G$-fusion in $H$, if whenever two subgroups $A$ and $B$ are conjugate via a conjugation map $\varphi_g : A \to B$ for some $g \in G$, then there is some $k \in K$ such that $\varphi_g$ and $\varphi_k$ agree on $A$, i.e., if $\forall a \in A$, $\varphi_g(a) = \varphi_k(a)$.

**Definition 2.2.** Let $G$ be a finite group and $P$ a Sylow $p$-subgroup of $G$. The fusion system of $G$ on $P$ is the category $\mathcal{F}_P(G)$, whose objects are all subgroups of $P$ and whose morphisms are

$$\text{Hom}_{\mathcal{F}_P(G)}(Q, R) = \text{Hom}_G(Q, R), \quad Q, R \leq P,$$

i.e., the set of all group homomorphism from $Q$ to $R$ induced by conjugation with elements in $G$. The composition of morphisms is the composition of group homomorphisms.

**Definition 2.3.** Let $\mathcal{F}_P(G)$ be the fusion system of $G$ on $P$ and let $Q$ be a subgroup of $P$. Then we define the automorphism group $\text{Aut}_P(Q)$ by $\text{Aut}_P(Q) = \text{Hom}_P(Q, Q)$.

**Remark.** $\text{Aut}_P(Q)$ is isomorphic to $N_P(Q)/C_P(Q)$, which can be seen by applying the first isomorphism theorem on the natural homomorphism $N_P(Q) \to \text{Aut}_P(Q)$ defined by $x \mapsto c_x$.

**Definition 2.4.** Let $G$ be a finite group and $P$ a Sylow $p$-subgroup of $G$. Let $\mathcal{F} = \mathcal{F}_P(G)$ be the fusion system of $G$ on $P$ and let $Q, R$ be any two subgroups of $P$. We say that $Q$ and $R$ are $\mathcal{F}$-isomorphic if there is a morphism $\phi : Q \to P$ in $\mathcal{F}$ such that $\phi(Q) = R$, i.e., if there is a $g \in G$ such that $gQg^{-1} = R$.

One may also say that two $\mathcal{F}$-isomorphic subgroups are $\mathcal{F}$-conjugate.

**Definition 2.5.** Let $G$ be a finite group and $P$ a Sylow $p$-subgroup of $G$. Let $\mathcal{F} = \mathcal{F}_P(G)$ be the fusion system of $G$ on $P$ and $Q$ a subgroup of $P$. The $\mathcal{F}$-conjugacy class containing $Q$ is the class of subgroups of $P$ that are $\mathcal{F}$-isomorphic to $Q$. 
**Definition 2.6.** Let $G$ be a finite group and $P$ a Sylow $p$-subgroup of $G$. Let $\mathcal{F} = \mathcal{F}_P(G)$ be the fusion system of $G$ on $P$. The skeleton of $\mathcal{F}$ is the category $\mathcal{F}_{sc}$, whose objects are representatives for the $\mathcal{F}$-conjugacy classes. For any two objects $A, B$ in $\mathcal{F}_{sc}$ we put $\text{Hom}_{\mathcal{F}_{sc}}(A, B) = \text{Hom}_\mathcal{F}(A, B)$.

In category theory, one would say that $\mathcal{F}_{sc}$ is equivalent to $\mathcal{F}$.

**Example 2.7.** Let $G$ be the symmetric group $S_4$ and let

$$P = \{1, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}.$$  

Then $P \in \text{Syl}_2(G)$ and $P \cong D_8$. In Figure 1 we see the subgroup lattice of $P$.

![Figure 1: Fusion system on $S_4$](image)

Conjugation by $(13)(24)$ is a morphism in $\mathcal{F} = \mathcal{F}_P(G)$ which maps $\{1, (12)\}$ and $\{1, (34)\}$ onto each other and we say that

$$\{1, (12)\} \text{ and } \{1, (34)\} \text{ are } \mathcal{F}\text{-isomorphic.}$$

We also have that the groups $\{1, (12)(34)\}$, $\{1, (13)(24)\}$ and $\{1, (14)(23)\}$ are $\mathcal{F}$-isomorphic. These isomorphisms are induced by conjugation by $(123)$ and $(132)$. Hence we see that $\{1, (12)(34)\}$ and $\{1, (13)(24)\}$ are fused in $G$ but not in $P$ and equally for $\{1, (12)(34)\}$ and $\{1, (14)(23)\}$.

The skeleton $\mathcal{F}_{sc}$ of the fusion system $\mathcal{F}_P(G)$ is shown in Figure 2.
The number on the edge between two subgroups $Q$ and $R$ is the size of $\text{Hom}_F(Q,R)$. Note that Figure 2 does not cover all information of the fusion system since it does not tell what $\text{Hom}_F(Q,R)$ are explicitly.

We have that $\text{Aut}_F(Q)$ acts on $\text{Hom}_F(Q,R)$ on the right and that $\text{Aut}_F(R)$ acts on $\text{Hom}_F(Q,R)$ on the left. For example, Let $Q = \{1, (12)(34)\} \cong \mathbb{Z}/2\mathbb{Z}$ and $R = \{1, (12)(34), (13)(24), (14)(23)\} \cong V_4 \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then $\text{Aut}_F(R) \cong S_3$ acts on $\text{Hom}_F(Q,R)$ from the left as $S_3$ on the set of three letters while $\text{Aut}_F(Q) = 1$ acts trivially from the right on $\text{Hom}_F(Q,R)$.

Denote $1, (12)(34), (13)(24), (14)(23)$ by $a, b, c, d$ respectively. Then $\text{Hom}_F(R,P)$ have as domain, $\{a, b, c, d\}$ since $a$ is always fixed under $\text{Hom}_F(R,P)$, we may denote the elements of $\text{Hom}_F(R,P)$ as cycles by $\alpha = \text{Id}_R, \beta = (bc), \gamma = (bd), \delta = (cd), \epsilon = (bcd), \zeta = (bdc)$.

Now consider the group $\text{Aut}_F(P)$ with action restricted to $R$. For any $\psi \in \text{Aut}_F(P)$ there are only two possibilities for $\psi|_R$. Either $\psi|_R = \text{Id}_R$ or $\psi|_R$ swaps $(13)(24)$ and $(14)(23)$. Hence the left action of $\text{Aut}_F(P)$ on $\text{Hom}_F(R,P)$ will induce only two maps, the identity map and the following

\[
\begin{align*}
\alpha &= \text{Id} \mapsto (cd) = \delta \text{ and } \delta = (cd) \mapsto \text{Id} = \alpha \\
\beta &= (bc) \mapsto (bdc) = \zeta \text{ and } \zeta = (bdc) \mapsto (bc) = \beta \\
\gamma &= (bd) \mapsto (bcd) = \epsilon \text{ and } \epsilon = (bcd) \mapsto (bd) = \gamma 
\end{align*}
\]

This map can be written in cycle form as $(\alpha\delta)(\beta\zeta)(\gamma\epsilon)$. Hence we conclude that $\text{Aut}_F(P)$ will act on $\text{Hom}_F(R,P)$ as $\langle (12)(34)(56) \rangle$ acts on the set $\{1, 2, 3, 4, 5, 6\}$.

Figure 2: The skeleton of the fusion system on $S_4$
Note that if $Q$ is not $\mathcal{F}$-isomorphic to any other subgroup in $\mathcal{F}$, then if $\alpha \in \text{Hom}_\mathcal{F}(Q, R)$ is the inclusion map of $Q$ into $R$ then $\text{Hom}_\mathcal{F}(Q, R) = \alpha \circ \text{Aut}_\mathcal{F}(Q)$ and we have $|\text{Hom}_\mathcal{F}(Q, R)| = |\text{Aut}_\mathcal{F}(Q)|$.

The next theorem is a classical theorem of Burnside which gives information on the fusion in $\mathcal{F}_p(G)$ when the $p$-group $P$ is abelian.

**Theorem 2.8** (Burnside). Let $G$ be a finite group and let $P$ be a Sylow-$p$-subgroup of $G$. If $P$ is abelian, then $\mathcal{F}_p(G) = \mathcal{F}_p(N_G(P))$.

**Proof.** Let $Q, R \leq P$ and let $\varphi: Q \to R$ be a morphism in $\mathcal{F}_p(G)$ such that $\varphi(q) = xq = xq^{-1}$ for some $x \in G$. Since $P$ is abelian, everything in $P$ centralizes $xQ \leq P$. Also, if $xpx^{-1} \in xP$ and $xqx^{-1} \in xQ$, then

$$xpx^{-1}xq^{-1} = xq^{-1} = xq^{-1}x^{-1}xp^{-1}$$

so that $xP$ centralizes $xQ$. Thus, both $P$ and $xP$ are Sylow-$p$-subgroups of $C_G(xQ)$ and hence we can find a $c \in C_G(xQ)$ such that $P = cxP$. This means that $cx \in N_G(P)$ and since $cxu = c(xux^{-1})^{-1} = xux^{-1} = \varphi(u)$ for $u \in Q$, we are done.

**Definition 2.9.** Let $p$ be a prime and let $G$ be a finite group. Then we say that a subgroup of $G$ is a $p'$-group if its order is coprime to $p$.

**Theorem 2.10.** Any two normal $p'$-subgroups of a finite group $G$ generate a normal $p'$-subgroup.

**Proof.** Let $H$ and $K$ be two normal $p'$-subgroups of the finite group $G$. Then, since $H$ (or $K$) is normal, the generated subgroup is just $HK$. But by the second isomorphism theorem, we know that $|HK| = |K||H|/|H \cap K|$ and so the order of $|HK|$ cannot possibly be divisible by $p$. Thus $HK$ is a $p'$-group. To see that it is normal, take any $g \in HK$ and write it as $g = hk$, where $h \in H$ and $k \in K$. If $x \in G$, then

$$xg^{-1} = xhx^{-1} = xhx^{-1}xhx^{-1} \in HK,$$

since $H$ and $K$ are normal.

**Remark.** The above implies that the subgroup generated by all the normal $p'$-subgroups is itself a normal $p'$-subgroup, that is, $G$ has a unique maximal normal $p'$-subgroup. This maximal normal $p'$-subgroup will be denoted by $O_{p'}(G)$.

**Definition 2.11.** Let $G$ be a finite group and let $P$ be a Sylow-$p$-subgroup of $G$. Then $G$ is said to be $p$-nilpotent if $P$ has a normal complement $K$; that is, $K$ is a normal subgroup of $G$ such that $G = KP$ and $K \cap P = 1$. That is $G \cong K \rtimes P$. Note that $K = O_{p'}(G)$.

$K$ is called the normal $p$-complement of $P$ in $G$.

Next is another classical theorem, known as; Frobenius' normal $p$-complement theorem. This gives a criterion for when $P$ controls $G$-fusion in $P$, i.e., when every morphism in $\mathcal{F}_p(G)$ is induced by conjugation in $P$. The last implication of this theorem will be proved in section 4.

**Theorem 2.12** (Frobenius). Let $G$ be a finite group and let $P$ be a Sylow $p$-subgroup of $G$. The following are equivalent:

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1. $G$ is $p$-nilpotent.

2. $N_G(Q)$ is $p$-nilpotent for any non-trivial $Q \leq P$.

3. We have $F_P(G) = F_P(P)$.

4. For any $Q \leq P$, the group $\text{Aut}_G(Q) = N_G(Q)/C_G(Q)$ is a $p$-group.

Proof. 1) $\implies$ 2): Since $G$ is $p$-nilpotent, we can write $G = KP$, where $K \trianglelefteq G$, $P \in \text{Syl}_p(G)$ and $K \cap P = 1$. As $P \cong G/K$ is a $p$-group, we see that $K = \{x \in G : \gcd(|x|, p) = 1\}$. It follows that for any $H \leq G$, $H \cap K = \{x \in H : \gcd(|x|, p) = 1\} = O_{p'}(H)$. Hence $H = O_{p'}(H)(P \cap H)$, and so $H$ is $p$-nilpotent. Thus, every subgroup of a $p$-nilpotent subgroup is $p$-nilpotent and so in particular, $N_G(Q)$ is $p$-nilpotent for any subgroup $Q$ of $P$.

1) $\implies$ 3): Let $Q, R$ be subgroups of $P$ and $\varphi \in \text{Hom}_G(Q, R)$. Then $\varphi = c_x$ for some $x \in G$. We want to show that $c_x(u) = xux^{-1} = pup^{-1} = c_p(u)$ for some $p \in P$ and every $u \in Q$. Since $G = KP$, we can write $x = yz$ where $y \in K$ and $z \in P$. But then by Lemma A.1

$$[y, zuz^{-1}] = \left[y, \underbrace{zu^{-1}}_{\in K}, \underbrace{zw^{-1}z^{-1}}_{\in K}\right] = \left[y, \underbrace{z}_{\in P}, \underbrace{zw^{-1}z^{-1}}_{\in P}\right] \in P \cap K = 1,$$

so that $xux^{-1} = zu^{-1}$.

3) $\implies$ 4): This follows from the fact that

$$\text{Aut}_G(Q) = \text{Hom}_G(Q, Q) = \text{Hom}_P(Q, Q) = N_P/Q/C_P(Q)$$

is a $p$-group.

2) $\implies$ 4):

Let $Q$ be a non-trivial subgroup of $P$. Then we can write $N_G(Q) = K_1 \times P_1$. Since $C_G(Q)$ is a subgroup of $N_G(Q)$, $C_G(Q)$ is also $p$-nilpotent by the first part of the proof. Thus we can write $C_G(Q) = K_2 \times P_2$. It follows that $|N_G(Q)|/|C_G(Q)| = \frac{|K_1||P_1|}{|K_2||P_2|}$, so we are done if we can show that $|K_1| = |K_2|$. Obviously, $|K_2| \leq |K_1|$. Now, $Q$ and $K_1$ are both normal in $N_G(Q)$ and have coprime orders, so that by the lemma, $K_1$ commutes with every element in $Q$. Thus $K_1 \leq C_G(Q) = K_2P_2$, and so $K_1 \leq K_2$ and finally $|K_1| = |K_2|$.

