Graph Theory
The Four Color Theorem

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Abstract

In this degree project I cover the history of the four color theorem, from the origin, to the first proof by Appel and Haken in 1976. Some basic graph theory is featured to ensure that the reader can follow and understand the proofs and procedures in the project.
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1. Basic Graph theory

A graph is a pair $G = (V, E)$ of sets such that $E \subseteq [V^2]$; thus the elements of $E$ are two-element subsets of $V$. The elements $v \in V$ are called vertices of the graph, while the $e \in E$ are the graph’s edges. An edge between vertices $v_i$ and $v_j$, are denoted $v_i v_j$. If we consider a planar graph $G$ we define its faces to be $\mathbb{R}^2 \setminus G$, this means that every face is an open subset of $\mathbb{R}^2$ that are bounded by the graph $G$. Since $G$ is bounded and lies inside a sufficiently large disc $D$, exactly one face is unbounded, the face that contains $\mathbb{R}^2 \setminus D$, this face is called the outer face, and every other face of $G$ is called an inner face.

If we consider two graphs, $G$ and $G'$ which have the property that $V' \subseteq V$ and $E' \subseteq E$, then $G'$ is a subgraph of $G$ which is written $G' \subseteq G$. If $G' \subseteq G$ and $G' \neq G$ we call $G'$ a proper subgraph of $G$.

We need to define cycles and trees. A cycle is a path, which starts and ends at the same vertex, and a tree is a connected graph not containing any cycles. Now we need to define a path and then we can move on to proving Euler’s formula. A path is a non-empty graph $P = (V, E)$ on the form.

$$V = \{v_0, v_1 \ldots v_{k-1}, v_k\} \quad E = \{v_0 v_1, v_1 v_2 \ldots v_{k-2} v_{k-1}, v_{k-1} v_k\}$$

The coloring of planar graphs stems originally from coloring countries on a map. A map, with countries adjacent to each other can be represented as a planar graph. If we take a map and let every country be a vertex, we connect the vertices with edges if the countries are adjacent to each other. So when talking about coloring graphs you can interpret it as coloring a map; the same way you can interpret coloring a graph as coloring a map, if you take every vertex in the graph, and turn it into a face in the map, an edge in the graph means that the faces in the map are connected.

If you have a graph, and you create a new graph where every face in the original graph is a vertex in the new one. And every every pair of faces that are adjacent in the original graph are connected by an edge in the new graph. We have created the dual graph of the original. It is called the dual graph because of symmetry, if I have a connected planar graph $G$, and the dual of $G$ is $G'$ then the dual of $G'$ is $G$. When creating the dual of a plane graph we are not guaranteed that there is only one edge connecting two vertices, but we might get a so called multigraph as the dual to the plane graph.

\footnote{In this paper I don’t consider loops(edges from a vertex onto itself), unless stated otherwise}
1.1. **Euler's formula.** In order to later on prove the five color theorem, we first need a formula known as “Euler’s formula”.

**Theorem 1.1.** If $G$ is a connected, planar graph and it contains $v$ vertices, $e$ edges and $f$ faces (including the outer face), then

$$v - e + f = 2$$

**Proof.** If the graph $G$ is not a tree, it contains a cycle, remove an edge $e$ that completes a cycle, this reduces the number of edges by one, and the number of faces by one, thus leaving $v - e + f = 2$ constant, continue until you have a tree. Trees have the property that $v = e + 1$, otherwise it’s either not connected or contains a cycle, and there is only one face (the outer face). This means that $v - e + f = 2$ now becomes $(e + 1) - e + 1 = 2$ which obviously always holds true. □

This theorem gives us a corollary which will be used to prove the five color theorem, and was used by Kempe in his attempt to prove the four color theorem.

**Corollary 1.2** (Only five neighbours theorem). *Every map has at least one face with five or fewer neighbours*

**Proof.** We consider a map with $f$ faces, $e$ edges and $v$ vertices and use Euler’s formula. We also assume that every vertex has at least three edges connected to it, this is because if we have a vertex with only two edges connected to it we can simply remove it and turn the two edges connected to it into one edge; the case where we cannot remove that vertex because it defines a new space can be disregarded in this proof, since if this occurs we have a face with three neighbours, and thus the theorem is true. We start by counting the edges in the map, we get that there is at least $3v$ edges, but since every edge is counted twice, since every edge is counted twice, we divide the number by two. Thus we have at least $e = \frac{3}{2}v$ edges. This can be written as $e \geq \frac{3}{2}v$, or $v \leq \frac{2}{3}e$.

