The wave equation and redshift in Bianchi type I spacetimes

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Abstract

The thesis consists of two independent parts. In the first part, we show how the solution to the scalar wave equation on 3-torus-Bianchi type I spacetimes can be written as a Fourier decomposition. We present results on the behaviour of these Fourier modes and apply them to the case of 3-torus-Kasner spacetimes. In the second part, we first consider the solution to the scalar wave equation, with special initial data, as a model for light in Bianchi type I spacetimes. We show that the obtained redshift coincides with the cosmological redshift. We also consider the Cauchy problem for Maxwell's vacuum equations, with special initial data, in order to model light in Bianchi type I spacetimes. We calculate the redshift of the solution and show that, also in this case, the obtained redshift coincides with the cosmological redshift.

Contents

Chapter 1

Introduction.

In this thesis, we study a certain type of semi-Riemannian manifolds. We will therefore assume that the reader is familiar with semi-Riemannian geometry, in particular with Lorentz geometry. The basic definitions are recalled in Chapter 2. For a more profound introduction on semi-Riemannian geometry, see e.g. [11]. In general relativity, one models universe as a 4-dimensional connected Lorentz manifold. One also assumes extra structure, called time-orientation. Such a manifold is called a spacetime. For a detailed introduction to the theory of general relativity, see e.g. [13]. It is customary to assume that the spacetime is globally hyperbolic, spatially isotropic and spatially homogeneous. Loosely speaking, globally hyperbolic means that one can define a global time-coordinate. Many properties of globally hyperbolic manifolds are known and can be found in e.g. [11]. A globally hyperbolic spacetime is called spatially homogeneous if the universe "looks the same around each point" and a spacetime is spatially isotropic if "space looks the same in all directions". Bianchi type I spacetimes are certain types of globally hyperbolic and spatially homogeneous spacetimes, that in general are spatially anisotropic (not spatially isotropic). A special case of spatially anisotropic Bianchi type I spacetimes are the so called Kasner spacetimes, that is, the (non-trivial) Bianchi type I spacetimes that qualify as a vacuum model for the universe. For a discussion of properties of Kasner spacetimes, see e.g. [9].

In Chapter 3, we study the linear scalar wave equation $\Box\varphi = 0$ on Bianchi type I spacetimes, where $\Box$ is the d'Alembert operator. This equation generalizes the usual wave equation in non-relativistic physics. The Cauchy problem for linear wave equations on globally hyperbolic manifolds has been solved. A good introduction can be found in [3]. This theory has applications in mathematical physics, for instance in algebraic quantum field theory (see [3]) and to model light (see [4]). In this thesis, we look at the the unique solution on Bianchi type I spacetime with periodic initial data. The questions discussed are:

- When can we write down the solution on Bianchi type I spacetimes explicitly?
- In the cases that we cannot write down the solution explicitly, what can we say about the solution?

In Section 3.1, we show that the solution can be written as a Fourier series with coefficients depending on time. In this way, we get ODE's for these modes. In two types of Kasner spacetimes, the ODE for these modes are explicitly solvable, see Section 3.2. In Sections 3.3-3.4, we give general results about the ODE for each mode that can be applied to Bianchi type I spacetimes that satisfy some additional assumptions. In Section 3.5, we apply these results to the case of a general Kasner spacetime. We are able to bound the growth of the modes for small times and bound the decrease of the modes for large times. Moreover, we show that, for each mode, the set of zeros is unbounded and there exists a smallest zero.

In Chapter 4, we compare the redshift of light obtained from three different methods of describing light. The standard way to model light in general relativity is to model a photon as a lightlike geodesic, see e.g. [11]. If one assumes that the energy of the photon is proportional to the frequency of the light, one can compute the so called cosmological redshift, observed between two different points in the universe. An independent way to model light is as a solution to the scalar wave equation with certain initial data. See [4] for a treatment of this method on Robertson-Walker spacetimes. Since there is a notion of Maxwell's equations on general globally hyperbolic spacetimes, see [7], one can also model light as a solution to Maxwell's vacuum equations. The goal of Chapter 4 is to apply these three methods to the case of Bianchi type I spacetimes and compare the resulting redshifts. As one can expect, since Bianchi type I spacetimes in general are anisotropic, the redshift will depend on the direction we send out the light. The question we discuss is:

- If we fix the direction of the light initially, do the 3 different ways of describing its propagation give the same redshift?

In Section 4.1, we fix the initial direction of light. We compute the cosmological redshift in the case of Bianchi type I spacetimes in Section 4.2. In Section 4.3, we model light by the solution of the scalar wave equation with initial data that correspond to the initial direction we have assumed on the light. We will generalize the notion of frequency to certain non-trigonometric functions and show that the redshift obtained in this case coincides with the cosmological redshift. In Section 4.4-4.8, we study the solutions to a certain Cauchy problem for Maxwell's vacuum equations. The idea is based on the fact that a constant-time-slice in a Bianchi type I spacetime is isometric to a constant-time-slice in Minkowski space. Therefore, we can assume that light initially, i.e. on the constant-time-slice, is described by the solution of Maxwell's equations, describing light in Minkowski space. We show that under extra assumptions on the metric (depending on the direction of the light), we obtain explicit solutions to the corresponding Cauchy problem. These explicit solutions gives again redshift that coincides with the cosmological redshift. For general Bianchi type I spacetimes, we again generalize the notion of frequency to another class of non-trigonometric functions, and show that this frequency gives once more the same redshift.

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Chapter 2

Preliminaries.

We will assume that the reader is familiar with the notion of smooth manifolds, tensor fields and connections on vector bundles. For an introduction on these topics, see for example [11]. We begin by fixing some notation.

Notation 2.0.1. We denote the space of smooth vector fields on N by $\mathfrak{X}(N)$. We let TN and TN^* denote the tangent bundle and cotangent bundle respectively. Moreover, we will always assume that curves are smooth unless stated otherwise.

2.1 Semi-Riemannian geometry.

All definitions in this chapter are based on [11].

2.1.1 Semi-Riemannian metrics and manifolds.

Semi-Riemannian geometry is a subfield of differential geometry, in the sense that one always consider smooth manifolds. The difference now is that more structure on the manifold is assumed, namely the existence of a *semi-Riemannian metric* on the manifold.

Definition 2.1.1 (Semi-Riemannian metric)**.** A *semi-Riemmanian metric* h on a smooth manifold N is a nondegenerate $(0,2)$-tensor field on N.

Definition 2.1.2 (Semi-Riemannian manifold)**.** A *semi-Riemannian manifold* is a smooth manifold N furnished with a semi-Riemannian metric h. One denotes it by (N, h) or N (when it is clear what metric is used).

As with all tensor fields, we can write the semi-Riemannian metric in local coordinates on N. Let $(x_1, ..., x_n) : U \subset N \to \mathbb{R}^n$ be a coordinate chart on N. Then we can write the metric as

$$h = \sum_{i,j=1}^{n} h_{ij} dx^i \otimes dx^j.$$

The next definition shows that one can actually speak of different types of metrics.

Definition 2.1.3 (Index of a symmetric bilinear form)**.** Let V be a real vector space and $b : V \times V \to \mathbb{R}$ a symmetric bilinear form. Then the index ν of b is defined by

$$\nu := \max\{\dim W | W \subset V \text{ is a subspace and } b|_{W\times W} \text{ is negative definite.}\}$$

One can actually show that for each $p \in N$, the index of $h_p : T_pN \times T_pN \to \mathbb{R}$ is the same. Therefore we can speak of an *index* ν *of* h.

Remark 2.1.4 (Riemannian geometry)**.** Readers familiar with Riemannian geometry have maybe already noted that these definitions generalize the notion of a Riemannian metric and a Riemannian manifold, where we set the index $\nu = 0$ in the above definitions. Hence Riemannian geometry is a special case of semi-Riemannian geometry, as the name indeed suggests.

Definition 2.1.5 (Isometry)**.** Two semi-Riemannian manifolds (N_1, h_1) and (N_2, h_2) are called *isometric* if there exists a diffeomorphism

$$f : N_1 \to N_2$$

such that for all $p \in N_1$

$$h_1|_p(x, y) = h_2|_{f(p)}(df(x), df(y)),$$

for all $x, y \in T_pM$. The map f is then called an *isometry* between N_1 and N_2.

It turns out to be very useful to know what directions "the derivative of the metric " vanishes. These directions are called "Killing vector fields" or just "Killing fields". Recall, from elementary differential geometry, the notion of the tensor derivation *Lie derivative.*

Definition 2.1.6 (Lie derivative on a function or a vector field)**.** Let $V \in \mathfrak{X}(N)$ and define the *Lie derivative* L_V to be the tensor derivation such that

$$L_V(f) := Vf \quad \forall f \in C^\infty(N),$$
$$L_V(X) := [V, X] \quad \forall X \in \mathfrak{X}(N).$$

We are now in shape to define *Killing fields* on a semi-Riemannian manifold.

Definition 2.1.7 (Killing field)**.** Let (N, h) be a semi-Riemannian manifold. $X \in \mathfrak{X}(N)$ is a *Killing vector field* on N if

$$L_X h = 0,$$

where L_X is the Lie derivative.

When explicitly calculating the Lie derivative of the metric, the following formula is useful.

Remark 2.1.8. If D is a tensor derivation acting on a $(0, 2)$-tensor A, then

$$DA(X, Y) = D(A(X, Y)) - A(DX, Y) - A(X, DY) \tag{2.1}$$

for all $X, Y \in \mathfrak{X}(N)$.

2.1.2 Levi-Civita connection, geodesics and curvature.

The natural question becomes: Is there a natural way to identify a unique connection to a semi-Riemannian metric? For this, we need to demand some property on the connection depending on the metric.

Definition 2.1.9 (Metric connection)**.** Let (N, h) be a semi-Riemannian manifold. A connection $\nabla : \mathfrak{X}(N)^2 \to \mathfrak{X}(N)$ is called metric if

$$X(h(Y, Z)) = h(\nabla_X Y, Z) + h(Y, \nabla_X Z),$$

for all $X, Y, Z \in \mathfrak{X}(N)$.

In fact, given a metric h, there is in general more than one metric connection. Nevertheless, there is exactly one metric connection, that is *torsion-free.*

Definition 2.1.10 (Torsion-free connection)**.** Let N be a smooth manifold. Then a connection $\nabla : \mathfrak{X}(N)^2 \to \mathfrak{X}(N)$ is called torsion-free if

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for all $X, Y \in \mathfrak{X}(N)$.

Each semi-Riemannian manifold will have this connection as its canonical one. Therefore it has got a special name.

Definition 2.1.11 (Levi-Civita connection)**.** Let (N, h) be a semi-Riemannian manifold. The unique torsion free metric connection $\nabla : \mathfrak{X}(N)^2 \to \mathfrak{X}(N)$ is called the Levi-Civita connection.

Since we now have a preferred connection on the manifold, we can define what we mean by *curvature* on a semi-Riemannian manifold.

Definition 2.1.12 (Riemannian curvature tensor)**.** Let (N, h) be a semi-Riemannian manifold with Levi-Civita connection ∇. The Riemannian curvature tensor on N is the $(1, 3)$-tensor defined by

$$R_{XY}Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where $X, Y, Z \in \mathfrak{X}(N)$.

In order to formulate Einstein's equations, we will need the so called "Ricci curvature" and "scalar curvature".

Definition 2.1.13 (Ricci curvature)**.** Let R be the Riemannian curvature tensor. Then we define the Ricci tensor as

$$Ric(X, Y) := \mathrm{tr}_h\{V \mapsto R_{XV}Y\}.$$

Remark 2.1.14 (In an orthonormal frame.)**.** Assume that $(E_i)_{i=1,\dots,n}$ is an orthonormal frame for (N, h). Then the Ricci curvature is given by

$$Ric(X, Y) = \sum_{i=1}^{n} \epsilon_i h(R_{XE_i}Y, E_i),$$

where $X, Y \in \mathfrak{X}(N)$ and $\epsilon_i := h(E_i, E_i)$.

Definition 2.1.15 (Scalar curvature)**.** Let Ric be the Ricci curvature. Then we define the scalar curvature as

$$S = \sum_{i=1}^{n} \epsilon_i Ric(E_i, E_i),$$

for any orthonormal frame $E_1, ..., E_n$.

We are now going to turn to curves on semi-Riemannian manifolds. We are going to aim for the very important notion of "curve of zero acceleration". These curves are so called geodesics. As we will see later, in general relativity, any photon or freely falling particle will be modelled by a geodesic. But first, the notion of a vector field along a curve.

Definition 2.1.16 (Vector field along a curve)**.** Let $c : J \subset \mathbb{R} \to N$ a smooth curve. We define a vector field along c to be smooth map

$$\begin{aligned} &Y_c(\cdot) : J \to TN \\ &\text{s.t. } Y_c(t) \in T_{c(t)}N, \ \forall t \in J. \end{aligned}$$

The set of vector fields along c is denoted by $\mathfrak{X}(c)$.

Remark 2.1.17. Note that for each $X \in \mathfrak{X}(N)$, we get a vector field along c by restricting to c, i.e. $X_c := X|_c$.

A basic example of a vector field along c is the *velocity vector field $c'(t)$ along c*. Since this is a vector field, we cannot just take a second derivative to obtain the acceleration. A natural attempt is therefore to use our notion of derivative of vector fields, namely the Levi-Civita-connection. But a connection requires the vector field to be defined on an open set. Since the extension of a vector field along a curve is not unique in general, this method does not work. Instead, we have to define the "induced covariant derivative" and show that it is indeed unique.

Definition 2.1.18 (Induced covariant derivative)**.** Let N be a smooth manifold with connection ∇ on TN. The unique function $\nabla_t : \mathfrak{X}(c) \to \mathfrak{X}(c)$ satisfying

1. ∇_t is $\mathbb{R}$-linear,
2. $\nabla_t(f \cdot X) = \frac{df}{dt} \cdot X + f \cdot \nabla_t X$ for all $f \in C^\infty(J)$ and $X \in \mathfrak{X}(c)$,
3. $\nabla_t X_c(t) = \nabla_{c'(t)} X$ for all $X \in \mathfrak{X}(N)$ and $t \in J$,

is called the *induced covariant derivative.*

In general, we can define the induced covariant derivative for any connection ∇ on N. From this point we are therefore going to assume that ∇ is the Levi-Civita connection and ∇_t is the (by the Levi-Civita connection) induced covariant derivative. Therefore, we now have a way to take the derivative of a vector field along a curve, and in particular we can generalize "acceleration" of a curve. The curves that have zero acceleration, we are going to call "geodesics".

Definition 2.1.19 (Geodesics)**.** Let ∇ be the Levi-Civita connection on a semi-Riemannian manifold (N, h). Then a smooth curve $\gamma : J \subset \mathbb{R} \to N$ is called a geodesic if

$$\nabla_t \gamma' = 0.$$

Notation 2.1.20. From this point on, we will let ∇ denote the Levi-Civita connection and ∇_t the induced covariant derivative.

The following lemma is of fundamental importance in semi-Riemannian geometry.

Lemma 2.1.21 (Local existence of geodesics)**.** *Let (N, h) be a semi-Riemannian manifold. Given $p \in N$ and $v \in T_pN$, there exists a unique geodesic $\gamma_v : J_v \to N$, such that*

1. $\gamma_v(0) = p$
2. $\gamma_v'(0) = v$
3. *J_v is maximal, i.e. if $\alpha : J_\alpha \to N$ is a geodesic satisfying (i) and (ii), then $J_\alpha \subset J_v$.*

We call γ_v a maximal geodesic.

2.1.3 Div, grad and the d'Alembert operator.

An important differential operator in semi-Riemannian geometry is the so called "d'Alembert operator". To define the d'Alembert operator, we need generalizations of the divergence of a vector field and gradient of a scalar function on $\mathbb{R}^n$.

Definition 2.1.22 (Divergence)**.** Let (N, h) be a semi-Riemannian manifold with Levi-Civita connection ∇. Let X be a vector field on N. Then we define the *divergence of X at $p \in N$* as

$$\operatorname{div}(X)(p) = \operatorname{tr}_h(Y \mapsto \nabla_Y X), Y \in T_pN.$$

Remark 2.1.23. If $(E_\alpha)_{\alpha=0}^3$ is a local orthonormal frame, then the divergence is given by

$$\operatorname{div}(X)(p) = \sum_{\alpha=0}^{3} \epsilon_\alpha h_p(\nabla_{E_\alpha} X, E_\alpha),$$

where $\epsilon_\alpha := h(E_\alpha, E_\alpha)$.

Definition 2.1.24 (Gradient)**.** Let (N, h) be a semi-Riemannian manifold. Let $\phi \in C^\infty(N)$. Then the gradient of ϕ is the unique vector field such that

$$h(\operatorname{grad}(\phi), X) = d\phi(X)$$

for all $X \in \mathfrak{X}(N)$.

We are now in shape to define the d'Alembert operator.

Definition 2.1.25 (d'Alembert operator)**.** Let (N, h) be a semi-Riemannian manifold. Let $\phi \in C^\infty(N)$. Then the d'Alembert operator is defined by

$$\Box\phi := -\text{div}(\text{grad}(\phi)).$$

2.2 Lorentz geometry and general relativity.

Loosely speaking, general relativity models the universe as a manifold with 3 spatial dimensions and 1 time dimension. This motivates the definition of a Lorentz manifold.

2.2.1 Lorentz manifold.

Recall that a Riemannian manifold is a semi-Riemannian manifold with a metric with index 0. Analogously, we now treat the case where the index is 1.

Definition 2.2.1 (Lorentz manifold)**.** A *Lorentz manifold* is a semi-Riemannian manifold (N, h) with index $\nu = 1$.

In the rest of this chapter, let (N, h) denote a Lorentz manifold.

Definition 2.2.2 (Causality of tangent vectors)**.** Let $p \in N$ and $v \in T_pN$. Then v is called

1. *timelike* if $h(v, v) < 0$,
2. *lightlike* if $h(v, v) = 0$ and $v \neq 0$,
3. *spacelike* if $h(v, v) > 0$ or $v = 0$,
4. *causal* if v is lightlike or timelike.

Example 2.2.3. Assume that the dimension of N is 3 and let $\varphi := (t, x_1, x_2) : U \to \mathbb{R}^3$ be local coordinates around $p \in N$ such that the metric at p is given by

$$h_p = -dt^2 + (dx^1)^2 + (dx^2)^2.$$

Then the set of the timelike vectors in T_pN (pushed forward under the diffeomorphism φ) form an upper and a lower cone in $\mathbb{R}^3$. This motivates the notion of *timecone.*

Definition 2.2.4 (Timecone)**.** Let $p \in N$ and let $v \in T_pN$ be timelike. The *timecone containing* $v \in T_pN$ is defined by

$$C(v) := \{w \in T_pN|\ w \text{ is timelike and } h(w, v) < 0\}.$$

The *opposite timecone* is defined by

$$C(-v) := \{w \in T_pN|\ w \text{ is timelike and } h(w, v) > 0\} = -C(v).$$

Remark 2.2.5. Note that if $w \in T_pN$ is timelike, then $C(w) = C(v)$ or $C(w) = C(-v)$. Hence the set of timelike vectors is the disjoint union of the two timecones $C(v)$ and $C(-v)$.

This gives a notion of time-orientation of a Lorentz manifold.

Definition 2.2.6 (Time-orientation)**.** A *time-orientation of* T_pN for $p \in N$, is a choice of one of the 2 timecones given in the previous remark. A *time-orientation for* N is a consistent choice of such timecones, i.e. a function τ that for each $p \in N$ assigns a choice of timecone τ_p such that there exists a timelike vector field V on a neighbourhood U containing p with $V_q \in \tau_q$ for all $q \in U$.

The following lemma might make it more intuitive what time-orientation means.

Lemma 2.2.7 (Equivalent definition of time-orientation)**.** *A Lorentzian manifold* (N, g) *is time-orientable if and only if there exists a timelike vector field* $X \in \mathfrak{X}(N)$*.*

For time-orientable Lorentz manifolds, we get a notion of future and past by calling one choice of time-orientation the *future* and the opposite time-orientation the *past.* A causal vector is said to be *future-pointing* (resp. *past-pointing*) if it lies in the closure of the *future timecone* (resp. *past timecone*). A causal vector field is *future-pointing* (resp. *past-pointing*) if, at each point, it is future-pointing (resp. past-pointing). Similarly, a piecewise smooth curve is called lightlike/spacelike/causal/future-pointing/past-pointing if all its tangent vectors satisfy the above definitions respectively. The definition of a piecewise timelike curve needs a separate definition, since we demand the curve to stay in the same timecone at each break.

Definition 2.2.8 (Piecewise smooth timelike curve)**.** A piecewise smooth curve $\alpha : J \to N$ is called timelike if α' is timelike and if

$$h(\alpha'(t_i^-), \alpha'(t_i^+) < 0,$$

at each break $t_i \in J$.

Hence we get a local notion of time. But what still is missing, is a global notion of time, i.e. a coordinate defined everywhere such that its coordinate vector field is timelike. This does not exist in general. Therefore, it is common in general relativity, to demand this as an extra property on the manifold. We will define the property of global hyperbolicity on a Lorentz manifold and state that this is equivalent to the existence of such a global "time coordinate".

Notation 2.2.9. Let $p, q \in N$.

- $p \ll q \Leftrightarrow$ there exists a future-pointing timelike curve in N from p to q,
- $p < q \Leftrightarrow$ there exists a future pointing causal curve in N from p to q.

Definition 2.2.10. Let A be a subset of the Lorentz manifold N. The *chronological future of* A is defined by

$$I_+^N(A) := \{q \in N | \exists p \in A \text{ s.t. } p \ll q\} \subset N.$$

The *causal future of* A is defined by

$$J_+^N(A) := \{q \in N | \exists p \in A \text{ s.t. } p < q\} \subset N.$$

The *chronological* and *causal past*, $I_-^N(A)$ and $J_-^N(A)$, are defined analogously. The *causal cone of* A is defined by

$$J^N(A) := J_-^N(A) \cup J_+^N(A).$$

In order to get existence of a global notion of time, it seems reasonable to demand that $p < p$ can never happen on the manifold, i.e. that there does not exist a causal future-pointing curve from p to itself. This is called the causality condition.

Definition 2.2.11 (Causality condition)**.** A Lorentz manifold N satisfies the causality condition if there are no closed causal curves in N.

In fact, we will need the property that says that no causal curve can get "arbitrarily close" to where it started.

Definition 2.2.12 (Strong causality condition)**.** A Lorentz manifold N satisfies the *strong causality condition* if and only if for every $p \in N$ and every neighbourhood $U \ni p$, there exists an open neighbourhood $V \subset U$ of p such that the image of every causal curve starting and ending in V lies entirely in U.

Note that the strong causality condition implies the causality condition. We are finally ready for the definition of a globally hyperbolic manifold.

Definition 2.2.13 (Globally hyperbolic manifold)**.** Let (N, h) be a connected and time-oriented Lorentz manifold. If N satisfies the strong causality condition, and if $J_+^N(p) \cap J_-^N(q)$ is compact for all $p, q \in N$ with $p < q$, then N is called a *globally hyperbolic manifold*.

The following definitions and theorem shows that existence of a global time coordinate is equivalent to global hyperbolicity.

Definition 2.2.14. A piecewise smooth curve $\alpha : (a, b) \to N$ (for $b \leq \infty$) is *extendible* if it has a continous extension $\bar{\alpha} : (a, b] \to N$ and α is called *inextendible* otherwise.

Definition 2.2.15 (Cauchy hypersurface)**.** Let N be a connected time-oriented Lorentz manifold. A subset $S \subset N$ is called a *Cauchy hypersurface* if every inextendible timelike curve in N meets S exactly once.

The proof of the following theorem can be found in [3, Theorem 1.3.10].

Theorem 2.2.16. *Let (N, h) be a connected time-oriented Lorentz manifold. The following are equivalent*

1. *N is globally hyperbolic.*
2. *N has a Cauchy hypersurface S.*
3. *(N, h) is isometric to $(\mathbb{R} \times S, \bar{h})$, where*

$$\bar{h}((v_0, v_1), (w_0, w_1)) = -\beta v_0 w_0 + \bar{h}_t(v_1, w_1),$$

for $(v_0, v_1), (w_0, w_1) \in T_{(t,x)}(\mathbb{R} \times S)$ where $\beta > 0$ is a smooth function on N and $\bar{h}_t$ is a Riemannian metric on S depending on $t \in \mathbb{R}$. Furthermore, $\{t\} \times S$ is a Cauchy hypersurface in N for each $t \in \mathbb{R}$.

Definition 2.2.17 (Foliation of a globally hyperbolic manifold)**.** Let (N, h) be a globally hyperbolic manifold. Assume that $(J \times S, \bar{h})$ with $J \subseteq \mathbb{R}$ a connected interval and the metric satisfying the properties described in point "3.", with $\mathbb{R}$ replaced by J, is isometric to (N, h). Then $(J \times S, \bar{h})$ is called a foliation of (N, h).

2.2.2 General relativity.

We continue by stating the most elementary definitions and results in general theory of relativity.

Definition 2.2.18 (Spacetime)**.** A *spacetime* is a connected time-oriented Lorentzian manifold of dimension four.

Remark 2.2.19. Note that all globally hyperbolic manifolds are spacetimes, but not the other way around.

Not all spacetimes are considered physically relevant. Albert Einstein stated his famous Einstein's equation as a criterion for physical relevance of a spacetime. In order to state that equation, we first define Einstein's gravitational tensor.