The fact that 4) implies 1) will be proved in section 4. \hfill \square

3 Transfer and the focal subgroup theorem

To simplify the proof of the last part of Frobenius’ normal $p$-complement theorem we want to use Alperin’s fusion theorem. For this we need a property of the so-called focal subgroup which was first introduced by Higman [9]. To prove the focal subgroup theorem we use the transfer homomorphism. If $G$ is a group, $H$ a subgroup of $G$ and $A$ is any abelian group, the transfer is a way of extending a homomorphism $\phi : H \to A$ to a homomorphism $\tau : G \to A$.

**Definition 3.1.** Let $G$ be a finite group and $H \leq G$. Let $\phi : H \to A$ be a homomorphism of $H$ into an abelian group $A$. Let $X$ be a set of right coset representatives for $H$ in $G$ and let $I$ be the index set of $X$. For each $g \in G$, $x_i \in X$ we have that $x_ig \in Hx_j$ for a unique $x_j \in X$. Define $\sigma_g : I \to I$ by
\(\sigma_g(i) = j\). Then \(x_i g x_j^{-1} = h_{i,g} \in H\) where \(h_{i,g}\) depends on \(i\) and \(g\). Define \(\tau: G \to A\) by

\[
\tau(g) = \prod_{i \in I} \phi(h_{i,g}) = \prod_{i \in I} \phi(x_i g x_{\sigma_g(i)}^{-1})
\]

We say that \(\tau\) is the transfer of \(G\) into \(A\) via \(\phi\).

**Theorem 3.2.** Let \(G, H\) and \(\tau\) be chosen as in definition 3.1. Then we have the following:

1. The transfer \(\tau\) is a homomorphism of \(G\) into \(A\).
2. \(\tau\) is independent on the choice of coset representatives of \(H\) in \(G\).

For a proof of this theorem, see [8].

The next theorem will be necessary in our proof of the focal subgroup theorem.

**Theorem 3.3.** Let \(\tau\) be the transfer of \(G\) into an abelian group \(A\) via \(H \leq G\) and the homomorphism \(\phi: H \to A\). For any \(g \in G\), \(\exists\{x_1, \ldots, x_t\} \subseteq G\) with \(t\) and \(x_i\) depending on \(g\), with the following properties:

1. \(x_i g^r_{x_i} x_i^{-1} \in H\) for some positive integers \(r_i\), \(0 \leq i \leq t\).
2. \(\sum_{i=1}^t r_i = n = |G : H|\)
3. \(\tau(g) = \phi(\prod_{i \in I} x_i g^r_{x_i} x_i^{-1})\)

In the proof of this theorem we follow the proof in Daniel Gorenstein [8].

**Proof.** Let \(y_i\) be coset representatives of \(H\) in \(G\), \(0 \leq i \leq n\). Let \(\sigma_g \in S_n\) be defined by \(y_i g \in H y_{\sigma_g(i)}\). Decompose \(\sigma_g\) into disjoint cycles and reorder the \(y_i\) such that the decomposition assumes the form:

\[(12...r_1)(r_1 + 1...r_1 + r_2)(r_1 + r_2 + 1...r_1 + r_2 + r_3)...(r_1 + r_2 + ... + r_t).\]

Then the \(i\)th cycle has length \(r_i\), \(0 \leq i \leq t\) and hence

\[
\sum_{i=1}^t r_i = n = |G : H|
\]

Hence (2) is proved.

Now let \(x_1, \ldots, x_t\) be coset representatives for the cosets labeled 1, \(r_1 + 1, r_1 + r_2 + 1, \ldots, r_1 + r_2 + \ldots + r_{i-1} + 1\) respectively. Then by definition of \(\sigma_g\), \(x_i g^j\) is a coset representative of \(H\) in \(G\) corresponding to the \((j+1)th\) coset of the \(i\)th cycle of \(\sigma_g\). Hence

\[
R := \{x_i g^j | 1 \leq i \leq t, 0 \leq j \leq r_i - 1\}
\]

form a complete set of coset representatives for \(H\) in \(G\). But then \(x_i g^r_{x_i} \in H x_i\) by definition of \(r_i\) and hence \(x_i g^r_{x_i} x_i^{-1} \in H\) which proves (1).

Now we prove (3). Lets use \(R\) as coset representatives and compute \(\tau(g)\).

Let \(y_k = x_i g^j\) (\(k\) depend on \(i\) and \(j\)) and consider \(y_k g = h_{k,g} y_k', h_{k,g} \in H\). If
\( \tau \mid P \) Since \( x_i g^{j+1} \) is a coset representative in \( R \) and the \( x_i g^{j+1} \in H \), if and only if \( y_k = x_i g^{j+1} \), by definition of \( h_{k,g} \).

This implies that \( h_{k,g} = 1 \) whenever \( j < r_i - 1 \). Hence \( \tau(g) \) is the product of those \( \phi(h_{k,g}) \) which corresponds to the elements \( y_k = x_i g^{j+1} \) for which we have

\[
y_k g = x_i g^{j+1} \in H \Rightarrow y_k g = (x_i g^{j+1}) x_i, \ (x_i g^{j+1}) \in H.
\]

Hence \( x_i = y_k \) and we get \( h_{k,g} = x_i g^{j+1} \) for each \( y_k \). Therefore we get that

\[
\tau(g) = \phi \left( \prod_{i=1}^{t} x_i g^{i} x_i^{-1} \right)
\]

and since \( \phi \) is a homomorphism we get (3).

\[ \Box \]

**Theorem 3.4** (The Focal Subgroup Theorem, [9]). Let \( G \) be a finite group, \( P \in \text{Syl}_p(G) \) and let \( G' \) be the commutator subgroup of \( G \). Then

\[
P \cap G' = \langle \{x, g \} = x^{-1} x^g : x \in P, \ g \in G, \ x^g \in P \rangle
\]

\[\equiv \langle x^{-1} \phi(x) : x \in P, \ \phi \in \text{Hom}_{F[G]}(\langle x \rangle, P) \rangle\]

Note that ones the first equality is proved the second one is trivial, since it is just a matter of translation into the setting of fusion systems.

**Proof.** Let \( P^* = \langle x^{-1} x^g : x \in P, \ g \in G, \ x^g \in P \rangle \). We want to show that \( P^* = P \cap G' \). Since \( x^{-1} x^g = [x, g] \), we obviously have \( P^* \subseteq P \cap G' \) and since \( P' \leq P^* \), \( P/P^* \) is abelian.

Let \( \phi : P \to P/P^* \) be the natural homomorphism and let \( \tau : G \to P/P^* \) be the transfer of \( G \) into \( P/P^* \) relative to \( P \) and \( \phi \).

**Claim:** If \( G/\ker \tau \cong P/P^* \), then \( P^* = P \cap G' \).

pf. Let \( K = \ker \tau \). \( G' \leq K \) as \( G/K \) is abelian. Also, \( G/K \) is a \( p \)-group and hence \( G/P = G/K \cong P/(P \cap K) \) and \( P \cap G' \leq K \Rightarrow |P \cap G'| \leq |P^*| \). Thus, since \( P^* \leq P \cap G' \) we have \( P^* = P \cap G' \).

Now we use the same notation as in Theorem 3.2. Let \( x \in P \) and choose elements \( x_i \in G \) and integers \( r_i \), \( 0 \leq i \leq t \) such that

\[
\tau(x) = \phi \left( \prod_{i=1}^{t} x_i x_i^{-1} \right) \equiv \prod_{i=1}^{t} x_i x_i^{-1} \left( \text{mod } P^* \right).
\]

Since \( P/P^* \) is abelian, we have

\[
\tau(x) = \phi \left( \prod_{i=1}^{t} x_i x_i^{-1} \right) \equiv \left( \prod_{i=1}^{t} x_i \right) \left( \prod_{i=1}^{t} x_i x_i^{-1} \right) \left( \text{mod } P^* \right).
\]

But since \( x_i x_i^{-1} = [x_i, x_i] \in P' \leq P^* \) we have that

\[
\tau(x) \equiv \prod_{i=1}^{t} x_i \left( \text{mod } P^* \right) \equiv x^\sum_{i=1}^{t} \left( \text{mod } P^* \right) \equiv x^n \left( \text{mod } P^* \right).
\]

But since \( |G : P| = n \), we get that \( \gcd(p,n) = 1 \) and hence, if \( x \notin P^* \), then \( \tau(x) \notin P^* \). Thus \( \tau \) maps \( P \) onto \( P/P^* \) and hence it also maps \( G \) onto \( P/P^* \) and we have

\[
\tau(G) = P/P^* \Rightarrow G/\ker \tau \cong P/P^*.
\]

\[ \Box \]
4 Proof of Frobenius \((4) \Rightarrow (1)\)

The goal of this section is to prove that \((4)\) implies \((1)\) in Frobenius’ normal \(p\)-complement theorem. To do this we use Alperin’s fusion theorem which is a strong result about conjugation in finite groups. The theorem was first stated and proved by Alperin in \([1]\) in 1967.

**Definition 4.1.** Let \(P\) and \(Q\) be Sylow \(p\)-subgroups of \(G\). \(R = P \cap Q\) is called the *tame intersection* of \(P\) and \(Q\) in \(G\) if both \(N_P(R)\) and \(N_Q(R)\) are Sylow \(p\)-subgroups of \(N_G(R)\).

We will later see that in a fusion system \(\mathcal{F}_P(G)\) of a finite group \(G\) on a Sylow \(p\)-subgroup \(P\), a tame intersection of \(P\) with any \(Q \in \text{Syl}_p(G)\) will be so-called *fully \(\mathcal{F}\)-normalized* which is an important property of a group in a fusion system.

**Example 4.2.** Every Sylow \(p\)-subgroup is a tame intersection, as we see from \(Q = Q \cap Q\).

**Theorem 4.3** (Alperin’s Fusion Theorem, \([1]\)). Let \(G\) be a finite group and \(P \in \text{Syl}_p(G)\). Let \(A, A^g \subseteq P\), for some \(g \in G\).

Then there exists elements \(x_1, x_2, \ldots, x_n\), subgroups \(Q_1, Q_2, \ldots, Q_n \in \text{Syl}_p(G)\) and an \(y \in N_G(P)\) such that

1. \(g = x_1 x_2 \cdots x_n y\),
2. \(P \cap Q_i\) is a tame intersection, \(0 \leq i \leq n\),
3. \(x_i\) is a \(p\)-element of \(N_G(P \cap Q_i)\), \(0 \leq i \leq n\),
4. \(A \subseteq P \cap Q_1\) and \(A^{x_1 x_2 \cdots x_i} \subseteq P \cap Q_{i+1}\), \(0 \leq i \leq n - 1\).

**Proposition 4.4** (Frobenius \(4 \Rightarrow 3\)). Suppose for any \(Q \leq P\), the group \(\text{Aut}_G(Q) = N_G(Q)/C_G(Q)\) is a \(p\)-group. Then \(\mathcal{F}_P(G) = \mathcal{F}_P(P)\).

**Proof.** We want to show that for any morphism in \(\mathcal{F}_P(G)\) induced by conjugation with some \(g \in G\), \(g\) can be written as a product of elements \(x_1, x_2, \ldots, x_n, y \in P\). From Theorem 4.3 we know we can find \(Q_i \in \text{Syl}_p(G)\) such that \(P \cap Q_i\) is a tame intersection and such that \(x_i \in N_G(P \cap Q_i)\), for \(0 \leq i \leq n\). So if we can show that \(x_i \in P\) for \(1 \leq i \leq n\) and \(y \in P\), we are done.

Since \(P \cap Q_i\) is a tame intersection we have that \(N_P(P \cap Q_i) \subseteq \text{Syl}_p(N_G(P \cap Q_i))\), thus \(N_P(P \cap Q_i)/C_G(P \cap Q_i) \subseteq \text{Syl}_p(N_G(P \cap Q_i)/C_G(P \cap Q_i))\). Now by assumption, \(\text{Aut}_G(P \cap Q_i) = N_G(P \cap Q_i)/C_G(P \cap Q_i)\) is a \(p\)-group \(\Rightarrow N_G(P \cap Q_i) = N_P(P \cap Q_i)C_G(P \cap Q_i)\) and since \(N_P(P \cap Q_i) \subseteq C_G(P \cap Q_i)\) we have that

\[
\frac{N_P(P \cap Q_i)C_G(P \cap Q_i)}{C_G(P \cap Q_i)} \cong \frac{N_P(P \cap Q_i)}{N_P(P \cap Q_i) \cap C_G(P \cap Q_i)} \cong \text{Aut}_P(P \cap Q_i) \quad (1)
\]

Hence each for \(1 \leq i \leq n\), \(x_i \in P\).

Now we use the same argument again to show that \(y \in P\). \(P\) is a tame intersection with itself and hence \(N_P(P) \subseteq \text{Syl}_p(N_G(P))\). But \(N_G(P)\) was assumed to be a \(p\)-group and hence \(N_G(P) = N_P(P)\) which implies that \(y \in P\).
Now to the proof of that (4) implies (1), in Frobenius Theorem. We will denote by $O^p(G)$, the smallest normal subgroup of $G$ such that $G/O^p(G)$ is a $p$-group.

**Proposition 4.5** (Frobenius 4 $\Rightarrow$ 1). Suppose for any $Q \leq P$, the group $\text{Aut}_G(Q) = N_G(Q)/C_G(Q)$ is a $p$-group. Then $G$ is $p$-nilpotent.

**Proof.** We will proceed by induction on $G$. The proof will be in two steps. First we will show that if $G$ has a proper normal subgroup $H$ such that $G/H$ is a $p$-group, then the statement is true. The second step is to show that $G$ actually contains such an $H$.