In order to prove that every map has a face with five or fewer neighbours we assume the opposite and show that this leads to an absurdity. So now we assume that every face is surrounded by six or more faces. Counting the edges we get that we have at least $6f$ edges, since every edge is counted twice, since it has a “beginning” and an “end” we divide the number by two. Thus we have at least $e = \frac{3}{2}v$ edges. This can be written as $e \geq \frac{3}{2}v$, or $f \leq \frac{1}{3}e$.

We now use the two inequalities we have produced, $v \leq \frac{2}{3}e$ and $f \leq \frac{1}{3}e$, in Euler’s formula. We then get the following: $f - e + v \leq \frac{1}{3}e - e + \frac{2}{3}e$ which is equivalent to $f - e + v \leq 0$. But since we know that
\[ f - e + v = 2 \] by Euler’s formula, we have now reached the conclusion that \( 2 \leq 0 \), which is obviously false, and shows that our assumption is false. Thus there exists a face with five or fewer neighbours in every map.

\[ \Box \]

2. The Four Color Theorem

2.1. History. The origin of the problem is credited to Francis Guthrie, whose brother Frederick Guthrie brought it up to Augustus DeMorgan during a lecture, somewhere around the year 1840. DeMorgan thought about this problem, the more time he spent thinking about the problem, the more he believed it to be true. On 23 October 1852 DeMorgan wrote to his friend Sir Hamilton, who was a mathematician in Ireland. This was not a rare thing, since the two of them were friends and were regular correspondants. It is however this letter that was the first mention of the four color theorem that exists today. From this until 1880, there was limited progress in proving the four color theorem, but in 1880 Alfred Bray Kempe published his proof of the four color theorem. This proof turned out to be fallacious, and Kempe is remembered mostly for this fallacious proof, which is a shame since his contributions to society were enough to get him knighted.

In January 1887, the “Journal of Education” published an article where they asked the readers to prove the four color theorem. This problem came from the headmaster, Reverend James Maurice Wilson, of Clifton College which was a boys’ school in the Bristol area. One of the “institutions” at Clifton College was that the headmaster sets a challenge problem to the entire school once a term, and this was the problem. The article caused much interest and several mathematicians wrote to him and said that they could not solve the problem and wanted to see a proof, which the headmaster claimed he had. The proof was created by Frederick Temple, bishop of London and later archbishop of Canterbury, and revolved around showing that it is impossible to have five mutually neighbouring faces in a map. While it is true that it is impossible to have five neighbouring countries in a map, this is not sufficient to prove the four color theorem.

Kempe’s proof revolved around using “Kempe chains” to color the graph. The term “minimal criminal” is in this context a minimal counterexample, with respect to the number of vertices. This means that if the four color theorem is false, then there exists a number of graphs that cannot be colored with four colors and out of these graphs one of them contains the fewest number of vertices. If we
can prove that this minimal criminal doesn’t exist, then there can be no minimal counterexample, and thus no counterexamples at all. Kempe's proof for the four color theorem follows below.

2.1.1. Kempe’s proof of the four color theorem.

Proof. We assume that there exists a minimal graph that is not four colorable, thus every smaller graph can be four colored, for coloring graphs we will use the colors: red, green, blue and yellow. Should we need a fifth color, we will use orange. We plan to cover one face of the graph, which is equivalent to shrinking it to a point. Since it has less faces than our minimal criminal, it can be four colored. We use the corollary of Euler’s formula, only five neighbours theorem, to guarantee that there exists at least one square(face with four neighbours) or pentagon(face with five neighbours). Should there be a face with fewer than four neighbours, for instance a triangle(face with three neighbours) or a digon(face with two neighbours) we will shrink it down, color the graph and then reinstate the face. Since there are only three or less neighbours to this face, we can color it with one of the remaining colors.

The Square. If our minimal criminal contains a square we shrink it down, color the rest of the graph, and the reinstate the face. Let us assume that the colors red and green are not adjacent. Two cases can occur, either the red and green face are connected by a so called Kempe chain, or they are not. A Kempe chain is a chain of faces with only two colors, say we look at a red-green Kempe chain. We start with a red face, the chain contains that face, every green face adjacent to that face, every red face adjacent to any of the green faces and so on.

If the red and green are not connected by a Kempe chain (case 1) we can take the red-green Kempe chain that is connected to one of the faces(but not both) and invert the colors of the chain, i.e every red face becomes green and vice versa. Then we have only three colors
adjacent to the square and we can color the square with the fourth color.

If they are connected by a Kempe chain (case 2) we take the yellow-blue Kempe chain connected to the blue or yellow face adjacent to the square (but not both) and invert the colors of that chain. This means that we only have three colors adjacent to the square and we can color the square with the fourth color. Since we have dealt with all possible cases containing a square, a minimal criminal cannot contain a square.