Definition 2.2.20 (Einstein's gravitational tensor)**.** Let (N, h) be a spacetime with Ricci tensor Ric and scalar curvature S. Then *Einstein's gravitational tensor* is defined as the $(0, 2)$-tensor G given by

$$G := Ric - \frac{1}{2} S g.$$

In general relativity, all energy and matter in the universe is assumed to be described by a $(0, 2)$-tensor T, called the *stress-energy tensor*. The choice of the tensor T depends on the matter distribution one wants to study. Let us define Einstein's equation.

Definition 2.2.21 (Einstein's equation)**.** Let (N, h) be a spacetime with Einstein's gravitational tensor G and stress-energy tensor T. Then the (N, h) satisfies *Einstein's equation* if

$$G = 8\pi T.$$

We proceed by the notion of observers in general relativity.

Definition 2.2.22 (Observers and Particles)**.** An *observer* or a *material particle* in a spacetime (N, h) is a timelike future-pointing curve $\alpha : J \subset \mathbb{R} \to N$ such that $h(\alpha'(s), \alpha'(s)) = -1$ for all $s \in J$. The parameter s is called the *proper time* of α. If the particle is a geodesic, the particle is said to be *freely falling.*

A *lightlike particle* is a future-pointing lightlike geodesic $\gamma : J \to N$.

Intuitively, a freely falling particle should be one that is not subject of any force, i.e. does not accelerate. As mentioned in the discussion of the induced covariant derivative, such curves are exactly those we call geodesics (see Definition 2.1.19).

Definition 2.2.23 (Observer field)**.** An *observer field* on an arbitrary spacetime N is a timelike, future-pointing, unit vector field. (Sometimes observer fields are also called *reference frames.*)

Remark 2.2.24. Each integral curve of an observer field U is an observer, parametrized by its proper time.

Definition 2.2.25 (Geodesic observer field)**.** An observer field U is called a *geodesic observer field* if all integral curves are geodesic, i.e.

$$\nabla_U U = 0.$$

Remark 2.2.26. Each integral curve of a *geodesic* observer field U is a *freely falling* observer, parametrized by its proper time.

In general relativity, the tangent vector of an observer contains information needed to define both the energy *and* the momentum of the particle.

Definition 2.2.27 (Energy-momentum vector field)**.** For an observer $\alpha : J \subset \mathbb{R} \to N$ of *mass* m, the *energy-momentum vector field* along α is defined by

$$P := m\alpha'.$$

The vector field α' is called the *4-velocity vector field* of α.

Note that the energy-momentum vector field of an observer is timelike. The following definition shows how the observer locally interprets time and space.

Definition 2.2.28 (Instantaneous observer)**.** An *instantaneous observer* at $p \in N$, is a timelike future-pointing unit vector $U_p \in T_pN$. The split of the tangent space at p into

$$T_pN = \mathbb{R}U_p + U_p^{\perp}$$

gives a notion of an *observer's time axis* $\mathbb{R}U_p$, and an observer's *rest space* $U_p^{\perp}$.

Having defined the necessary properties of observers, we come to an essential point in general relativity, how measurements *depend* on the observers. Let $p =: \alpha(s_0) \in N$ and let $U_p \in T_pN$ be an instantaneous observer. The energy momentum P of α at p, splits into the orthogonal sum

$$P = EU_p + \vec{P}, \quad E > 0.$$

Definition 2.2.29. We define the *energy measured by* U_p to be E and the *(classical) momentum measured by* U_p to be $\vec{P}$.

2.3 The framework of the thesis.

It is very common to assume that the spacetime is globally hyperbolic, spatially homogeneous and spatially isotropic. We define the last two concepts in what follows.

Definition 2.3.1 (Spatial homogeneity)**.** A globally hyperbolic manifold (N, h) is called *spatially homogeneous* if there exists a foliation $(J \times S, \bar{h})$ (see Definition 2.2.17) such that for an arbitrary $t \in J$ and for each $p, q \in \{t\} \times S$, there exists an isometry

$$f : (J \times S, \bar{h}) \to (J \times S, \bar{h})$$

such that

$$f(p) = q,$$

and

$$(s, y) \in \{s\} \times S \Rightarrow f(s, y) \in \{s\} \times S.$$

Definition 2.3.2 (Spatial isotropy)**.** A globally hyperbolic manifold (N, h) is called *spatially isotropic* if there exists a foliation $(J \times S, \bar{h})$ such that for each $p = (t, x) \in J \times S$ and for each $v_p, w_p \in T_p(\{t\} \times S)$ with

$$h(v_p, v_p) = h(w_p, w_p),$$

there exists a local isometry ϕ defined on a neighbourhood $U \subset J \times S$ of p such that

- $d\phi(v_p) = w_p$,
- $(s, y) \in U \Rightarrow \phi(s, y) \in \{s\} \times S$.

2.3.1 Bianchi type I spacetimes.

In this thesis, we will work with Bianchi type I spacetimes, which are spatially homogeneous, but in general spatially anisotropic (not spatially isotropic).

Definition 2.3.3. Let $I \subset \mathbb{R}$ be a connected open interval. A *Bianchi type I spacetime* is the manifold

$$M := I \times \mathbb{R}^3,$$

equipped with a metric of the form

$$g := -dt^2 + \sum_{i=1}^{3} a_i(t)^2 (dx^i)^2,$$

where $a_i : I \to (0, \infty)$ are smooth.

Notation 2.3.4. Whenever we write (M, g) in this thesis, we will refer to a Bianchi type I spacetime.

There are a few useful propositions to be noted about Bianchi type I spacetimes.

Proposition 2.3.5. *The vector field $\partial_t \in \mathfrak{X}(M)$ gives a canonical time orientation of M.*

Proof. This follows by Lemma 2.2.7, since ∂_t is timelike. □

Proposition 2.3.6. *(M, g) is globally hyperbolic.*

Proof. By Theorem 2.2.16, this is equivalent to the existence of a spacelike Cauchy hypersurface. Since $S := \{t_0\} \times \mathbb{R}^3$, for $t_0 \in I$, with induced metric is obviously spacelike, we claim that S is a Cauchy hypersurface. Assume hence that $\alpha : (a, b) \to M$ is a timelike inextendible curve.

Assume first that α does not intersect S. W.l.o.g. we assume that α is future pointing and that $t_{-1} := t \circ \alpha(s_{-1}) < t_0$ for some $s_{-1} \in (a, b)$. Note that $t \circ \alpha$ is monotone increasing and bounded, hence converging. Let $s_n \to b$ as $n \to \infty$. Then

$$|x_i \circ \alpha(s_m) - x_i \circ \alpha(s_n)| \leq \int_{s_n}^{s_m} |\partial_s(x_i \circ \alpha)(s)|\, ds \leq A \int_{s_n}^{s_m} \partial_s(t\circ\alpha)(s) ds = A(t\circ\alpha(s_m) - t\circ\alpha(s_n)),$$

where $A := (\min_{t\in[t_{-1},t_0]} a_i(t))^{-1}$. But $(t \circ \alpha(s_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, hence $(x \circ \alpha(s_n))_{n\in\mathbb{N}}$ is a Cauchy sequence too. Therefore, for each $i = 1, 2, 3$, the sequence converges in $\mathbb{R}^3$ and α is extendible. We conclude that α does intersect S.

Since

$$t \circ \alpha$$

is strictly increasing, α cannot intersect S more than once. Hence the proposition is proven. □

Proposition 2.3.7 (Spatially homogeneous)**.** (M, g) *is spatially homogeneous for any choice of* a_i*.*

Proof. Let $p = (t, x_p), q = (t, x_q) \in M$. Then the isometry defined by

$$f : M \to M$$
$$f(t, x) = (t, x - x_p + x_q),$$

satisfies $f(p) = q$ and

$$f(s, y) = (s, y - x_p + x_q) \in \{s\} \times S.$$

By Definition 2.3.1, (M, g) is spatially homogeneous. □

The explicit isometry used in the proof of spatial homogeneity, gives the following corollary.

Corollary 2.3.8. *The coordinate vector fields* ∂_i *are Killing vector fields.*

The question of spatial isotropy is more involved. Intuitively it is clear that the so called "spatially flat Robertson-Walker spacetimes" are isotropic.

Definition 2.3.9 (Spatially flat Robertson-Walker spacetime)**.** Let (M, g) be a Bianchi type I spacetime and assume that there is a function f such that

$$f(t) = a_1(t) = a_2(t) = a_3(t)$$

for all $t \in I$. Then (M, g) is called a *spatially flat Robertson-Walker spacetime.*

We will show that indeed, (M, g) is spatially isotropic exactly when it is isometric to a spatially flat Robertson-Walker spacetime.

Proposition 2.3.10 (Spatially isotropic)**.** (M, g) *is spatially isotropic, if and only if it is isometric to a spatially flat Robertson-Walker spacetime.*

Proof. By an intermediate step in the proof of [9, Proposition 41], wee see that

$$\frac{\dot{a}_1}{a_1} = \frac{\dot{a}_2}{a_2} = \frac{\dot{a}_3}{a_3},$$

is equivalent to spatial isotropy. This is the same as

$$\partial_t(\ln(a_1)) = \partial_t(\ln(a_2)) = \partial_t(\ln(a_3)).$$

Integrating and taking exponential, gives

$$a_1 D_1 = a_2 D_2 = a_3,$$

where $D_1, D_2 > 0$ are constant. It is not difficult to see that this condition is equivalent to the condition that (M, g) is isometric to a spatially flat Robertson Walker spacetime. □

We conclude by defining the natural orthonormal frame on (M, g).

Definition 2.3.11 (Orthonormal frame on (M, g))**.** Define the orthonormal frame

$$E_0 = \partial_t,$$
$$E_i = \frac{1}{a_i}\partial_i, \text{ for } i = 1, 2, 3.$$

The following can easily be calculated.

Lemma 2.3.12. *The only non-zero components of the Levi-Civita connection with respect to this orthogonal frame is*

$$\nabla_{E_i} E_0 = \frac{\dot{a}_i}{a_i} E_i,$$
$$\nabla_{E_i} E_i = \frac{\dot{a}_i}{a_i} E_0.$$

2.3.2 3-torus-Bianchi type I spacetimes.

Define the action

$$a : \mathbb{Z}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$$

as

$$a(z, r) = r + z.$$

This induces naturally a quotient Lie group

$$\mathrm{T} := \mathbb{R}^3/\mathbb{Z}^3.$$

Definition 2.3.13 (3-torus-Bianchi type I spacetime)**.** Define the *3-torus-Bianchi type I spacetime* as

$$M_{\mathrm{T}} := I \times \mathrm{T},$$

where $\mathrm{T} := \mathbb{R}^3/\mathbb{Z}^3$ with metric

$$g := -dt^2 + \sum_{i=1}^{3} a_i(t)^2 (dx^i)^2.$$

Notation 2.3.14. Whenever we write (M_{T}, g) in this thesis, we will refer to the 3-torus-Bianchi type I spacetime with induced Bianchi type I metric. In order to shorten the name of this class, we will in this thesis use the shorter name *3-torus spacetime* for a 3-torus-Bianchi type I spacetime.

Remark 2.3.15. All properties like time-orientation, Killing fields, global hyperbolicity, isotropy/anisotropy and homogeneity are inherited from the general Bianchi type I spacetimes.

2.3.3 Kasner spacetimes.

A particular case of Bianchi type I spacetimes, is the subclass of (non-trivial) spacetimes that satisfy Einstein's vacuum equations. One can show (see e.g. [9, Chapter 3]) that each such solution is isometric to a *Kasner spacetime*, defined as follows.

Definition 2.3.16 (Kasner spacetime)**.** Let

$$K := \mathbb{R}_+ \times \mathbb{R}^3,$$

be equipped with a *Kasner metric* g_K, i.e.

$$g_K = -dt^2 + \sum_{i=1}^{3} t^{2p_i} (dx^i)^2,$$

where the $p_i \in \mathbb{R}$ satisfy the following relations:

$$\sum_{i=1}^{3} p_i^2 = 1, \tag{2.2}$$

$$\sum_{i=1}^{3} p_i = 1. \tag{2.3}$$

Then (K, g_K) is called a *Kasner spacetime*. Substituting $\mathbb{R}^3$ with the 3-torus T in the above definition, defines the *3-torus-Kasner spacetime*.

Notation 2.3.17. Whenever we write (K, g_K) and (K_T, g_K) in this thesis, we will refer to the Kasner spacetime and the 3-torus-Kasner spacetime respectively.

Remark 2.3.18 (Spatial homogeneity/isotropy)**.** Note that since Kasner spacetimes are special cases of Bianchi type I spacetimes, they are *spatially homogeneous*. From the conditions (2.3) and (2.2), one can conclude that a Kasner spacetime cannot be isometric to a Flat Robertson-Walker spacetime. Hence Kasner spacetimes are *spatially anisotropic*.

We see immediately that at least one p_i must satisfy

$$0 \leq p_i \leq 1.$$

A simple calculation gives the formula for the other two.

Lemma 2.3.19. *Let $s := p_j$ such that $0 \leq s \leq 1$. Then*

$$p_k = \frac{1-s}{2} \pm \sqrt{-\frac{3s^2}{4} + \frac{s}{2} + \frac{1}{4}}.$$

for $k \neq j$. Moreover, one $p_i \in [-\frac{1}{3}, 0]$ and the other two are contained in $[0, 1]$.

The lemma implies that if some $p_i = 1$, then the other two must be 0. Also, if all $p_i \neq 1$, then all p_i's are non-zero. Therefore, it is natural to distinguish between two different cases of Kasner spacetimes.

Definition 2.3.20 (Flat/non-flat Kasner spacetimes)**.** (K, g) is called a *flat Kasner spacetime* with a *flat Kasner metric* if one $p_i = 1$ (and the other two vanish) and it is called *non-flat Kasner spacetime* with a *non-flat Kasner metric* if all $p_i \neq 1$.

Chapter 3

The wave equation on 3-torus-Bianchi type I spacetimes.

In this chapter we will look at solutions to the scalar wave equation on 3-torus-Bianchi type I spacetimes. See the previous chapter for a definition of this, and the fixing of notation.

3.1 The Fourier decomposition of the solution.

We start with the precise problem formulation, and note that existence and uniqueness follows from a theorem in [3].

3.1.1 Problem formulation.

Recall the definition of d'Alembert operator (Definition 2.1.25). Let (M_{T}, g) be the 3-torus spacetime with induced metric. We will assume that initial data are given on a slice $\{t_0\} \times \mathrm{T}$. The problem is formulated as follows.

Problem 3.1.1 (The scalar wave equation)**.** Given $t_0 \in I$ and $\varphi_0, \varphi_1 \in C^\infty(\{t_0\} \times \mathrm{T})$, can we then solve the following equation explicitly:

$$\Box\, \varphi = 0,\ \varphi|_{\{t_0\}\times \mathrm{T}} = \varphi_0,\ \partial_t\varphi|_{\{t_0\}\times \mathrm{T}} = \varphi_1?$$

If we cannot solve it explicitly, what can we say about the behaviour of the solution?

3.1.2 The existence and uniqueness theorem.

The following theorem is proved in [3, Theorem 3.2.11]:

Theorem 3.1.2. *On a globally hyperbolic Lorentzian manifold N, let $S \subset N$ be a spacelike Cauchy hypersurface. Let ν be the future directed timelike unit normal field along S. Let E be a vector bundle over N and let P be a normally hyperbolic operator acting on sections in E.*

Then for each $u_0, u_1 \in C_c^\infty(S, E)$ and for each $f \in C_c^\infty(N, E)$ there exists a unique $u \in C^\infty(N, E)$ satisfying $Pu = f$, $u|_S = u_0$, and $(\nabla_\nu u)|_S = u_1$.

Moreover, $supp(u) \subset J^N(K)$ where $K = supp(u_0) \cup supp(u_1) \cup supp(f)$.

This theorem applies to Problem 3.1.1 as:

Corollary 3.1.3. *Let (M_{T}, g) be a 3-torus spacetime. Then for any $t_0 \in I$ and $\varphi_0, \varphi_1 \in C^\infty(\{t_0\} \times \mathrm{T})$, the Cauchy problem*

$$\Box\, \varphi = 0,\ \varphi|_{\{t_0\}\times \mathrm{T}} = \varphi_0,\ \partial_t \varphi|_{\{t_0\}\times \mathrm{T}} = \varphi_1 \tag{3.1}$$

has a unique solution $\varphi \in C^\infty(M_{\mathrm{T}})$.

Proof. First we recall from Proposition 2.3.6, that Bianchi type I spacetimes are globally hyperbolic. Hence it follows that the 3-torus spacetime is globally hyperbolic.

It is clear that ∂_t is a future directed timelike unit normal vector field along $S = \{t_0\} \times \mathrm{T}$. Define $E := M_{\mathrm{T}} \times \mathbb{R}$. Then the space of sections in E is just $C^\infty(M_{\mathrm{T}})$. Note also, since S is compact, $C^\infty(S) \cong C_c^\infty(S, E)$. What remains in order to use Theorem 3.1.2, is the claim that $P := \Box$ is a normally hyperbolic operator. This fact is proven in [3, Example 1.5.1].

Hence we can apply Theorem 3.1.2 with $f = 0$, and conclude that there exists $\varphi \in C^\infty(M_{\mathrm{T}}, M_{\mathrm{T}} \times \mathbb{R}) \cong C^\infty(M_{\mathrm{T}}, \mathbb{R})$ such that

$$\Box \varphi = 0, \quad \varphi|_{\{t_0\}\times \mathrm{T}} = \varphi_0, \quad (\nabla_{\partial_t} \varphi)|_{\{t_0\}\times \mathrm{T}} = \varphi_1.$$

Since $\nabla_{\partial_t} \varphi = \partial_t \varphi$, we are done. □

Hence we know that a smooth and unique solution φ exists to our problem. We proceed by writing φ as a Fourier series.

3.1.3 The solution as a Fourier series.

Since we can express the d'Alembert operator in terms of a local orthonormal frame (recall Definition 2.1.25), we can write it in coordinates as follows.

Lemma 3.1.4. *Let $\varphi \in C^\infty(M_{\mathrm{T}})$. Then*

$$\Box\, \varphi = \left(\partial_{tt} + \sum_{i=1}^{3} \frac{\dot{a}_i}{a_i} \partial_t - \sum_{i=1}^{3} \frac{1}{a_i^2} \partial_{ii} \right) \varphi.$$

Proof. The result follows by a simple calculation:

$$\begin{aligned}
\Box\, \varphi &= -\mathrm{div}(\mathrm{grad}\varphi) = -\sum_{\alpha=0}^{3} \epsilon_\alpha g(\nabla_{E_\alpha} \mathrm{grad}\varphi, E_\alpha) \\
&= -\sum_{\alpha=0}^{3} \epsilon_\alpha \left(E_\alpha g(\mathrm{grad}\varphi, E_\alpha) - g(\mathrm{grad}\varphi, \nabla_{E_\alpha} E_\alpha) \right) \\
&= -\sum_{\alpha=0}^{3} \epsilon_\alpha \left(E_\alpha (E_\alpha \varphi) - (\nabla_{E_\alpha} E_\alpha)\varphi \right) = -\left(-\partial_{tt} + \sum_{i=1}^{3} \frac{1}{a_i^2} \partial_{ii} - \sum_{i=1}^{3} \frac{\dot{a}_i}{a_i} \partial_t \right) \varphi \\
&= \left(\partial_{tt} + \sum_{i=1}^{3} \frac{\dot{a}_i}{a_i} \partial_t - \sum_{i=1}^{3} \frac{1}{a_i^2} \partial_{ii} \right) \varphi
\end{aligned}$$

□

The previous Lemma implies that the equation splits into a sum of differential operators with respect to time in one term and differential operators with respect to space in the other term. Since each spacelike slice is diffeomorphic $\mathbb{R}^3/\mathbb{Z}^3$, this motivates a product ansatz, where the factors depending on the space coordinates are terms in a Fourier series. The next step is therefore to write the solution given by Corollary 3.1.3 as a linear combination of Fourier terms with time dependent coefficients.

Remark 3.1.5 (Fourier series)**.** For $z \in \mathbb{Z}^3$ and $x \in \mathbb{R}^3$, define

$$\phi_z(x) := \exp(2\pi i z \cdot x).$$

It is a standard result of functional analysis, that $(\phi_z)_{z\in\mathbb{Z}^3}$ is an orthonormal basis for $L^2(\mathrm{T})$.

Lemma 3.1.6. *Let $z = (z_1, z_2, z_3) \in \mathbb{Z}^3$. Define the complex valued function*

$$\varphi_z(t,x) := \alpha_z(t)\phi_z(x),$$

for all $(t,x) \in M$. Then

$$\Box\varphi_z(t,x) = \left(\alpha_z''(t) + \sum_{j=1}^{3} \frac{\dot{a}_j(t)}{a_j(t)}\alpha_z'(t) + 4\pi^2\alpha_z(t)\sum_{j=1}^{3}\frac{z_j^2}{a_j(t)^2}\right)\phi_z(x),$$

for all $(t,x) \in M$.

Proof.

$$\begin{aligned}
\Box\varphi_z(t,x) &= \left(\partial_{tt} + \sum_{j=1}^{3}\frac{\dot{a}_j(t)}{a_j t)}\partial_t - \sum_{j=1}^{3}\frac{1}{a_j(t)^2}\partial_{jj}\right)\varphi_z(t,x) \\
&= \alpha_z''(t)\phi_z(x) + \sum_{j=1}^{3}\frac{\dot{a}_j(t)}{a_j(t)}\alpha_z'(t)\phi_z(x) - \alpha_z(t)\sum_{j=1}^{3}\frac{1}{a_j(t)^2}\left(2\pi i z_j\right)^2\phi_z(x) \\
&= \left(\alpha_z''(t) + \sum_{j=1}^{3}\frac{\dot{a}_j(t)}{a_j(t)}\alpha_z'(t) + 4\pi^2\alpha_z(t)\sum_{j=1}^{3}\frac{z_j^2}{a_j(t)^2}\right)\phi_z(x).
\end{aligned}$$

□

The following theorem shows that it is enough to study the ODE for each mode.

Theorem 3.1.7. *The unique solution to Equation 3.1 is given by*

$$\varphi(t,x) = \sum_{z\in\mathbb{Z}^3}\alpha_z(t)\phi_z(x)$$

for all $(t,x) \in M$, where $\alpha_z : I \to \mathbb{C}$ is the unique solution to

$$\alpha_z''(t) + \sum_{j=1}^{3}\frac{\dot{a}_j(t)}{a_j(t)}\alpha_z'(t) + 4\pi^2\alpha_z(t)\sum_{j=1}^{3}\frac{z_j^2}{a_j(t)^2} = 0, \ \forall t \in I, \tag{3.2}$$

$$\alpha_z(t_0) = \langle\varphi_0, \phi_z\rangle_{L^2(\mathrm{T})}, \tag{3.3}$$

$$\alpha_z'(t_0) = \langle\varphi_1, \phi_z\rangle_{L^2(\mathrm{T})}. \tag{3.4}$$

Remark 3.1.8. Note that one would obtain analogous results by using the (continuous) Fourier transform on a general Bianchi type I spacetime instead of the Fourier series on the 3-torus-Bianchi type I spacetime. Many results in the rest of this chapter would also apply in that case. Nevertheless, there are some technical details that we will avoid using Fourier series, i.e. looking at solutions that are periodic in space.

Proof. We think of the real valued φ as complex valued with vanishing imaginary part. Fix $t \in I$. Since T has finite Lebesgue measure and $\varphi(t,\cdot)$ is bounded on T, $\varphi(t,\cdot) \in L^2(\mathrm{T})$. Hence $\varphi(t,\cdot)$ can be written in terms of the orthonormal basis as

$$\varphi(t,\cdot) = \sum_{z\in\mathbb{Z}^3}\langle\varphi(t,\cdot),\phi_z\rangle_{L^2(\mathrm{T})}\phi_z.$$

Define now for all $t \in I$,

$$\alpha_z : I \to \mathbb{R}$$

by

$$\alpha_z(t) := \langle \varphi(t,\cdot), \phi_z \rangle_{L^2(\mathrm{T})},$$

and smoothness of α_z follows immediately from the dominated convergence theorem, since ϕ_z is bounded. Therefore, for all $t \in I$, we have

$$\varphi(t,x) = \sum_{z\in\mathbb{Z}^3} \alpha_z(t)\phi_z(x),$$

where α_z is smooth. Note now that

$$\begin{aligned}
0 = \langle 0, \phi_{z'} \rangle = \langle \Box\, \varphi, \phi_{z'} \rangle &\overset{\text{dom. convergence}}{=} \langle \sum_{z\in\mathbb{Z}^3} \Box \left(\alpha_z(t)\phi_z\right), \phi_{z'})\rangle \\
&\overset{\text{Lemma 3.1.6}}{=} \langle \sum_{z\in\mathbb{Z}^3} \left(\alpha_z''(t) + \sum_{j=1}^{3} \frac{\dot{a}_j(t)}{a_j(t)} \alpha_z'(t) + 4\pi^2 \alpha_z(t) \sum_{j=1}^{3} \frac{z_j^2}{a_j(t)^2} \right) \phi_z, \phi_{z'} \rangle \\
&\overset{\text{continuity of } \langle\cdot,\cdot\rangle}{=} \sum_{z\in\mathbb{Z}^3} \left(a_z''(t) + \sum_{j=1}^{3} \frac{\dot{a}_j(t)}{a_j(t)} \alpha_z'(t) + 4\pi^2 \alpha_z(t) \sum_{j=1}^{3} \frac{z_j^2}{a_j(t)^2} \right) \langle \phi_z, \phi_{z'} \rangle \\
&= \sum_{z\in\mathbb{Z}^3} \left(a_z''(t) + \sum_{j=1}^{3} \frac{\dot{a}_j(t)}{a_j(t)} \alpha_z'(t) + 4\pi^2 \alpha_z(t) \sum_{j=1}^{3} \frac{z_j^2}{a_j(t)^2} \right) \delta_{z,z'} \\
&= a_{z'}''(t) + \sum_{j=1}^{3} \frac{\dot{a}_j(t)}{a_j(t)} \alpha_{z'}'(t) + 4\pi^2 \alpha_{z'}(t) \sum_{j=1}^{3} \frac{{z'_j}^2}{a_j(t)^2}, \ \forall t \in I.
\end{aligned}$$

This is equation (3.2). The initial conditions of (3.1) translate into initial conditions for $\alpha_z(t)$ as

$$\begin{aligned}
\alpha_z(t_0) &= \langle \varphi(t_0,\cdot), \phi_z \rangle_{L^2(\mathrm{T})} = \langle \varphi_0, \phi_z \rangle_{L^2(\mathrm{T})}, \\
\alpha_z'(t_0) &= \partial_t|_{t=t_0} \langle \varphi(t,\cdot), \phi_z \rangle_{L^2(\mathrm{T})} = \langle (\partial_t \varphi)(t_0,\cdot), \phi_z \rangle_{L^2(\mathrm{T})} \\
&= \langle \varphi_1(\cdot), \phi_z \rangle_{L^2(\mathrm{T})}.
\end{aligned}$$

This completes the proof. □

From now on, we will study solutions to the ODE's in the previous theorem for a fixed $z \in \mathbb{Z}^3$. The method used is sometimes called the *mode solution* and each α_z is then called a *mode.*

Remark 3.1.9 (Existence and uniqueness of solution)**.** In the proof of the theorem, we have actually shown the existence and uniqueness of solutions to the problem (3.2, 3.3, 3.4) as a corollary from the existence and uniqueness theorem (Thm 3.1.2). (Note that the existence and uniqueness also follows from standard results on linear ODE's, see for instance [14, Theorem 19.I].)