So first assume that there is an $H \triangleleft G$ such that $G/H$ is a $p$-group. Let $Q$ be a proper $p$-subgroup of $H$. By assumption $N_G(Q)/C_G(Q)$ is a $p$-group. We have that $H \cap N_G(Q) = N_H(Q)$ and $H \cap C_G(Q) = C_H(Q)$ and hence $N_H(Q)/C_H(Q)$ is a $p$-group. Hence, by induction $H$ has a normal $p$-complement $K = O^p(H)$. By Theorem 2.10, $K$ is the unique maximal normal $p^i$-subgroup and hence $K$ is characteristic in $H$. Thus, since $H \triangleleft G$, we have that $K \triangleleft G$, by Lemma A.4. But both $G/H$ and $H/K$ are $p$-groups and hence, so is $G/K$. Hence $K$ is a normal $p$-complement in $G$ and the induction is complete.

Now we prove the second step. From Proposition 4.4 and the focal subgroup theorem we have that

$$P \cap G' = \langle x^{-1} x^g | x, g \in P, g \in G \rangle = \langle x^{-1} x^g | x, x^g, g \in P \rangle = P'$$

(2)

and we have that $P' < P$ since $P$ is a $p$-group. Now consider the group $G'O^p(G) \leq G$. Let $\phi : P \to P/P'$ be the natural homomorphism and let $\tau$ be the transfer from $G$ into $P/P'$ via $\phi$. Then we have that

$$G/\ker \tau \cong P/P' \neq 1.$$  

(3)

This means that $G' \leq \ker \tau$ since $P/P'$ is abelian and $O^p(G) \leq \ker \tau$ since $P/P'$ is a $p$-group. Thus $G'O^p(G) \leq \ker \tau$ and hence $G'O^p(G)$ is a proper normal subgroup of $G$. Obviously $G/(G'O^p(G))$ is a $p$-group as $G/O^p(G)$ is by definition of $O^p(G)$. So if we put $H = G'O^p(G)$ in part one, we are done.  

## 5 Abstract fusion systems

In this section we will introduce a more general definition of a fusion system. Instead of defining it on a finite group $G$ we will define a fusion system on a $p$-group $P$ directly, without requiring that $P$ is a subgroup of some larger group $G$. We will also loosen the requirement of every morphism being induced by conjugation. Instead we are satisfied if every morphism, induced by conjugation in $P$, is in the fusion system.

It is however hard to work with the definition of an abstract fusion system alone and hence we introduce the notion of a saturated fusion system. We will prove that for any finite group $G$ with a Sylow $p$-subgroup $P$, $F_P(G)$ is saturated.

Since we may construct a fusion system on any finite group but not every fusion system is realisable on a finite group, the concept of an abstract fusion system implies that the class of abstract fusion systems is bigger than the class of finite groups. This observation gives a prospect that the theory of fusion
systems could help in simplifying the the classification of finite simple groups. The fusion systems that are not realisable on any finite group $G$, are called exotic fusion systems.

We begin with the definition of an abstract fusion system.

**Definition 5.1.** Let $P$ be a finite $p$-group. A fusion system on $P$ is a category $\mathcal{F}$, whose objects are all subgroups of $P$ and whose morphisms $\text{Hom}_\mathcal{F}(Q, R)$ are sets of injective homomorphisms having the following three properties:

1. For each $g \in P$ such that $gQ \leq R$, $c_g : Q \to R$ defined by $c_g(x) = gx$ is in $\text{Hom}_\mathcal{F}(Q, R)$.
2. For each $\phi \in \text{Hom}_\mathcal{F}(Q, R)$, the induced isomorphism $Q \to \phi(Q)$ and its inverse lies in $\text{Hom}_\mathcal{F}(Q, \phi(Q))$ and $\text{Hom}_\mathcal{F}(\phi(Q), Q)$ respectively.
3. Composition of morphisms in $\mathcal{F}$ is the usual composition of group homomorphisms.

The axioms are inspired from $G$-conjugacy and Definition 2.2. The first axioms guarantees agreement with Definition 2.2 in the case one has an underlying group. The second and third axioms are there to make $\mathcal{F}$-conjugacy into an equivalence relation, as it is for $G$-conjugacy.

**Example 5.2.** Let $P$ be a $p$-group and $\mathcal{F}$ the fusion system on $P$ such that, for any subgroups $Q, R$ of $P$, $\text{Hom}_\mathcal{F}(Q, R)$ consists of all injective homomorphisms from $Q$ to $R$. Then $\mathcal{F}$ is called the universal fusion system on $P$.

Consider the fusion system on $D_8$ in Example 2.7. For the Klein 4-group $V = \{\text{id}, (12), (34), (12)(34)\}$ we have $\text{Aut}_\mathcal{F}(V) \subset \text{Aut}(V)$ and hence, this is not the universal fusion system on $D_8$. However, if we let $G = A_6$ an construct the fusion system on $D_8$ we see that $\mathcal{F}_{D_8}(G)$ is the universal fusion system on $D_8$ as in this fusion system, both Klein 4-groups have their full automorphism group.

To define a saturated fusion system we need the following definitions.

**Definition 5.3.** Let $\mathcal{F}$ be a fusion system on a finite $p$-group $P$. A subgroup $Q$ of $P$ is said to be fully $\mathcal{F}$-automized if $\text{Aut}_P(Q) \cong N_P(Q) / C_P(Q)$ is a Sylow $p$-subgroup of $\text{Aut}_\mathcal{F}(Q)$.

**Definition 5.4.** Let $\mathcal{F}$ be a fusion system on a finite $p$-group $P$ and let $Q \leq P$. For any $\phi : Q \to P$ in $\mathcal{F}$ we set

$$N_\phi = \{ y \in N_P(Q) \mid \exists z \in N_P(\phi(Q)) \text{ such that } \phi(yu) = z\phi(u), \forall u \in Q \}$$  \hspace{1cm} (4)

Note that $QC_P(Q) \leq N_\phi \leq N_P(Q)$.

**Definition 5.5.** Let $\mathcal{F}$ be a fusion system on a finite $p$-group $P$. A subgroup $Q$ of $P$ is said to be receptive if every morphism $\phi$, whose image is $Q$, is extensible to $N_\phi$.

To get insight into what these definitions mean we look at the following proposition.

**Proposition 5.6.** Let $\mathcal{F}$ be a fusion system on a finite group $G$ and let $P \leq G$ such that $P \in \text{Syl}_p(G)$. If $Q \leq P$ is such that $N_P(Q) \in \text{Syl}_p(N_G(Q))$ then $Q$ is receptive.
Proof. Suppose that \( \phi : R \to Q \) is an isomorphism in \( \mathcal{F} \). Now

\[
N_\phi = \{ x \in N_P(R) \mid \exists y \in N_P(Q) \text{ such that } \phi(xrx^{-1}) = y\phi(r)y^{-1}, \forall r \in R \},
\]

thus \( c_x \circ \phi^{-1} \circ c_y \circ \phi \) centralizes \( R \) and \( c_y \circ \phi \circ c_{y^{-1}} \circ \phi^{-1} \) centralizes \( \phi(R) = Q \Rightarrow \phi \circ c_x = \phi' \circ \phi, \) for some \( \phi' \) that is induced by conjugation with some element \( g \in C_G(Q) \). Thus

\[
\phi(N_\phi) \leq N_P(Q)C_G(Q)
\]

and since \( N_\phi \) is a \( p \)-group and \( N_P(Q) \in \text{Syl}_p(N_G(Q)) \Rightarrow \) there exists a \( \psi \) in \( \mathcal{F} \) induced by some \( c \in C_G(Q) \) such that \( \psi(\phi(N_\phi)) \leq N_P(Q) \). Thus we can define \( \theta = \psi \circ \phi \) such that \( \theta : N_\phi \to N_P(Q) \), and the proof is done.

\[
\text{Definition 5.7.} \quad \text{Let } \mathcal{F} \text{ be a fusion system on a finite } p\text{-group } P. \text{ A subgroup } Q \text{ of } P \text{ is said to be fully } \mathcal{F}\text{-centralized if } |C_P(R)| \leq |C_P(Q)| \text{ for any } R \leq P \text{ and } \mathcal{F}\text{-isomorphic to } Q. \text{ Q is called fully } \mathcal{F}\text{-normalized if } |N_P(R)| \leq |N_P(Q)| \text{ for any } R \leq P \text{ and } \mathcal{F}\text{-isomorphic to } Q. \]

Note that if a subgroup is fully \( \mathcal{F} \)-normalized and fully \( \mathcal{F} \)-centralized it must be fully \( \mathcal{F} \)-automized since \( |N_P(Q)| = |C_P(Q)|\text{Aut}_P(Q) \), for some subgroup \( Q \).

\[
\text{Example 5.8.} \quad \text{Take } G = S_4 \text{ and let } P = \langle(1243), (12) \rangle \text{ then } P \in \text{Syl}_p(G) \text{ and } P \cong D_4. \text{ Now look at the subgroup } Q = \{id, (12)(34), (13)(24), (14)(23)\} \text{ which is a subgroup of } P \text{ and which is isomorphic to the Klein-4-group. Since } Q \text{ is normal in } G \text{ and conjugation by } (123) \text{ permutes } (12)(34), (13)(24) \text{ and } (14)(23) \text{ transitively, we know that } H_1 = \{id, (12)(34)\}, H_2 = \{id, (13)(24)\} \text{ and } H_3 = \{id, (14)(23)\} \text{ make a conjugacy class in } G. \text{ We note that } H_1 \triangleleft P \text{ while } H_2 \text{ and } H_3 \text{ are not, hence } H_1 \text{ is fully } \mathcal{F}\text{-normalized. Furthermore, } N_P(H_1) = P \in \text{Syl}_p(N_G(H_1)) \text{ and thus, by Proposition 5.6, } H_1 \text{ is receptive.}
\]

\[
\text{Proposition 5.9.} \quad \text{Let } \mathcal{F} \text{ be a fusion system on a finite group } G \text{ and let } P \leq G \text{ such that } P \in \text{Syl}_p(G). \text{ Let } Q \leq P
\]

1. \( Q \) is fully \( \mathcal{F} \)-centralized if and only if \( C_P(Q) \in \text{Syl}_p(C_G(Q)) \)

2. \( Q \) is fully \( \mathcal{F} \)-normalized if and only if \( N_P(Q) \in \text{Syl}_p(N_G(Q)) \)

\[
\text{Proof.} \quad \text{First we prove 1. Let } S \in \text{Syl}_p(C_G(Q)) \text{ such that } C_P(Q) \leq S. \text{ By Sylow’s theorem there is a } g \in G \text{ such that } (QS)^g \leq P \text{ and we have that } Q \cong (Q^g)^2. \text{ For any } y \in S^2, ggg^{-1} \in C_G(Q) \text{ which implies that } (ggy^{-1}zg^{-1}y^{-1}) = z \Leftrightarrow g(y^{-1}zg)y^{-1} = g^{-1}zy \text{ for all } z \in Q. \text{ Hence } S \leq C_G(Q^2) \cap P = C_P(Q^2) \text{ and we conclude that } |C_P(Q)| \leq |S| \leq |C_P(Q^2)|. \text{ From here it is easy to see that } Q \text{ is fully } \mathcal{F}\text{-centralized if and only if } |C_P(Q)| = |S|.
\]

To prove 2, just use the same argument for normalizers insted of centralizers.

We will now define what is meant by a saturated fusion system. The axioms for fusion systems are quite hard to work with alone whereas the concept saturation solves a lot of these problems.

\[
\text{Definition 5.10.} \quad \text{Let } \mathcal{F} \text{ be a fusion system on a finite } p\text{-group } P. \text{ We say that } \mathcal{F} \text{ is saturated if every } \mathcal{F}\text{-conjugacy class of subgroups of } P \text{ contains a subgroup that is both receptive and fully } \mathcal{F}\text{-automized.}
\]
Theorem 5.11. Let $G$ be a finite group and let $P$ be a Sylow $p$-subgroup of $G$. The fusion system $\mathcal{F}_P(G)$ is saturated.

Proof. We want to show that every $\mathcal{F}_P(G)$-conjugacy class of $P$ contains a fully $\mathcal{F}$-automized and receptive subgroup. Take any $R \leq P$, then there is some $Q \leq P$ $\mathcal{F}$-conjugate to $R$ such that $Q$ is fully $\mathcal{F}$-normalized. Hence, by Proposition 5.9, $N_P(Q) \in \text{Syl}_p(N_G(Q))$ and then by Proposition 5.6, $Q$ is receptive. Now notice that $N_P(Q)/C_G(Q) \in \text{Syl}_p(N_G(Q)/C_G(Q))$ and $\text{Aut}_G(Q) \cong N_G(Q)/C_G(Q)$ and

$$\frac{N_P(Q)C_G(Q)}{C_G(Q)} \cong \frac{N_P(Q)}{N_P(Q) \cap C_G(Q)} = \frac{N_P(Q)}{C_P(Q)} \cong \text{Aut}_P(Q). \quad (7)$$

Hence $\text{Aut}_P(Q)$ is a Sylow $p$-group of $\text{Aut}_G(Q) = \text{Aut}_{\mathcal{F}_P(G)}(Q)$, which shows that $Q$ is fully automized.

Theorem 5.11 implies that the class of saturated fusion systems is at least as big as the class of finite groups.

Proposition 5.12. Let $\mathcal{F}$ be a fusion system on a finite $p$-group $P$ and let $Q, R$ be $\mathcal{F}$-isomorphic subgroups of $P$ such that $R$ is fully $\mathcal{F}$-automized. There exists an isomorphism $\psi : Q \to R$ in $\mathcal{F}$ such that $N_\psi = N_P(Q)$, i.e. if $R$ is receptive then $\psi$ extends to a morphism from $N_P(Q)$ to $N_P(R)$ in $\mathcal{F}$.