(Case 1)

The Pentagon. If our minimal criminal contains a pentagon, we shrink it down, color the rest of the graph, and then reinstate the face. Two cases can occur, since we have a pentagon and only four colors to color the faces adjacent to it, two of the faces need to have the same color after the coloring. Since two adjacent faces cannot have the same color there must be a face between the two faces with the same color, we can without loss of generality assume that two faces adjacent to the pentagon are colored blue, as in the pictures below, likewise the face between the blue faces is colored red.

Case 1: At most one of the two following statements are true: “The red and green faces are connected by a Kempe chain.” and “The red and yellow faces are connected by a Kempe chain.”

If this is the case then first consider the yellow and red faces adjacent to the pentagon. If the yellow-red part connected to the red face adjacent to the pentagon does not link up with the
yellow-red part connected to the yellow face adjacent to the pentagon, then we can simply interchange the colors of one of these chains. This ensures that one color is now available to use on one of the faces adjacent to the pentagon, and thus gives us a four coloring.

If the yellow and red faces are connected by a yellow-red chain, as shown here

We are then guaranteed that the green and red faces are not connected and then we simply interchange the colors of one of the green-red chains. This ensures that there exists one color that no face adjacent to the is colored with, and thus we can color the pentagon with that color, which gives us a four coloring of the graph.
Case 2: Both statements “The red and green faces are connected by a Kempe chain.” and “The red and yellow faces are connected by a Kempe chain.” are true.

If this is the case then we take the blue-green chain connected to the blue face that is not adjacent to the green face and the blue-yellow chain connected to the blue face that is not adjacent to the yellow face and interchange the colors in these chains.

This ensures that no face adjacent to the pentagon is colored blue and thus we color the pentagon blue and we have a four
coloring of the graph.

Since we have dealt with all possible cases, we have proved the four color theorem.

□

As was stated earlier, this proof was fallacious, and this was proven in 1889 by Percy John Heawood who managed to find a counter-example which showed that Kempe’s method was fallacious. The problem with this method of proving the four color theorem comes when dealing with a pentagon. What Heawood found was that the problem arises when inverting the colors of two Kempe chains simultaneously, when we invert the colors of one chain we are guaranteed that it will not interfere with the coloring of the graph, but when inverting two chains at the same time we are no longer guaranteed that it will not interfere with the coloring. This method was however sufficient to prove the five color theorem.

2.2. The Five Color Theorem.

Theorem 2.1 (The five color theorem). Every graph can be colored with at most five colors in such a way that two neighbouring faces use different colors.

Proof. We start by observing that a minimal criminal cannot contain a square. If this were true, we would simply shrink it, color the graph, reinstate it and have a fifth color available. This in conjunction with the only five neighbours theorem shows that we only need to deal with the case of a pentagon.
(Dealing with a square)

We shrink the pentagon down, color the map and then reinstate the pentagon. We now assume that every color has been used to color a face adjacent to the pentagon, otherwise we use one of the colors available. As shown in the picture below, the colors we use for the faces adjacent to the pentagon are: red, orange, blue, green and yellow.

Now let’s look at a Kempe chain between two faces that are not adjacent, red and green, say. This gives us two possible cases. Either there is a red-green chain connecting the red and green faces adjacent to the pentagon, or there isn’t.

If there isn’t we take one of the two chains and invert the colors of it. This ensures us that we only have four colors adjacent to the pentagon, and we use the fifth to color it.
(If the red-green chain isn’t connected)

If the Kempe chains are connected we look at another Kempe chain. We take the face that is in between the red and green, and one of the two remaining faces and look at that Kempe chain. For instance the blue-yellow if, say, the yellow was between red and green. Since the blue-yellow chain connected to the yellow face cannot be connected to the blue-yellow face connected to the blue face, we invert the colors of the blue-yellow chain connected to the blue face. This ensures that we only use four colors to color the faces adjacent to the pentagon, and we use the fifth color to color the pentagon.