Remark 3.1.10 (Sufficient to look for real valued solutions)**.** Recall that the a_i's are all real valued, and hence all coefficients in the differential equation (3.2) are real. Therefore, if α_z is a solution to (3.2) with initial conditions $\alpha_z(t_0)$ and $\alpha_z'(t_0)$, then the complex conjugate $\overline{\alpha}_z$ is a solution to (3.2) with initial conditions $\alpha_{-z}(t_0)$ and $\alpha_{-z}'(t_0)$. By uniqueness of solution, $\overline{\alpha}_z = \alpha_{-z}$.

Hence $\overline{\alpha_z \phi_z} = \alpha_{-z}\phi_{-z}$ and hence

$$\alpha_z\phi_z + \alpha_{-z}\phi_{-z} = 2\Re(\alpha_z\phi_z)$$

is a solution to (3.2) with initial data

$$2\Re(\alpha_z(t_0)), \quad 2\Re(\alpha_z'(t_0)).$$

This shows that indeed, it is enough to look for real solutions (also denoted by) α_z with real initial data $\alpha_z(t_0)$ and $\alpha_z'(t_0)$.

3.2 Two cases of explicit solutions.

Note first that the differential equation (3.2) has the following solution when $z = (0,0,0) \in \mathbb{Z}^3$.

Proposition 3.2.1. *If $z = (0,0,0) \in \mathbb{Z}^3$, then*

$$\alpha_z(t) = \alpha_z'(t_0) a_1 a_2 a_3(t_0) \int_{t_0}^{t} \frac{1}{a_1 a_2 a_3(w)} dw + \alpha_z(t_0),$$

for all $t \in I$.

In the rest of this section we look at two cases where we can solve equation (3.2) explicitly also for $z \neq (0,0,0) \in \mathbb{Z}^3$. Recall that the Kasner metric is a Bianchi type I metric, where we have chosen the a_i's as

$$a_i : I = \mathbb{R}_+ \to \mathbb{R}, \quad a_i(t) := t^{p_i},$$

where the p_i's satisfy

$$\sum_{i=1}^{3} p_i = 1, \quad \sum_{i=1}^{3} p_i^2 = 1.$$

Note that if (K_{T}, g_K) is the 3-torus spacetime equipped with a Kasner metric, then equation (3.2) becomes

$$\alpha_z''(t) + \frac{\alpha_z'(t)}{t} + 4\pi^2 \alpha_z(t) \sum_{j=1}^{3} z_j^2 t^{-2p_j} = 0,$$

for all $t \in \mathbb{R}_+$. We will see that we can get an explicit solution to the previous equation if two of the p_i's are chosen to be the same. One easily verifies the following Lemma.

Lemma 3.2.2. *If $p = (p_1, p_2, p_3)$ satisfies*

$$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} p_i^2 = 1,$$

and two of the p_i's are equal, then

$$\{p_1, p_2, p_3\} = \{1, 0, 0\} \text{ (Flat Kasner metrics)}, \quad \textit{or} \quad \{p_1, p_2, p_3\} = \left\{-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\}.$$

3.2.1 The case of Flat Kasner metric.

Since the other cases are completely analogous, we assume in this section that

$$p_1 = 1, \quad p_2 = p_3 = 0.$$

In this case equation (3.2) becomes

$$\alpha_z''(t) + \frac{\alpha_z'(t)}{t} + 4\pi^2 \alpha_z(t) \left(z_1^2 t^{-2} + z_2^2 + z_3^3\right) = 0, \tag{3.5}$$

for all $t \in \mathbb{R}_+$. We divide this subsection into different cases for $z \in \mathbb{Z}^3$. In 2 of the 4 cases below, the solution will be expressed by the so called "Bessel functions". We start by introducing them. All definitions follow [12, p. 227].

Definition 3.2.3 (Bessel equation for the parameter $\nu \in \mathbb{C}$.)**.** Let $\nu \in \mathbb{C}$ be a fixed parameter. We define *Bessel equation* for ν as

$$x^2 y''(x) + x y'(x) + (-\nu^2 + x^2) y(x) = 0. \tag{3.6}$$

We will only need the solution when $\nu \in \mathbb{C} \backslash \mathbb{Z}_-$, where $\mathbb{Z}_-$ denotes the negative integers.

Definition 3.2.4 (Bessel functions of first and second kind for $\nu \in \mathbb{C}\backslash\mathbb{Z}$.)**.** Let $\nu \in \mathbb{C}\backslash\mathbb{Z}$ be a fixed parameter. We define the *Bessel function of the first kind for* ν as

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k}$$

and the *Bessel function of the second kind for* ν as

$$Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}.$$

Definition 3.2.5 (Bessel functions of first and second kind for $n \in \mathbb{Z}_{\geq 0}$.)**.** Let $n \in \mathbb{Z}_{\geq 0}$ be a fixed parameter. We define the *Bessel function of the first kind for* n as

$$J_n(x) := \frac{1}{\pi} \int_0^\pi \cos(x\sin(t) - nt)dt$$

and the *Bessel function of the second kind for* n as

$$Y_n(x) := \lim_{\nu \to n} Y_\nu(x),$$

where Y_ν for $\nu \in \mathbb{C}\backslash\mathbb{Z}_-$ is as in the previous definition.

The following statement can be found in [12, p. 227].

Lemma 3.2.6. *If* $\nu \in \mathbb{C}\backslash\mathbb{Z}_-$*, then the general solution to* (3.6) *is given by*

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x).$$

Case 1: $z = (0, 0, 0)$

Proposition 3.2.7.

$$\alpha_z(t) = \alpha_z'(t_0)t_0 \ln\left(\frac{t}{t_0}\right) + \alpha_z(t_0).$$

Proof. Immediate from Proposition 3.2.1. □

Case 2: $z_1 = 0, (z_2, z_3) \neq (0, 0)$

Proposition 3.2.8.

$$\alpha_z(t) = c_1 J_0\left(2\pi\sqrt{z_2^2 + z_3^2}t\right) + c_2 Y_0\left(2\pi\sqrt{z_2^2 + z_3^2}t\right)$$

Proof. By Definition 3.2.5 and Lemma 3.2.6. □

Case 3: $z_1 \neq 0, (z_2, z_3) = (0, 0)$

Proposition 3.2.9.

$$\alpha_z(t) = C_1 \sin(2\pi z_1 \ln(t)) + C_2 \cos(2\pi z_1 \ln(t))$$

Proof. A simple calculation. □

Case 4: $z_1 \neq 0, (z_2, z_3) \neq (0,0)$

Proposition 3.2.10.

$$\alpha_z(t) = c_1 J_{2i\pi z_1}\left(2\pi t\sqrt{z_2^2+z_3^2}\right) + c_2 Y_{2i\pi z_1}\left(2\pi t\sqrt{z_2^2+z_3^2}\right)$$

Proof. After the variable substiution $x(t) = 2\pi t\sqrt{z_2^2+z_3^2}$ and with the choice $\nu = 2\pi i z_1$, equation (3.6) becomes

$$t^2\alpha_z''(t) + t\alpha_z'(t) + \alpha_z(t)\left(-(2i\pi z_1)^2 + \left(2\pi t\sqrt{z_2^2+z_3^2}\right)^2\right) = 0.$$

Simplifying and dividing by t^2 gives equation (3.7). This completes the proof. □

3.2.2 The case when $\{p_1, p_2, p_3\} = \left\{-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\}$.

Similarly to the previous section, without loss of generality, we assume

$$p_1 = -\frac{1}{3}, \quad p_2 = p_3 = \frac{2}{3}.$$

In this case equation 3.2 becomes

$$\alpha_z''(t) + \frac{\alpha_z'(t)}{t} + 4\pi^2\alpha_z(t)\left(z_1^2 t^{2/3} + (z_2^2+z_3^2)t^{-4/3}\right) = 0, \tag{3.7}$$

for all $t \in \mathbb{R}_+$. In addition to the Bessel functions, defined in the previous subsection, we will need the so called "Heun Biconfluent function" in order to express the solutions. The definition of the Heun Biconfluent functions follows the documentation of Maple, which in turn refers to [5].

Definition 3.2.11 (Heun Biconfluent function)**.** Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ be constants. Then we define the *Heun Biconfluent function* of the parameters $\alpha, \beta, \gamma, \delta$, denoted as $HeunB(\alpha, \beta, \gamma, \delta, z)$ as the solution to

$$y''(z) - \frac{\left(\beta z + 2z^2 - \alpha - 1\right)(y'(z))}{z} - \frac{1}{2}\frac{((2\alpha - 2\gamma + 4)z + \beta\alpha + \beta + \delta)y(z)}{z} = 0,$$

such that

$$y(0) = 1, \quad y'(0) = \frac{\alpha\beta + \beta + \delta}{2\alpha + 2}.$$

Case 1: $z = (0,0,0)$

Proposition 3.2.12.

$$\alpha_z(t) = \alpha_z'(t_0)t_0 \ln\left(\frac{t}{t_0}\right) + \alpha_z(t_0).$$

Proof. Immediate from Proposition 3.2.1. □

Case 2: $z_1 = 0, (z_2, z_3) \neq (0,0)$

Proposition 3.2.13.

$$\alpha_z(t) = c_1 J_0\left(6\pi\sqrt{z_2^2+z_3^2}\,t^{-1/3}\right) + c_2 Y_0\left(6\pi\sqrt{z_2^2+z_3^2}\,t^{-1/3}\right)$$

Proof. By Definition 3.2.5 and Lemma 3.2.6. □

Case 3: $z_1 \neq 0, (z_2, z_3) = (0,0)$

Proposition 3.2.14.

$$\alpha_z(t) = C_1 J_0\left(\frac{3}{2}\pi z_1 t^{4/3}\right) + C_2 Y_0\left(\frac{3}{2}\pi z_1 t^{4/3}\right)$$

Proof. By Definition 3.2.5 and Lemma 3.2.6. □

Case 4: $z_1 \neq 0, (z_2, z_3) \neq (0,0)$

According to the "dSolve" method in Maple, the solution in this case is given by the following.

Proposition 3.2.15.

$$\alpha_z(t) = e^{\frac{3}{2}\pi\sqrt{-z_3^2}t^{4/3}} HeunB\left(0,0,0,\frac{6i\pi^{3/2}\sqrt{3}(z_1^2+z_2^2)}{(-z_3^2)^{1/4}}, i\sqrt{3}\sqrt{-z_3^2}^{1/4} t^{2/3}\right)\times$$
$$\left(\int \frac{e^{-3\pi\sqrt{-z^2}t^{4/3}}}{tHeunB\left(0,0,0,\frac{6i\pi^{3/2}\sqrt{3}(z_1^2+z_2^2)}{(-z_3^2)^{1/4}}, i\sqrt{3}\sqrt{-z_3^2}^{1/4} t^{2/3}\right)} dt \cdot c_1 + c_2\right)$$

3.3 Rewriting the ODE for the Fourier modes.

The previous chapter shows that even in easy special cases, explicit solutions get quite complicated. We devote the rest of the chapter to a qualitative study of the solutions in general Bianchi type I spacetimes. In particular, we want to study the behaviour of the solutions for large and small $t \in I$. We therefore rewrite the equation in two different ways, which both will turn out to be useful.

3.3.1 By writing the solution as a product.

Let α_z be the solution of equation(3.2) and define

$$\gamma_z := \alpha_z\sqrt{a_1a_2a_3}.$$

Recall that the equation (3.2) can be written as

$$\frac{1}{a_1a_2a_3}\partial_t(a_1a_2a_3(\partial_t\alpha_z)) + 4\pi^2\alpha_z\sum_{i=1}^{3}\frac{z_i^2}{a_i^2} = 0. \tag{3.8}$$

Now define $f := a_1a_2a_3$. The following lemma is a simple calculation.

Lemma 3.3.1. *The equation for γ_z becomes*

$$\gamma_z'' + \left(\frac{1}{4}\frac{(f')^2}{f^2} - \frac{1}{2}\frac{f''}{f} + 4\pi^2\sum_{i=1}^{3}\frac{z_i^2}{a_i^2}\right)\gamma_z = 0.$$

We will indeed use this result on the Kasner space later, the next remark shows how the equation looks in this case.

Example 3.3.2 (Kasner metrics)**.** We take as an example, the Kasner metric, given by

$$a_i(t) := t^{p_i}.$$

Since we demand that $\sum_{i=1}^{3} p_i = 1$,

$$f(t) = a_1a_2a_3(t) = t \Rightarrow f'(t) = 1 \Rightarrow f''(t) = 0,$$

the differential equation for the $\gamma_z(t)$ becomes

$$\gamma_z''(t) + \left(\frac{1}{4t^2} + 4\pi^2 \sum_{i=1}^{3} \frac{z_i^2}{t^{2p_i}}\right) \gamma_z(t) = 0, \tag{3.9}$$

for all $t \in I$.

3.3.2 By a change of variables.

In this section, we will perform a variable substitution from t to s, by defining

$$\beta_z(s(t)) := \alpha_z(t),$$

for all $t \in I$, where

$$s(t) := \int_{t'}^{t} \frac{1}{a_1a_2a_3(w)} dw, \tag{3.10}$$

for some arbitrarily chosen $t' \in I$. Note that

$$\partial_t s = \frac{1}{a_1a_2a_3}.$$

Since $a_i > 0$, s is a smooth strictly increasing function, hence a C^1-diffeomorphism. The chain rule implies that

$$\partial_t \alpha_z = \partial_t s \partial_s \beta_z = \frac{1}{a_1a_2a_3} \partial_s \beta_z$$

and hence

$$\frac{1}{a_1a_2a_3} \partial_t(a_1a_2a_3\partial_t\alpha_z) = \frac{1}{a_1a_2a_3} \partial_t s \partial_s(\partial_s\beta_z) = \frac{1}{(a_1a_2a_3)^2} \partial_s^2 \beta_z.$$

Again, equation (3.2) can be written as

$$\frac{1}{a_1a_2a_3} \partial_t(a_1a_2a_3(\partial_t\alpha_z)) + 4\pi^2\alpha_z \sum_{i=1}^{3} \frac{z_i^2}{a_i^2} = 0. \tag{3.11}$$

which is now equivalent to

$$\frac{1}{(b_1b_2b_3)^2} \partial_s^2 \beta_z + 4\pi^2 \beta_z \sum_{i=1}^{3} \frac{z_i^2}{b_i^2} = 0,$$

where $b_i(s) := a_i(t(s))$. This can be written as

$$\partial_s^2 \beta_z + 4\pi^2 \beta_z \sum_{\substack{i,j,k=1 \\ i\neq j \neq k \neq i}}^{3} z_i^2 b_j^2 b_k^2 = 0, \tag{3.12}$$

Example 3.3.3 (Kasner metrics)**.** Again we take the Kasner metric as the example, given by

$$a_i(t) := t^{p_i}.$$

Since we demand that $\sum_{i=1}^{3} p_i = 1$, the formula for variable substitution, equation (3.10) with $t_0 := 1$, is given by

$$s(t) = \int_1^t \frac{1}{w} dw = \ln(t).$$

Therefore $b_i(s) = a_i(e^s) = e^{p_i s}$ and hence $b_j(s)b_k(s) = e^{(p_j+p_k)s} = e^{(1-p_i)s}$ if $i \neq j \neq k \neq i$. Therefore equation 3.12 will be given by

$$\beta_z''(s) + 4\pi^2 \beta_z(s) \sum_{i=1}^{3} z_i^2 e^{(2-2p_i)s} = 0, \tag{3.13}$$

for all $s \in \mathbb{R}$, where "prime" now means derivative with respect to s.

3.4 Methods to describe the general solution.

Motivated by the previous section, we treat in this section the differential equation

$$v'' + Kv = 0, \tag{3.14}$$

where $K : J \subset \mathbb{R} \to \mathbb{R}_+$ is a continuous function and $J \subset \mathbb{R}$ is an interval. The theory for such differential equations is well known. The reader can find a treatment of the theory in for example [14, §27] and [1, §27]. In this chapter we give a more detailed description relevant to equation coming from previous sections of the thesis.

Remark 3.4.1 (Existence and uniqueness.)**.** Since the equation we treat is linear, and K is continuous, by for instance [14, Theorem 19.I], we conclude that there exists a unique solution $v \in C^2(J)$ satisfying (3.14) for given initial data $v(s_0)$ and $v'(s_0)$.

We first note that trigonometric functions are special cases.

Remark 3.4.2 (Trigonometric solutions)**.** Assume $K(s) = k$. The solution to equation (3.14) is given by a trigonometric function on the form

$$f(s) = A\cos(\sqrt{k}s) + B\sin(\sqrt{k}s),$$

for all $s \in J$. If $s_0 \in J$ is a zero of f, then $\{s_0 + \frac{\pi}{\sqrt{k}}n : n \in \mathbb{Z}\} \cap J$ is the set of zeros for f. A and B are determined by the initial data for f.

3.4.1 Existence of zeros.

The idea now is to bound the general solution locally by trigonometric functions.

Theorem 3.4.3 (The upper and lower bounds)**.** *Let $J \subset \mathbb{R}$ be an interval. Assume that $K : J \to \mathbb{R}_+$ is a continuous function and that $v : J \to \mathbb{R}$ satisfies*

$$v''(s) + K(s)v(s) = 0. \tag{3.15}$$

Assume that $s_0 \in J$ is such that

$$v'(s_0) = 0, \quad v(s_0) > 0,$$

and assume that $k \in \mathbb{R}_+$ such that

$$J_k := \left(s_0 - \frac{\pi}{2\sqrt{k}}, s_0 + \frac{\pi}{2\sqrt{k}}\right) \subset J,$$

Let $f : J \to \mathbb{R}$ be the unique (trigonometric) solution to

$$f''(s) + kf(s) = 0. \tag{3.16}$$

such that $f'(s_0) = 0$ and $f(s_0) = v(s_0) > 0$.

- *If $0 < k < K(s)$ for all $s \in J_k$, then there exist $s_{-1}, s_1 \in J_k$ such that*

$$s_{-1} < s_0 < s_1, \quad v(s_{-1}) = v(s_1) = 0$$

  and

$$0 < v(s) < f(s),$$

  for all $s \in [s_{-1}, s_1]\backslash\{s_0\}$.

- *If $0 < K(s) < k$ for all $s \in J_k$, then*

$$0 < f(s) < v(s),$$

  for all $s \in J_k\backslash\{s_0\}$.

From the theorems, one can for instance say something about the amount of zeros given certain conditions on $K(s)$. The following result is maybe the most immediate one.

Corollary 3.4.4 (Amount of zeros)**.** *Assume that $v : (a, \infty) \to \mathbb{R}$ is a non-trivial solution to*

$$v''(s) + K(s)v(s) = 0$$

where there exist a $k \in \mathbb{R}_+$ such that

$$K(s) > k$$

for all $s \in (a, \infty)$. Then v has infinitely many zeros, pairwise within a distance bounded from above by $\frac{\pi}{\sqrt{k}}$.

Proof. We want to use the Theorem 3.4.3 inductively. For this, we need to prove that for each zero s_i of v, there exists $s_m > s_i$ such that $v'(s_m) = 0$. Assume therefore w.l.o.g. that $v'(s) > 0$ for all $s \in (s_i, \infty)$. Then there exists a $\delta > 0$ and a $s' > s_i$ such that $v(s) > \delta$ for all $s > s'$. Hence

$$v''(s) = -K(s)v(s) \leq -k\delta, \ \forall s > s',$$

which is a contradiction. Hence we can use Theorem 3.4.3 inductively, and the proof is complete. □

We now prove the lemma, which essentially proves the theorem.

Lemma 3.4.5 (First upper bound)**.** *Let $J \subset \mathbb{R}$ be an interval. Assume that $K : J \to \mathbb{R}_+$ is a continuous function and that $v : J \to \mathbb{R}$ satisfies*

$$v''(s) + K(s)v(s) = 0. \tag{3.17}$$

Assume that $s_0 \in J$ is such that

$$v'(s_0) = 0, \quad v(s_0) > 0,$$

and that

$$\left[s_0, s_0 + \frac{\pi}{2\sqrt{k}}\right] \subset J,$$

where k satisfies

$$0 < k < K(s),$$

for all $s \in \left[s_0, s_0 + \frac{\pi}{2\sqrt{k}}\right]$. Let $f : J \to \mathbb{R}$ be the unique (trigonometric) solution to

$$f''(s) + kf(s) = 0$$

such that $f'(s_0) = 0$ and $f(s_0) = v(s_0) > 0$. Then there exists $s_1 \in \left[s_0, s_0 + \frac{\pi}{2\sqrt{k}}\right]$ such that

$$v(s_1) = 0$$

and

$$0 < v(s) < f(s),$$

for all $s \in (s_0, s_1)$.

Proof. W.l.o.g, we can assume that $s_0 = 0$, $f(0) = v(0) = 1$ and $k = 1$. The general case just follows by scaling and translating f. Note that in this case

$$f(s) = \cos(s).$$

Note that $f(\frac{\pi}{2}) = 0$. We write $K(s) = \epsilon(s) + k$, where $\epsilon(s) > 0$ for all $s \in \left[0, \frac{\pi}{2}\right]$. Then we see that

$$\frac{d}{ds}(v'(s)^2 + kv(s)^2) = 2v''(s)v'(s) + 2v'(s)kv(s) = 2v'(s)\left(v''(s) + K(s)v(s)\right) - 2\epsilon(s)v'(s)v(s)$$

$$= -2\epsilon(s)v(s)v'(s)$$

Now assume that $[0, s_v) \subset \mathbb{R}$ is the largest subinterval such that $v(s)$ is positive for all $s \in [0, s_v)$. By equation (3.17), this implies that v'' is negative on $[0, s_v)$. This, together with the fact that $v'(0) = 0$ implies that v' is negative on $[0, s_v)$, so $v'v < 0$ on this interval. Hence

$$\frac{d}{ds}(v'(s)^2 + kv(s)^2) > 0 \tag{3.18}$$

for all $s \in (0, s_v)$. Since

$$\frac{d}{ds}(f'(s)^2 + kf(s)^2) = 2f'(s)(f''(s) + kf(s)) = 0,$$

we conclude that

$$f'(s)^2 + kf(s)^2 = v'(0)^2 + kv(0)^2,$$

for all $s \in [0, s_v)$. Therefore, with equation (3.18) we see that

$$v'(s)^2 + kv(s)^2 > v'(0)^2 + kv(0)^2 = f'(s)^2 + kf(s)^2, \quad \forall s \in (0, s_v).$$

Note first that $v''(0) < f''(0)$. Hence $v''(s) < f''(s)$ for all $s \in (0, s')$ for some $s' > 0$ and hence $v(s) < f(s)$ for all $s \in (0, s')$. Assume now that $t \in [0, s_v)$ such that $v(s) < f(s)$ for all $s \in (0, t)$ and $f(t) = v(t)$. Then the above equation shows that

$$|v'(t)| > |f'(t)|.$$

We know that $v'(s) < 0$ for all $s \in (0, s_v)$. But since

$$0 < v(s) < f(s)$$

for all $s \in [0, t)$, we conclude (recalling that $f(s) = \cos(s)$) that $f'(t) < 0$. Hence

$$v'(t) < f'(t).$$

This means that there is an open neighbourhood $(t - \delta, t + \delta) \subset [0, s_v)$ such that

$$v'(s) < f'(s)$$

for all $s \in (t - \delta, t + \delta)$. But hence

$$f(t - \delta) = f(t) - \int_{t-\delta}^{t} f'(s)ds < v(t) - \int_{t-\delta}^{t} v'(s)ds = v(t - \delta).$$

This contradicts the assumption that $v(s) < f(s)$ for all $s \in (0, t)$. Hence

$$v(s) < f(s)$$

for all $s \in [0, s_v)$. Since $f(\pi/2) = 0$, we see that $s_v \leq \pi/2$, and by assumption $v(s_v) = 0$. We define hence $s_1 := s_v$ and the proof is complete.