Proof. Since $R$ is fully $\mathcal{F}$-automized $\text{Aut}_P(R)$ is a Sylow $p$-subgroup of $\text{Aut}_\mathcal{F}(R)$. If $\phi : Q \to R$ is an isomorphism in $\mathcal{F}$, then $\phi \circ \text{Aut}_P(Q) \circ \phi^{-1}$ is a $p$-subgroup of $\text{Aut}_\mathcal{F}(R)$ and hence there is an $\alpha \in \text{Aut}_\mathcal{F}(R)$ such that $\alpha \circ \phi \circ \text{Aut}_P(Q) \circ \phi^{-1} \circ \alpha^{-1} \leq \text{Aut}_P(R)$. Now let $\psi = \alpha \circ \phi$. Now this says exactly that for any $x \in N_P(Q)$ there is an $y \in N_P(R)$ such that $\psi \circ c_x \circ \psi^{-1} = c_y \iff \psi \circ c_x = c_y \circ \psi$ as $\psi$ is injective. But then $N_P(R)$ satisfies the definition of $N_\psi$ and we get $N_\psi = N_P(Q)$.

We denote by $c_Q$, the map $N_P(Q) \to \text{Aut}_P(Q)$ defined by $c_Q(g) = c_g$ for $g \in N_P(Q)$.

Proposition 5.13. Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $P$. Let $Q$ and $R$ be $\mathcal{F}$-isomorphic subgroups of $P$ and let $\phi : Q \to R$ be a morphism in $\mathcal{F}$. Suppose $\phi$ extends to a morphism $\bar{\phi} : S \to P$ for some $S \leq N_P(Q)$. Then the image of $\bar{\phi}$ is contained within $N_P(R)$ and we have that $c_Q(S) \leq \text{Aut}_P(Q) \cap \phi^{-1}\text{Aut}_P(R)$.

Proof. Let $x \in S$. For all $g \in Q$ we have that $x^g \in Q$ and thus $\bar{\phi}(x^g) = \bar{\phi}(xg\bar{x}^{-1}) \in R \Rightarrow \bar{\phi}(x) \in N_P(R)$. Thus the image of $\bar{\phi}$ is in $N_P(R)$. Also, we have that $c_R(\bar{\phi}(x)) \in \text{Aut}_P(R)$, therefore $\phi(c_Q(S)) = c_R(\phi(S)) \leq \text{Aut}_P(R)$. And since $c_Q(S) \leq \text{Aut}_P(Q)$ we have that $c_Q(S) \leq \text{Aut}_P(Q) \cap \phi^{-1}\text{Aut}_P(R)$.

Proposition 5.14. Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $P$. $Q \leq P$ is receptive if and only if it is fully $\mathcal{F}$-centralized.

Proof. Suppose a subgroup $Q \leq P$ is receptive and not fully $\mathcal{F}$-centralized. There exists a fully $\mathcal{F}$-centralized subgroup $R$ of $P$ which is $\mathcal{F}$-conjugate to $Q$. Now define $\phi : R \to Q$ to be an isomorphism in $\mathcal{F}$. Since $Q$ is receptive there
is a morphism \( \psi : N_\phi \to N_P(Q) \) such that \( \psi|_R = \phi \). Now \( RC_P(R) \leq N_\phi \) and \( \psi(R) = Q \). Now suppose that \( x \in \psi(C_P(R)) \). Any \( q \in Q \) may be written as \( \psi(q) \). Hence \( \psi^{-1}(x)(\psi^{-1}(x))^{-1} = x \lhd x\psi(r)x^{-1} = \psi(r) \) which implies that \( x \in C_P(Q) \). Hence \( \psi(C_P(R)) \leq C_P(Q) \Rightarrow |C_P(R)| \leq |C_P(Q)| \), which is a contradiction.

Conversely, suppose \( Q \) is fully \( \mathcal{F} \)-centralized but not receptive. Since \( \mathcal{F} \) is saturated there exists some \( R \leq P \), \( \mathcal{F} \)-conjugate to \( Q \) and such that \( R \) is fully \( \mathcal{F} \)-automized and receptive. Then by Proposition 5.14 there is an isomorphism \( \psi : Q \to R \) that can be extended to a morphism \( \psi : N_P(Q) \to N_P(R) \). Hence \( \psi \) is fully \( \mathcal{F} \)-centralized but not receptive. Since \( \psi \) is fully \( \mathcal{F} \)-centralized, \( \psi \) is also fully \( \mathcal{F} \)-automized, and thus \( \psi \) has an extension \( \bar{\psi} \) extending \( \psi \) such that \( \bar{\psi}(N_\phi) \leq \psi(N_P(Q)) \). Now, by part 1, both \( Q \) and \( R \) are fully \( \mathcal{F} \)-centralized and hence we have that \( \psi : C_P(Q) \to C_P(R) \) is an isomorphism so by Proposition 5.13 we have that \( \psi(N_P(Q)) \) is the full preimage in \( N_P(R) \) of \( \psi(C_P(Q)) \). Also \( \bar{\theta}(N_\phi) \) is the preimage in \( N_P(R) \) of \( c_R(\bar{\theta}(N_\phi)) = \bar{\theta}(c_S(N_\phi)) \leq \psi(C_P(Q)) \) since \( c_S(N_\phi) \) is fully \( \mathcal{F} \)-centralized and thus \( \bar{\theta}(N_\phi) \leq \psi(N_P(Q)) \) so that \( \psi^{-1}|_{\bar{\theta}(N_\phi)} \) is a map from \( N_\phi \) to \( N_P(Q) \) extending \( \phi = \psi^{-1}\bar{\theta} \).

**Theorem 5.15.** Let \( \mathcal{F} \) be a saturated fusion system on a finite \( p \)-group \( P \), and let \( Q \leq P \). Then \( Q \) is fully \( \mathcal{F} \)-normalized if and only if \( Q \) is fully \( \mathcal{F} \)-automized and receptive.

**Proof.** Suppose that \( Q \) is fully \( \mathcal{F} \)-automized and receptive. Then we have from Proposition 5.14 that \( Q \) is fully \( \mathcal{F} \)-centralized. Thus, since \( Q \) is both fully \( \mathcal{F} \)-centralized and fully \( \mathcal{F} \)-automized we have from \( |N_P(Q)| = |C_P(Q)||\text{Aut}_P(Q)| \) that \( Q \) is fully \( \mathcal{F} \)-normalized.

Now suppose that \( Q \) is fully \( \mathcal{F} \)-normalized. Then, since \( \mathcal{F} \) is saturated, we can find an \( R \leq P \), \( \mathcal{F} \)-isomorphic to \( Q \), such that \( R \) is fully \( \mathcal{F} \)-automized and receptive. Now from Proposition 5.14 we know that \( R \) is fully \( \mathcal{F} \)-centralized since it is receptive. Thus, from the arguments above, we have that \( R \) is fully \( \mathcal{F} \)-normalized. This implies that \( |N_P(R)| = |N_P(Q)| \), hence

\[
|N_P(Q)| = |\text{Aut}_P(Q)||C_P(Q)| = |\text{Aut}_P(R)||C_P(R)| = |N_P(R)|
\]  

Now since \( R \) is fully \( \mathcal{F} \)-automized and fully \( \mathcal{F} \)-centralized we must have that \( |C_P(R)| \geq |C_P(Q)| \) and \( |\text{Aut}_P(R)| \geq |\text{Aut}_P(Q)| \) as \( |\text{Aut}_\mathcal{F}(R)| = |\text{Aut}_\mathcal{F}(Q)| \). Hence we must have equality for both atomizers and centralizers. And thus \( Q \) is fully \( \mathcal{F} \)-automized and receptive, since, from Proposition 5.14, fully \( \mathcal{F} \)-centralized implies receptive.

\[\square\]

### 6 Alperins fusion theorem

In this chapter we will look at Alperin’s fusion theorem [1], which tells that conjugation in a finite group can be carried out in a series of conjugations by elements with certain properties. In Chapter 9 we will look at a reformulation
of Alperin’s fusion theorem which is more suitable for fusion systems. We will also bring up some examples to show what consequences this has on the fusion systems on finite groups.

We start with some definitions and then a series of lemmas which we will use to prove Alperin’s fusion theorem

**Definition 6.1.** Let $P$ be Sylow $p$-subgroups of $G$. For $R, Q \in \text{Syl}_p(G)$ write $R \sim Q$ if there exists Sylow $p$-subgroups $Q_1, Q_2, \ldots, Q_n$ and elements $x_1, x_2, \ldots, x_n$ s.t.

1. $P \cap Q_i$ is a tame intersection, $1 \leq i \leq n$,
2. $x_i$ is a $p$-element in $N_G(P \cap Q_i)$, $1 \leq i \leq n$,
3. $P \cap R \leq P \cap Q_1$ and $(P \cap R)^{x_1x_2 \ldots x_i} \leq P \cap Q_{i+1}, 1 \leq i \leq n - 1$
4. $R^x = Q$ where $x = x_1x_2 \ldots x_n$

we say $R \sim Q$ via $x$ and that the set $\{Q_i, x_i : 1 \leq i \leq n\}$ accomplish $R \sim Q$.

**Lemma 6.2.** The relation $\sim$ is transitive.

**Proof.** If $\{R_i, y_i : 1 \leq i \leq m\}$ and $\{Q_i, x_i : 1 \leq i \leq n\}$ accomplish $S \sim R$ and $R \sim Q$ respectively, then $R_1, \ldots, R_m, Q_1, \ldots, Q_n$ accomplish $S \sim Q$. $\square$

**Lemma 6.3.** If $S \sim P$ via $x$, $Q^x \sim P$ and $P \cap Q \leq P \cap S$, then $Q \sim P$.

**Proof.** By Lemma 6.2 it is enough to show that $Q \sim Q^x$.

Claim: If $\{S_i, x_i : 1 \leq i \leq n\}$ accomplish $S \sim P$ then $\{S_i, x_i : 1 \leq i \leq n\}$ also accomplish $Q \sim Q^x$. pf: (1),(2) and (4) in Definition 6.1 are trivial and (3) is clear as $P \cap Q \leq P \cap S \leq P \cap S_i$ since $S \sim P$. $\square$

**Lemma 6.4.** Assume $R, Q \in \text{Syl}_p(G)$ with $R \sim P$ and $P \cap Q < R \cap Q$. Assume further, for all $S \in \text{Syl}_p(G)$ with $|S \cap P| > |Q \cap P|$, that $S \sim P$. Then $Q \sim P$.

**Proof.** By assumption $\exists x \in G$ such that $R \sim P$ via $x$. Now $P \cap Q^x = R^x \cap Q^x = (R \cap Q)^x$, so $|P \cap Q| = |P \cap Q^x| > |P \cap Q|$.

Hence $Q^x \sim P$ and by lemma 6.3 $Q \sim P$. $\square$

**Lemma 6.5.** Assume $P$ and $Q$ be Sylow $p$-subgroups of $G$, $P \cap Q$ is a tame intersection, and $S \sim P$, $\forall S \in \text{Syl}_p(G)$ such that $|P \cap S| > |P \cap Q|$. Then $Q \sim P$.

**Proof.** By Lemma 6.2 we may assume that $Q \neq P$. Thus $P \cap Q < P_0 = N_P(P \cap Q) \in \text{Syl}_p(N_G(P \cap Q))$ and $N_Q(P \cap Q) \in \text{Syl}_p(N_G(P \cap Q))$.

Hence there is an $x \in N_G(P \cap Q)$ such that $N_Q(P \cap Q)^x = N_P(P \cap Q)$ and thus $Q \sim Q^x$ is accomplished by $\{Q, x\}$.

Furthermore $P \cap Q < P_0 \leq P \cap Q^x$, so by hypothesis $Q^x \sim P$. Hence by Lemma 6.2 $Q \sim Q^x \sim P \Rightarrow Q \sim P$. $\square$

**Lemma 6.6.** If $Q, P \in \text{Syl}_p(G)$ then $Q \sim P$. 

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Proof. Proceed by induction on $|P : P \cap Q| = n$. If $n = 1 \Rightarrow P \cap Q = P \Rightarrow Q = P$ and $Q \sim P$.

Now assume $Q \neq P$ and that $S \sim P, \forall S \in \text{Syl}_p(G)$ such that $|S \cap P| > |Q \cap P|$. Take $S \in \text{Syl}_p(N_G(P \cap Q))$ containing $N_P(P \cap Q)$, and let $R \in \text{Syl}_p(G)$ such that $S \leq R \Rightarrow P \cap R \geq P \cap S \geq N_P(P \cap Q) \Rightarrow P \cap Q \Rightarrow R \sim P$ via some $x \in G$, by induction. Thus, since $P \cap R \geq P \cap Q$ and $R \sim P$ via some $x \in G$ we need only show that $Q^x \sim P$ to establish $Q \sim P$.

First, $(P \cap Q)^x \leq S^x \leq P$ so that $P \cap Q^x \geq P \cap (P \cap Q)^x = (P \cap Q)^x$. However, if $|P \cap Q^x| > |P \cap Q|$ then $Q^x \sim P$ by induction. Thus we may assume $|P \cap Q^x| = |P \cap Q| \Rightarrow P \cap Q^x = (P \cap Q)^x$.

Claim: $N_P(P \cap Q^x) \in \text{Syl}_p(N_G(P \cap Q^x))$.

To prove this take $S \in \text{Syl}_p(N_G(P \cap Q)) \Rightarrow S^x \in \text{Syl}_p(N_G(P \cap Q^x))$ but $N_G(P \cap Q^x) = N_G(P \cap Q^x) = N_G(P \cap Q^x) \Rightarrow S \in \text{Syl}_p(N_G(P \cap Q^x))$. However $S^x \leq R^x = P$ so that $S^x \leq N_P(P \cap Q^x)$ but $N_P(P \cap Q^x)$ is a $p$-subgroup of $N_G(P \cap Q^x)$ containing a Sylow $p$-subgroup of $N_G(P \cap Q^x)$ thus $N_P(P \cap Q^x) \in \text{Syl}_p(N_G(P \cap Q^x))$.