(If the red-green chain is connected)

Since we have dealt with all possible cases, we have proven the five color theorem. □

Another corollary to Euler’s formula is a result called the “Counting formula for cubic maps”. We will start by defining a cubic map. A cubic map is a map where every vertex is connected by exactly three edges. We limit ourselves to cubic maps for simplicity and because in order to deal with later progress on proving the four color theorem, cubic maps are used. Do remember that the outer face is considered a face. Suppose that our map has \( C_2 \) two-sided countries (digons), \( C_3 \) three-sided countries (triangles), \( C_4 \) four-sided countries (squares), and so on. Then \( F \), the number of faces in our map is the sum of these numbers. \( F = C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + \ldots \)
Next we count the borders that define these countries. Since every digon is surrounded by two lines, the $C_2$ digons are surrounded by a total of $2C_2$ edges. Since each triangle is surrounded by three lines, the $C_3$ triangles are surrounded by a total of $3C_3$ edges. Since each square is surrounded by four lines, the $C_4$ squares are surrounded by a total of $4C_4$ edges, and so on. If we add up all these edges it appears likely that we will get $E$, the total number of edges, but this is not true. Since every edge helps define two faces, every edge is counted twice, and thus

$$2E = 2C_2 + 3C_3 + 4C_4 + 5C_5 + 6C_6 + 7C_7 + ...$$

which we rewrite as

$$E = C_2 + \frac{3}{2}C_3 + \frac{5}{2}C_4 + \frac{5}{2}C_5 + \frac{5}{2}C_6 + \frac{5}{2}C_7 + ...$$

Through the same reasoning we count the number of vertices in our map. Since it’s a cubic map there are exactly three edges connected to any one vertex. This gives us a total of $3V$ edges, but as in the above argument, every edge has been counted twice, so we have $3V = 2E$. Since $2E$ could be written as $2E = 2C_2 + 3C_3 + 4C_4 + 5C_5 + 6C_6 + 7C_7 + ...$ $3V$ can be written as

$$3V = 2C_2 + 3C_3 + 4C_4 + 5C_5 + 6C_6 + 7C_7 + ...$$

Which, when divided by three, gives us the expression for $V$.

$$V = \frac{2}{3}C_2 + \frac{4}{3}C_3 + \frac{5}{3}C_4 + \frac{5}{3}C_5 + 2C_6 + \frac{7}{3}C_7 + ...$$

We have now defined $F$, $E$ and $V$ in terms of $C_2$, $C_3$, $C_4$ and so on. If we substitute these equations into Euler’s formula, which we recall is $2 = F - E + V$, we get the following.

$$2 = (C_2 + C_3 + C_4 + ...) - (C_2 + \frac{3}{2}C_3 + 2C_4 + \frac{5}{2}C_5 + 3C_6 + \frac{5}{2}C_7 + ...) +$$

$$+ (\frac{2}{3}C_2 + C_3 + \frac{4}{3}C_4 + \frac{5}{3}C_5 + 2C_6 + \frac{7}{3}C_7 + ...)$$

Rearranging this gives us

$$2 = C_2(1 - 1 + \frac{2}{3}) + C_3(1 - \frac{3}{2} + 1) + C_4(1 - 2 + \frac{4}{3}) + C_5(1 - \frac{5}{2} + \frac{5}{3}) +$$

$$+ C_6(1 - 3 + 2) + C_7(1 - \frac{7}{2} + \frac{7}{3}) + ...$$

$$= \frac{2}{3}C_2 + \frac{1}{2}C_3 + \frac{1}{3}C_4 + \frac{1}{6}C_5 + 0C_6 - \frac{1}{6}C_7 - ...$$

We multiply this by 6 to remove the fractions, and then we arrive at the counting formula.
Corollary 2.2 (Counting formula for cubic maps). \[ 4C_2 + 3C_3 + 2C_4 + \]
\[ C_5 - C_7 - 2C_8 - 3C_9 - \ldots = 12 \]

We note that the coefficients successively decrease by one, and this means that the coefficient of \( C_6 \) is 0 and thus that term does not appear in our formula. What this means, is that for a cubic map we have a way of describing the different types of faces that appear.

2.3. Unavoidable sets of reducible configurations. The end of the 19th century marked a new dawn for the four color theorem, as Kempe’s proof was faulty, and no new proofs had taken its place. This was a time for new ideas and new ways of attacking the problem. Some people felt that the true reason no correct proof of the four color theorem had been presented was due to the fact that the best mathematicians had not been occupying themselves with the problem.

A story is told about the German number-theorist Hermann Minkowski, who one day during a topology lecture arrogantly said: “This theorem has not yet been proved, but that is because only mathematicians of the third rank have occupied themselves with it, I believe I can prove it.” He began to sketch out a proof at that very moment, and kept at it during these lectures for several weeks. One rainy morning, Minkowski entered the hall, followed by a crash of thunder. He turned to the class, with a deeply serious expression on his face and said “Heaven is angered by my arrogance. My proof of the four color theorem is also defective”. He then took up the lecture on topology at the point where he dropped it several weeks earlier.