□

Remark 3.4.6. Note that the variable substitution given by $t := -s$ and $w(t) := v(-t)$ translates the equation

$$v''(s) + K(s)v(s) = 0$$

into the analogous equation

$$w''(t) + \bar{K}(t)w(t) = 0,$$

where $\bar{K}(t) := K(-t)$. Hence the analogous results as those of Lemma 3.4.5 also holds "backwards", i.e. after this substitution of variables.

The proof of the theorem is basically done at this point.

Proof of Theorem 3.4.3. The first statement follows immediately by Lemma 3.4.5 and Remark 3.4.6. The second statement follows by an analogous lemma. □

3.4.2 Improved bound on distance between zeros.

The main result of this section improves the bounds of the previous section, given the existence of zeros.

Theorem 3.4.7 (Local bound by trigonometric functions)**.** *Let $J \subset \mathbb{R}$ be an interval. Assume that $K : J \to \mathbb{R}_+$ is a continuous function and that $v : J \to \mathbb{R}$ satisfies*

$$v''(s) + K(s)v(s) = 0. \tag{3.19}$$

Assume that $s_{-1}, s_0, s_1 \in J$ are such that $s_{-1} < s_0 < s_1$, $[s_{-1}, s_1] \subset J$ and

$$\begin{aligned} &v'(s_0) = 0, \quad v(s_0) > 0, \quad v(s_{-1}) = v(s_1) = 0,\\ &v(s) > 0, \quad \forall s \in (s_{-1}, s_1). \end{aligned}$$

Define

$$K_{min} := \min_{s\in[s_{-1},s_1]} K(s), \quad K_{max} := \max_{s\in[s_{-1},s_1]} K(s).$$

Let $f, g : J \to \mathbb{R}$ be the unique solutions to

$$\begin{aligned} &f''(s) + K_{min} f(s) = 0,\\ &g''(s) + K_{max} g(s) = 0,\\ &f(s_0) = g(s_0) = v(s_0),\\ &f'(s_0) = g'(s_0) = v'(s_0) = 0. \end{aligned}$$

Then

$$v(s) \le f(s)$$

for all $s \in [s_{-1}, s_1]$ and

$$g(s) \le v(s)$$

for all $s \in \left[s_0 - \frac{\pi}{2\sqrt{K_{max}}}, s_0 + \frac{\pi}{2\sqrt{K_{max}}}\right]$.

Hence we can improve the bound on the distance between zeros.

Corollary 3.4.8 (Bound on distance between zeros)**.** *Let $J, K, v, K_{min}, K_{max}, s_{-1}, s_0, s_1$ be as in the previous theorem. Then the distance between s_{-1} and s_1 satisfies*

$$\frac{\pi}{\sqrt{K_{max}}} \le s_1 - s_{-1} \le \frac{\pi}{\sqrt{K_{min}}}.$$

Proof. The general solutions to equations for f, g are given by

$$\begin{aligned} f(s) &= A\cos(\sqrt{K_{\min}}s) + B\sin(\sqrt{K_{\min}}s),\\ g(s) &= C\cos(\sqrt{K_{\max}}s) + D\sin(\sqrt{K_{\max}}s). \end{aligned}$$

Since the period of a trigonometric function is known, this proves the corollary. □

We now prove Theorem 3.4.7.

Proof of Theorem 3.4.7. In the proof of the Lemma 3.4.5, one does not need the assumption that $K(s) > k$ for all $s \in [s_0, s_0 + \frac{\pi}{2\sqrt{k}}]$, given that one knows existence of zeros.

If we instead know that v has zeros s_{-1}, s_1 and a maximum s_0 such that $s_{-1} < s_0 < s_1$ and $v(s) > 0$ for all $s \in (s_{-1}, s_1)$, then it is enough to assume that $K(s) > k$ for all $s \in [s_0, s_1]$.

Let $\epsilon > 0$. Apply this result to the functions $f_\epsilon, g_\epsilon : J \to \mathbb{R}$ defined by

$$\begin{aligned}
&f_\epsilon' + (K_{min} - \epsilon)f = 0,\\
&g_\epsilon' + (K_{max} + \epsilon)g = 0,\\
&f_\epsilon(s_0) = g_\epsilon(s_0) = v(s_0),\\
&f_\epsilon'(s_0) = g_\epsilon'(s_0) = v'(s_0) = 0.
\end{aligned}$$

Hence

$$v(s) < f_\epsilon(s)$$

for all $s \in [s_{-1}, s_1] \backslash \{s_0\}$ and

$$g_\epsilon(s) < v(s)$$

for all $s \in \left[s_0 - \frac{\pi}{2\sqrt{K_{max}+\epsilon}}, s_0 + \frac{\pi}{2\sqrt{K_{max}+\epsilon}}\right] \backslash \{s_0\}$.

Since the solution of a trigonometric function does continuously depend on the coefficient in the differential equation,

$$\begin{aligned}
&\lim_{\epsilon \to 0} f_\epsilon(s) = f(s),\\
&\lim_{\epsilon \to 0} g_\epsilon(s) = g(s),
\end{aligned}$$

pointwise. This proves the theorem. □

3.4.3 A bound on the amplitude of the solution for increasing K.

In this section, we prove that, under the assumption that K is increasing, the amplitude is bounded.

Theorem 3.4.9 (Amplitude decrease)**.** *Let $J \subset \mathbb{R}$ be an interval. Assume that $K : J \to \mathbb{R}_+$ is a increasing continuous function and that $v : J \to \mathbb{R}$ satisfies*

$$v''(s) + K(s)v(s) = 0. \tag{3.20}$$

Assume that $s_0, s_2 \in J$ are such that $[s_0, s_2] \subset J$ and

$$\begin{aligned}
&v'(s_0) = v'(s_2) = 0, \quad v(s_0) > 0, \ v(s_2) < 0,\\
&v'(s) < 0, \quad \forall s \in (s_0, s_2).
\end{aligned}$$

Then

$$\sqrt{K(s)}\,|v(s)| \leq \sqrt{K(s_2)}\,|v(s_2)|\,, \tag{3.21}$$

for all $s \in [s_0, s_2)$ and

$$|v(s_0)| \geq |v(s_2)|\,.$$

Before we prove the theorem, we prove the following corollary.

Corollary 3.4.10. *Let $v : (-\infty, a]$ satisfy*

$$v''(s) + K(s)v(s) = 0$$

where $K : (-\infty, a] \to \mathbb{R}_+$ is smooth, increasing and satisfies

$$\begin{aligned}
&K \in L^{1/2}(-\infty, a),\\
&\left(x \mapsto \int_{-\infty}^{x} \sqrt{K(t)}dt\right) \in L^1(-\infty, a).
\end{aligned}$$

Then v has finitely many zeros.

Proof. Assume that v has infinitely many zeros. By Corollary 3.4.8, we see that the set of zeros is unbounded. Hence, if we use Theorem 3.4.9 inductively, one concludes that

$$|v(s)|\sqrt{K(s)} \leq |v(s_0)|\sqrt{K(s_0)} := C$$

for all $s \leq s_0$. Hence

$$|v''(s)| = |v(s)|\,K(s) \leq C\sqrt{K(s)},$$

for all $s \leq s_0$. The assumption on K implies now that

$$v'' \in L^1(-\infty, s_0).$$

Hence

$$v'(s) = \int_{s_0}^{s} v''(t)dt + v'(s_0) = \int_{s_0}^{s} v''(t)dt \to \int_{s_0}^{-\infty} v''(t)dt =: D$$

as $s \to -\infty$. If $D \neq 0$, we arrive at a contradiction to the assumption on infinitely many zeros. Therefore, assume $D = 0$. Similar to before,

$$|v'(s)| \leq \int_{s}^{s_0} |v''(t)|\,dt \leq |C| \int_{s}^{s_0} \sqrt{K(t)}dt \leq |C| \int_{-\infty}^{s_0} \sqrt{K(t)}dt < \infty,$$

and hence

$$v(s) = \int_{s_0}^{s} v'(t)dt + v(s_0) \to \int_{s_0}^{-\infty} v'(t)dt + v(s_0) =: E,$$

as $s \to -\infty$. Now, if $E \neq 0$, we are done. Hence assume that $E = 0$. This contradicts the second statement of Theorem 3.4.9, i.e. that the amplitude does not decrease when $s \to -\infty$. Hence, by contradiction, we conclude the corollary.

□

As before, we start by proving a lemma, and proceed with the proof of the theorem afterwards.

Lemma 3.4.11 (Another bound)**.** *Let $J \subset \mathbb{R}$ be an interval. Assume that $K : J \to \mathbb{R}_+$ is a continuous function and that $v : J \to \mathbb{R}$ and satisfies*

$$v''(s) + K(s)v(s) = 0.$$

Assume that $s_0, s_1 \in J$ are such that

$$v(s_0) = 0, \quad v'(s_0) > 0, \quad v'(s_1) = 0,$$
$$v'(s) > 0, \quad \forall s \in (s_0, s_1).$$

Moreover, we choose k such that

$$0 < k < K(s),$$

for all $s \in (s_0, s_1)$. Let $f : J \to \mathbb{R}$ be the unique (trigonometric) solution to

$$f''(s) + kf(s) = 0$$

such that $f(s_0) = 0$ and $f'(s_0) = v'(s_0) > 0$. Then

$$s_1 < s_0 + \frac{\pi}{2\sqrt{k}}$$

and

$$0 < v(s) < f_{max} := f\left(s_0 + \frac{\pi}{2\sqrt{k}}\right),$$

for all $s \in (s_0, s_1]$.

Proof. Note first that $s_1 < s_0 + \frac{\pi}{2\sqrt{k}}$. For if not, it is easy to see that this contradicts Theorem 3.4.7 by translating and scaling the function f so that the maxima of f and v coincide.

Note that

$$f'(s_0)^2 = f'(s)^2 + kf(s)^2, \quad \forall s \in \left(s_0, s_0 + \frac{\pi}{2\sqrt{k}}\right), \tag{3.22}$$

since

$$\frac{d}{ds}\left(f'(s)^2 + kf(s)^2\right) = 2f'(s)\left(f''(s) + kf(s)\right) = 0, \quad \forall s \in \left(s_0, s_0 + \frac{\pi}{2\sqrt{k}}\right).$$

Hence

$$f_{max} := f\left(s_0 + \frac{\pi}{2\sqrt{k}}\right) = \frac{f'(s_0)}{\sqrt{k}},$$

where the last equality follows by (3.22).

Similar to the proof of Lemma 3.4.5, we see that

$$f'(s_0)^2 > v'(s)^2 + kv(s)^2, \quad \forall s \in (s_0, s_1], \tag{3.23}$$

since

$$\frac{d}{ds}\left(v'(s)^2 + kv(s)^2\right) = 2v'(s)\left(v''(s) + kv(s)\right) = 2v'(s)v(s)(k - K(s)) < 0,$$

for all $s \in (s_0, s_1)$ and

$$v'(s_0)^2 + v(s_0)^2 = v'(s_0)^2 = f'(s_0)^2.$$

If we now choose $s = s_1$ in equation (3.23), we see that

$$v(s_1) < f_{max}.$$

We conclude that

$$v(s) \leq v(s_1) < f_{max},$$

for all $s \in (s_0, s_1]$. □

Proof of Theorem 3.4.9. Let $s_1 \in (s_0, s_2)$ be such that $v(s_1) = 0$. Note that the inequality (3.21) is clear for $s \in [s_1, s_2)$. Therefore, fix $t \in [s_0, s_1)$. Let $f, g : J \to \mathbb{R}$ satisfy

$$\begin{aligned} f''(s) + K(t)f(s) &= 0, \\ g''(s) + K(s_2)g(s) &= 0, \end{aligned}$$

and $f(s_1) = g(s_1) = 0, f'(s_1) = g'(s_1) = v'(s_1)$. Since

$$\begin{aligned} K(s) &\geq K(t), \ \forall s \in (t, \infty) \cap J, \\ K(s) &\leq K(s_2), \ \forall s \in (-\infty, s_2) \cap J, \end{aligned}$$

we get by (a slight modification of) Lemma 3.4.11 and Remark 3.4.6 that

$$\begin{aligned} &f(s) < v(s) < 0, \ \forall s \in [t, s_1), \\ &0 < g(s) < v(s), \ \forall s \in \left(s_1, s_1 + \frac{\pi}{2\sqrt{K(s_2)}}\right). \end{aligned}$$

As before,

$$f'(s)^2 + K(t)f(s)^2$$

and

$$g'(s)^2 + K(s_2)g(s)^2$$

are constant. By the initial conditions, this constant must be the same for both expressions, i.e.

$$f'(s)^2 + K(t)f(s)^2 = g'(w)^2 + K(s_2)g(w)^2$$

for all $s, w \in J$. In particular, since

$$f'\left(s_1 - \frac{\pi}{2\sqrt{K(t)}}\right) = 0,$$
$$g'\left(s_1 + \frac{\pi}{2\sqrt{K(s_2)}}\right) = 0,$$

it follows that

$$\begin{aligned} K(t)(f_{\min})^2 &= K(t)f\left(s_1 - \frac{\pi}{2\sqrt{K(t)}}\right)^2 \\ &= K(s_2)g\left(s_1 + \frac{\pi}{2\sqrt{K(s_2)}}\right)^2 = K(s_2)(g_{\max})^2. \end{aligned}$$

Hence

$$|v(s_2)| \geq |g_{\max}| = \sqrt{\frac{K(t)}{K(s_2)}}\,|f_{\min}| \geq \sqrt{\frac{K(t)}{K(s_2)}}\,|v(t)|\,,$$

and the proof of (3.21) is done. Define the function h as the solution to

$$h''(s) + K(s_1)h(s) = 0$$

such that $h(s_1) = v(s_1) = 0$ and $h'(s_1) = v'(s_1)$ and using similar inequalities shows

$$|v(s_0)| \geq |v(s_2)|\,.$$

□

3.5 Application: The behaviour of the solution for Non-flat Kasner metrics.

Since the explicit solution for the case of Flat Kasner metrics has been found, we study in this section the solution in the case of Non-flat Kasner metrics. Recall that for a Kasner metric,

$$a_i(t) = t^{p_i}, \quad \sum_{i=1}^{3} p_i = \sum_{i=1}^{3} p_i^2 = 1$$

and

$$I = \mathbb{R}_+.$$

Remark 3.5.1. For $z = (0,0,0)$, the solution is given by

$$\alpha_z(t) = \alpha_z'(t_0)t_0 \ln\left(\frac{t}{t_0}\right) + \alpha_z(t_0),$$

for all $t \in I$, by Proposition 3.2.1. Assume therefore in what follows, that $z \neq (0,0,0)$.

3.5.1 Distribution of zeros.

Equation (3.13) shows that the rewritten differential equation is

$$\beta_z''(s) + 4\pi^2\beta_z(s)\sum_{j=1}^{3} z_j^2 e^{(2-2p_j)s} = 0 \tag{3.24}$$

for all $s \in \mathbb{R}$, where $(2 - 2p_j) > 0$ for each p_i. Recall that $\beta_z(s) = \alpha_z(t(s))$, where $t(s) = e^s$. We will in this section prove results for β_z and translate them to $\alpha_z(s)$.

Theorem 3.5.2. *The solution α_z has a smallest zero. Moreover, the set of zeros is discrete and unbounded from above.*

Proof. We claim that the statement holds for β_z. Then the statement is true for α_z, since $s(t) = \ln(t)$ is monotone.

Indeed, one notes that if

$$K(s) := 4\pi^2 \sum_{j=1}^{3} z_j^2 e^{(2-2p_j)s} = 0, \ \forall s \in \mathbb{R},$$

it is increasing and

$$\begin{aligned} &K \in L^{1/2}(-\infty, 0), \\ &\left(x \mapsto \int_{-\infty}^{x} \sqrt{K(t)} dt \right) \in L^1(-\infty, 0), \end{aligned}$$

so the assumptions of Corollary 3.4.10 are fullfilled, and it follows that β_z has a smallest zero.

Also, since $K(s)$ is bounded from below for all $s \in (0, \infty)$, Corollary 3.4.4 implies that there exist infinitely many zeros of β_z. □

Therefore, we can denote the ordered set of zeros of α_z by

$$(t_i)_{i \in \mathbb{N}},$$

where t_1 is the smallest zero. The following theorem gives a bound on the quotient between the zeros.

Theorem 3.5.3 (Zeros in case of Non-flat Kasner metric)**.** *If $(t_i)_{i\in\mathbb{N}}$ is the ordered set of zeros of α_z. Let $t_i < t_{i+1}$ be neighbouring zeros for α_z. Then*

$$\exp\left(\frac{1}{\sqrt{\sum_{k=1}^{3} z_k^2 (t_{i+1})^{2-2p_k}}} \right) \le \frac{t_{i+1}}{t_i} \le \exp\left(\frac{1}{\sqrt{\sum_{k=1}^{3} z_k^2 (t_i)^{2-2p_k}}} \right).$$

Proof. Since the logarithm function is monotone and

$$\begin{aligned} b_z(s_i) &= \alpha_z(t(s_i)) = \alpha_z(t_i) = 0, \\ b_z(s_{i+1}) &= \alpha_z(t(s_{i+1})) = \alpha_z(t_{i+1}) = 0, \end{aligned}$$

it follows that $s_i = \ln(t_i)$ and $s_{i+1} := \ln(t_{i+1})$ are neighbouring zeros for β_z. By Corollary 3.4.8,

$$\frac{1}{\sqrt{\sum_{k=1}^{3} z_k^2 e^{(2-2p_k)s_{i+1}}}} \le s_{i+1} - s_i \le \frac{1}{\sqrt{\sum_{k=1}^{3} z_k^2 e^{(2-2p_k)s_i}}}.$$

This translates to

$$\frac{1}{\sqrt{\sum_{k=1}^{3} z_k^2 (t_{i+1})^{2-2p_k}}} \le \ln(t_{i+1}) - \ln(t_i) \le \frac{1}{\sqrt{\sum_{k=1}^{3} z_k^2 (t_i)^{2-2p_k}}},$$

and the proof is complete. □

Corollary 3.5.4. *If $(t_i)_{i\in\mathbb{N}}$ is the ordered set of zeros of α_z. Then*

$$\lim_{i\to\infty} \frac{t_{i+1}}{t_i} = 1.$$

We know that one of the p_k, say p_j is negative. In the case where the coefficient in front of t^{2-2p_j} is nonzero, we get a stronger result.

Corollary 3.5.5. *Assume that $p_j < 0$ and $z_j \neq 0$. Then*

$$\lim_{i\to\infty} |t_{i+1} - t_i| = 0.$$

Proof. The right inequality from Theorem 3.5.3 is

$$\frac{t_{i+1}}{t_i} \leq \exp\left(\frac{1}{\sqrt{\sum_{k=1}^{3} z_k^2 (t_i)^{2-2p_k}}}\right).$$

Subtracting 1 from both expressions and multiplying by $t_i > 0$ gives, for large t_i,

$$|t_{i+1} - t_i| \leq \left(\exp\left(\frac{1}{\sqrt{\sum_{k=1}^{3} z_k^2 (t_i)^{2-2p_k}}}\right) - 1\right) t_i \leq \left(\exp\left(\frac{1}{\sqrt{z_j^2 + 1}(t_i)^{1-p_j}}\right) - 1\right) t_i.$$

By the standard limit ($a \neq 0, \alpha > 0$)

$$\left(\exp\left(\frac{1}{ax^\alpha}\right) - 1\right) x^\alpha \to \frac{1}{a}$$

as $x \to \infty$ and since $1 - p_j > 1$, we are done. □

3.5.2 Decay of the amplitude for large times.

Since

$$K(s) := 4\pi^2 \sum_{j=1}^{3} z_j^2 e^{(2-2p_j)s} = 0, \ \forall s \in \mathbb{R},$$

is increasing, we know by Theorem 3.4.9 that the amplitude of β_z is decreasing when $s \to \infty$. Therefore also the amplitude of α_z is decreasing as $t \to \infty$. In fact, the following statement is true.

Theorem 3.5.6. *Let $z \neq 0$ and let α_z be a real valued solution to equation* (3.2)*, i.e.*

$$\alpha_z''(t) + \frac{\alpha_z'(t)}{t} + 4\pi^2 \alpha_z(t) \sum_{j=1}^{3} z_j^2 t^{-2p_j} = 0,$$

for all $t \in \mathbb{R}_+$. Then there exists a $C > 0$ and a $t' \in \mathbb{R}_+$ such that

$$|\alpha_z(t)| \leq \frac{C}{\left(4\pi^2 \sum_{i=1}^{3} z_i^2 t^{(2-2p_i)}\right)^{1/4}}.$$

as for all $t \geq t'$.

The proof of this theorem uses a lemma proven in [10]. Before stating this lemma, we start with some necessary definitions. We look for **complex** solutions to the equation

$$f''(s) + K(s)f(s) = 0, \quad \forall s \in (a, \infty), \tag{3.25}$$

with extra assumptions on K. Let K_1, K_2 be defined on (a, ∞) such that $K = K_1 + K_2$. In order to state the theorem, we define

$$\psi_{K_1,K_2} := |K_1|^{-\frac{1}{4}} \left(-\frac{d^2}{ds^2} + K_2\right) |K_1|^{-\frac{1}{4}}.$$

Lemma 3.5.7. *Assume that $K_1(s) > 0$ for all $s \in (a, \infty)$, $|K_1|^{1/2} \notin L^1(a, \infty)$ and $\psi_{K_1,K_2} \in L^1(a, \infty)$. Then there exists solutions f_1 and f_2 to (3.25) such that*

$$e^{-i\int_a^s |K_1(r)|^{1/2} dr} |K_1(s)|^{1/4} f_1(s) \to 1 \tag{3.26}$$

$$e^{i\int_a^s |K_1(r)|^{1/2} dr} |K_1(s)|^{1/4} f_2(s) \to 1 \tag{3.27}$$

when $s \to \infty$.

Proof. See [10, Proposition 3.2]. □

In particular, we get the following result for a general solution to (3.25).

Lemma 3.5.8. *Let K_1, K_2 and ψ_{K_1,K_2} be as in Lemma 3.5.8. Assume that f satisfies (3.25). Then there exists a $C > 0$ such that*

$$|f(s)| |K_1(s)|^{1/4} \to C$$

as $s \to \infty$.

Proof. Since the solution space of the equation (3.25) is a complex vector space of dimension 2, f_1 and f_2 form a basis, if they are not linearly dependent. That f_1 and f_2 are linearly independent follows immediately from equations (3.26) and (3.27). Therefore, there exist constants $a \in \mathbb{C}$ and $b \in \mathbb{C}$, such that

$$f(s) = af_1(s) + bf_2(s).$$

Recall the following triangle inequality on $\mathbb{C}$,

$$||a| - |b|| \leq |a - b|.$$

We use this inequality, together with Lemma 3.5.7,

$$\begin{aligned}
&\left| |f(s)| |K_1(s)|^{1/4} - \left| e^{i\int_a^s |K_1(r)|^{1/2} dr} a + e^{-i\int_a^s |K_1(r)|^{1/2} dr} b \right| \right| \\
&\leq \left| f(s) |K_1(s)|^{1/4} - (e^{i\int_a^s |K_1(r)|^{1/2} dr} a + e^{-i\int_a^s |K_1(r)|^{1/2} dr} b) \right| \\
&= \left| af_1(s) |K_1(s)|^{1/4} + bf_2(s) |K_1(s)|^{1/4} - (e^{i\int_a^s |K_1(r)|^{1/2} dr} a + e^{-i\int_a^s |K_1(r)|^{1/2} dr} b) \right| \\
&\leq |a| \left| f_1(s) |K_1(s)|^{1/4} - e^{i\int_a^s |K_1(r)|^{1/2} dr} \right| + |b| \left| f_2(s) |K_1(s)|^{1/4} - e^{-i\int_a^s |K_1(r)|^{1/2} dr} \right| \\
&\leq |a| \left| e^{-i\int_a^s |K_1(r)|^{1/2} dr} f_1(s) |K_1(s)|^{1/4} - 1 \right| + |b| \left| e^{i\int_a^s |K_1(r)|^{1/2} dr} f_2(s) |K_1(s)|^{1/4} - 1 \right| \to 0
\end{aligned}$$

as $s \to \infty$. Chose

$$C := \left| e^{i\int_a^s |K_1(r)|^{1/2} dr} a + e^{-i\int_a^s |K_1(r)|^{1/2} dr} b \right|,$$

and the statement is proven. □

We want to apply this proposition to the real solutions of (3.13). We formulate the result in the following lemma.

Lemma 3.5.9. *Let $z \neq 0 \in \mathbb{Z}^3$ and let β_z be a **real** solution to equation (3.13), i.e.*

$$\beta_z''(s) + \left(4\pi^2 \sum_{i=1}^{3} z_i^2 e^{(2-2p_i)s} \right) \beta_z(s) = 0. \tag{3.28}$$

Then there exists a $C > 0$ and a $s' \in \mathbb{R}$ such that

$$|\beta_z(s)| \leq \frac{C}{\left(4\pi^2 \sum_{i=1}^{3} z_i^2 e^{(2-2p_i)s} \right)^{1/4}}$$

for all $s \geq s'$.