Let $T \in \text{Syl}_p(N_G(P \cap Q^x))$ containing $N_Q(P \cap Q^x)$ and let $U \in \text{Syl}_p(G)$ such that $T \leq U$.

Claim: It is enough to show that $U \sim P$ to complete the proof.

Since $P \cap Q^x < Q$ as $|P \cap Q^x| = |P \cap Q|$ so that $U \cap Q^x \geq N_U(P \cap Q^x) > P \cap Q^x \Rightarrow$ if we show $U \sim P$ then by Lemma 6.4 $Q^x \sim P$ and we are done.

$P \cap U \geq P \cap T \geq P \cap Q^x$ so if $P \cap U > P \cap Q^x$ we are done by induction.

Assume $P \cap U = P \cap Q^x$. $T = N_U(P \cap Q^x)$ by choice of $T$ and $U \Rightarrow$ since $P \cap Q^x = P \cap Q$ we have $N_U(P \cap U) \in \text{Syl}_p(N_G(P \cap U)) \Rightarrow P \cap U$ is a tame intersection. Finally, $|P \cap U| = |P \cap Q|$ so that $S \sim P, \forall S \in \text{Syl}_p(G)$ with $|P \cap S| > |P \cap U|$ and by Lemma 6.6, $U \sim P$.

Now the proof of Alperin’s fusion theorem will be easy.

Theorem 6.7 (Alperin’s Fusion Theorem, [1]). Let $G$ be a finite group and $P \in \text{Syl}_p(G)$. Let $A, A^g \subseteq P$, for some $g \in G$.

Then there exists elements $x_1, x_2, \ldots, x_n$, subgroups $Q_1, Q_2, \ldots, Q_n \in \text{Syl}_p(G)$ and an $y \in N_G(P)$ such that

1. $g = x_1x_2\ldots x.ny$,
2. $P \cap Q_i$ is a tame intersection, $0 \leq i \leq n$,
3. $x_i$ is a $p$-element of $N_G(P \cap Q_i)$, $0 \leq i \leq n$,
4. $A \subseteq P \cap Q_i$ and and $A^{x_1x_2\ldots x_i} \subseteq P \cap Q_i + 1$, $0 \leq i \leq n - 1$.

Proof. $A \subseteq P \Rightarrow A^g \subseteq P^g \Rightarrow A \subseteq P^{g^{-1}} \cap P$. By Lemma 6, there is an $x \in G$ such that $P^{g^{-1}} \sim P$ via $x$. Let $\{Q_i, x_i : 1 \leq i \leq n\}$ accomplish $P^{g^{-1}} \sim P \Rightarrow P^{g^{-1}x_i} = P$ which implies that $y = x^{-1}g \in N_G(P)$. (2) and (3) are clear and for (1) and (4) we have

1. $g = x(x^{-1}g) = x_1x_2\ldots x_ny$.
2. $A \subseteq P \cap P^{g^{-1}} \leq P \cap Q_i$, since $P^{g^{-1}} \sim P$ and $A^{x_1\ldots x_n} \subseteq (P \cap P^{g^{-1}})^{x_1\ldots x_n} \leq P \cap Q_i + 1$.
Alperin’s theorem deals with the collection of pairs \((H_i, T_i)\) where \(H = P \cap Q_i\) is a tame intersection for some \(Q_i \in \text{Syl}_p(G)\) and \(T_i\) consists of all \(p\)-elements of \(N_G(H)\). This observation lead naturally to the following definition.

**Definition 6.8.** Let \(G\) be a finite group and \(P\) a Sylow \(p\)-subgroup of \(G\).

1. A family is a collection of subgroups of \(P\). We denote a family by \(C\).

2. A family \(C\) is a conjugation family if whenever \(A, A^g\) are subgroups of \(P\) for some \(g \in G\), then there are elements \(H_i \in C\), elements \(x_i \in N_G(H_i)\), for \(1 \leq i \leq n\), and \(y \in N_G(P)\) such that \(g = x_1x_2 \ldots x_ny\) and \(A^{x_1x_2 \ldots x_i} \in H_i + 1\) for \(0 \leq i \leq n\).

3. A family \(C\) is a weak conjugation family if whenever \(A, A^g\) are subgroups of \(P\) for some \(g \in G\), then there elements \(H_i \in C\), elements \(x_i \in N_G(H_i)\), for \(1 \leq i \leq n\), and \(y \in N_G(P)\) such that \(A^g = A^{x_1x_2 \ldots x_n}y\) and \(A^{x_1x_2 \ldots x_i} \in H_i + 1\) for \(0 \leq i \leq n - 1\).

Alperin’s theorem clearly implies that if \(C\) is the family consisting of all \(H \leq P\) such that \(H = P \cap Q\) is a tame intersection for some \(Q \in \text{Syl}_p(G)\), then \(C\) is a conjugation family.

Note that in a weak conjugation family we do not require the elements \(g\) and \(x_1x_2 \ldots x_ny\) to be equal, only that they induce the same map in \(\text{Hom}_{G}(A, P)\).

We see that weak conjugation families are the right concept for fusion systems, as we do not care about the particular element, but only the conjugation map it induces. An equivalent definition of a weak conjugation family, which is better suited for fusion systems, is the following one.

**Definition 6.9.** Let \(P\) be a \(p\)-group and \(\mathcal{F}\) a saturated fusion system on \(P\). A weak conjugation family \(\mathcal{C}\) is a collection of subgroups of \(P\) such that \(\mathcal{F} = \langle \text{Aut}(Q) : Q \in \mathcal{C} \rangle\), i.e. every morphisms in \(\mathcal{F}\) can be written as a finite composition of automorphisms in \(\mathcal{F}\), of subgroups in \(\mathcal{C}\).

Alperin showed in [1], that the family \(\mathcal{C}_A\), consisting of subgroups \(H\) of \(P\) where \(H = P \cap Q\) is a tame intersection for some \(Q \in \text{Syl}_p(G)\) and such that \(C_G(H) \leq H\), is a weak conjugation family. This will be investigated further in the next section.

### 7 Centric, radical and essential subgroups

The aim of Section 8 is to prove Alperin’s fusion theorem for fusion systems. To describe the set of subgroups which controls conjugation in a fusion system we need the concept of centric, radical and essential subgroups.

Goldschmidt showed in [7], as a refinement of Alperin’s fusion theorem, that if \(\mathcal{F}\) is a saturated fusion system on a \(p\)-group \(P\), then the class of essential subgroups of \(P\) determines the fusion system. Puig showed the class of essential subgroups is also the smallest class which will determine the fusion system. As Linckelmann writes in [10]; "The class of essential subgroups are essential”.

We denote by \(O_p(G)\), the largest normal \(p\)-subgroup of \(G\).

**Definition 7.1.** Let \(G\) be a finite group and \(p\) a prime dividing \(|G|\). A proper subgroup \(M\) is strongly \(p\)-embedded in \(G\), if \(M\) contains a Sylow \(p\)-subgroup \(P\) of \(G\) and \(M \cap P^g = 1\) for any \(g \in G \setminus M\).
Example 7.2. Let $G = S_4$ and $P$ a Sylow $2$-subgroup of $G$. $G$ has a normal subgroup $N$ of order $2^2$ and hence $G$ can not have a strongly $p$-embedded subgroup, since for any $M \leq G$, $M \cap M^g$ contains $N$ for all $g \in G$. In fact we have the following:

Remark. If $G$ contains a strongly $p$-embedded subgroup, then $O_p(G) = 1$. We will prove this in the proof of Proposition 7.8.

Definition 7.3. Let $\mathcal{F}$ be a fusion system on a finite $p$-group $P$. A subgroup $Q$ is $\mathcal{F}$-centric if for every subgroup $R$, $\mathcal{F}$-isomorphic to $Q$, $C_P(R) \leq R$.

With this definition it is relevant to look back at the family $C_A$, described at the end of Section 6.

Proposition 7.4. Let $G$ be a finite group, $P$ a Sylow $p$-subgroup of $G$ and $\mathcal{F} = F_P(G)$. If $H$ is a subgroup of $P$ such that $H = P \cap Q$ is a tame intersection for some $Q \in \text{Syl}_p(G)$ and such that $C_G(H) \leq H$, then $H$ is fully $\mathcal{F}$-normalized and $\mathcal{F}$-centric.

Proof. Since $N_P(H) \in \text{Syl}_p(N_G(H))$, $H$ is fully $\mathcal{F}$-normalized by Proposition 5.9.

We have that $C_P(H) \leq C_G(H) \leq H$. So we want to show that for any $g \in G$, $C_P(qH) \leq qH$. If $qH = H$ then this is obvious so suppose $qH \neq H$. Let $x$ be any element in $C_G(qH)$. Then $\forall h \in H$, $xghg^{-1}x^{-1} = ghg^{-1}$ and hence $g^{-1}xg \in C_G(H)$. This implies that $x \in qC_G(H)$ and hence $C_P(qH) \leq C_G(qH) \leq qC_G(H) \leq qH$.

From this result, it is clear that Alperin’s theorem implies that the collection of all fully normalized, centric subgroups determines the hole fusion system. This is however not the smallest class of subgroups which determines the fusion system.

Lemma 7.5. Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $P$. If $Q \leq P$ is fully $\mathcal{F}$-centralized then $QCP(Q)$ is $\mathcal{F}$-centric.

Proof. Suppose $Q$ is fully $\mathcal{F}$-centralized and let $\tilde{Q} = QCP(Q)$. Since $Q \leq \tilde{Q}$ and $C_P(Q) \leq \tilde{Q}$, $C_P(Q)$ centralizes both $Q$ and $C_P(Q)$ and hence $C_P(\tilde{Q}) \leq C_P(Q) \cap C_P(C_P(Q)) \leq \tilde{Q}$.

Let $\phi : Q \to R$ be an $\mathcal{F}$-isomorphism. If $\phi(a) \in \phi(C_P(Q))$, then $\forall \phi(q) \in \phi(Q)$, $\phi(a)\phi(q)\phi(a)^{-1} = \phi(q) \to \phi(a) \in C_P(\phi(Q))$. Hence we have that $R = \phi(QCP(Q)) = \phi(Q)\phi(C_P(Q)) \leq \phi(Q)\phi(C_P(\phi(Q)))$.

Now, as $Q$ is fully centralized, $|C_P(Q)| \leq |C_P(\phi(Q))|$ and hence $C_P(Q) \leq R$, by the same argument that $C_P(\tilde{Q}) \leq \tilde{Q}$. Thus $\tilde{Q}$ is $\mathcal{F}$-centric.

Definition 7.6. Let $\mathcal{F}$ be a fusion system on a finite $p$-group $P$. A subgroup $Q$ is $\mathcal{F}$-radical if

$$O_p(\text{Aut}_\mathcal{F}(Q)) = \text{Inn}(Q).$$

Note in particular that if $\mathcal{F}$ is a saturated fusion system on a $p$-group $P$, then $P$ is $\mathcal{F}$-radical as $\text{Inn}(P) = \text{Aut}_P(P) \in \text{Syl}_p(\text{Aut}_\mathcal{F}(P))$.

Definition 7.7. Let $\mathcal{F}$ be a fusion system on a finite $p$-group $P$. A subgroup $Q$ is $\mathcal{F}$-essential if $Q$ is $\mathcal{F}$-centric and $\text{Out}_\mathcal{F}(Q) \cong \text{Aut}_\mathcal{F}(Q)/\text{Inn}(Q)$ contains a strongly $p$-embedded subgroup.
Here it is good to make the following observation.

**Proposition 7.8.** Let $F$ be a fusion system on a finite $p$-group $P$. If $Q$ is an $F$-essential subgroup of $P$, then $Q$ is $F$-radical.

**Proof.** Suppose $Q \leq P$ is $F$-essential and let $R = \Out_F(Q)$. Then $R$ contains a strongly $p$-embedded subgroup $M$, i.e. $M < R$ contains a Sylow $p$-subgroup $S$ of $R$ such that $p \nmid |M \cap S^p|$ for all $g \in R \setminus M$. But then $R$ can not contain a non-trivial normal $p$-group since if it did, this group would be contained in every Sylow $p$-subgroup of $R$ and hence also in $M \cap S^p$ which contradicts the fact that $p \nmid |M \cap S^p|$. Hence $O_p(R) = 1$ which implies that $O_p(\Aut_F(Q)/\Inn(Q)) = 1$. But this is the same as saying that $O_p(\Aut_F(Q)) = \Inn(Q)$. \qed

As we will see in Section 8, the family of fully $F$-normalized, $F$-essential subgroups is a weak conjugation family. Hence by Proposition 7.8 and Proposition 7.4, we can make the following observation. If $\mathcal{C}$ is a family of subgroups $H$ of $P$, such that for every $H \in \mathcal{C}$, $H = P \cap Q$ is a tame intersection for some $Q \in \text{Syl}_p(G)$, $C_G(H) \leq H$ and $N_G(H)/H$ contains no non-trivial normal $p$-subgroup, then $\mathcal{C}$ is a weak conjugation family.

Next Proposition will be necessary in the proof of Alperin’s fusion theorem for fusion systems. Furthermore, it gives a picture of what it means for a subgroup to be strongly $p$-embedded.

**Proposition 7.9.** Let $G$ be a finite group and let $p$ be a prime dividing $|G|$. Let $X$ be the partially ordered set of non-trivial $p$-subgroups of $G$, where the partial order is given by inclusion of one subgroup into the next. Let $A_p(G)$ be the undirected graph corresponding to $X$ in the natural way. Then $G$ contains a strongly $p$-embedded subgroup if and only if $A_p(G)$ is disconnected.