As shown earlier with the “only five neighbours theorem”, any map contains a face with five or fewer neighbours, so any cubic map must contain one of the following: a digon, a triangle, a square or a pentagon. We can call this an unavoidable set, since a part of this set of faces must appear in any cubic map.

Kempe’s proof of the four color theorem was fallacious, but the proof that a minimal criminal cannot contain a square, or face with fewer neighbours, was correct. This means that the face with the fewest neighbours in a minimal criminal is a pentagon, and with the counting formula for cubic graphs, we can show that there must be at least twelve pentagons, but the map containing only twelve pentagons is called the dodecahedron, and this map can be colored with four colors, so a minimal criminal must contain thirteen or more faces. We can improve our unavoidable set with a method called “discharging”. With this method we can prove that our unavoidable set can be
extended to the following set: a digon, a triangle, a square, a pentagon adjacent to another pentagon or a pentagon adjacent to a hexagon.

In order to prove this, we assume the opposite, that we have a cubic map without any of the above configurations of faces, and derive a contradiction. By assumption a pentagon cannot be adjacent to a digon, triangle or square, since there are none, or to a pentagon or hexagon. Thus each pentagon is only adjacent to faces with seven or more edges. We now assign to every face a number, based on the number of edges that define the face. To a face with $k$ edges defining it, we assign the number $6 - k$, so that each pentagon ($k = 5$) receives a charge of $1$; each hexagon ($k = 6$) receives a charge of $0$; each heptagon ($k = 7$) receives a charge of $-1$; each octagon ($k = 8$) receives a charge of $-2$; and so on...

If the map has $C_5$ pentagons, $C_6$ hexagons, $C_7$ heptagons, $C_8$ octagons and so on, and no digons, triangles or squares per our assumption, the total charge on the map is

$$(1 \cdot C_5) + (0 \cdot C_6) + ((-1) \cdot C_7) + ((-2) \cdot C_8) + ((-3) \cdot C_9) + \ldots$$

This result is the same as the counting formula for cubic maps, where the number of digons, triangles and squares all equal zero. These results combined leads us to the conclusion that the total charge on the map is 12. But if we now take all the positive charges, the charge in the pentagons, and spread it equally to all adjacent faces, in a way that no charge is removed or added. This procedure is called “discharging” the map. Each neighbour to the pentagon receives a charge of $\frac{1}{5}$, since we have not created or removed any charge on the map, the total is still 12, and each pentagon now have a charge of zero. Let us look at the heptagons, since they have initial charge -1, it needs at least six neighbouring pentagons to receive enough charges to have a positive charge after the discharge, however, this would require that two of the pentagons are adjacent, which is not allowed per our assumption. Thus after the discharge each heptagon retains a negative charge.

Let us now look at an octagon, since it initially had a charge of -2, it would require eleven neighbouring pentagons in order to receive a positive charge after the discharge, which, because the octagon only has eight neighbours, is impossible.

Any hexagon in the map has a charge of zero before the discharge, and has no neighbouring pentagons per our assumption, so its charge can only be negative after the discharge.
So our assumption earlier was false, and any cubic map contains at least one of the following: a digon, a triangle, a square, two adjacent pentagons, or a pentagon adjacent to a hexagon.

The details on the size of the charge and how you redistribute it can vary in order to prove certain sets are unavoidable; this is just one example of a fairly simple unavoidable set and how one proves that it is, in fact, unavoidable.

2.4. **Rings in graphs.** The next part of the history regarding the four color theorem is based mostly on the work of George David Birkhoff. Earlier, finding reducible configurations of unavoidable sets was very difficult, and very little success were made with the sets that was already proven to be unavoidable. What Birkhoff did was considering rings of faces in graphs. In the proof of the five color theorem, and in Kempe’s “proof” of the four color theorem, a central part was removing one face from the graph, and since this new graph has fewer faces, it can be colored; then you reinstate the face. Birkhoff’s technique was to, instead of removing one face from the graph, remove a ring of faces and every face inside the ring. He showed without much difficulty that every ring of three faces is reducible, which means that if you have a chain of three faces in a map, you can color it with four colors. The procedure is very similar to Kempe’s and Heawood’s when dealing with the four color theorem. You cover the faces inside the ring, ignoring them for the moment, and color everything outside of it; since we’re looking for a minimal criminal, this can be done by assumption. Then you color the faces inside the ring, and the ring again, and after possibly recoloring the ring, you have your coloring of the whole map.