Proof. Following the notation in the beginning of the section,

$$K(s) := 4\pi^2 \sum_{i=1}^{3} z_i^2 e^{(2-2p_i)s},$$

for all $s \in \mathbb{R}$. Define

$$K_1(s) := 4\pi^2 \sum_{i=1}^{3} z_i^2 e^{(2-2p_i)s}, \quad K_2(s) := 0.$$

This satisfies of course $K = K_1 + K_2$, and $K_1 > 0$. We see that ψ_{K_1,K_2} becomes

$$\psi_{K_1,K_2}(s) = -K_1(s)^{-1/4} \frac{d^2}{ds^2}\left(K_1(s)^{-1/4}\right) = -K_1(s)^{-1/2}\left(\frac{5}{16}\left(\frac{K_1'(s)}{K_1(s)}\right)^2 - \frac{1}{4}\left(\frac{K_1''(s)}{K_1(s)}\right)\right)$$

First we note that ψ_{K_1,K_2} does not have singularities. Since $p_i \in (-1/2, 1)$ for $i = 1, 2, 3$, we see that $2 - 2p_i > 0$. We claim that the second factor is bounded. For this we show that

$$\frac{K_1''(s)}{K_1(s)}$$

is bounded. That the first term is bounded follows by a similar argument. Note that

$$\frac{K_1''(s)}{K_1(s)} = \frac{\sum_{i=1}^{3}(2-2p_i)^2 z_i^2 \exp((2-2p_i)s)}{\sum_{i=1}^{3} z_i^2 \exp((2-2p_i)s)}.$$

Assume now that p_j is the smallest (or one of the smallest) p_i such that $z_i \neq 0$. We factor out $\exp((2-2p_j)s)$ from both numerator and denominator.

$$\frac{K_1''(s)}{K_1(s)} = \frac{\sum_{i=1}^{3}(2-2p_i)^2 z_i^2 \exp(2(p_j-p_i)s)}{\sum_{i=1}^{3} z_i^2 \exp(2(p_j-p_i)s)}$$

Note that the denominator is bounded from below by z_j^2. Also, since $p_j - p_i \leq 0$, we see that both denominator and numerator converge as $s \to \infty$. This implies that the quotient converges and in particular that there exists a constant $C > 0$ and $s_0 \in (a, \infty)$ such that $\left|\frac{K_1''(s)}{K_1(s)}\right| < C$ for all $s > s_0$.

Similarly, one proves that the first term is bounded. We conclude that there exists a constant $D > 0$ and a $s_1 \in (a, \infty)$ such that

$$\left|\frac{5}{16}\left(\frac{K_1'(s)}{K_1(s)}\right)^2 - \frac{1}{4}\left(\frac{K_1''(s)}{K_1(s)}\right)\right| \leq D$$

for all $s \geq s_1$.

Hence it suffices to show that $K_1^{-1/2} \in L^1(a, \infty)$, in order to prove that $\psi_{K_1,K_2} \in L^1(a, \infty)$. We use a similar argument as before. Note that

$$K_1(s)^{-1/2} = \frac{1}{2\pi} \exp((p_j - 1)s)\left(\sum_{i=1}^{3} z_i^2 \exp(2(p_j - p_i)s)\right)^{-1/2}.$$

We know that the second factor converges. Since $p_j - 1 < 0$, the first factor is in $L^1(a, \infty)$ and hence $K^{-1/2} \in L^1(a, \infty)$. Therefore we have proven that $\psi_{K_1,K_2} \in L^1(a, \infty)$.

What remains in order to apply Lemma 3.5.7 is to show that $K_1(s)^{1/2} \notin L^1(a, \infty)$. This follows by noting that

$$K_1(s)^{1/2} = 2\pi\left(\sum_{i=1}^{3} z_i^2 e^{(2-2p_i)s}\right)^{1/2} \geq 2\pi |z_j|$$

for all $s \in (a, \infty)$. Hence we can apply Lemma 3.5.8 and get that the **complex** solution β_z^c to equation (3.28) satisfies

$$|\beta_z^c(s)| \left(4\pi^2 \sum_{i=1}^{3} z_i^2 e^{(2-2p_i)s}\right)^{1/4} \to C'$$

as $s \to \infty$, for some $C' > 0$. Since

$$|\beta_z| = |\Re(\beta_z^c)| \leq |\beta_z^c|,$$

the statement follows. □

The proof of the theorem is just a translation of the result under the change of variable $s(t) := \ln(t)$.

Proof of Theorem 3.5.6. We recall that

$$\alpha_z(t) := b_z(s(t)), \ \forall t \in \mathbb{R}_+,$$

where

$$s(t) := \ln(t).$$

Note that

$$s(t) \to \infty \Leftrightarrow t \to \infty.$$

Furthermore,

$$|\alpha_z(t)| \left(4\pi^2 \sum_{i=1}^{3} z_i^2 t^{(2-2p_i)}\right)^{1/4} = |\beta_z(s(t))| \left(4\pi^2 \sum_{i=1}^{3} z_i^2 e^{(2-2p_i)s(t)}\right)^{1/4},$$

for all $t \in \mathbb{R}_+$ and the statement follows by Lemma 3.5.9. □

3.5.3 Growth of the amplitude for small times.

We get a similar bound for small times.

Theorem 3.5.10. *Let $z \neq 0$ and let α_z be a real valued solution to equation (3.2), i.e.*

$$\alpha_z''(t) + \frac{\alpha_z'(t)}{t} + 4\pi^2 \alpha_z(t) \sum_{j=1}^{3} z_j^2 t^{-2p_j} = 0,$$

for all $t \in \mathbb{R}_+$. Then there exists a $C > 0$ and an $\delta \in \mathbb{R}$ such that

$$|\alpha_z(t)| \leq \frac{C}{\sqrt{t}}$$

for all $t \leq \delta$.

Proof. By Remark 3.3.2, we know that if we write the solution as

$$\alpha_z(t) = \frac{\gamma_z(t)}{\sqrt{t}},$$

we get the following differential equation for γ_z:

$$\gamma_z''(t) + \left(\frac{1}{4t^2} + 4\pi^2 \sum_{i=1}^{3} z_i^2 t^{-2p_i}\right) \gamma_z(t) = 0,$$

for all $t \in \mathbb{R}_+$. Let

$$K(t) := \left(\frac{1}{4t^2} + 4\pi^2 \sum_{i=1}^{3} z_i^2 t^{-2p_i}\right).$$

Note that there is a $\delta > 0$ (depending on z) such that K is increasing on $(0, \delta)$ for $t \to 0$. By Remark 3.4.6, and Theorem 3.4.9 this means that γ_z is bounded on $(0, \delta)$. This proves the theorem. □

Chapter 4

Redshift in Bianchi type I spacetimes.

In general relativity, there is a standard way to describe light, namely as a lightlike geodesic. This gives a notion of the so called cosmological redshift. We describe two other methods to describe light in Bianchi type I spacetime and compare the resulting redshift in respective case.

4.1 The general setting.

In all models, we start on a hypersurface $S_{Bia}(t_0) := \{t_0\} \times \mathbb{R}^3$. The following lemma shows that we can without loss of generality, assume that $a_j(t_0) = 1,\ j = 1, 2, 3$.

Lemma 4.1.1. *Let (M, g_a) and (M, g_b) be Bianchi type I spacetimes such that*

$$g_a = -dt^2 + \sum_{j=1}^{3} a_j(t)^2 (dx^j)^2,$$
$$g_b = -dt^2 + \sum_{j=1}^{3} b_j(t)^2 (dx^j)^2,$$

and $b_j(t) := \frac{a_j(t)}{a_j(t_0)}$. Then

$$f : (M, g_a) \to (M, g_b)$$
$$f(t, x_1, x_2, x_3) = (t, a_1(t_0)x_1, a_2(t_0)x_2, a_3(t_0)x_3)$$

is a time coordinate preserving isometry and $b_j(t_0) = 1$.

We therefore assume that $a_j(t_0) = 1$. Before define the notion of redshift, let us fix an initial spatial direction for the light.

Definition 4.1.2 (Initial direction for the light ray)**.** We define the initial direction of the light to be

$$C = (C_1, C_2, C_3) \in \mathbb{R}^3.$$

Whenever we write C or the components C_j for $j = 1, 2, 3$, in the rest of the thesis, we refer to this definition.

This direction will be interpreted in each model respectively. We define the redshift of light between two Cauchy hypersurfaces $S_{Bia}(t_1) := \{t_1\} \times \mathbb{R}^3$ and $S_{Bia}(t_2) := \{t_2\} \times \mathbb{R}^3$. Note that we have not defined "wavelength" yet, this will be differently defined in the three models.

Definition 4.1.3 (Redshift in Bianchi type I spacetimes)**.** Assume that light has wavelength $\lambda(t_1)$ on $S_{Bia}(t_1)$ and wavelength $\lambda(t_2)$ on $S_{Bia}(t_2)$. Then the redshift from t_1 to t_2 is given by

$$z(t_1, t_2) := \frac{\lambda(t_2) - \lambda(t_1)}{\lambda(t_1)}.$$

We will use units where the speed of light is 1. Hence we assume the following relation between the wavelength of light λ and the frequency of light ω:

$$\lambda\omega = 1.$$

Lemma 4.1.4 (Equivalent definition of redshift in Bianchi type I spacetimes.)**.** *Assume that light has frequency $\omega(t_1)$ on $S_{Bia}(t_1)$ and frequency $\omega(t_2)$ on $S_{Bia}(t_2)$. Then the redshift from t_1 to t_2 is given by*

$$z(t_1, t_2) = \frac{\omega(t_1)}{\omega(t_2)} - 1.$$

4.2 Cosmological redshift in Bianchi type I spacetimes.

The cosmological redshift for Robertson-Walker spacetimes is presented in the chapter "Geodesics and Redshift" starting on page 353 in [11]. The following definition in Bianchi type I spacetimes is made completely analogously.

4.2.1 Definition of cosmological redshift in Bianchi type I spacetimes.

The main assumption when modelling light as a geodesic is the following definition of frequency for a lightlike geodesic.

Definition 4.2.1 (Frequency - in cosmological redshift)**.** Let $\gamma : J \subset \mathbb{R} \to M$ be a lightlike geodesic. The *frequency $\nu : J \to \mathbb{R}$ measured by ∂_t* is defined as

$$\nu(s) := \frac{E(s)}{h},$$

for all $s \in J$, where h is the Planck constant and $E(s)$ is the energy measured by ∂_t (recall Definition 2.2.29).

Using

$$\lambda\nu = 1,$$

we get the following definition of wavelength.

Definition 4.2.2 (Wavelength - in cosmological redshift)**.** The wavelength $\lambda : J \to \mathbb{R}$ of a lightlike geodesic is given by

$$\lambda(s) = \frac{h}{E(s)}.$$

We now have the tools to define *cosmological redshift* in Bianchi type I spacetimes.

Definition 4.2.3 (Cosmological redshift)**.** Assume (M, g) is a Bianchi type I spacetime and $\gamma : I \subset \mathbb{R} \to M$ is a lightlike future pointing geodesic. Let $\gamma(s_1) = p$ and $\gamma(s_2) = q$ and let ∂_t be the observer field on M. The *redshift parameter of γ from s_1 to s_2 relative to ∂_t* is defined as

$$z := \frac{\lambda(s_2) - \lambda(s_1)}{\lambda(s_1)}.$$

Remark 4.2.4. This definition seems to depend on the proper time of the geodesic, and would therefore not be compatible with Definition 4.1.3. We will show that the result will only depend on the time coordinate, so the definitions do indeed coincide. Note also that using ∂_t as observer is analogous to measure the wavelength at a hypersurface $S_{Bia}(t)$ for $t \in I$, since ∂_t is normal to $S_{Bia}(t)$.

4.2.2 Calculating the cosmological redshift.

Theorem 4.2.5 (Redshift in Bianchi type I spacetimes.)**.** *Let (M, g) be a Bianchi type I spacetime. Assume that $\gamma : J \subset \mathbb{R} \to M$ is a lightlike geodesic, defined by*

$$\gamma'(t_0) = (\|C\|, C_1, C_2, C_3), \tag{4.1}$$

where $C \in \mathbb{R}^3$ is as in Definition 4.1.2 (this is lightlike since $a_j(t_0) = 1$) for all j. Let $t_1 := t \circ \gamma(s_1)$ and $t_2 := t \circ \gamma(s_2)$ for $s_1 < s_2$. Then the redshift parameter relative to ∂_t is given by

$$z(t_1, t_2) = \left(\frac{\sum_{j=1}^3 C_j^2 a_j(t_1)^{-2}}{\sum_{j=1}^3 C_j^2 a_i(t_2)^{-2}} \right)^{1/2} - 1.$$

Proof. In coordinates, we write

$$\gamma(s) = (t(s), x_1(s), x_2(s), x_3(s)),$$

for all $s \in J$. Hence we can now write the energy-momentum tensor as

$$P = \gamma' = (t', x_1', x_2', x_3').$$

and the energy is given by

$$E = -\langle P, \partial_t \rangle = t'.$$

By definition of lightlike geodesic,

$$0 = \langle \gamma'(s), \gamma'(s) \rangle = -\left(t'(s)\right)^2 + \sum_{j=1}^3 a_j(t(s))^2 \left(x_j'(s)\right)^2, \tag{4.2}$$

for all $s \in J$. Since ∂_j for $j = 1, 2, 3$ are Killing fields (Corollary 2.3.8), we conclude by [11, Lemma 9.27], that

$$C_j = \langle \gamma'(s), \partial_j \rangle = x_j'(s) a_j(t(s))^2,$$

for all $s \in J$, where C_j are the constants in equation (4.1). Hence

$$x_j'(s) = C_j a_j(t(s))^{-2}. \tag{4.3}$$

Combining equation (4.2) and equation (4.3) gives

$$-\left(t'(s)\right)^2 + \sum_{i=1}^3 C_i^2 a_i(t(s))^{-2} = 0,$$

for all $s \in J$. This implies that the energy is

$$E(s) = t'(s) = \left(\sum_{j=1}^3 C_j^2 a_i(t(s))^{-2} \right)^{1/2}, \ \forall s \in J.$$

By Defintion 4.2.2,

$$\lambda(s_1) = \frac{h}{E(s_1)}, \quad \lambda(s_2) = \frac{h}{E(s_2)}.$$

Hence the cosmological redshift is

$$z(t_1, t_2) = \frac{\frac{h}{E(s_2)} - \frac{h}{E(s_1)}}{\frac{h}{E(s_1)}} = \frac{E(s_1)}{E(s_2)} - 1 = \left(\frac{\sum_{j=1}^3 C_j^2 a_j(t_1)^{-2}}{\sum_{j=1}^3 C_j^2 a_j(t_2)^{-2}} \right)^{1/2} - 1.$$

This completes the proof. □

Since the Kasner spacetimes are the subclass of (non-trivial) Bianchi type I spacetimes that satisfy the Einstein's vacuum equation, it is interesting to see how the formula for the cosmological redshift looks in this case.

Corollary 4.2.6 (Redshift in Kasner spacetimes.). *Assume that $g = g_K$, i.e. that the spacetime is a Kasner spacetime. Then the redshift parameter is given by*

$$z(t_1, t_2) = \left(\frac{\sum_{j=1}^{3} C_j^2 t_1^{-2p_j}}{\sum_{j=1}^{3} C_j^2 t_2^{-2p_j}} \right)^{1/2} - 1,$$

where the p_j's are the usual parameters in the Kasner metric.

One can also expect the equation to simplify when the motion is parallel to a spatial coordinate. The next corollary will show that this is indeed the case.

Corollary 4.2.7 (Motion parallel to a spatial coordinate.). *Assume that $C_i = 1$ and C_k for $k \neq i$. Then the redshift parameter is given by*

$$z(t_1, t_2) = \frac{a_i(t_2)}{a_i(t_1)} - 1.$$

Proof. Applying Theorem 4.2.5 gives

$$z(t_1, t_2) = \left(\frac{\sum_{j=1}^{3} C_j^2 a_j(t_1)^{-2}}{\sum_{j=1}^{3} C_j^2 a_j(t_2)^{-2}} \right)^{1/2} - 1 = \left(\frac{a_i(t_1)^{-2}}{a_i(t_2)^{-2}} \right)^{1/2} - 1 = \frac{a_i(t_2)}{a_i(t_1)} - 1.$$

The last step follows since the a_j's are assumed to be positive. □

We get a similar result in the spatially flat Robertson-Walker spacetime.

Corollary 4.2.8 (Spatially flat Robertson-Walker spacetimes as a special case.). *Assume that (M, g) is a Spatially flat Robertson-Walker spacetime. Then the redshift parameter is given by*

$$z(t_1, t_2) = \frac{f(t_2)}{f(t_1)} - 1,$$

where $f(t) = a_1(t) = a_2(t) = a_3(t)$.

Proof. Apply Theorem 4.2.5 to the case when $f(t) := a_1(t) = a_2(t) = a_3(t)$. Then

$$\begin{aligned} z(t_1, t_2) &= \left(\frac{\sum_{j=1}^{3} C_j^2 a_j(t_1)^{-2}}{\sum_{j=1}^{3} C_j^2 a_j(t_2)^{-2}} \right)^{1/2} - 1 = \left(\frac{f(t_1)^{-2}}{f(t_2)^{-2}} \right)^{1/2} \left(\frac{\sum_{j=1}^{3} C_j^2}{\sum_{j=1}^{3} C_j^2} \right)^{1/2} - 1 \\ &= \frac{f(t_2)}{f(t_1)} - 1. \end{aligned}$$

□

4.3 Redshift obtained by solving the scalar wave equation.

In this section, we will model light as a solution to the wave equation. In Robertson-Walker spacetimes, this can be done directly by looking at the solution to the general wave equation expanded into modes. In the anisotropic Bianchi type I spacetimes, we must look at initial data modelling light in the direction C, since the cosmological redshift depends on this direction.

We start by defining the frequency.

Definition 4.3.1 (Definition of frequency of a function $f : J \subset \mathbb{R} \to \mathbb{C}$)**.** Assume that a function $f : J \to \mathbb{C}$ where the vectors $(\Re(f(t_0)), \Re(f'(t_0)))$ and $(\Im(f(t_0)), \Im(f'(t_0)))$ are linearly independent. Assume furthermore that there exist continuous functions $a : J \to \mathbb{R}$ and $b : J \to \mathbb{R}_+$ such that

$$f''(t) + a(t)f'(t) + b(t)f(t) = 0,$$

for all $t \in J$. Then we define the *frequency of f* as

$$\omega_f := \sqrt{b}.$$

See Appendix B for a motivation of this definition.

4.3.1 The isotropic case.

Recall from the previous chapter, that the solution to the wave equation that is 1-periodic in all space direction, i.e. solution on the 3-Torus-Bianchi type I spacetime, could be written

$$\varphi(t, x) = \sum_{z \in \mathbb{Z}3} \alpha_z(t)\phi_z(x)$$

where $\phi_z(x) = \exp(2\pi iz \cdot x)$ for all $x \in \mathbb{R}^3$ and $\alpha_z(t) = \langle \varphi(t, \cdot), \phi_z \rangle$ for all $t \in I$. In Remark 3.1.10, we noted that it is enough to consider real solutions if one wants to study behaviour of the solutions, but here we want to look at the original (complex) mode. In other words, in the next proposition we are **not** going to assume that α_z is real.

Proposition 4.3.2. *Assume that $f := a_1 = a_2 = a_3$, i.e. (M_{T}, g) is a spatially flat 3-torus-Robertson-Walker spacetime. Let α_z, for $z \in \mathbb{Z}^3$, be a complex mode solution to the scalar wave equation discussed in the previous chapter, i.e. a solution to* (3.2)*. Then the frequency of α_z is*

$$\omega_{\alpha_z} = 2\pi \frac{\|z\|}{f},$$

where $f := a_1 = a_2 = a_3$.

Proof. Note that equation (3.2) becomes

$$\ddot{\alpha}_z + 3\frac{\dot{f}}{f}\dot{\alpha}_z + \alpha_z \frac{4\pi^2}{f^2} \|z\|^2 = 0.$$

By Definition 4.3.1, the frequency for α_z is

$$\omega_{\alpha_z} = \sqrt{\frac{4\pi^2 \|z\|^2}{f^2}} = 2\pi \frac{\|z\|}{f}.$$

The proof is complete. □

Corollary 4.3.3. *The redshift in the case of spatially flat 3-torus-Robertson-Walker spacetime obtained by modelling light as a solution to the* ***general scalar wave equation*** *coincides with the cosmological redshift (Corollary 4.2.8).*

Proof. Insert the above result in Lemma 4.1.4. □

4.3.2 The general case.

Remark 4.3.4 (A consequence of anisotropy)**.** In general Bianchi type I spacetimes, we expect from Theorem 4.2.5, that the redshift depends on the initial spatial direction of the light, i.e. the constants (C_1, C_2, C_3). Therefore we cannot look at the frequency of an arbitrary wave, as is possible in Robertson-Walker spacetimes.

We define the initial data that is natural to use. As is customary with the wave equation, we will specify the initial data on a hypersurface $S_{Bia}(t_0)$.

Notation 4.3.5. We denote by $C^\perp$ the complement of span(C) in $\mathbb{R}^3$ w.r.t. the Euclidean inner product.

Definition 4.3.6 (Translation invariance on $C^\perp$)**.** A function $f \in C^\infty(M, \mathbb{R})$ is *translation invariant on* $C^\perp$ if for all $(t, x) \in M$ and for all $v \in C^\perp \subset \mathbb{R}^3$,

$$f(t, x + v) = f(t, x).$$

For convenience we are going to assume that our functions are periodic in C. In this case we will be able to write the solution as a Fourier series. With some modifications, one could get analogous results using the (continuous) Fourier transform instead of Fourier series.

Definition 4.3.7 (C-periodic function)**.** A function $f \in C^\infty(M, \mathbb{R})$ is C-periodic if

$$f(t, x + C) = f(t, x),$$

for all $x \in \mathbb{R}^3$.

Notation 4.3.8. We denote the space of all smooth C-periodic functions that are translation invariant on $C^\perp$ by $C^\infty_C(M, \mathbb{R})$.

We can now state the model of light, using the scalar wave equation.

Problem 4.3.9 (Scalar wave equation for light)**.** Let $C = (C_1, C_2, C_3) \in \mathbb{R}^3$ be given. Let $\varphi_1, \varphi_2 \in C^\infty_C(M, \mathbb{R})$. Then a function $\varphi \in C^\infty_C(M, \mathbb{R})$ solves the *scalar wave equation for light in direction* C if

$$\Box\varphi = 0, \quad \varphi|_{S_{Bia}(t_0)} = \varphi_0, \quad (\partial_t\varphi)|_{S_{Bia}(t_0)} = \varphi_1.$$

Lemma 4.3.10 (Well-posed problem)**.** *There exists a unique solution to Problem 4.3.9.*

Proof. Let $\mathrm{T}^1 := \mathbb{R}/\mathbb{Z}$ denote the 1-torus. Note that

$$L : C^\infty_C(M, \mathbb{R}) \to C^\infty(I \times \mathrm{T}^1, \mathbb{R}),$$
$$L(f)(t, s) = f(t, sC),$$

is an isomorphism. But on $(I \times \mathrm{T}^1, g)$ with induced Bianchi type I metric g, we can apply Theorem 3.1.2, similarly to the proof of Corollary 3.1.3, using that T^1 is compact. This implies the existence of a unique solution φ_{T^1} to the corresponding problem on $I \times \mathrm{T}^1$. Hence $\varphi := L^{-1}(\varphi_{\mathrm{T}^1})$ is the unique solution to Problem 4.3.9. □

Remark 4.3.11. Since for each $t \in I$, the function $\varphi(t, \cdot)$ is C-periodic and smooth, we can write it as a Fourier series

$$\varphi(t, x) = \sum_{n \in \mathbb{Z}} \alpha_n(t)\phi^C_n(x),$$

where for all $(t, x) \in M$ with $\phi^C_n(x) := \exp(2\pi i C \cdot xn)$.

Lemma 4.3.12. *The equations for each mode in the previous remark are*

$$\ddot{\alpha}_n + \dot{\alpha}_n \sum_{j=1}^{3} \frac{\dot{a}_i}{a_i} + \alpha_n \left(4\pi^2 n^2 \sum_{j=1}^{3} \frac{C_j^2}{a_j^2} \right) = 0,$$
$$\alpha_n(t_0) = \langle \varphi_0, \phi^C_n \rangle,$$
$$\alpha'_n(t_0) = \langle \varphi_1, \phi^C_n \rangle.$$

Proof. Follows analogously to Theorem 3.1.7. □

Theorem 4.3.13. *Let $n \in \mathbb{Z}$. The frequency of α_n is*

$$\omega_{\alpha_n} = 2\pi n \left(\sum_{j=1}^{3} C_j^2 a_j^{-2} \right)^{1/2}.$$

Proof. The frequency for α_n becomes, using Definition 4.3.1,

$$\omega_{\alpha_n} = \left(4\pi^2 n^2 \sum_{j=1}^{3} C_j^2 a_j^{-2} \right)^{1/2} = 2\pi n \left(\sum_{j=1}^{3} C_j^2 a_j^{-2} \right)^{1/2}.$$

□

The redshift coincides with the cosmological redshift in this case also in this case.

Corollary 4.3.14. *The redshift obtained by modelling light as a solution to the* ***scalar wave equation with solution and initial data in*** $C_C^\infty(M, \mathbb{R})$ *coincides with the cosmological redshift (Theorem 4.2.5).*

Proof. Insert the above result in Lemma 4.1.4. □

Remark 4.3.15. Note that the result *is not* dependent on the fact that we chose C-periodic initial data. We obtain the same redshift with pC-periodic initial data, for any $p \in \mathbb{R}_+$.

4.4 The Cauchy problem for Maxwell's equations.

We now turn to the third method, modelling light as certain solutions to Maxwell's equations in vacuum. We begin by formulating Maxwell's equations on a general Lorentz manifold. For a deeper introduction to the theory, see for example [7].