**Proof.** Suppose that $G$ has a strongly $p$-embedded subgroup $M$ and $P$ a Sylow $p$-subgroup of $G$ such that $P \leq M$. Let $g \in G \setminus M$. If $P$ and $P^g$ are in different components in $A_p(G)$ then of course, $A_p(G)$ is disconnected. Hence suppose, to get a contradiction, that $P$ and $P^g$ are in the same component in $A_p(G)$. Denote by $S$, the set of all Sylow $p$-subgroups of $G$ and let $S_M$ be the set of all Sylow $p$-subgroups of $M$. Let the distance on $S$ be the function 

$$d : S \times S \to \mathbb{N}, \quad (R_1, R_2) \mapsto d(R_1, R_2)$$

defined as follows; for any $R_1, R_2 \in S$, if $K_1 \leq K_2 \leq \cdots \leq K_k$ is a shortest path from $R_1$ to $R_2$, then $d(R_1, R_2) = k$.

Now choose $R \in S_M$ such that for any $R' \in S_M$, $d(R, P^m) \leq d(R', P^m)$. Let $Q_1 \leq Q_2 \leq \cdots \leq Q_n$ be a path of minimal length connecting $Q_1 \leq R$ and $Q_n \leq P^m$, i.e. $d(R, P^m) = n$. Since $R$ is a Sylow $p$-subgroup of $M$, there is an $h \in M$ such that $R = P^h$. Hence $R \cap P^g = 1$, since if it is not, then $1 \neq P \cap P^{gh^{-1}} \leq M \cap P^{gh^{-1}}$ which is a contradiction since $gh^{-1} \notin M$. This implies that $n \geq 3$.

Now $Q_3$ is contained in some Sylow $p$-subgroup of $G$ and hence there is an $m \in G$ such that $Q_3 \leq P^m$. If $m \in G \setminus M$, then since $Q_1 \leq M \cap P^m$, this contradicts that $M$ is strongly $p$-embedded. Hence $m \in M$. But then $P^m \in S$ and $Q_3 \leq Q_4 \leq \cdots \leq Q_n$ is a shorter path then the one from $R$ to $P^g$. Then $d(R, P^g) > d(P^m, P^g)$, which is a contradiction since $P^m \in S_M$ and hence $P$ and $P^g$ must be in different components in $A_p(G)$. 21
Conversely, suppose that $A_p(G)$ is disconnected and let $P$ be a Sylow $p$-subgroup of $G$. Let $M$ be the set of all $g \in G$ such that $P^g$ is in the same component of $A_p(G)$ as $P$. Then $M$ is a proper subgroup of $G$ containing $P$. By definition of $M$, $P$ and $P^g$ are in different components of $A_p(G)$ and hence $P \cap P^g = 1$ for any $g \in G \setminus M$.

Now suppose, to get a contradiction, that $x \in G \setminus M$ and $M \cap P^x$ contains a $p$-group $Q$. Then there is a Sylow $p$-subgroup $S$ of $M$ containing $Q$ and hence, for some $y \in M$, $S^y = P$. Also $S^{x^y} = P$ which means that $Q \leq S^{x^y}$ and we get that $Q \subseteq S \cap S^{x^y}$. But this implies that $Q^y \subseteq S^y \cap S^{x^y} = P \cap P^y$. Since $M$ is a group, $y^{-1} \in M$ and since $x \notin M$ we have that $xy \notin M$. Hence $P \cap P^y \geq Q^y \neq 1$ is a contradiction and we conclude that $M$ is strongly $p$-embedded in $G$.

8 Alperin’s theorem for fusion systems

In this section we will prove a version of Alperin’s fusion theorem more suitable for abstract fusion systems. The formulation given here is due to Puig. Instead of conjugation by an element $g$ we will consider an $F$-isomorphism $\theta$ and prove that it can always be written as a composition series of $F$-automorphisms of subgroups of fully $F$-normalized essential subgroups of $P$.

All $F$-essential subgroups are $F$-centric radical and hence Alperin’s fusion theorem implies that a fusion system is determined by its subcategory of $F$-centric radical subgroups [10].

We will begin by a proposition followed by some lemmas that we will will use in the proof of the theorem.

Definition 8.1. If $M$ is a set of morphisms and $\phi$ is a morphism we write $\phi^{-1} \circ M \circ \phi = M^\phi$.

The following proposition makes it clear that if $F$ is a fusion system on a finite $p$-group $P$ and $Q \leq P$ is $F$-essential, then so is every subgroup $F$-isomorphic to $Q$.

Proposition 8.2. Let $F$ be a fusion system on a finite $p$-group $P$. If $E$ is an $F$-essential subgroup of $P$ and $\theta \in \operatorname{Aut}_F(P)$, then $\theta(E)$ is an $F$-essential subgroup of $P$.

Proof. Since $\theta(E)$ is $F$-isomorphic to $E$, $\theta(E)$ is $F$-centric as $E$ is. So we want to show that $\operatorname{Out}_F(\theta(E))$ contains a strongly-embedded subgroup.

We know that there is an $M \in \operatorname{Out}_F(E)$ such that $M$ contains a Sylow $p$-subgroup of $\operatorname{Out}_F(E)$ and $M \cap M^\phi$ is a $p'$-group $\forall \phi \in \operatorname{Out}_F(E) \setminus M$.

Observe that if $\alpha \in M$, then $\theta \circ \alpha \circ \theta^{-1} \in \operatorname{Out}_F(\theta(E))$.

Conjugating $M \cap M^\phi$ by $\theta$ is an isomorphism and hence $(M \cap M^\phi)^\theta = M^\theta \cap M^{\phi \circ \theta}$ which is then a $p'$-group. But we have that $M^{\phi \circ \theta} = \theta^{-1} \circ \phi^{-1} \circ M \circ \phi \circ \theta = (M^\theta)_{\theta \circ \phi \circ \theta^{-1}}$.

So we want to show that $\forall \beta \in \operatorname{Out}_F(\theta(E)) \setminus M^\theta$, $\exists \phi_\beta \in \operatorname{Out}_F(E) \setminus M$ such that $\beta = \theta^{-1} \circ \phi_\beta \circ \theta$. But this is obvious since, if $\beta \in \operatorname{Out}_F(\theta(E)) \setminus M^\theta$, then $\theta \circ \beta \circ \theta^{-1} \notin M$ and $\theta \circ \beta \circ \theta^{-1} \in \operatorname{Out}_F(E)$, and hence we put $\phi_\beta = \theta \circ \beta \circ \theta^{-1}$.

Thus $M^\theta \cap (M^\theta)^\beta$ is a $p'$-group $\forall \beta \in \operatorname{Out}_F(\theta(E)) \setminus M^\theta$ and hence $\theta(E)$ is $F$-essential.

□
Now we will prove a few lemmas that will be necessary in the proof of Alperin’s theorem for fusion systems.

**Lemma 8.3.** Let $F$ be a fusion system on a finite $p$-group $P$ and let $Q, R$ be subgroups of $P$. Let $\phi : Q \to R$ be an isomorphism in $F$ such that $R$ is fully $F$-normalized. Then there is an isomorphism $\psi : Q \to R$ in $F$ such that $N_\psi = N_P(Q)$, i.e. $\psi$ extends to a morphism from $N_P(Q)$ to $P$ in $F$.

**Proof.** From Theorem 5.15, we see that $R$ is fully $F$-automized and receptive. Hence by Proposition 5.12, there is an isomorphism $\psi : Q \to R$ in $F$ such that $N_\psi = N_P(Q)$. \hfill \Box

**Proposition 8.4.** Let $P$ be a finite group and $Q$ a subgroup of $P$. If $R$ and $S$ are both subgroups of $N_P(Q)$, containing $Q \cap C_P(Q)$ and such that $\text{Aut}_R(Q) = \text{Aut}_S(Q)$, then $R = S$.

**Proof.** Let $x \in S$. Since $\text{Aut}_R(Q) = \text{Aut}_S(Q)$, there is an $y \in R$ such that $xq^{-1} = yq^{-1}$, $\forall q \in Q$. Hence $x^{-1}y \in C_P(Q)$ which is a subgroup of both $R$ and $S$. But since $x \in S$ and $y^{-1} \in R$ we have that $x \in R$ and $y \in S$, and thus $S = R$. \hfill \Box

**Lemma 8.5.** Let $F$ be a fusion system on a finite $p$-group $P$. Let $Q$ be a fully $F$-normalized subgroup of $P$ and let $\alpha \in \text{Aut}_F(Q)$. Then $R = N_\alpha$ is the unique subgroup of $N_P(Q)$ such that $Q \cap C_P(Q) \leq R$ and $\text{Aut}_R(Q) = \text{Aut}_P(Q) \cap (\alpha^{-1} \circ \text{Aut}_P(Q) \circ \alpha)$.

**Proof.** The uniqueness follows from Proposition 8.4.

To prove that $R \leq N_\alpha$, let $r \in R$. Then $r \in N_P(Q)$ and the conjugation map $c_r : Q \to Q$ is in $\text{Aut}_R(Q)$, and hence in $\text{Aut}_P(Q) \cap (\alpha^{-1} \circ \text{Aut}_P(Q) \circ \alpha) \Rightarrow c_r = \alpha^{-1} \circ c_\alpha \circ \alpha$ for some $c_\alpha \in \text{Aut}_P(Q)$. But then $a \in N_P(Q)$ and we have $\alpha(q) = \alpha^{-1}(\alpha(q))$, $\forall q \in Q$. Hence $r \in N_\alpha$ and we have $R \subseteq N_\alpha$.

Now we prove that $N_\alpha \leq R$. Let $a \in N_\alpha$, then $\exists z \in N_R(\alpha(Q))$ such that $\alpha(\alpha(q)) = \alpha(z \alpha(q))$, $\forall q \in Q$. Hence $c_\alpha \in \text{Aut}_R(Q)$ and thus there is an $r \in R$ such that $r^{-1}a = q \in Q \Rightarrow r^{-1}a \in C_P(Q) \leq R$ and hence $rr^{-1}a = a \in R \Rightarrow N_\alpha \subseteq R$ and we thus have $R = N_\alpha$. \hfill \Box

Note that we do not use the property of the fusion system that $\text{Hom}_P(A, B) \subseteq \text{Hom}_F(A, B)$, for any subgroups $A, B$ of $P$. In [10] Linckelmann uses another definition that he calls a category on a finite $p$-group, which is similar to what we call a fusion system on a finite $p$-group except that it misses the property $\text{Hom}_P(A, B) \subseteq \text{Hom}_F(A, B)$ for subgroups $A, B$ of $P$.

**Lemma 8.6.** Let $F$ be a saturated fusion system on a finite $p$-group $P$ and let $Q \leq P$ be a fully $F$-normalized. Then there is a unique subgroup $R$ of $N_P(Q)$ such that $Q \cap C_P(Q) \leq R$ and $\text{Aut}_R(Q) = \text{Aut}_F(Q)$. Furthermore, every automorphism $\alpha \in \text{Aut}_F(Q)$ extends to an automorphism $\beta \in \text{Aut}_F(R)$.

**Proof.** The uniqueness follows from Proposition 8.4.

For every $\alpha \in \text{Aut}_F(Q)$ we have $\alpha^{-1} \circ \text{Aut}_R(Q) \circ \alpha = \text{Aut}_R(Q)$ and hence, by Lemma 8.5, $R \subseteq N_\alpha$. Thus $\alpha$ extends to a morphism $\beta : R \to P$. Hence, for any $x \in R$ and any $q \in Q$ we have $\alpha(q) = \beta(q) = \beta(x)\alpha(q)$ which is equal to saying $\alpha^{-1} \circ c_x \circ \alpha = c_\beta(x)$. Since $\text{Aut}_R(Q)$ is normal in $\text{Aut}_F(Q)$ we have that $c_\beta(x) \in \text{Aut}_R(Q)$. Hence for every $q \in Q$ we have $c_\beta(x)(q) = \beta(x)q\beta(x)^{-1} = \cdots$
Let $M$ be the set of all isomorphisms in $\mathcal{F}$ that satisfy the claim of the theorem. Then $M$ is closed under composition and if $\alpha : A \to B$ is a morphism in $M$, so is $\alpha^{-1}$. We also have that the restriction of $\alpha$ to any subgroup $H$ of $A$ is in $M$.

Let $\phi : Q \to R$ be any isomorphism in $\mathcal{F}$. We proceed by induction on the index of $Q$ in $P$. If $Q = P$, there is nothing to prove so suppose that $Q$ is a proper subgroup of $P$.

We are going to use the fact that if $Q$ is a proper subgroup of $N_\alpha$ then $\phi$ extends to $N_\alpha$ and hence $\phi \in M$ by induction.

Let $\psi : R \to T$ be an isomorphism in $\mathcal{F}$ such that $T$ is fully $\mathcal{F}$-normalized. By Lemma 8.3, $\psi$ may be chosen so that it can be extended to $N_P(R)$ and hence $\psi \in M$ by induction. Then, as $\psi^{-1} \in M$, we only have to prove that $\psi \circ \phi \in M$.

We have that $\psi \circ \phi : Q \to T$ and since $T$ is fully $\mathcal{F}$-normalized, again by Lemma 8.3, there is an isomorphism $\beta : Q \to T$ in $\mathcal{F}$, which extends to $N_P(Q)$. Hence, by induction, $\beta \in M$ and it will suffice to show that $\psi \circ \phi \circ \beta^{-1} \in M$.

Note that $\psi \circ \phi \circ \beta^{-1} \in \text{Aut}_\mathcal{F}(T)$ and that $T$ is fully $\mathcal{F}$-normalized. Let $\xi = \psi \circ \phi \circ \beta^{-1}$. By Lemma 8.6 there is a unique subgroup $S$ of $N_P(T)$ containing $TC_P(T)$ and every automorphism in $\text{Aut}_\mathcal{F}(T)$ extends to a morphism in $\text{Aut}_\mathcal{F}(S)$. If $S > T$ then we are done by induction, so assume that $S = T$. Then by Lemma 8.6 $O_p(\text{Aut}_\mathcal{F}(T)) = \text{Aut}_S(T) = \text{Aut}_\mathcal{T}(T) = \text{Inn}(T)$ and $TC_P(T) \leq T$ implies $C_P(T) \leq T$. Hence $T$ is $\mathcal{F}$-radical centric.