For rings with four faces, things get more difficult, since it can be colored with two, three or four colors, and this may greatly complicate the eventual recoloring of the ring. However, by switching the colors in appropriate two-colored Kempe chains, Birkhoff showed that these difficulties can always be overcome, which led to the deduction that every ring of four faces is reducible, and thus can’t appear in a minimal criminal. Birkhoff managed to show that every ring of five faces is reducible except for one special case: a single pentagon surrounded by a ring of five countries, the same case that Kempe never managed to prove. Thirty years later, Arthur Bernhart managed to show the reducibility for rings of six faces. The story is told, that shortly after Bernhart married, his wife encountered Birkhoff’s wife at a mathematics meeting. Mrs Birkhoff asked Mrs Bernhart: “Tell
me, did your husband make you draw maps for him to color on your honeymoon, as mine did?"

At this point in time (early 1900's) it became popular among mathematicians to research for degrees, and many mathematicians received doctoral degrees for dissertations on map coloring. For instance Philip Franklin, who managed to show that the following configurations are reducible and can not appear in a minimal criminal:

• a pentagon in contact with three pentagons and one hexagon
• a pentagon surrounded by two pentagons and three hexagons
• a hexagon surrounded by four pentagons and two hexagons
• any \( n \)-sided polygon in contact with \( n - 1 \) pentagons.

By applying the counting formula, he was able to deduce that every map with up to 25 faces can be colored with four colors, and therefore any minimal criminal must have at least 26 faces. Alfred Errera managed to prove that a minimal criminal must contain at least thirteen pentagons and that it cannot contain only pentagons and hexagons. Clarence Reynolds managed to prove in 1926 that every map with 26 faces can be colored with four colors, which Franklin upped to 31 in 1938. Two years later C. E. Winn increased it to 35, where it remained for a quarter of a century.

The most famous ring of faces through the history of the four color theorem is probably the Birkhoff diamond. As you can see in the picture, it consists of four pentagons surrounded by a ring of six faces. Next I will exhibit the method used to show that the Birkhoff diamond is reducible.

If the faces in the ring surrounding the diamond are numbered 1 through 6 as shown in the picture, it turns out that there are essentially 31 different ways that they can be colored with the four colors red, green blue and yellow. The colorings are as follows (the reason for
the asterisks is given below):

\[
\begin{align*}
\text{rgrg} & \quad \text{rgrb} & \quad \text{rgrb} & \quad \text{rgby} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgby} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} \\
\text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} \\
\text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} \\
\text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} & \quad \text{rgb} \\
\end{align*}
\]

Notice that we cannot include coloring such as \text{rgygrb} in which the faces colored \text{red} are adjacent to each other, and I have also omitted colorings such as \text{rgrrgry} since this is essentially the same coloring as \text{rgrrgrb} (on recoloring the final \text{yellow} as \text{blue}).

Consider the coloring \text{rgrrgrb}. This coloring can be extended directly to the diamond, as shown below, and for this reason it is called a \text{good coloring}. In the same way all the asterisked colorings above are all good colorings.

![Diagram of a diamond with colors]

The coloring \text{rgrrgrb} is not a good coloring, but by using Kempe-chain changes of color that changes the colors \text{red} and \text{yellow}, or \text{green} and \text{blue}, it can be converted into one of the good colorings \text{rgrrgrb}, \text{rgrrgrb} or \text{rgrrbyb}. For example if there is a \text{red-yellow} chain connecting faces 3 and 5, then we can interchange the colors in the \text{blue-green} chain connecting face 4 so as to recolor face 4 \text{green}. Similarly, if there is a \text{red-yellow} chain connecting faces 1 and 5, then we can interchange the colors in the \text{blue-green} chain containing face 6 so as to recolor face 6 \text{green}. However, if there is no \text{red-yellow} chain connecting faces 3 and 5, or 1 and 5, then we can interchange the colors in the \text{red-yellow} chain containing face 5 so as to recolor face 5 \text{yellow}. (These situations are illustrated below.) Thus, the coloring \text{rgrrgrb} can be converted into a good coloring.
The coloring $rgrbry$ is also not a good coloring, but by using Kempe-chain changes that interchange the colors red and green, or blue and yellow, it can be converted into either $rgrbgy$ (which is good) or into $rgrbrb$ (which can be made good, as shown earlier). This is because, if there is a blue-yellow chain connecting faces 4 and 6, then we can interchange the colors in the red-green chain containing face 5 so as to recolor face 5 green. However, if there is no blue-yellow chain connecting faces 4 and 6, we can interchange the colors in the blue-yellow chain containing face 6 so as to recolor face 6 blue, as illustrated below. Thus, the coloring $rgrbry$ can be converted into a good coloring.
It is proven in a similar way to this that all of the 15 possible colorings that do not result in a good coloring can be modified and turned into good colorings with the use of Kempe chain arguments. Thus, the Birkhoff diamond is reducible.