Definition 4.4.1 (Maxwell's equations)**.** Assume that (N, h) is a Lorentz manifold of dimension n. Let $J \in \Lambda^1(N)$ be given. $F \in \Lambda^2(N)$ is said to satisfy the *Maxwell's equations* if

$$dF = 0, \quad \delta F = J,$$

where d is the exterior derivative and δ is the codifferential (see Appendix A). Furthermore, F is called the *electromagnetic field.*

The 1-form J in the above definition is called the *4-current.* In vacuum, there is no charge and current, therefore we set $J = 0$.

Definition 4.4.2 (Maxwell's vacuum equations)**.** $F \in \Lambda^2(M)$ is said to satisfy the *Maxwell's vacuum equations* if

$$dF = 0, \quad \delta F = 0.$$

In order to define the electric and magnetic fields on a hypersurface, we need to have a notion of "restricting" a 2-form to a hypersurface.

Definition 4.4.3. Let (N, h) be a globally hyperbolic manifold and let S be a spacelike Cauchy hypersurface in N. Denote the embedding by

$$\iota : S \hookrightarrow N,$$

and define the functions

$$\begin{aligned} p_{(0)} :& \Lambda^2(N) \to \Lambda^2(S) \\ & U \mapsto \iota^*(U), \\ p_{(n)} :& \Lambda^2(N) \to \Lambda^1(S) \\ & U \mapsto - * \iota^*(*U), \end{aligned}$$

where $\iota^* : \Lambda^2(N) \to \Lambda^2(S)$ is the induced pullback.

Definition 4.4.4 (Electric and magnetic field)**.** Let (N, h) be globally hyperbolic and let $S \subset N$ be a Cauchy hypersurface. Assume that F is a solution to Maxwell's equations. The corresponding forms $E \in \Lambda^1(S)$ and $B \in \Lambda^2(S)$, defined by

$$E := p_{(n)}(F)$$
$$B := p_{(0)}(F)$$

are called the *electric* respectively *magnetic field* on S induced by F.

We continue by defining the Cauchy Problem for Maxwell's vacuum equations, following [8]. From now on, we denote the Hodge-star operator on N and on a submanifold $S \subset N$ with the symbol $*$.

Definition 4.4.5 (The Cauchy Problem for Maxwell's vacuum equations)**.** Let (N, h) be a globally hyperbolic manifold. Let $X \in \Lambda^2(S)$ and $Y \in \Lambda^1(S)$ satisfy

$$dX = 0, \quad \delta Y = 0.$$

Then $F \in \Lambda^2(N)$ is said to solve the Cauchy problem for Maxwell's equations with initial data X and Y if

$$dF = 0, \quad \delta F = 0,$$
$$X = p_{(0)}(F), \quad Y = p_{(n)}(F).$$

Lemma 4.4.6. *There exists a unique solution to the Cauchy Problem for Maxwell's vacuum equations.*

Proof. This is proven in [8, Section 4.2.3]. □

Remark 4.4.7. Assume that $F \in \Lambda^2(N)$ solves the Cauchy problem for Maxwell's equations with initial $X \in \Lambda^2(S)$ and $Y \in \Lambda^1(S)$. If E_S and B_S are the electric and magnetic fields on S, then

$$E_S = X,$$
$$B_S = Y.$$

If there is an isometry between two Cauchy hypersurfaces S_{N_1} and S_{N_2} in two globaly hyperbolic manifolds N_1 and N_2, then for each Cauchy problem in N_1 with initial data defined on S_{N_1} there is a corresponding Cauchy problem on N_2 with initial data defined on S_{N_2}.

Lemma 4.4.8 (Pullback of initial data.)**.** *Assume (N_1, h_1) and (N_2, h_2) are globally hyperbolic manifolds and $S_{N_1} \subset N_1$ and $S_{N_2} \subset N_2$ are hypersurfaces with induced metrics. Assume*

$$f : S_{N_1} \to S_{N_2}$$

is an isometry . Assume that X_{N_2} and Y_{N_2} are the initial data for the Cauchy Problem for light on N_2. Then

$$X_{N_1} := f^* X_{N_2}, \quad Y_{N_1} := f^* Y_{N_2},$$

are well-defined initial data for a Cauchy Problem for Maxwell's vacuum equations on M.

Proof. What needs to be shown is

$$dX_{N_1} = 0, \quad \delta Y_{N_1} = 0.$$

By assumption,

$$dX_{N_2} = 0, \quad \delta Y_{N_2} = 0.$$

Hence

$$dX_{N_1} = d\left(f^* X_{N_2}\right) = f^* dX_{N_2} = f^* 0 = 0,$$

and

$$\delta Y_{N_1} = \delta\left(f^* Y_{N_2}\right) = f^* \delta Y_{N_2} = f^* 0 = 0,$$

since d and δ commute with f^*. □

4.5 The Cauchy problem for light in Bianchi type I spacetimes.

We start this section by identifying the initial data one uses on Minkowski space in order to describe light. Then we define the Cauchy problem for light in Bianchi type I spacetime by the Cauchy problem in vacuum with the initial data from Minkowski space.

4.5.1 Light in Minkowski space.

In physics (see e.g. [6]), one usually writes the solutions of Maxwell's equations in Minkowski space describing light as

$$E_{Min}(t,x) = E^0 e^{i(C\cdot x - \|C\| t)}, \quad B_{Min}(t,x) = B^0 e^{i(C\cdot x - \|C\| t)}, \tag{4.4}$$

where $E^0, B^0, C \in \mathbb{R}^3$ form an orthogonal basis of $\mathbb{R}^3$ and and $\|E^0\| = \|B^0\|$. We will for simplicity assume that $\|E^0\| = \|B^0\| = 1$. Furthermore,

$$\frac{C}{\|C\|} = E^0 \times B^0.$$

The question is, do there exist well-defined initial data, such that the above equations is the solution of the corresponding Cauchy problem for Maxwell's vacuum equations in Minkowski space?

The Cauchy problem for light in Minkowski space.

To answer this question, we need to translate the equation (4.4) into a two-form F on the Minkowski space. The usual way to translate the result into the electromagnetic two-form F on Minkowski space is given by

$$\begin{pmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{pmatrix} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}. \tag{4.5}$$

The next step is to calculate the initial conditions X_{Min} and Y_{Min} such that the above two-form is the solution to the corresponding Cauchy problem for Maxwell's vacuum equations.

Lemma 4.5.1 (Initial data in Minkowski space). *Let $(\mathbb{R}^4_1, \eta)$ be the Minkowski space and let $S_{Min}(t_0) := \{t_0\} \times \mathbb{R}^3 \subset \mathbb{R}^4_1$. The above defined $F \in \Lambda^2(\mathbb{R}^4_1)$ is the unique solution to the Cauchy problem for Maxwell's vacuum equations with initial data $X_{Min} \in \Lambda^2(S_{Min}(t_0))$ and $Y_{Min} \in \Lambda^1(S_{Min}(t_0))$ given by*

$$X_{Min} := p_{(0)}(F) = (-{B^0}_1 dx^2 \wedge dx^3 + {B^0}_2 dx^1 \wedge dx^3 - {B^0}_3 dx^1 \wedge dx^2) e^{i(C\cdot x - \|C\| t_0)},$$
$$Y_{Min} := p_{(n)}(F) = ({E^0}_1 dx^1 + {E^0}_2 dx^2 + {E^0}_3 dx^3) e^{i(C\cdot x - \|C\| t_0)}.$$

Proof. The formula for X_{Min} and Y_{Min} follows directly by equation (4.5) and Lemma A.2.5 in Appendix. □

4.5.2 Light in Bianchi type I spacetimes with initial data from Minkowski space.

The idea now is to use Lemma 4.4.8 to generalize the Cauchy problem for light from Minkowski space to general Bianchi type I spacetimes. This relies on the following proposition, recall that $a_j(t_0) = 1$ for all j.

Proposition 4.5.2. *Let (M, g) be a Bianchi type I spacetime, and $(\mathbb{R}^4_1, \eta)$ be the Minkowski space. Let $t_0 \in I$. Then $S_{Bia}(t_0) := \{t_0\} \times \mathbb{R}^3 \subset M$ and $S_{Min}(t_0) := \{t_0\} \times \mathbb{R}^3 \subset \mathbb{R}^4_1$ are Cauchy hypersurfaces in the respective case. Moreover, the two Cachy hypersurfaces with induced metric are isometric.*

Proof. The induced metric on $S_{Bia}(t_0)$ is given by

$$g_{t_0} = \sum_{i=1}^{3} a_i(t_0)^2 dx^i \otimes dx^i = \sum_{i=1}^{3} dx^i \otimes dx^i,$$

and on $S_{Min}(t_0)$ by $\eta_{t_0} = \sum_{i=1}^3 dx^i \otimes dx^i$. Note that the (identity) map

$$f : S_{Bia}(t_0) \to S_{Min}(t_0),$$
$$f(t_0, x) = (t_0, x),$$

is the desired isometry. □

Hence we can calculate the pullback of these initial data to the Bianchi type I hypersurface. This is a trivial calculation, since the isometry of the hypersurfaces is the identity.

Lemma 4.5.3 (Corresponding initial data in Bianchi type I spacetimes)**.** *Let X_{Min} and Y_{Min} be as in Lemma 4.5.1. Then their pullbacks under the isometry f from the previous proposition are*

$$f^*X_{Min} =(-B^0{}_1 dx^2 \wedge dx^3 + B^0{}_2 dx^1 \wedge dx^3 - B^0{}_3 dx^1 \wedge dx^2) \cdot e^{i(C\cdot x - \|C\| t_0)},$$
$$f^*Y_{Min} =(E^0{}_1 dx^1 + E^0{}_2 dx^2 + E^0{}_3 dx^3) \cdot e^{i(C\cdot x - \|C\| t_0)},$$

Stating the Cauchy problem for light in Bianchi type I spacetimes.

We can now state the the Cauchy problem for light in Bianchi type I spacetimes. Lemma 4.5.3 and Lemma 4.1.1 imply that we should define our initial conditions as follows.

Definition 4.5.4 (Initial conditions in Bianchi type I spacetimes)**.** We define the initial data for the Cauchy problem for light in Bianchi type I spacetimes as

$$X_{Bia} :=(-B^0{}_1 dx^2 \wedge dx^3 + B^0{}_2 dx^1 \wedge dx^3 - B^0{}_3 dx^1 \wedge dx^2) \cdot e^{i(C\cdot x - \|C\| t_0)},$$
$$Y_{Bia} :=(E^0{}_1 dx^1 + E^0{}_2 dx^2 + E^0{}_3 dx^3) e^{i(C\cdot x - \|C\| t_0)}.$$

Problem 4.5.5 (The Cauchy Problem for light in Bianchi type I spacetimes)**.** Let (M, g) be a Bianchi type I spacetime. Assume that $t_0 \in I$ and let $S_{Bia}(t_0) := \{t_0\} \times \mathbb{R}^3$. Let $X_{Bia} \in \Lambda^2(S_{Bia}(t_0))$ and $Y_{Bia} \in \Lambda^1(S_{Bia}(t_0))$ be as above. Then $F \in \Lambda^2(M)$ is said to satisfy the Cauchy Problem for light in M if

$$dF = 0, \quad \delta F = 0,$$
$$X_{Bia} = p_{(0)}(F), \quad Y_{Bia} = p_{(n)}(F).$$

From now on, we discuss solutions to this problem.

4.5.3 The equations in components of the electromagnetic tensor.

We rewrite the Cauchy problem for light in Bianchi type I spacetimes as equations for the components of F.

The initial data.

Lemma 4.5.6 (Initial conditions on the electromagnetic field tensor.). *Let (M, g) be a Bianchi type I spacetime. The initial conditions on $F \in \Lambda^2(M)$ are given by*

$$\begin{aligned}(F_{01}, F_{02}, F_{03})(t_0, x) =& ({E^0}_1, {E^0}_2, {E^0}_3)e^{i(C\cdot x - \|C\|t_0)},\\ (F_{23}, F_{13}, F_{12})(t_0, x) =& (-{B^0}_1, {B^0}_2, -{B^0}_3)e^{i(C\cdot x - \|C\|t_0)}.\end{aligned}$$

Proof. The initial conditions need to satisfy

$$\begin{aligned}X_{Bia} &= p_{(0)}(F),\\ Y_{Bia} &= p_{(n)}(F),\end{aligned}$$

from Definition 4.5.4. By Lemma A.2.5,

$$\begin{aligned}p_{(0)}(F) &= \iota^*(F) = F_{12}dx^1 \wedge dx^2 + F_{13}dx^1 \wedge dx^3 + F_{23}dx^2 \wedge dx^3,\\ p_{(n)}(F) &= - * \iota^*(*F) = F_{01}dx^1 + F_{02}dx^2 + F_{03}dx^3.\end{aligned}$$

Identifying the components from Lemma 4.5.3, gives the result. □

The differential equations.

Lemma 4.5.7 (Maxwell's equations in components). *Let $F \in \Lambda^2(M)$ be given in coordinates as*

$$F = \sum_{\substack{\alpha,\beta=0\\ \alpha<\beta}}^{3} F_{\alpha\beta}dx^\alpha \wedge dx^\beta.$$

Then the Maxwell's equations for light in Bianchi type I spacetimes are given by

$$\begin{aligned}0 =& \partial_1 F_{01}\frac{a_2a_3}{a_1} + \partial_2 F_{02}\frac{a_1a_3}{a_2} + \partial_3 F_{03}\frac{a_1a_2}{a_3},\\ 0 =& -\partial_t\left(F_{03}\frac{a_1a_2}{a_3}\right) + \partial_1 F_{13}\frac{a_2}{a_1a_3} + \partial_2 F_{23}\frac{a_1}{a_2a_3},\\ 0 =& \partial_t\left(F_{02}\frac{a_1a_3}{a_2}\right) - \partial_1 F_{12}\frac{a_3}{a_1a_2} + \partial_3 F_{23}\frac{a_1}{a_2a_3},\\ 0 =& -\partial_t\left(F_{01}\frac{a_2a_3}{a_1}\right) - \partial_2 F_{12}\frac{a_3}{a_1a_2} - \partial_3 F_{13}\frac{a_2}{a_1a_3}.\end{aligned}$$

together with

$$\begin{aligned}0 &= \partial_1 F_{23} - \partial_2 F_{13} + \partial_3 F_{12},\\ 0 &= \partial_t F_{12} - \partial_1 F_{02} + \partial_2 F_{01},\\ 0 &= \partial_t F_{13} - \partial_1 F_{03} + \partial_3 F_{01},\\ 0 &= \partial_t F_{23} - \partial_2 F_{03} + \partial_3 F_{02}.\end{aligned}$$

Proof. This follows immediately by the Definition 4.4.2 together with Lemma A.2.2 and Lemma A.2.3 in Appendix. □

4.5.4 The rewritten problem.

The goal of this subsection is to rewrite the equations for a two-form into a differential equation for its components, and then simplify it further to an equation of two functions

$$\begin{aligned}v &: M \to \mathbb{C}^3,\\ w &: M \to \mathbb{C}^3.\end{aligned}$$

Definition 4.5.8. We define the functions $w, v \in C^\infty(M, \mathbb{C}^3)$ through

$$v = (v_1, v_2, v_3) := \left(F_{01}\frac{a_2a_3}{a_1}, F_{02}\frac{a_1a_3}{a_2}, F_{03}\frac{a_1a_2}{a_3} \right),$$
$$w = (w_1, w_2, w_3) := (-F_{23}, F_{13}, -F_{12}).$$

The next step is to translate the initial conditions and differential equation respectively to the new functions.

The initial conditions.

Since $a_j(t_0) = 1$ for $j = 1, 2, 3$, we get the following initial conditions for v and w.

Lemma 4.5.9 (The rewritten initial conditions)**.** *Let v and w be as above. Then the initial conditions become*

$$v_0(x) := v(t_0, x) = E^0 e^{i(C\cdot x - \|C\| t_0)}, \tag{4.6}$$
$$w_0(x) := w(t_0, x) = B^0 e^{i(C\cdot x - \|C\| t_0)}. \tag{4.7}$$

The differential equations.

Notation 4.5.10. For $f, g : M \to \mathbb{C}^3$, define

$$[f_i g_i] := (f_1g_1, f_2g_2, f_3g_3),$$
$$\nabla \times (f_1, f_2, f_3) := (\partial_2 f_3 - \partial_3 f_2, \partial_3 f_1 - \partial_1 f_3, \partial_1 f_2 - \partial_2 f_1),$$
$$\nabla \cdot (f_1, f_2, f_3) := \sum_{j=1}^{3} \partial_j f_j.$$

Lemma 4.5.11 (The rewritten differential equations)**.** *Let v and w be as above. Then the differential equations become*

$$\nabla \cdot v = 0, \tag{4.8}$$
$$\partial_t v = \frac{1}{a_1a_2a_3} \nabla \times [{a_i}^2 w_i], \tag{4.9}$$
$$\nabla \cdot w = 0, \tag{4.10}$$
$$\partial_t w = -\frac{1}{a_1a_2a_3} \nabla \times [{a_i}^2 v_i]. \tag{4.11}$$

Proof. By Lemma 4.5.7, the equations now become

$$\begin{aligned}
0 =& \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3 \\
\partial_t v_3 =& \partial_1 w_2 \frac{a_2}{a_1a_3} - \partial_2 w_1 \frac{a_1}{a_2a_3} \\
\partial_t v_2 =& -\partial_1 w_3 \frac{a_3}{a_1a_2} + \partial_3 w_1 \frac{a_1}{a_2a_3} \\
\partial_t v_1 =& \partial_2 w_3 \frac{a_3}{a_1a_2} - \partial_3 w_2 \frac{a_2}{a_1a_3}.
\end{aligned}$$

together with

$$\begin{aligned}
0 &= \partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3, \\
\partial_t w_1 &= -\partial_1 v_2 \frac{a_2}{a_1a_3} + \partial_2 v_1 \frac{a_1}{a_2a_3}, \\
\partial_t w_2 &= \partial_1 v_3 \frac{a_3}{a_1a_2} - \partial_3 v_1 \frac{a_1}{a_2a_3}, \\
\partial_t w_1 &= -\partial_2 v_3 \frac{a_3}{a_1a_2} + \partial_3 v_2 \frac{a_2}{a_1a_3}.
\end{aligned}$$

Using the notation defined above, finishes the proof. ☐

Stating the rewritten problem.

We can now state the Cauchy problem for light in Bianchi type I spacetimes in terms of v and w.

Problem 4.5.12 (The rewritten Cauchy Problem for Maxwell's equations for light in Bianchi type I spacetimes)**.** Let (M, g) be a Bianchi type I spacetime. Assume that $t_0 \in I$. Then $v : M \to \mathbb{C}^3$ and $w : M \to \mathbb{C}^3$ is said to satisfy the *rewritten Cauchy Problem for light in Bianchi type I spacetimes* if they satisfy equations (4.8 - 4.11) and if

$$\begin{aligned} v(t_0, x) &= v_0(x), \\ w(t_0, x) &= w_0(x), \end{aligned}$$

where v_0, w_0 are as above.

4.6 Explicit solutions to the Cauchy problem for light.

Definition 4.6.1. Define the following time dependent inner product on $\mathbb{C}^3$ as

$$\begin{aligned} &(\cdot, \cdot)_t : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C} \\ &(y, z)_t := \sum_{j=1}^{3} y_j a_j^2(t) \overline{z_j}. \end{aligned}$$

for all $y, z \in \mathbb{C}^3$.

The following theorem gives necessary and sufficient conditions on the a_i, such that the solution is of a certain explicit form.

Theorem 4.6.2. *Let, for all* $(t, x) \in M$*,*

$$\begin{aligned} v(t, x) &= E^0 e^{i(C\cdot x - \Omega(t))}, \\ w(t, x) &= B^0 e^{i(C\cdot x - \Omega(t))}, \end{aligned}$$

with

$$\Omega(t) = \int_{t_0}^{t} \left(\sum_{j=1}^{3} C_j^2 a_j(w)^{-2} \right)^{1/2} dw + \|C\| \, t_0.$$

Then v, w is a solution to the rewritten Cauchy problem for Bianchi type I spacetimes (Problem 4.5.12) if and only if the a_j satisfy

$$(E^0, B^0)_t = 0, \tag{4.12}$$

$$(E^0, E^0)_t = \frac{a_1 a_2 a_3(t)}{\|C\|} \left(\sum_{j=1}^{3} C_j^2 a_j(t)^{-2} \right)^{1/2}, \tag{4.13}$$

$$(B^0, B^0)_t = \frac{a_1 a_2 a_3(t)}{\|C\|} \left(\sum_{j=1}^{3} C_j^2 a_j(t)^{-2} \right)^{1/2}, \tag{4.14}$$

for all $t \in I$. Moreover, each integral curve γ of the vector field grad(S)*, where S is the exponent*

$$S(t, x) := C \cdot x - \Omega(t)$$

is a lightlike geodesic satisfying

$$\gamma'(t_0) = (\|C\|, C_1, C_2, C_3).$$

Remark 4.6.3 (Generalized phase)**.** Note that since E^0 and B^0 are real, the solution of the above form has a time dependent phase

$$e^{-i\Omega}$$

with frequency

$$\omega = |\Omega'| = \left(\sum_{j=1}^{3} C_j^2 a_j^{-2}\right)^{1/2}.$$

Remark 4.6.4 (Connection to the cosmological redshift)**.** Since any integral curve γ to $\mathrm{grad}(S)$ is a lightlike geodesic, we can define the frequency ν (depending on the time coordinate) as in cosmological redshift (Definition 4.2.1). Note that this is given by

$$\nu = \frac{1}{h}\left(\sum_{j=1}^{3} C_j^2 a_j^{-2}\right)^{1/2}.$$

Hence it coincides, up to multiplication by Planck's constant, with the frequency from the solutions v and w, namely

$$\omega = \left(\sum_{j=1}^{3} C_j^2 a_j^{-2}\right)^{1/2}.$$

We now turn to the proof of the theorem. Recall that

$$\begin{aligned}
&\langle v, C\rangle = \langle w, C\rangle = 0,\\
&v(t_0, x) = E^0 e^{i(C\cdot x - \|C\| t_0)},\\
&w(t_0, x) = B^0 e^{i(C\cdot x - \|C\| t_0)}.
\end{aligned}$$

Hence we can write the solution as

$$\begin{aligned}
v(t,x) &= \left(\nu_{E^0}(t)E^0 + \nu_{B^0}(t)B^0\right) e^{i(C\cdot x - \|C\| t_0)},\\
w(t,x) &= \left(\mu_{E^0}(t)E^0 + \mu_{B^0}(t)B^0\right) e^{i(C\cdot x - \|C\| t_0)},
\end{aligned}$$

where

$$\begin{aligned}
\nu &:= \begin{pmatrix} \nu_{E^0} \\ \nu_{B^0} \end{pmatrix} : I \to \mathbb{C}^2,\\
\mu &:= \begin{pmatrix} \mu_{E^0} \\ \mu_{B^0} \end{pmatrix} : I \to \mathbb{C}^2.
\end{aligned}$$

The next lemma shows that the solution can indeed be written on this form.

Lemma 4.6.5. *We can write the solutions v, w of the above form with ν, μ determined by*

$$\partial_t \nu = iM\mu, \tag{4.15}$$
$$\partial_t \mu = -iM\nu, \tag{4.16}$$

where

$$M := \frac{\|C\|}{a_1 a_2 a_3}\begin{pmatrix} -(E^0, B^0)_t & -(B^0, B^0)_t \\ (E^0, E^0)_t & (B^0, E^0)_t \end{pmatrix},$$

and the initial data

$$\nu(t_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.17}$$

$$\mu(t_0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.18}$$

Proof. The equations (4.9) and (4.11) become (using that $E^0, B^0, C \in \mathbb{R}^3$) for each $x \in \mathbb{R}^3$

$$\begin{aligned}
\partial_t v(\cdot,x) &= \langle \frac{i}{a_1a_2a_3} C \times [{a_j}^2 w_j(\cdot,x)], E^0\rangle E^0 + \langle \frac{i}{a_1a_2a_3} C \times [{a_j}^2 w_j(\cdot,x)], B^0\rangle B^0 \\
&= \frac{i}{a_1a_2a_3}(E^0 \times C)\cdot [{a_j}^2 w_j(\cdot,x)]E^0 + \frac{i}{a_1a_2a_3}(B^0 \times C)\cdot [{a_j}^2 w_j(\cdot,x)]B^0 \\
&= \frac{i\,\|C\|}{a_1a_2a_3}\left(-(w(\cdot,x),B^0)_t E^0 + (w(\cdot,x),E^0)_t B^0\right), \\
\partial_t w(\cdot,x) &= \frac{i\,\|C\|}{a_1a_2a_3}\left((v(\cdot,x),B^0)_t E^0 - (v(\cdot,x),E^0)_t B^0\right).
\end{aligned}$$

Identifying the components from the two second calculations gives the following system of equations,

$$e^{i(C\cdot x - \|C\| t_0)}\partial_t \begin{pmatrix} \nu_{E^0} \\ \nu_{B^0} \\ \mu_{E^0} \\ \mu_{B^0} \end{pmatrix} = \frac{i\,\|C\|}{a_1a_2a_3}\begin{pmatrix} -(w(\cdot,x),B^0)_t \\ (w(\cdot,x),E^0)_t \\ (v(\cdot,x),B^0)_t \\ -(v(\cdot,x),E^0)_t \end{pmatrix}.$$

We further note that

$$\begin{aligned}
(w(\cdot,x),B^0)_t &= (\mu_{E^0}E^0 + \mu_{B^0}B^0, B^0)_t e^{i(C\cdot x - \|C\|t_0)} \\
&= (\mu_{E^0}(E^0,B^0)_t + \mu_{B^0}(B^0,B^0)_t)e^{i(C\cdot x - \|C\|t_0)}.
\end{aligned}$$

Similar equations hold for the other ones. Hence

$$\partial_t \begin{pmatrix} \nu_{E^0} \\ \nu_{B^0} \\ \mu_{E^0} \\ \mu_{B^0} \end{pmatrix} = \frac{i\,\|C\|}{a_1a_2a_3}\begin{pmatrix} -\mu_{E^0}(E^0,B^0)_t - \mu_{B^0}(B^0,B^0)_t \\ \mu_{E^0}(E^0,E^0)_t + \mu_{B^0}(B^0,E^0)_t \\ \nu_{E^0}(E^0,B^0)_t + \nu_{B^0}(B^0,B^0)_t \\ -\nu_{E^0}(E^0,E^0)_t - \nu_{B^0}(B^0,E^0)_t \end{pmatrix}.$$

A standard existence theorem for linear ordinary differential equations gives the lemma. □

We now prove Theorem 4.6.2 and conclude with a remark connecting the form of this solution to the method used in cosmological redshift.