If $T$ is essential, we are done so suppose it is not. Then by Proposition 7.9, the graph $\mathcal{A}_p(\text{Out}_\mathcal{F}(T))$ is connected. Put $A = \text{Aut}_\mathcal{F}(T)$ and $B = \xi \circ \text{Aut}_\mathcal{F}(T) \circ \xi^{-1}$. Since $T$ is fully $\mathcal{F}$-normalized, both $A$ and $B$ are Sylow $p$-subgroups of

$rq^{-1}$ for some $r \in R$ and we get that $r^{-1}\beta(x) \in C_P(Q)$. Since $C_P(Q)$ is a subgroup of $R$ we see that $\beta(x) \in R$ and hence $\beta \in \text{Aut}_\mathcal{F}(R)$.
\[\text{Aut}_\tau(T). \text{ Since } A_P(\text{Out}_\tau(T)) \text{ is connected and } \text{Out}_\tau(T) = \text{Aut}_\tau(T)/\text{Aut}(T) \]

there is a sequence

\[\text{Aut}_\tau(T) = S_1, S_2, \ldots, S_n = \xi \circ \text{Aut}_\tau(T) \circ \xi^{-1}\]

such that \(\text{Aut}_\tau(T) < S_i \cap S_{i+1}\) for \(1 \leq i \leq n - 1\). For \(1 \leq j \leq n\), choose \(\theta_j \in \text{Aut}_\tau(T)\) such that \(\theta_j \circ \text{Aut}_\tau(T) \circ \theta_j^{-1} = S_j\). Choose \(\theta_1 = \text{Id}_R\), then obviously \(\theta \in \mathcal{M}\). Now we proceed by induction over \(j\) to show that \(\theta_j \in \mathcal{M}\) for \(1 \leq j < n\).

Let \(1 \leq j < n\). Then we have that

\[\text{Aut}_\tau(T) < S_j \cap S_{j+1} = (\theta_j \circ \text{Aut}_\tau(T) \circ \theta_j^{-1}) \cap (\theta_{j+1} \circ \text{Aut}_\tau(T) \circ \theta_{j+1}^{-1})\]

and hence, as \(\text{Aut}_\tau(T)\) is normal in \(\text{Aut}_\tau(T)\),

\[\text{Aut}_\tau(T) = \theta_j^{-1} \circ \text{Aut}_\tau(T) \circ \theta_j < \text{Aut}_\tau(T) \cap (\theta_j^{-1} \circ \theta_{j+1} \circ \text{Aut}_\tau(T) \circ \theta_{j+1}^{-1} \circ \theta_j).\]

But then, by Lemma 8.7 and induction, \(\theta_j^{-1} \circ \theta_{j+1} \in \mathcal{M}\). Since \(\theta_1 \in \mathcal{M}\) we get, by induction over \(j\), that \(\theta_j^{-1} \in \mathcal{M}\) and as \(\mathcal{M}\) is closed under compositions, so is \(\theta_{j+1}\). Finally, we have that

\[S_n = \theta_n \circ \text{Aut}_\tau(T) \circ \theta_n^{-1} = \xi \circ \text{Aut}_\tau(T) \circ \xi^{-1}\]

which means that \(\xi \circ \theta_n \in N_{\text{Aut}_\tau(T)}(\text{Aut}_\tau(T))\). Hence \(\xi \circ \theta_n\) extends to \(N_P(T)\) and we have that \(\xi \circ \theta_n \in \mathcal{M}\). But \(\theta_n \in \mathcal{M}\) which means that \(\theta_n^{-1} \in \mathcal{M}\). Hence \(\xi \in \mathcal{M}\) which implies that \(\phi \in \mathcal{M}\). \(\square\)

As a consequence of Theorem 8.8 we have the following result.

**Example 8.9.** Let \(P\) be a dihedral, semidihedral or generalized quaternion group of order \(2^n \geq 16\). These groups have the following presentations.

\[D_{2^n} = \langle r, s : r^{2^{n-1}} = s^2 = 1, sr = r^{-1}s \rangle\]

\[SD_{2^n} = \langle r, s : r^{2^{n-1}} = s^2 = 1, sr = r^{2^{n-2}}s \rangle\]

\[Q_{2^n} = \langle r, s : r^{2^{n-1}} = s^4 = 1, sr = r^{-1}s, r^{2^{n-2}} = s^2 \rangle\]

Let \(r, s\) be generators of \(P\), such that \(|r| = 2^{n-1}, |s| = 2\) if \(P\) is semidihedral. Consider the order of the element \(r^i s\).

i) If \(P\) is dihedral, then \((r^i s)^2 = r^i s r^i s = r^i s r^{-i} s s = id\) for all \(i \in \mathbb{Z}\).

ii) If \(P\) is semidihedral, then \((r^i s)^2 = r^i s r^i s = r^i r^{2^{n-2}-1} s r^{i-1} s = \ldots = r^i r^{(2^{n-2}-1)i} = r^i 2^{n-2} i = id\) for all \(i \in \mathbb{Z}\).

iii) If \(P\) is generalized quaternion, then \((r^i s)^2 = r^i s r^i s = h^i h^{-i} x x = x^2\) and hence \((r^i s)^4 = id\) for all \(i \in \mathbb{Z}\).

Hence we conclude that \(|r^i s|\) is either 2 or 4 for all \(i \in \mathbb{Z}\). Note also the following three facts.

a) If \(P\) is dihedral, then every subgroup of \(P\) is either dihedral, cyclic or a Klein 4-group. This is easy to see since \(D_{2^n} \cong (\mathbb{Z}/2^{n-1}\mathbb{Z}) \rtimes_{\phi} (\mathbb{Z}/2\mathbb{Z})\), where \(\phi(0)\) is the identity map and \(\phi(1)\) is the inversion map. Observe that \(\mathbb{Z}_2 \rtimes_{\phi} \mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong V_4\) as the inverse of 1 in \(\mathbb{Z}/2\mathbb{Z}\) is 1.
b) If $P$ is generalized quaternion, then every subgroup of $P$ is either cyclic or generalized quaternion. This follows since

$$Q_{2^n} \cong ((\mathbb{Z}/2^{n-1}\mathbb{Z}) \rtimes_{\alpha} (\mathbb{Z}/4\mathbb{Z}))/((2^{n-2}, 2))$$

where we define $(a, b)(c, d) = (a + \alpha(a)c, b + d) = (a + (-1)^b c, b + d)$. Every subgroup of this quotient corresponds to a subgroup of $(\mathbb{Z}/2^{n-1}\mathbb{Z}) \rtimes_{\alpha} (\mathbb{Z}/4\mathbb{Z})$ containing $((2^{n-1}, 2))$ and hence we conclude that there are at most 4 $F$.

This implies that there are, up to conjugates, at most two different $F$-essential subgroup, there are only two possibilities for the outer automorphism group and hence we conclude that all subgroups must be either cyclic or generalized quaternion.

c) If $P$ is semidihedral, then $P$ has three maximal subgroups. One cyclic, one dihedral and one generalized quaternion. The maximal cyclic group is obviously $\langle r \rangle$. The maximal dihedral group is $\langle r^2, s \rangle$ as $sr^2 = r^2(2^{n-1}-1)s = r^{-2}s = (r^2)^{-1}s$ which is the dihedral relation. Finally, it is easy to check that the maximal generalized quaternion group is $\langle r^2, rs \rangle$.

Now 2-groups which are either dihedral or cyclic will have automorphism groups of order, a power of 2, and hence can not be $\mathcal{F}$-essential.

For every $i \in \mathbb{Z}$, put $E_i = \langle r^{2^i}, r^i s \rangle \cong V_4$ if $|r^i| = 2$, and $E_i = \langle r^{2^i}, r^i s \rangle \cong Q_8$ if $|r^i s| = 4$. Let $\mathcal{F}$ be any saturated fusion system on $P$. Then we have the following.

1. If $\phi \in \text{Aut}(P)$ have odd order, then since $\langle r \rangle \cong \mathbb{Z}/2^{n-1}\mathbb{Z}$ we must have that $\phi|_{\langle r \rangle} = \text{Id}_{\langle r \rangle}$ and $P/\langle r \rangle \cong \langle s \rangle$. Hence, by Lemma A.5, $\phi = \text{Id}_P$. Thus $\text{Aut}(P)$ is a 2-group, and hence so is $\text{Aut}_\mathcal{F}(P)$. Note that this argument also holds for any subgroup $Q$ of $P$, such that $Q$ is either dihedral, semidihedral or generalized quaternion and $|Q| \geq 16$. Now, since $\text{Aut}_\mathcal{F}(P)$ is a Sylow 2-subgroup of $\text{Aut}_\mathcal{F}(P)$, we must have that $\text{Aut}_\mathcal{F}(P) = \text{Aut}_P(P) = \text{Inn}(P)$.

2. From a), b), c) and (1), one may conclude that the only subgroups of $P$ whose automorphism groups are not 2-groups are the $E_i$ and hence these are the only subgroups which could be $\mathcal{F}$-essential. For example, if $Q \leq P$ is generalized quaternion of order $\geq 16$, then by (1), $\text{Aut}(Q)$ is a 2-group and hence, so is $\text{Aut}_\mathcal{F}(Q)$.

3. For each $i$ we have that $\text{Out}(E_i) \cong S_3$ and $\text{Out}_\mathcal{F}(E_i) \cong \mathbb{Z}/2\mathbb{Z}$. Hence $\text{Out}_\mathcal{F}(E_i)$ must be one of these groups.

4. If $i \equiv j(\text{mod } 2)$, then $E_i, E_j$ are $P$-conjugate, and hence $\text{Out}_\mathcal{F}(E_i) \cong \text{Out}_\mathcal{F}(E_j)$. This is not hard to prove since if $i \equiv j(\text{mod } 2)$, then $j = 2m + i$ for some $m \in \mathbb{Z}$, and hence, if $P$ is dihedral or generalized quaternion, $r^m(r^j s)r^{-m} = r^{2m+i} s$. If $P$ is semidihedral, then $r^m(r^j s)r^{-m} = r^{m+(1-2^{n-2})m}s = r^{2m+i-2^{n-2}m} \in r^m(E_i)r^{-m}$ and since $2^{n-3} \notin r^m(E_i)r^{-m}$ we get that $(r^{2^m})^{2m(r^{2m+i-2^{n-2}m})} = r^{2m+i} \in r^m(E_i)r^{-m}$ and hence $r^m(E_i)r^{-m} = E_j$.

This implies that there are, up to conjugates, at most two different $\mathcal{F}$-essential subgroups. For each $\mathcal{F}$-essential subgroup, there are only two possibilities for the outer automorphism group and hence we conclude that there are at most 4 distinct fusion systems on $P$.

All these fusion systems are all realizable by finite groups [5, 4]. For example, if $P \cong SD_{2^n}$ and $q \equiv 2^{n-2} - 1(\text{mod } 2^{n-1})$, then an example of four groups.
realizing distinct fusion systems on $P$ are, $SD_{2^n}$, $GL_2(q)$, $PSL_3(q)$ and a specific extension of $PSL_2(q^2)$ by $\mathbb{Z}/2\mathbb{Z}$ [4].

In Section 9, we will study the case where $P$ is the semidihedral group of order 16. Then one fusion system is also realizable by $GL_2(3)$.

9 A fusion system on $GL_2(3)$

Let $G = GL_2(3)$ and let $P$ be the Sylow 2-subgroup of $G$ generated by

\[
\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}
\]

Then $P$ is isomorphic to the semidihedral group $SD_{16} = \langle s, r : s^2 = r^8 = 1, sr = r^3s \rangle$. Here $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $r = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. Let $\mathcal{F} = \mathcal{F}_P(G)$. In Figure 3, we see the subgroup lattice of $P$.

![Subgroup lattice of $SD_{16}$](image)

Figure 3: Subgroup lattice of $SD_{16}$

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Let us for simplicity denote the subgroup of $P$, isomorphic to $Q_8$, by $Q_8$ etc.

**$F$-automorphisms of $Q_8$**

$$Q_8 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \right\}$$

and these matrices can be represented as $1$, $-1$, $i$, $-i$, $j$, $-j$, $k$, $-k$ respectively.

By Proposition A.7, $\text{Aut}(Q_8) \cong S_4$ and hence $|\text{Aut}_F(Q_8)| = 24$. $\text{Inn}(Q_8) \cong Q_8/Z(Q_8)$ which has order 4. Conjugation by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$$

induces 9 distinct automorphisms not in $\text{Inn}(Q_8)$ and hence we have found 13 automorphisms and since $|\text{Aut}_F(Q_8)| = 24$ we have that $\text{Aut}_F(Q_8) \cong S_4$.

**$F$-automorphisms of $D_8$**

$$D_8 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \right\}$$

By Proposition A.6, $\text{Aut}(D_8) \cong D_8$ and $\text{Inn}(D_8) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and hence we have $|\text{Aut}_F(D_8)| \in \{4, 8\}$. Since $|\text{Inn}(D_8)| = 4$ we need only find one map not in $\text{Inn}(D_8)$ to show that $\text{Aut}_F(D_8) = D_8$. Since every map in $\text{Inn}(D_8)$ fixes a non-identity element and conjugation by $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ fixes only the id matrix, we have found a 5th map and hence $\text{Aut}_F(D_8) = D_8$.

**$F$-automorphisms of $\mathbb{Z}/8\mathbb{Z}$**

$$\mathbb{Z}/8\mathbb{Z} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \right\}$$

Every automorphism of $\mathbb{Z}_8$ is determined by how one of the generators are mapped and since there are 4 elements of order 8 in $\mathbb{Z}_8$ we have $|\text{Aut}_F(\mathbb{Z}_8)| \leq 4$.