In fact, we don't have to consider all possible colorings. If we modify the map by removing five of the edges, as shown below, then we obtain a new map with fewer faces, which therefore can be colored with four colors.

What we have done here corresponds to eliminating every coloring where faces 1 and 3, and faces 4 and 6, have different color. The result is that we're left with only six colorings: rgrgrb, rgrgby, rgrbrg, rgrbgy, rgrbyg and rgrbry. The first five of these are good colorings, and the last can be converted into a good coloring, as we saw above. Which shows that the configuration can be colored with four colors, so the Birkhoff diamond is reducible.

The German mathematician Heinrich Heesch introduced the term D-reducible for configurations of faces, for which every coloring of the surrounding ring of faces is a good coloring; or can be converted into a good coloring with the use of Kempe-chain arguments. Thus, a digon, a triangle and a square are all D-reducible configurations, as shown by Kempe; and so is the Birkhoff diamond, as shown above. Heesch also coined the term C-reducible for configurations that can be proven to be reducible after some modifications, as illustrated above.

Heinrich Heesch was the first to advocate a coordinated search for unavoidable sets and reducible configurations. Heesch created the method of discharging, and would be one of the people who paved the way for the proof of the theorem. Heesch is also known for contributing to the solution of Hilbert’s problem number 18.
David Hilbert, one of the greatest mathematicians of his time, gave a lecture in Paris at the second International Congress of Mathematics. During this lecture he set out the 23 mathematical problems that he hoped to see solved during the twentieth century. Heesch was working on the second part of the 18th problem, the *regular parquet problem*. This part of the problem revolved around constructing a particular type of tiling of the plane, Heesch solved this problem in 1932 by constructing a number of tilings that can be used to cover the plane according to the rules of the problem. One of his tiling patterns was later incorporated into the ceiling of Göttingen’s library.

2.5. **An alternate approach.** Birkhoff once remarked that almost every great mathematician had worked on the four color theorem at one time or another, and in one of his last papers he classified all previous investigations into two types: *qualitative approaches* and *quantitative approaches*.

The qualitative approach aims to show that every map of a certain type can be colored with four colors, this is the approach we’ve seen so far. In this approach Kempe chains play an important role.

The quantitative approach aims to show that the number of ways a map can be colored with four colors is greater than zero. The quantitative approach was introduced by Birkhoff while he was attending Princeton University, and his paper on this approach was published one year before his paper on reducible configurations and the Birkhoff diamond.

Consider the following map:

Birkhoff assigned the Greek letter \( \lambda \)(lambda) to the number of colors used. Face A can be colored with any of these \( \lambda \) colors. Since face B is adjacent to face A, it can be colored with any of the remaining \( \lambda - 1 \) colors. Finally, since face C and D are both neighbours with faces A and B, but not with each other, they can be colored with any of the remaining \( \lambda - 2 \) colors. Thus the total number of ways to color this map is

\[
\lambda \cdot (\lambda - 1) \cdot (\lambda - 2)^2
\]
For example, if $\lambda = 4$, meaning that we have four colors to use, the number of ways to color this map is $4 \cdot 3 \cdot 2^2 = 48$. If we, for instance, have seven colors ($\lambda = 7$), the number of ways to color the map is $7 \cdot 6 \cdot 5^2 = 1050$.

Birkhoff used the symbol $P(\lambda)$ to denote the number of ways of coloring the map with $\lambda$ colors; thus, for the above map,

$$P(\lambda) = \lambda \cdot (\lambda - 1) \cdot (\lambda - 2),$$

and if we remove the parentheses we get

$$P(\lambda) = \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda.$$

Birkhoff managed to prove that the expression for the number of ways you can color a map with $\lambda$ colors is always a polynomial, which he called the chromatic polynomial. Note that if $P(4)$ is a positive number for a map, then that map can be colored with four colors.

A result that was pointed out to him was that the coefficients of the chromatic polynomial alternate in sign, he later managed to prove that this is always the case. Birkhoff hoped that he would be able to prove the four color theorem by investigating the properties of the chromatic polynomials, and in one of the four papers he wrote about them, he managed to obtain the following inequality

$$P(\lambda) \geq \lambda \cdot (\lambda - 1) \cdot (\lambda - 2) \cdot (\lambda - 3)^{n-3},$$

which is true for every map with $n$ countries, and where $\lambda$ is any positive integer except 4. If he could also have proven this result for $\lambda = 4$, then he would have shown that

$$P(4) \geq 4 \cdot 3 \cdot 2 \cdot 1^{n-3} = 24,$$

which, since 24 is a positive number, would have proven that every map can be colored with four colors (in 24 different ways), hence proving the four color theorem.