Proof of Theorem 4.6.2. In this setting the conditions (4.12), (4.13) and (4.14) are equvalent to

$$M = \begin{pmatrix} 0 & -\left(\sum_{j=1}^3 C_j^2 a_j^{-2}\right)^{1/2} \\ \left(\sum_{j=1}^3 C_j^2 a_j^{-2}\right)^{1/2} & 0 \end{pmatrix}, \ \forall t \in I,$$

in Lemma 4.6.5. It is now easy to see that the unique solution to equations (4.15) and (4.16) with initial data (4.17) and (4.18) is given by

$$\begin{aligned}
\begin{pmatrix} \nu_{E^0}(t) \\ \nu_{B^0}(t) \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp\left(-i\int_{t_0}^t \left(\sum_{j=1}^3 C_j^2 a_j(w)^{-2}\right)^{1/2} dw\right), \\
\begin{pmatrix} \mu_{E^0}(t) \\ \mu_{B^0}(t) \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp\left(-i\int_{t_0}^t \left(\sum_{j=1}^3 C_j^2 a_j(w)^{-2}\right)^{1/2} dw\right).
\end{aligned}$$

This proves the first part of Theorem 4.6.2.

For the second claim, note that grad(S) is given by

$$\begin{aligned}\operatorname{grad}(S)|_{(t,x)} &= \left(\Omega'(t), \frac{C_1}{a_1(t)^2}, \frac{C_2}{a_2(t)^2}, \frac{C_3}{a_3(t)^2}\right)\\ &= \left(\left(\sum_{j=1}^{3} \frac{C_j^2}{a_j(t)^2}\right)^{1/2}, \frac{C_1}{a_1(t)^2}, \frac{C_2}{a_2(t)^2}, \frac{C_3}{a_3(t)^2}\right),\end{aligned}$$

and grad(S) is ligtlike. Now fix $p := (t_0, x_p) \in M$. Let $\gamma(s) := (t(s), x_1(s), x_2(s), x_3(s))$, for $s \in J \subset \mathbb{R}$, be the lightlike geodesic such that $t \circ \gamma(s_0) = t_0$, satisfying

$$\gamma'(s_0) = (\|C\|, C_1, C_2, C_3),$$

Now, by [11, Lemma 9.27],

$$C_j = \langle \gamma'(s), \partial_j \rangle = x_j'(s) a_j(t(s))^2$$

for all $s \in J$. Hence

$$\gamma'(s) = \left(\left(\sum_{j=1}^{3} \frac{C_j^2}{a_j(t(s))^2}\right)^{1/2}, \frac{C_1}{a_1(t(s))^2}, \frac{C_2}{a_2(t(s))^2}, \frac{C_3}{a_3(t(s))^2}\right)$$

for all $s \in J$. Hence γ is an integral curve of grad(S) and by uniqueness of integral curves the statement is proven. □

4.7 Frequency of the general solution to the Cauchy problem for light.

In this section, we show that the unique solution to the rewritten Cauchy problem for light in Bianchi type I spacetimes (Problem 4.5.12) will have a certain frequency. Since there is no reason to expect explicit solutions in general, the result relies on a definition of frequency for a *pair of functions*, generalizing the constant case.

Definition 4.7.1 (Frequency of a pair of functions $f, g : I \to \mathbb{C}^2$). Assume that $f, g : I \to \mathbb{C}^2$ such that there exist

$$f_+, f_-, g_+, g_- : I \to \mathbb{C}^2$$

and a family of real matrices $A : I \to \mathbb{R}^{2\times 2}$, such that for $t_0 \in I$,

$$\begin{aligned} i(\partial_t f)(t_0) &= A(t_0) g(t_0),\\ i(\partial_t g)(t_0) &= -A(t_0) f(t_0), \end{aligned}$$

and

$$\begin{aligned} f(t) &= f_+(t) + f_-(t),\\ f_+'(t) &= A(t) f_+(t),\\ f_-'(t) &= -A(t) f_-(t), \end{aligned}$$

for all $t \in I$ and

$$\begin{aligned} g(t) &= g_+(t) + g_-(t),\\ g_+'(t) &= A(t) g_+(t),\\ g_-'(t) &= -A(t) g_-(t), \end{aligned}$$

for all $t \in I$. Assume furthermore that the two eigenvalues of $A(t)$ are given by $\pm i\alpha(t) \neq 0$, for $t \in I$. Then the *frequency of the pair* (f, g) is defined as

$$\omega_{(f,g)}(t) := |\alpha(t)|.$$

See the Appendix B for a motivation of this definition of frequency. Recall from the previous section, that we can write the solution as

$$v(t,x) = \left(\nu_{E^0}(t)E^0 + \nu_{B^0}(t)B^0\right) e^{i(C\cdot x - \|C\| t_0)},$$
$$w(t,x) = \left(\mu_{E^0}(t)E^0 + \mu_{B^0}(t)B^0\right) e^{i(C\cdot x - \|C\| t_0)},$$

where

$$\nu := \begin{pmatrix}\nu_{E^0}\\ \nu_{B^0}\end{pmatrix} : I \to \mathbb{C}^2,$$
$$\mu := \begin{pmatrix}\mu_{E^0}\\ \mu_{B^0}\end{pmatrix} : I \to \mathbb{C}^2.$$

Theorem 4.7.2 (Frequency of the general solution to Problem 4.5.12)**.** *The unique solution to the rewritten Cauchy problem for light in Bianchi type I spacetimes is of the above form, where the pair (ν,μ) has frequency*

$$\omega_{(\nu,\mu)} = \left(\sum_{j=1}^{3} C_j^2 a_j^{-2}\right)^{1/2}.$$

We start by proving a lemma that gives decoupled systems for ν and μ.

Lemma 4.7.3. *The equations reduce further to*

$$M^{-1}\partial_t\left(M^{-1}\partial_t\nu\right) - \nu = 0,$$
$$\nu(t_0) = \begin{pmatrix}1\\0\end{pmatrix},$$
$$(\partial_t\nu)(t_0) = \begin{pmatrix}-i\,\|C\|\\0\end{pmatrix},$$

and

$$M^{-1}\partial_t\left(M^{-1}\partial_t\mu\right) - \mu = 0,$$
$$\mu(t_0) = \begin{pmatrix}0\\1\end{pmatrix},$$
$$(\partial_t\mu)(t_0) = \begin{pmatrix}0\\-i\,\|C\|\end{pmatrix},$$

where

$$M := \frac{\|C\|}{a_1a_2a_3}\begin{pmatrix}-(E^0,B^0)_t & -(B^0,B^0)_t\\ (E^0,E^0)_t & (B^0,E^0)_t\end{pmatrix}.$$

Proof. The equations (4.15) and (4.16) implies

$$\begin{aligned}
&\partial_t\nu = iM\mu\\
\Rightarrow\quad &M^{-1}\partial_t\nu = i\mu\\
\Rightarrow\quad &\partial_t\left(M^{-1}\partial_t\nu\right) = i\partial_t\mu = i(-i)M\nu = M\nu\\
\Rightarrow\quad &M^{-1}\partial_t\left(M^{-1}\partial_t\nu\right) - \nu = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\partial_t\mu = -Mi\nu\\
\Rightarrow\quad &M^{-1}\partial_t\mu = -i\nu\\
\Rightarrow\quad &\partial_t\left(M^{-1}\partial_t\mu\right) = -i\partial_t\nu = -iiM\mu = M\mu\\
\Rightarrow\quad &M^{-1}\partial_t\left(M^{-1}\partial_t\mu\right) - \mu = 0.
\end{aligned}$$

The initial data follows also using equations (4.15) and (4.16). □

Lemma 4.7.4. *The functions* $\nu_+, \nu_- : I \to \mathbb{C}^2$ *defined as the unique solutions to*

$$\partial_t \nu_+ = M\nu_+,$$
$$\partial_t \nu_- = -M\nu_-,$$

with initial condition

$$\nu_+(t_0) = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \nu_-(t_0) = \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

satisfy

$$\nu = \nu_+ + \nu_-.$$

The functions $\mu_+, \mu_- : I \to \mathbb{C}^2$ *defined as the unique solutions to*

$$\partial_t \mu_+ = M\mu_+,$$
$$\partial_t \mu_- = -M\mu_-,$$

with initial condition

$$\mu_+(t_0) = \frac{1}{2}\begin{pmatrix} -i \\ 1 \end{pmatrix}, \quad \mu_-(t_0) = \frac{1}{2}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

satisfy

$$\mu = \mu_+ + \mu_-.$$

The claim is now that the eigenvalues of M are given by

$$i(\det M)^{1/2} = i\left(\sum_{j=1}^{3} C_j^2 a_j^{-2}\right)^{1/2}.$$

Notation 4.7.5. We define, for all $t \in I$,

$$l(t) := \frac{\|C\|}{a_1a_2a_3(t)}(E^0, B^0)_t,$$
$$m(t) := \frac{\|C\|}{a_1a_2a_3(t)}(B^0, B^0)_t,$$
$$n(t) := \frac{\|C\|}{a_1a_2a_3(t)}(E^0, E^0)_t.$$

With this notation definition,

$$M = \begin{pmatrix} -l & -m \\ n & l \end{pmatrix}.$$

Note that $l(t) \in \mathbb{R}$ and $m(t), n(t) > 0$, since $(\cdot,\cdot)_t$ is positive definite for all $t \in I$.

Lemma 4.7.6. *For all* $t \in I$*, we have*

$$l(t)^2 < m(t)n(t).$$

Hence the eigenvalues

$$\lambda_1 = i(\det M)^{1/2} = i(mn - l^2)^{1/2}$$
$$\lambda_2 = -i(\det M)^{1/2} = -i(mn - l^2)^{1/2}$$

are imaginary.

Proof. Note by the Cauchy-Schwarz inequality,

$$(E^0, B^0)_t{}^2 < (E^0, E^0)_t(B^0, B^0)_t.$$

The calculation of the eigenvalues is straightforward, showing the lemma. □

Lemma 4.7.7. *Let M be as above. Then*

$$\det M = \sum_{j=1}^{3} C_j^2 a_j^{-2}.$$

Proof. We calculate

$$\begin{aligned}
-l^2 + mn &= -(B^0, E^0)_t (E^0, B^0)_t + (E^0, E^0)_t (B^0, B^0)_t \\
&= -\sum_j {a_j}^4 (E^0)_j^2 (B^0)_j^2 - \sum_{i \neq j} {a_i}^2 {a_j}^2 (E^0)_i (B^0)_i (E^0)_j (B^0)_j \\
&\quad + \sum_j {a_j}^4 (E^0)_j^2 (B^0)_j^2 + \sum_{i \neq j} {a_i}^2 {a_j}^2 (E^0)_i^2 (B^0)_j^2 \\
&= \sum_{i \neq j} {a_i}^2 {a_j}^2 \left((E^0)_i^2 (B^0)_j^2 - (E^0)_i (B^0)_i (E^0)_j (B^0)_j \right) \\
&= \sum_{i < j} {a_i}^2 {a_j}^2 \left((E^0)_i^2 (B^0)_j^2 - 2(E^0)_i (B^0)_i (E^0)_j (B^0)_j + (E^0)_j^2 (B^0)_i^2 \right) \\
&= \sum_{i < j} {a_i}^2 {a_j}^2 \left((E^0)_i (B^0)_j - (E^0)_j (B^0)_i \right) \left((E^0)_i (B^0)_j - (E^0)_j (B^0)_i \right) \\
&= \sum_{l \neq i < j \neq l} {a_i}^2 {a_j}^2 (E_0 \times B^0)_l (E_0 \times B^0)_l \\
&= \frac{1}{\|C\|^2} \sum_{l \neq i < j \neq l} {a_i}^2 {a_j}^2 C_l^2
\end{aligned}$$

To obtain an equation for the determinant, we need to multiply with $\|C\|^2 (a_1 a_2 a_3)^{-2}$.

$$\det M = \frac{\|C\|^2}{(a_1 a_2 a_3)^2} \frac{1}{\|C\|^2} \sum_{l \neq i < j \neq l} {a_i}^2 {a_j}^2 C_l^2 = \sum_{l=1}^{3} \frac{{C_l}^2}{{a_l}^2}.$$

□

The proof of Theorem 4.7.2 is now almost done.

Proof of Theorem 4.7.2. By Lemma 4.7.6 and Lemma 4.7.7, we see that the eigenvalues of M are given by

$$\pm i \left(\sum_{j=1}^{3} C_j^2 a_j^{-2} \right)^{1/2}.$$

Choosing

$$\begin{aligned}
f &:= \nu, \\
g &:= \mu, \\
A &:= M
\end{aligned}$$

in Definition 4.7.1 gives the result. □

4.8 Redshift obtained from Maxwell's equations.

We turn back to the electric and magnetic fields defined on a hypersurface (see Definition 4.4.4).

Remark 4.8.1. Let (M, g) be a Bianchi type I spacetime and let $S_t := \{t\} \times \mathbb{R}^3$ be a Cauchy hypersurface in M. Lemma A.2.5 implies that $E^t \in \Lambda^1(S_t)$ and $B^t \in \Lambda^2(S_t)$ are given by

$$\begin{aligned} E^t(x) &= F_{01}(t,x)dx^1 + F_{02}(t,x)dx^2 + F_{03}(t,x)dx^3, \\ B^t(x) &= F_{23}(t,x)dx^2 \wedge dx^3 + F_{13}(t,x)dx^1 \wedge dx^3 + F_{12}(t,x)dx^1 \wedge dx^2 \end{aligned}$$

for all $x \in \{t\} \times \mathbb{R}^3 \cong \mathbb{R}^3$.

4.8.1 Redshift obtained from the explicit solutions.

We first translate the exact solutions v and w into solutions for the electric field E^t and magnetic field B^t on $S_{Bia}(t)$.

Theorem 4.8.2. *Let (M, g) be a Bianchi type I spacetime and let $S_t := \{t\} \times \mathbb{R}^3$ be a Cauchy hypersurface. The a_i satisfy*

$$\begin{aligned} (E^0, B^0)_{t'} &= 0, \\ (E^0, E^0)_{t'} = (B^0, B^0)_{t'} &= \frac{a_1a_2a_3(t')}{\|C\|}\left(\sum_{j=1}^{3} C_j^2 a_j(t')^{-2}\right)^{1/2}, \end{aligned}$$

for all $t' \in I$ if and only if the electromagnetic fields $E \in \Lambda^1(S_t)$ and $B \in \Lambda^2(S_t)$ for the Cauchy problem for light are given by

$$\begin{aligned} E^t(x) &= \left((E^0)_1 \frac{a_1(t)}{a_2a_3(t)}dx^1 + (E^0)_2 \frac{a_2(t)}{a_1a_3(t)}dx^2 + (E^0)_3 \frac{a_3(t)}{a_1a_2(t)}dx^3\right) \cdot e^{i(C\cdot x - \Omega(t))}, \\ B^t(x) &= \left(-(B^0)_1 dx^2 \wedge dx^3 + (B^0)_2 dx^1 \wedge dx^3 - (B^0)_3 dx^1 \wedge dx^2\right) \cdot e^{i(C\cdot x - \Omega(t))}, \end{aligned}$$

where

$$\Omega(t) = \int_{t_0}^{t} \left(\sum_{j=1}^{3} C_j^2 a_j(u)^{-2}\right)^{1/2} du + \|C\|\, t_0.$$

Proof. The statement follows from Theorem 4.6.2 and Remark 4.8.1. □

Corollary 4.8.3 (Solution in spatially flat RW spacetime)**.** *The explicit solution to the Cauchy Problem 4.5.12 in spatially flat Robertson-Walker spacetime is given by*

$$\begin{aligned} E^t(x) &= \left((E^0)_1 dx^1 + (E^0)_2 dx^2 + (E^0)_3 dx^3\right) \frac{1}{f(t)} e^{i\left(C\cdot x - \|C\| \int_{t_0}^{t} \frac{1}{f(w)}dw - \|C\| t_0\right)}, \\ B^t(x) &= \left(-(B^0)_1 dx^2 \wedge dx^3 + (B^0)_2 dx^1 \wedge dx^3 - (B^0)_3 dx^1 \wedge dx^2\right) e^{i\left(C\cdot x - \|C\| \int_{t_0}^{t} \frac{1}{f(w)}dw - \|C\| t_0\right)}. \end{aligned}$$

Proof. Note that

$$(\cdot,\cdot)_{t'} = f(t')^2 \langle \cdot,\cdot \rangle.$$

Hence

$$\begin{aligned} (E^0, B^0)_{t'} &= f(t')^2 \langle E^0, B^0 \rangle = 0, \\ (E^0, E^0)_{t'} &= f(t')^2, \\ (B^0, B^0)_{t'} &= f(t')^2, \end{aligned}$$

and

$$\frac{a_1a_2a_3(t')}{\|C\|}\left(\sum_{j=1}^{3} C_j^2 a_j(t')^{-2}\right)^{1/2} = \frac{f(t')^3}{\|C\|}\frac{\|C\|}{f(t')} = f(t')^2.$$

This shows that assumptions of Theorem 4.8.2 are fullfilled. We calculate the function $\Omega(t)$ in the exponent. By definition,

$$\begin{aligned}\Omega(t) &= \int_{t_0}^{t} \left(\sum_{j=1}^{3} C_j^2 a_j(w)^{-2} \right)^{1/2} dw + \|C\|\, t_0 \\ &= \int_{t_0}^{t} \left(\sum_{j=1}^{3} {C_j}^2 \right)^{1/2} \frac{1}{f(w)} dw + \|C\|\, t_0 \\ &= \|C\| \int_{t_0}^{t} \frac{1}{f(w)} dw + \|C\|\, t_0\end{aligned}$$

and Theorem 4.8.2 proves the statement. □

Corollary 4.8.4 (Motion in one direction)**.** *Let $C = (C_1, C_2, C_3)$ and assume that $C_l \neq 0$ for some l, and $C_j = 0$ for $j \neq l$. The explicit solution to the Cauchy Problem 4.5.12 in this case is given by*

$$E^t(x) = \Big((E^0)_1 \frac{a_1(t)}{a_2 a_3(t)} dx^1 + (E^0)_2 \frac{a_2(t)}{a_1 a_3(t)} dx^2 + (E^0)_3 \frac{a_3(t)}{a_1 a_2(t)} dx^3 \Big) e^{i(C_l x_l - C_l \int_{t_0}^{t} \frac{1}{a_i(w)} dw - C_l t_0)},$$

$$B^t(x) = \big(-(B^0)_1 dx^2 \wedge dx^3 + (B^0)_2 dx^1 \wedge dx^3 - (B^0)_3 dx^1 \wedge dx^2 \big)\, e^{i(C_l x_l - C_l \int_{t_0}^{t} \frac{1}{a_i(w)} dw - C_l t_0)},$$

if and only if $a_j = a_k$ for $j, k \neq l$.

Proof. Assume w.l.o.g that $C_1 \neq 0$ and $C_2 = C_3 = 0$. Since $C \perp E^0, B^0$, $(E^0)_1 = (B^0)_1 = 0$. Hence

$$0 = \langle E^0, B^0 \rangle = (E^0)_2 (B^0)_2 + (E^0)_3 (B^0)_3.$$

Therefore, (4.12) is equivalent to

$$(E^0)_2 (B^0)_2 a_2(t)^2 + (E^0)_3 (B^0)_3 a_3(t)^2 = 0,$$

and (4.13) and (4.14) are equivalent to

$$((E^0)_2)^2 a_2(t)^2 + ((E^0)_3)^2 a_3(t)^2 = ((B^0)_2)^2 a_2(t)^2 + ((B^0)_3)^2 a_3(t)^2 = a_2(t) a_3(t).$$

One checks that these equalities are satisfied if and only if

$$a_2 = a_3.$$

We calculate

$$\begin{aligned}\Omega(t) &= \int_{t_0}^{t} \left(\sum_{j=1}^{3} C_j^2 a_j(w)^{-2} \right)^{1/2} dw + \|C\|\, t_0 \\ &= |C_1| \int_{t_0}^{t} \frac{1}{a_1(w)} dw + |C_1|\, t_0.\end{aligned}$$

Theorem 4.8.2 implies the statement. □

Conclusion: The redshift obtained for the explicit solution.

Corollary 4.8.5. *Let (M, g) be a Bianchi type I spacetime and let $S_t := \{t\} \times \mathbb{R}^3$ be a Cauchy hypersurface. Assume that a_j, for $j = 1, 2, 3$, satisfy*

$$\begin{aligned}&(E^0, B^0)_{t'} = 0, \\ &(E^0, E^0)_{t'} = (B^0, B^0)_{t'} = \frac{a_1 a_2 a_3(t')}{\|C\|} \left(\sum_{j=1}^{3} C_j^2 a_j(t')^{-2} \right)^{1/2},\end{aligned}$$

for all $t' \in I$. Then the redshift of $(E^t(x))_{t\in J}$ and $(B^t(x))_{t\in J}$ is given by

$$z(t_1, z_2) = \left(\frac{\sum_{j=1}^3 C_j^2 a_j(t_1)^{-2}}{\sum_{j=1}^3 C_j^2 a_j(t_2)^{-2}} \right)^{1/2} - 1.$$

Proof. The frequency of $E^t(x)$ and $B^t(x)$ is given by

$$\omega(t) = \Omega'(t) = \left(\sum_{j=1}^3 C_j^2 a_j(t)^{-2} \right)^{1/2}.$$

The result follows from Lemma 4.1.4 and Theorem 4.8.2. □

4.8.2 Redshift obtained from the general solutions.

We get the following result in general.

Theorem 4.8.6. *Let (M, g) be a Bianchi type I spacetime and let $S_t := \{t\} \times \mathbb{R}^3$ be a Cauchy hypersurface in M. The electric and magnetic fields $E^t \in \Lambda^1(S_t)$ and $B^t \in \Lambda^2(S_t)$ are given by*

$$E^t(x) = \left(\frac{a_1(t)}{a_2 a_3(t)} \tilde{\nu}_1(t) dx^1 + \frac{a_2(t)}{a_1 a_3(t)} \tilde{\nu}_2(t) dx^2 + \frac{a_3(t)}{a_1 a_2(t)} \tilde{\nu}_3(t) dx^3 \right) e^{i(C\cdot x - \omega t_0)},$$
$$B^t(x) = \left(-\tilde{\mu}_1(t) dx^2 \wedge dx^3 + \tilde{\mu}_2(t) dx^1 \wedge dx^3 - \tilde{\mu}_3(t) dx^1 \wedge dx^2 \right) e^{i(C\cdot x - \omega t_0)}$$

for all $x \in \{t\} \times \mathbb{R}^3 \cong \mathbb{R}^3$, where $\tilde{\nu}_j$ are the components of

$$\nu_{E^0} E^0 + \nu_{B^0} B^0 \in \mathbb{C}^3,$$

and $\tilde{\mu}_j$ are the components of

$$\mu_{E^0} E^0 + \mu_{B^0} B^0 \in \mathbb{C}^3.$$

The frequency of the pair (ν, μ) (as in Definition 4.7.1), defined by

$$\nu := \begin{pmatrix} \nu_{E^0} \\ \nu_{B^0} \end{pmatrix} : I \to \mathbb{C}^2,$$
$$\mu := \begin{pmatrix} \mu_{E^0} \\ \mu_{B^0} \end{pmatrix} : I \to \mathbb{C}^2,$$

is given by

$$\omega_{(\nu,\mu)} = \left(\sum_{j=1}^3 C_j^2 a_j^{-2} \right)^{1/2}.$$

Proof. This follows by Theorem 4.7.2. □

Conclusion: The redshift obtained in the general case.

If we interpret the frequency of E^t and B^t in the previous theorem as the frequency of the pair (ν, μ) (Definition 4.7.1), we arrive at the following result.

Corollary 4.8.7. *Let (M, g) be a Bianchi type I spacetime and let $S_t := \{t\} \times \mathbb{R}^3$ be a Cauchy hypersurface. The redshift of $(E^t(x))_{t\in J}$ and $(B^t(x))_{t\in J}$ is given by*

$$z(t_1, z_2) = \left(\frac{\sum_{j=1}^3 C_j^2 a_j(t_1)^{-2}}{\sum_{j=1}^3 C_j^2 a_j(t_2)^{-2}} \right)^{1/2} - 1.$$

Proof. The result follows from Lemma 4.1.4 and Theorem 4.8.6. □

Appendix A

Hodge-star operator.

A.1 Definition of the Hodge-star operator.

In this section, let (N, h) be a semi-Riemannian manifold of dimension n. We first follow [11].

Definition A.1.1 (Volume element.)**.** A volume element on an n-dimensional semi-Riemannian manifold N is a smooth n-form ω such that $\omega(e_1, ..., e_n) = \pm 1$ for every (local) orthonormal frame in N.