Conjugation by $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ maps $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. Suppose that conjugation by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$. Then if $\gamma = \left( \text{det} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)^{-1} = \frac{1}{ad-bc}$, we have

$$\gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \gamma \begin{pmatrix} ad - ac + 2bd - bc & a^2 - 2b^2 \\ 2d^2 - c^2 & ac - cb + ad - 2bd \end{pmatrix}$$

But if this is equal to $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ we get that

$$ad - ac + 2bd - bc = 2(ad - bc) \quad (10)$$

$$ac - bc + ad - 2bd = 2(ad - bc) \quad (11)$$

Adding (8) and (9) gives $2(ad - bc) = 4(ad - bc) \Rightarrow ad - bc = 0$ which is a contradiction. Hence $\text{Aut}_F(\mathbb{Z}_8) \cong S_2$. 28
\( \mathcal{F} \)-automorphisms of \( SD_{16} \)

\(|P| = 16\) and \(|Z(P)| = 2\). We have that \( \text{Aut}_P(P) \cong N_G(P)/C_G(P) \) and since \(|G| = 48\) and \( Z(P) \leq C_G(P) \), we must have that \(|\text{Aut}_P(P)| \in \{2, 3, 4, 6, 8, 12, 16, 24\} \).

Every automorphism of \( D_8 \) is induced by conjugation in \( P \) and hence \( |\text{Aut}_P(P)| \) is at least \( 8 \Rightarrow |\text{Aut}_P(P)| \in \{8, 12, 16, 24\} \).

Since \( \mathcal{F}_G(P) \) is saturated, \( \text{Aut}_P(P) \in \text{Syl}_2(\text{Aut}_P(P)) \) and hence \( |\text{Aut}_P(P)| \neq 16 \). Also, \( \text{Aut}_P(P) \cong P/Z(P) \) which has order 8. Hence \( 8|\text{Aut}_P(P)| \) and we have \( |\text{Aut}_P(P)| \in \{8, 24\} \).

Now, all automorphisms of \( SD_{16} \) is determined by how the generators \( s \) and \( r \) are mapped. There are 5 elements of order 2 of which one is central and there are 4 elements of order 8. Hence \( |\text{Aut}(SD_{16})| = 4 \cdot 4 = 16 \) and we have \( |\text{Aut}_P(P)| = 8 \).

The skeleton \( \mathcal{F}_{sc} \) of the fusion system \( \mathcal{F}_P(G) \) is shown in Figure 4.

![Figure 4: The skeleton of the fusion system on SD_{16} in GL_2(3)](image)

\( \mathcal{F} \)-essential subgroups

Since \( Q_8 \) is the only subgroup of \( P \) whose automorphism group is not a \( p \)-group, \( Q_8 \) is the only potential candidate for being \( \mathcal{F} \)-essential. \( \text{Aut}_\mathcal{F}(Q_8) \cong \)
$\text{NC}(Q_8)/\text{CG}(Q_8)$ and $|\text{Aut}_F(Q_8)| = 24$. Thus $|\text{CG}(Q_8)| = 2 \Rightarrow \text{CG}(Q_8) = Z(G) \leq Q_8$ and hence $Q_8$ is $F$-centric.

$$\text{Out}_F(Q_8) \cong \frac{\text{Aut}_F(Q_8)}{\text{Inn}(Q_8)}$$

and this group has order 6. Since $\text{Inn}(Q_8)$ has no element of order 4 it is not cyclic and hence $\text{Inn}(Q_8) \cong V_4$ and we get

$$\text{Out}_F(Q_8) \cong \frac{\text{Aut}_F(Q_8)}{\text{Inn}(Q_8)} \cong S_4/V_4 \cong S_3$$

We know that $S_3$ has 3 Sylow 2-subgroups of which none are fixed under conjugation by an element not in the Sylow 2-subgroup itself. Hence if we choose $M$ to be one of these Sylow 2-subgroups we get that $M$ is strongly 2-embedded in $\text{Out}_F(Q_8)$. Hence $Q_8$ is $F$-essential. $Q_8$ is also fully $F$-normalized as it is not isomorphic to any other subgroup of $P$.

Let $K_1, K_2$ denote the two subgroups isomorphic to the Klein-4-group. Since neither of these are included in $Q_8$ which is the only $F$-essential subgroup, Alperin’s fusion theorem implies that any isomorphism from $K_1$ to $K_2$ must be a restriction of some $\psi \in \text{Aut}_F(P)$.

The only subgroup of order 2 contained in $Q_8$ is $Z(P) = \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\rangle$. Hence any isomorphism $\alpha : Q \to R$, where $|Q| = |R| = 2$, must be a restriction to $Q$ of some $\alpha \in \text{Aut}_F(P)$.

If $A = \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \right\rangle$ and $B = \left\langle \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle$, then Alperin’s theorem implies for any isomorphism $\phi : A \to B$, there is a $\psi_1 \in \text{Aut}_F(Q_8)$ and a $\psi \in \text{Aut}_F(P)$ such that $\phi = \psi|_{\phi_1(A)} \circ \phi_1|_A$. Note that $\psi$ or $\phi_1$ or both, might be the identity map.

### A Some properties of finite groups

**Lemma A.1.** Let $G$ be a group and suppose $H$ and $K$ are normal subgroups of $G$ such that their orders are relatively prime. Then $hk = kh$ for all $h \in H$, $k \in K$.

**Proof.** First note that $H \cap K = 1$ since if $x \in H \cap K$, the order of $x$ divides both $H$ and $K$. But the orders are coprime, so it must be that $x = 1$. Take any $h \in H$ and $k \in K$. Then since conjugation preserves the order we get

$$[h, k] = hkh^{-1}k^{-1} = hkh^{-1}k^{-1} \in P \cap K = 1,$$

so that $hk = kh$. \qed

**Lemma A.2.** If $p$ is a prime and $P$ is a $p$-group, then $Z(P) \neq 1$.

**Proof.** This is an easy application of the class equation. \qed

**Lemma A.3.** Let $p$ be a prime and $P$ a $p$-group. If $H < P$ then $H < N_P(H)$.
Proof. We proceed by induction on $|P|$. If $|P| = p$, then $Z(P) = P$ so suppose $|P| > p$. Let $H$ be a proper subgroup of $P$. First suppose $Z(P) \not\subseteq H$, then $H < (Z(P), H)$. But $Z(P)$ normalizes every subgroup of $P$ and hence $(Z(P), H) \leq N_P(H)$. Hence we may assume that $Z(P) \leq H$. Since $Z(P)$ is non-trivial, $|P/Z(P)| < |P|$ and hence, by induction, $H/Z(P) < N_P(H)/Z(P)$. By the lattice isomorphism theorem we now have that $H < N_P(G)$ and the induction is complete. 

Lemma A.4. Let $G$ be a group, $H$ a normal subgroup of $G$ and $K$ a characteristic subgroup of $H$. Then $K$ is normal in $G$.

Proof. Let $H \leq G$ and $K \text{ char } H$. For any $g \in G$, we have that $gHg^{-1} = H$ and hence $g$ induces a map $c_g \in \text{Aut}(H)$. Since $K$ is characteristic in $H$ we have that $gHg^{-1} = c_g(H) = H$.

Lemma A.5. Let $P$ be a $p$-group and $\phi \in \text{Aut}(P)$ of order prime to $p$. If $1 = P_0 \leq P_1 \leq \ldots \leq P_n = P$ is a sequence of subgroups all normal in $P$, and such that $\phi|_{P_i} \equiv \text{Id}_{P_1}(\text{mod } P_{i-1})$. Then $\phi = \text{Id}_P$.

Proof. We use induction on $n$ and hence it suffice to prove the statement for $n = 2$.

Suppose that $1 \leq P_1 \leq P$ and $\phi \in \text{Aut}(P)$ such that $p \nmid |\phi|$. By assumption, $\phi|_{P_1} = \text{Id}_{P_1}$ and $\phi(g)P_1 = gP_1$ for all $g \in P$. Hence we have that $\phi(g) = gy$ for some $y \in P_1$ and since $y$ is fixed under $\phi$ we must have that $\phi^k(g) = gy^k$ for all $k \in \mathbb{Z}$.

Let $m = |\phi|$. We have that $g = \phi^m(g) = gy^m$ which implies that $y^m = 1$. But since $gcd(m, p) = 1$ and $y$ has order $p$, this implies that $y = 1$ and hence $\phi$ must be the identity map on $P$.

Proposition A.6. $\text{Aut}(D_8) \cong D_8$ and $\text{Inn}(D_8) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. $D_8 = \langle r, s : r^4 = s^2 = 1, sr = r^3s \rangle$. The only elements of order 4 are $r$ and $r^3$ and hence for any $\phi \in \text{Aut}(D_8)$, $\phi(r) \in \langle r, r^3 \rangle$. Hence $\phi(r^2) = \phi(r)^2 = r^2$, since $(r^3)^2 = r^2$. This means that $\langle r^2 \rangle = Z(D_8)$ is characteristic in $D_8$.

There are 5 elements of order 2, namely $\{r^2, s, rs, sr, r^2s\}$ and $r^2$ is fixed under any automorphism. Hence there are 2 possibilities for $\phi(r)$ and 4 possibilities for $\phi(s)$ and since every automorphism is determined by how the generators $r$ and $s$ are mapped, $|\text{Aut}(D_8)| \leq 8$.

Now consider $D_{16} = \langle a, b : a^8 = b^2 = 1, ba = a^7b \rangle$, which has a subgroup $D_8 = \langle a^2, b \rangle$. Since $|D_{16} : D_8| = 2$, $D_8 \leq D_{16}$. The only elements commuting with both $a^2$ and $b$ are 1 and $a^4$ and hence $C_{D_{16}}(D_8) = \langle a^4 \rangle$. Hence we have that $\text{Aut}_{D_{16}}(D_8) \cong N_{D_{16}}(D_8)/C_{D_{16}}(D_8) = D_{16}/a^4 \cong D_8$. Thus since $|\text{Aut}(D_8)| \leq 8$ we have $\text{Aut}(D_8) = \text{Aut}_{D_{16}}(D_8) \cong D_8$.

For the second part, $\text{Inn}(D_8) \cong D_8/Z(D_8) = D_8/\langle r^2 \rangle$. And since $D_8/\langle r^2 \rangle$ is a group of order 4 which is not cyclic, $D_8/\langle r^2 \rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proposition A.7. $\text{Aut}(Q_8) \cong S_4$.

Proof. $Q_8 = \{1, -1, i, -i, j, -j, k, -k\} = \langle i, j : i^4 = j^4 = 1, i^2 = j^2, ji = i^{-1}j \rangle$. Hence $Q_8$ has one element of order 1, one of order 2 and six of order 4. We first want to find an upper limit for $|\text{Aut}(Q_8)|$. Every automorphism of $\phi \in \text{Aut}(Q_8)$ is determined by how the generators $i$ and $j$ are mapped,
since $\phi(i)$ and $\phi(j)$ will generate $\phi(Q_S)$. Also, the conjugacy classes of $Q_S$ are \{1\}, \{-1\}, \{i, -i\}, \{j, -j\}, \{k, -k\} and any automorphism of $Q_8$ will permute these. So first we choose where to map \{i, -i\} (3 choices), then how to map \{j, -j\} (2 choices), then how to map $i$ inside \{i, -i\} (2 choices) and then how to map $j$ in \{j, -j\} (2 choices). This gives us that $|\text{Aut}(Q_S)| \leq 3 \cdot 2 \cdot 2 \cdot 2 = 24$.

If we can find a set $M$ of 4 elements such that $\text{Aut}(Q_S)$ acts on $M$ as $S_4$ acts on $\{1, 2, 3, 4\}$ then we are done. Let

$$M = \{S_1 = \{i, j, k\}, \{i, -j, -k\}\}, S_2 = \{i, -j, k\}, \{i, j, k\}\},$$

$$S_3 = \{i, -j, k\}, \{i, j, -k\}, \{i, j, k\}\}$$

and let $\text{Aut}(Q_S)$ act on $M$, by $\phi(S_1) = \{\phi(i), \phi(j), \phi(k)\}, \{\phi(i), \phi(-j), \phi(-k)\}$ and equally for $S_2$ and $S_3$, where $\phi \in \text{Aut}(Q_S)$.

We know that $S_4$ is generated by (12) and (1234) so we want to find the corresponding permutation of $M$ arising from the action of $\text{Aut}(Q_S)$. Note that any automorphism on $Q_S$ is a permutation of the elements of $Q_8$ and hence we can write it in cycle notation. Consider the map $\psi : S_4 \to \text{Aut}(Q_S)$ defined by

$$(12) \mapsto \sigma_1 = (i \cdot -i)(j \cdot -j)(k \cdot -k) = (S_1 S_2)$$

$$(1234) \mapsto \sigma_2 = (i \cdot -j \cdot i)(j \cdot -k)(k \cdot -k) = (S_1 S_2 S_3 S_4)$$

Now $(12)(1234) = (234)$ and $\psi((12))\psi((1234)) = (i \cdot -i)(j \cdot -k)(i \cdot -j \cdot i)(k \cdot -k) = (i \cdot -k \cdot j \cdot -i \cdot -j)$. It is not hard to see that the action of $(i \cdot -k \cdot j \cdot -i \cdot -j)$ on $M$ is exactly $(S_2 S_3 S_4)$ which implies that $\psi((12)(1234)) = \psi((12))\psi((1234))$. From here it is not hard to conclude that $\psi$ is a homomorphism and hence $\psi(S_4) = \langle \sigma_1, \sigma_2 \rangle$.

This shows that $\langle \sigma_1, \sigma_2 \rangle \cong S_4$ and since $\langle \sigma_1, \sigma_2 \rangle \leq \text{Aut}(Q_S)$ which has order $\leq 24$, we conclude that $\text{Aut}(Q_S) \cong S_4$. \hfill $\square$

References


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