The proofs of the following two lemmas about existence of volume elements can be found in [11, Lemma 7.19 and Lemma 7.20].

Lemma A.1.2 (Local existence of a volume element.)**.** *On the domain U of a coordinate system ξ, there is a (local) volume element ω_ξ such that*

$$\omega_\xi(\partial_1, ..., \partial_n) = |\det(g_{ij})|^{1/2}.$$

Lemma A.1.3 (Global existence of a volume element.)**.** *A semi-Riemannian manifold N has a global volume element if and only if N is orientable.*

The next three definitions follow [2].

Definition A.1.4. Assume that $e_1, ..., e_n$ is a local orthonormal basis defined on an open set $U \subset N$. Define the induced metric, also denoted by h, on the m-forms $\Lambda^m(U)$ for $1 \leq m \leq n$ by

$$h(e^*_{i_1} \wedge ... \wedge e^*_{i_m}, e^*_{j_1} \wedge ... \wedge e^*_{j_m}) := \delta_{i_1 j_1} ... \delta_{i_m j_m} \epsilon_{i_1} ... \epsilon_{i_m}$$

for any $1 \leq i_1 < ... < i_m \leq n$ and $1 \leq j_1 < ... < j_m \leq n$ and

$$\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases},$$
$$\epsilon_j = h(e_j, e_j).$$

Definition A.1.5 (Hodge-star operator)**.** For any $0 \leq p \leq n$, define the *Hodge operator* $*_p$ to be the unique vector bundle isomorphism

$$*_p : \Lambda^p(N) \to \Lambda^{n-p}(N)$$

such that

$$\omega \wedge (*_p \eta) = h(\omega, \eta) Vol$$

for all $\omega, \eta \in \Lambda^p(N)$.

The Hodge operator is used to define the codifferential.

Definition A.1.6 (Codifferential)**.** Let (N, h) be a semi-Riemannian manifold of index s and dimension n. The codifferential is defined by

$$\delta := (-1)^{np+1+s} *_{n-p+1} d*_p$$

where $*_p : \Lambda^p(N) \to \Lambda^{n-p}(N)$ is the Hodge-star operator.

A.2 Hodge operator for Bianchi type I spacetimes.

Let now (M, g) be a Bianchi type I spacetime. Note that we have a global volume element of the form

$$Vol := a_1a_2a_3 dt \wedge dx^1 \wedge dx^2 \wedge dx^3.$$

As before, (M, g) denotes the Bianchi type I spacetimes.

Lemma A.2.1. *The Hodge-star operator on $\Lambda^2(M)$, is given in coordinates by*

$$\begin{aligned}
*_2 (dt \wedge dx^1) &= -\frac{a_2a_3}{a_1} dx^2 \wedge dx^3, \\
*_2 (dt \wedge dx^2) &= \frac{a_1a_3}{a_2} dx^1 \wedge dx^3, \\
*_2 (dt \wedge dx^3) &= -\frac{a_1a_2}{a_3} dx^1 \wedge dx^2, \\
*_2 (dx^1 \wedge dx^2) &= \frac{a_3}{a_1a_2} dt \wedge dx^3, \\
*_2 (dx^1 \wedge dx^3) &= -\frac{a_2}{a_1a_3} dt \wedge dx^2, \\
*_2 (dx^2 \wedge dx^3) &= \frac{a_1}{a_2a_3} dt \wedge dx^1.
\end{aligned}$$

Proof. The formula for the Hodge-star operator becomes

$$\omega \wedge (*\eta) = h(\omega, \eta)\ a_1a_2a_3\ dt \wedge dx^1 \wedge dx^2 \wedge dx^3, \quad \forall \omega, \eta \in \Lambda^2(N).$$

We will calculate the first and the fourth equality, the rest follow by symmetry. We have

$$\begin{aligned}
(dt \wedge dx^1) \wedge *_2(dt \wedge dx^1) &= h((dt \wedge dx^1), (dt \wedge dx^1))\ a_1a_2a_3\ dt \wedge dx^1 \wedge dx^2 \wedge dx^3, \\
&= -\frac{1}{{a_1}^2} a_1a_2a_3\ dt \wedge dx^1 \wedge dx^2 \wedge dx^3.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(dx^1 \wedge dx^2) \wedge *_2(dx^1 \wedge dx^2) &= h((dx^1 \wedge dx^2), (dx^1 \wedge dx^2))\ a_1a_2a_3\ dt \wedge dx^1 \wedge dx^2 \wedge dx^3, \\
&= \frac{1}{{a_1}^2{a_2}^2} a_1a_2a_3\ dt \wedge dx^1 \wedge dx^2 \wedge dx^3.
\end{aligned}$$

The minus-signs in some of the equations above is due to the antisymmetry of the wedge product. □

Note that the Hodge-star is by construction a linear operator.

Lemma A.2.2. *Let $F \in \Lambda^2(M)$ be given in coordinates by*

$$F = \sum_{\substack{\alpha,\beta=0 \\ \alpha<\beta}}^{3} F_{\alpha\beta} dx^\alpha \wedge dx^\beta.$$

Then

$$\begin{aligned}
d(*_2 F) = &\left(-\partial_1 F_{01}\frac{a_2 a_3}{a_1} - \partial_2 F_{02}\frac{a_1 a_3}{a_2} - \partial_3 F_{03}\frac{a_1 a_2}{a_3}\right) dx^1 \wedge dx^2 \wedge dx^3 \\
&+ \left(-\partial_t\left(F_{03}\frac{a_1 a_2}{a_3}\right) + \partial_1 F_{13}\frac{a_2}{a_1 a_3} + \partial_2 F_{23}\frac{a_1}{a_2 a_3}\right) dt \wedge dx^1 \wedge dx^2 \\
&+ \left(\partial_t\left(F_{02}\frac{a_1 a_3}{a_2}\right) - \partial_1 F_{12}\frac{a_3}{a_1 a_2} + \partial_3 F_{23}\frac{a_1}{a_2 a_3}\right) dt \wedge dx^1 \wedge dx^3 \\
&+ \left(-\partial_t\left(F_{01}\frac{a_2 a_3}{a_1}\right) - \partial_2 F_{12}\frac{a_3}{a_1 a_2} - \partial_3 F_{13}\frac{a_2}{a_1 a_3}\right) dt \wedge dx^2 \wedge dx^3.
\end{aligned}$$

Proof. We start by computing $*_2 F$:

$$\begin{aligned}
*_2 F = *_2\left(\sum_{\substack{\alpha,\beta=0\\ \alpha<\beta}}^{3} F_{\alpha\beta} dx^\alpha \wedge dx^\beta\right) &= \sum_{\substack{\alpha,\beta=0\\ \alpha<\beta}}^{3} F_{\alpha\beta} *_2 (dx^\alpha \wedge dx^\beta) \\
= -F_{01}\frac{a_2 a_3}{a_1} dx^2 \wedge dx^3 &+ F_{02}\frac{a_1 a_3}{a_2} dx^1 \wedge dx^3 - F_{03}\frac{a_1 a_2}{a_3} dx^1 \wedge dx^2 \\
+ F_{12}\frac{a_3}{a_1 a_2} dt \wedge dx^3 &- F_{13}\frac{a_2}{a_1 a_3} dt \wedge dx^2 + F_{23}\frac{a_1}{a_2 a_3} dt \wedge dx^1.
\end{aligned}$$

Taking the exterior derivative of this form gives the result. □

Lemma A.2.3. *Let $F \in \Lambda^2(M)$ be given in coordinates by*

$$F = \sum_{\substack{\alpha,\beta=0\\ \alpha<\beta}}^{3} F_{\alpha\beta} dx^\alpha \wedge dx^\beta.$$

Then

$$\begin{aligned}
dF = &(\partial_1 F_{23} - \partial_2 F_{13} + \partial_3 F_{12}) dx^1 \wedge dx^2 \wedge dx^3 \\
&+ (\partial_t F_{12} - \partial_1 F_{02} + \partial_2 F_{01}) dt \wedge dx^1 \wedge dx^2 \\
&+ (\partial_t F_{13} - \partial_1 F_{03} + \partial_3 F_{01}) dt \wedge dx^1 \wedge dx^3 \\
&+ (\partial_t F_{23} - \partial_2 F_{03} + \partial_3 F_{02}) dt \wedge dx^2 \wedge dx^3.
\end{aligned}$$

Proof. An elementary calculation. □

In the next calculation, we calculate the Hodge-star operator on the Cauchy hypersurface $S_{Bia}(t) := \{t\} \times \mathbb{R}^3 \subset M$.

Lemma A.2.4. *The Hodge-star operator on $S_{Bia}(t)$ is given by*

$$\begin{aligned}
*_2 (dx^1 \wedge dx^2) &= \frac{a_3}{a_1 a_2} dx^3, \\
*_2 (dx^1 \wedge dx^3) &= -\frac{a_2}{a_1 a_3} dx^2, \\
*_2 (dx^2 \wedge dx^3) &= \frac{a_1}{a_2 a_3} dx^1, \\
*_1\, dx^1 &= \frac{a_2 a_3}{a_1} dx^2 \wedge dx^3, \\
*_1\, dx^2 &= -\frac{a_1 a_3}{a_2} dx^1 \wedge dx^3, \\
*_1\, dx^3 &= \frac{a_1 a_2}{a_3} dx^1 \wedge dx^2.
\end{aligned}$$

Proof. Let h_t denote the induced (Riemannian) metric on $S_{Bia}(t)$. The formula for the Hodge-star operator is now

$$\omega \wedge (*\eta) = h_t(\omega, \eta)\ a_1 a_2 a_3\ dx^1 \wedge dx^2 \wedge dx^3.$$

We calculate the first equality, the rest are analogous.

$$(dx^1 \wedge dx^2) \wedge *_2(dx^1 \wedge dx^2) = \frac{1}{{a_1}^2 {a_2}^2} a_1 a_2 a_3(t) dx^1 \wedge dx^2 \wedge dx^3 = \frac{a_3}{a_1 a_2} dx^1 \wedge dx^2 \wedge dx^3.$$

Hence

$$*(dx^1 \wedge dx^2) = \frac{a_3}{a_1 a_2} dx^3.$$

□

The following lemma is necessary to formulate the Cauchy problem for Maxwell's equations properly.

Lemma A.2.5. *Let (M, g) be a Bianchi type I spacetime and*

$$\iota : S_{Bia}(t) \hookrightarrow M$$

the natural embedding. Let $F \in \Lambda^2(M)$ be given in coordinates as

$$F = \sum_{\substack{\alpha,\beta=0 \\ \alpha<\beta}}^{3} F_{\alpha\beta} dx^\alpha \wedge dx^\beta.$$

Then

$$\begin{aligned} \iota^*(F) &= F_{12} dx^1 \wedge dx^2 + F_{13} dx^1 \wedge dx^3 + F_{23} dx^2 \wedge dx^3, \\ - *\iota^*(*F) &= F_{01} dx^1 + F_{02} dx^2 + F_{03} dx^3. \end{aligned}$$

(Note that the second Hodge-star is on $\Lambda(S_{Bia}(t))$.)

Proof. We prove only the second equality, the first is immediate by the same arguments. By the proof of Lemma A.2.2,

$$\begin{aligned} *_2 F = &- F_{01} \frac{a_2 a_3}{a_1} dx^2 \wedge dx^3 + F_{02} \frac{a_1 a_3}{a_2} dx^1 \wedge dx^3 - F_{03} \frac{a_1 a_2}{a_3} dx^1 \wedge dx^2 \\ &+ F_{12} \frac{a_3}{a_1 a_2} dt \wedge dx^3 - F_{13} \frac{a_2}{a_1 a_3} dt \wedge dx^2 + F_{23} \frac{a_1}{a_2 a_3} dt \wedge dx^1. \end{aligned}$$

Since ∂_t is the normal vector field on $S_{Bia}(t_0)$, every term including dt vanish when taking ι^* (we use the same notation for F and its restriction to $S_{Bia}(t_0)$):

$$\iota^*(*F) = -F_{01} \frac{a_2 a_3}{a_1} dx^2 \wedge dx^3 + F_{02} \frac{a_1 a_3}{a_2} dx^1 \wedge dx^3 - F_{03} \frac{a_1 a_2}{a_3} dx^1 \wedge dx^2.$$

By taking the 3-dimensional Hodge-star operator we conclude that

$$*\iota^*(*F) = -F_{01} dx^1 - F_{02} dx^2 - F_{03} dx^3.$$

□

Appendix B

Frequency of non-trigonometric functions.

It is not obvious how one should define frequency of a non-trigonometric function. The cosmological redshift relies on a definition of frequency (Definition 4.2.1) that postulates that the frequency is inverse proportional to the energy of a photon in general relativity. The models using the scalar wave equation (Section 4.3) and Maxwell's equations (Section 4.4-4.8) use definitions of frequency for non-trigonometric functions (Definition 4.3.1 and Definition 4.7.1). We discuss these definitions here.

B.1 Generalizing the definition of frequency for trigonometric functions.

We start out with a the basic example.

Example B.1.1 (For a constant matrix A)**.** Assume that $A \in \mathbb{C}^{2\times 2}$ and that the eigenvalues of A are $\pm i\alpha \neq 0$ are constant. Assume furthermore that $f : J \to \mathbb{C}^2$, where $J \subset \mathbb{R}$ is an interval, satisfies

$$f' = Af, \quad f(t_0) = f_0.$$

Then

$$f(t) = P^{-1} \begin{pmatrix} e^{i\alpha(t-t_0)} & 0 \\ 0 & e^{-i\alpha(t-t_0)} \end{pmatrix} Pf(t_0),$$

where P is the eigenvector matrix. In this case, it is natural to understand the frequency as $|\alpha|$.

The goal of the rest of the section is to understand what properties of the above example that can be generalized to the non-constant case. In the following, we consider a matrix-valued function

$$A : J \to \mathbb{C}^{2\times 2}.$$

Lemma B.1.2. *The eigenvalues of A are of the form $\lambda_1 = -\lambda_2 = i\alpha$, for $\alpha > 0$ if and only if A is on the form*

$$A = \begin{pmatrix} -l & -m \\ n & l \end{pmatrix}$$

where $l, m, n : J \to \mathbb{C}$ such that

$$mn - l^2 > 0.$$

Moreover, in this case

$$A^{-1} = -A\frac{1}{\det A},$$

and

$$\det A = \alpha^2.$$

Proof. We start with a general matrix

$$A = \begin{pmatrix} -l & -m \\ n & o \end{pmatrix}$$

and want to show that $o = l$. The characteristic equation for A is

$$0 = (\lambda + l)(\lambda - o) + mn = \lambda^2 + (l - o)\lambda - ol + mn = 0.$$

Therefore the first claim is proven. The second claim follows by a simple calculation. □

The following proposition is of great importance to show the difference between the constant and non-constant case.

Proposition B.1.3. *Assume that A is of the form*

$$A = \begin{pmatrix} -l & -m \\ n & l \end{pmatrix},$$

and

$$f' = Af. \tag{B.1}$$

If A is constant, then the eigenvalues are uniquely determined by f. This is not true in general for a non-constant A.

Proof. When A is constant, then

$$f'' = (Af)' = A'f + Af' = Af' = A^2 f = -\det Af = -\alpha^2 f.$$

This does not work when A is not constant. A counter example is given by

$$A_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$
$$A_2 := \begin{pmatrix} 2i & -ie^{2it} \\ ie^{-2it} & -2i \end{pmatrix}.$$

If we define

$$f(t) := \begin{pmatrix} e^{it} \\ e^{-it} \end{pmatrix},$$

then A_1 and A_2 satisfy

$$f' = A_1 f,$$
$$f' = A_2 f.$$

But the eigenvalues of A_1 are $\pm i$ for all $t \in J$ and the eigenvalues of A_2 are given by $\pm i\sqrt{3}$. □

The previous proposition implies that the frequency of a system of equation of the form (B.1) is not in a natural way determined by f, in general. But if we assume that A is real valued and that the real part and the imaginary part of the initial data of f are linearly independent, then the frequency is indeed well-defined.

Proposition B.1.4. *Assume that $Re(f(t'))$ and $\Im(f(t'))$ are linearly independent for some $t' \in J$. Then there exists at most one matrix valued function $A : J \to \mathbb{R}^{2\times 2}$ such that*

$$f' = Af. \tag{B.2}$$

Proof. Note that if A is real-valued, the real part and the imaginary part of $f(t')$ satisfy equation (B.2) respectively. Since they are linearly independent initially, they will be linearly independent for all $t \in J$. This determines A uniquely. □

B.2 Well-definedness of the generalized frequency.

We are going to demand that the frequency of a function scales under coordinate changes in the following sense. The reader can note that this holds in the case demonstrated in Example B.1.1.

Assume that $t = \varphi(u)$, where $\varphi : J' \to J$ is a diffeomorphism between the intervals $J, J' \subset \mathbb{R}$, then for $f : J \to \mathbb{R}$,

$$\omega_{f\circ\varphi}(u) = |\varphi'(u)|\, \omega_f(\varphi(u)),\ \forall u \in J'.$$

B.2.1 The first definition.

Definition B.2.1 (Definition of frequency of a function $f : J \subset \mathbb{R} \to \mathbb{C}$)**.** Assume that a function $f : J \to \mathbb{C}$ where the vectors $(\Re(f(t_0)), \Re(f'(t_0)))$ and $(\Im(f(t_0)), \Im(f'(t_0)))$ are linearly independent. Assume furthermore that there exist continuous functions $a : J \to \mathbb{R}$ and $b : J \to \mathbb{R}_+$ such that

$$f''(t) + a(t)f'(t) + b(t)f(t) = 0,$$

for all $t \in J$. Then we define the *frequency of* f as

$$\omega_f := \sqrt{b}.$$

We check that the above definition is well-defined and invariant under coordinate changes.

Proposition B.2.2. *Assume that* $(\Re(f(t_0)), \Re(f'(t_0)))$ *and* $(\Im(f(t_0)), \Im(f'(t_0)))$ *are linearly independent. Assume that* $a_1, a_2, b_1, b_2 : J \to \mathbb{R}$

$$f''(t) + a_1(t)f'(t) + b_1(t)f(t) = 0,$$
$$f''(t) + a_2(t)f'(t) + b_2(t)f(t) = 0.$$

Then $a_1 = a_2$ *and* $b_1 = b_2$*. In particular, the frequency of* f *is well-defined.*

Moreover, if $t = \varphi(u)$*, where* $\varphi : J' \to J$ *is a diffeomorphism between the intervals* $J, J' \subset \mathbb{R}$*, then*

$$\omega_{f\circ\varphi}(u) = |\varphi'(u)|\, \omega_f(\varphi(u)),\ \forall u \in J'.$$

Proof. We define

$$\vec{x}(t) := \begin{pmatrix} f(t) \\ f'(t) \end{pmatrix},\ \forall t \in I,$$

and note that

$$\partial_t \vec{x} = \begin{pmatrix} 0 & 1 \\ -b_1 & -a_1 \end{pmatrix} \vec{x}. \tag{B.3}$$

By assumption, the real and imaginary parts of $\vec{x}(t_0)$ are linearly independent. It is easy to see that this implies that the real and imaginary parts of $\vec{x}(t)$ are linearly independent for all $t \in I$. This implies in turn that the matrix in the equation (B.3) is uniquely determined. In particular, $a_1 = a_2$ and $b_1 = b_2$.

An easy calculation shows that

$$\omega_{f\circ\varphi} = \sqrt{\varphi'^2 b_1 \circ \varphi} = |\varphi'|\, \sqrt{b_1 \circ \varphi} = |\varphi'|\, \omega_f \circ \varphi.$$

□

B.2.2 The second definition.

Definition B.2.3 (Frequency of a pair of functions $f, g : J \subset \mathbb{R} \to \mathbb{C}^2$)**.** Assume that $f, g : J \to \mathbb{C}^2$ such that for some $t' \in J$,

$$\operatorname{span}\{\Re(g(t')), \Im(g(t')), \Re(f(t')), \Im(f(t'))\}$$

has real dimension 2. Assume moreover that

$$f_+, f_-, g_+, g_- : J \to \mathbb{C}^2$$

and a family of real matrices $A : J \to \mathbb{R}^{2\times 2}$, such that

$$\begin{aligned} if'(t_0) &= A(t_0)g(t_0), \\ ig'(t_0) &= -A(t_0)f(t_0), \end{aligned}$$

and

$$\begin{aligned} f(t) &= f_+(t) + f_-(t), \\ f_+'(t) &= A(t)f_+(t), \\ f_-'(t) &= -A(t)f_-(t), \end{aligned}$$

for all $t \in J$ and

$$\begin{aligned} g(t) &= g_+(t) + g_-(t), \\ g_+'(t) &= A(t)g_+(t), \\ g_-'(t) &= -A(t)g_-(t), \end{aligned}$$

for all $t \in J$. Assume furthermore that the two eigenvalues of $A(t)$ are given by $\pm i\alpha(t) \neq 0$, for $t \in J$. Then the *frequency of the pair* (f, g) is defined as

$$\omega_{(f,g)}(t) := |\alpha(t)| \,.$$

Proposition B.2.4. *Assume that $A_1, A_2 : J \to \mathbb{R}^{2\times 2}$ satisfy the above definition. Then*

$$A_1 = A_2.$$

Moreover, if $t = \varphi(u)$, where $\varphi : J' \to J$ is a diffeomorphism between the intervals $J, J' \subset \mathbb{R}$, then

$$\omega_{f\circ\varphi}(u) = |\varphi'(u)|\, \omega_f(\varphi(u)), \ \forall u \in J'. \tag{B.4}$$

Proof. We claim that

$$i(f_+ - f_-) = g.$$

We start by showing the equality for t_0. Indeed,

$$i(f_+(t_0) - f_-(t_0)) = i(A(t_0)^{-1}f_+'(t_0) + A(t_0)^{-1}f_-'(t_0)) = iA(t_0)^{-1}f'(t_0) = g(t_0).$$

The next step is to show that the first derivatives coincide,

$$i(f_+'(t_0) - f_-'(t_0)) = i(A(t_0)f_+(t_0) + A(t_0)f_-(t_0)) = iA(t_0)f(t_0) = g'(t_0).$$

Note now that both $i(f_+ - f_-)$ and g are solutions to the linear second order differential equation

$$A^{-1}\partial_t(A^{-1}\partial_t X) - X = 0,$$

with the same initial data, by the previous calculations. Hence it follows that

$$i(f_+ - f_-) = g.$$

We conclude that

$$-ig' = (f'_+ - f'_-) = A(f_+ + f_-) = Af.$$

Similarly one shows that

$$if' = Ag.$$

By assumption, we can pick two of the vectors $\Re(g(t')), \Im(g(t')), \Re(f(t')), \Im(f(t')) \in \mathbb{R}^2$ that are linearly independent. By a similar result as Proposition B.1.4, the proof is finished.

Equation (B.4) follows by a simple calculation. □

B.2.3 Frequency of trigonometric functions as a special case.

We will in this section show that when the solutions in the above definitions are trigonometric functions, then the traditional frequency coincides with the above defined frequencies. We start by defining traditional frequency.

Definition B.2.5 (Traditional frequency)**.** We say that $f : I \to \mathbb{C}^n$ has *traditional frequency* $|\alpha|$ if it is periodic with smallest period $\frac{2\pi}{|\alpha|}$.

Proposition B.2.6 (The case of Definition B.2.1)**.** *Assume that* $b \in \mathbb{R}_+$ *is constant. Assume that* $f : I \to \mathbb{C}$ *satisfies*

$$f'' + bf = 0.$$

Then the traditional frequency is

$$\omega = \sqrt{b}.$$

Proof. This is clear, since

$$f(t) = c_1 e^{i\sqrt{b}(t-t_0)} + c_2 e^{-i\sqrt{b}(t-t_0)}.$$

□

Proposition B.2.7 (The case of Definition B.2.3)**.** *Assume that* $f, g : I \to \mathbb{C}^2$ *such that there exist*

$$f_+, f_-, g_+, g_- : I \to \mathbb{C}^2$$

and a constant matrix $A \in \mathbb{R}^{2\times 2}$*, such that*

$$\begin{aligned} if'(t_0) &= Ag(t_0), \\ ig'(t_0) &= -Af(t_0), \end{aligned}$$

and

$$\begin{aligned} f(t) &= f_+(t) + f_-(t), \\ f'_+(t) &= Af_+(t), \\ f'_-(t) &= -Af_-(t), \end{aligned}$$

for all $t \in I$ *and*

$$\begin{aligned} g(t) &= g_+(t) + g_-(t), \\ g'_+(t) &= Ag_+(t), \\ g'_-(t) &= -Ag_-(t), \end{aligned}$$

for all $t \in I$*. Assume furthermore that the two eigenvalues of* A *are given by* $\pm i\alpha \neq 0$*. Then* f *and* g *are periodic with traditional frequency*

$$\omega = |\alpha|.$$

Proof. Note that

$$f(t) = e^{A(t-t_0)} f_+(t_0) + e^{-A(t-t_0)} f_-(t_0),$$
$$g(t) = e^{A(t-t_0)} g_+(t_0) + e^{-A(t-t_0)} g_-(t_0),$$

where

$$f_+(t_0) = -ig_+(t_0),$$
$$f_-(t_0) = ig_-(t_0).$$

Since there exists a constant $P \in \mathbb{C}^{2\times 2}$ such that

$$e^{A(t-t_0)} = P^{-1} \begin{pmatrix} e^{i\alpha(t-t_0)} & 0 & \\ 0 & e^{-i\alpha(t-t_0)} & 0 \end{pmatrix} P,$$

we see that the solution is periodic with traditional frequency

$$\omega = |\alpha| .$$

□

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