2.6. The addition of the computer. Wolfgang Haken, a German mathematician, invited Heesch to the University of Illinois, where Haken worked, to give a lecture on the four color theorem. Haken raised the question whether computers could be of assistance in examining a large number of configurations.

Heesch had already considered this, and in the mid 1960’s he had enlisted the help of Karl Dürre, who had managed to develop a method for testing D-reducibility in configurations, which was algorithmic enough to be implemented on a computer, even though it might take a long time.
By November 1965, Dürrre was able to confirm that the Birkhoff diamond is D-reducible, and soon established the D-reducibility of several configurations of varying complexity. The complexity of a configuration is measured by its ring-size, which is the number of faces surrounding the configuration (the Birkhoff diamond has ring-size 6, for instance). As we saw earlier there were 31 different colorings for the Birkhoff diamond, and the number of possible colorings grows quickly as the ring-size increases. Note that when we color the rings, we only color the faces in the ring, not the faces outside or inside of the ring.

<table>
<thead>
<tr>
<th>Ring-size</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>Colorings</td>
<td>31</td>
<td>91</td>
<td>274</td>
<td>820</td>
<td>2461</td>
<td>7381</td>
<td>22144</td>
<td>64430</td>
<td>199291</td>
</tr>
</tbody>
</table>

Their initial calculations indicated that they would need to check a lot of configurations for reducibility, even configurations of ring-size up to 18. For ring-sizes that large, the number of colorings exceed 18 million. At this point in time, with the computer at the University of Hanover, a typical configuration of ring-size 12 might take around six hours to analyse; some configurations of ring-size 13 took anything between sixteen and sixty-one hours, and ring-sizes 14 and above were out of the question. The estimation that there would be ten thousand different configurations to analyse would take anything between three thousand and fifty thousand computer hours, which was not a realistic proposition for the Hanover computer, or any computer at that time.

It soon became clear that the Hanover computer wasn’t powerful enough to do the work that was needed, but Heesch and Dürrre came into contact with Yoshio Shimamoto, Director of the Computer Center at the Atomic Energy Commission’s Brookhaven Laboratory in Upton, Long Island. At this laboratory there was the most powerful computer of its day, a Cray Control Data 6600. Shimamoto was interested in the four color theorem, and invited Heesch and Dürrre over to continue their testing on that computer. Now some configurations of ring-size 14 were able to be checked, and smaller configurations were not a big hassle for this computer.

Meanwhile, Shimamoto was doing his own research on the four color theorem, and using a different but related method, he was able to show that if he could find a single configuration with certain properties, and if this configuration was D-reducible, then the four color theorem would follow, meaning that the entire proof might rest on a single configuration. On September 30th 1971 he found
what he was looking for. During a rather boring meeting, he was
doodling maps and eventually produced the following configuration
of ring-size 14, that became known as the Shimamoto Horseshoe.

(The Shimamoto horseshoe)

Shimamoto presented his findings to Heesch and Dürre, and asked
them to check if the horseshoe was D-reducible. The first attempt
took the computer more than one hour longer than expected, and was
terminated. The second attempt took even longer than that. For the
third attempt, the computer was allowed to run the entire weekend,
and after grinding on for twenty-six hours, it confirmed that the
horseshoe was, in fact, not D-reducible.

In 1970 Heesch had managed to show, after a discharging experi-
ment, that would result in approximately 8900 configurations that
would need to be checked, with configurations of ring-size up to 18.
Haken was pessimistic about having to deal with such a large amount
of configurations, and since some of them were enormous he made
some calculations on how long it would take to check all of these
cases. He knew that the required time for examining a configuration
of ring-size 14 was at this point in time 25 minutes on average, mul-
tiplied by a factor of four for every new face in the ring, this would
imply that the average configuration of ring-size 18 would take over a
hundred hours of computer time, not to mention more storage space
than existed on any computer. If there was a thousand configurations
of ring-size 18, the whole process would take about eleven years on a
fast computer.

Haken decided that a new approach was needed, and deviated
somewhat from the method that most mathematicians used at this
time. Instead of collecting reducible configurations by the hundreds
and then package them up into an unavoidable set, Haken’s new
method was to aim directly for an unavoidable set. Instead of check-
ing configurations to see if you could make an unavoidable set out of
them, he searched for an unavoidable set and wanted to show that
it was in fact reducible. He felt that it was inappropriate to spend
a lot of expensive computer time checking the configurations that
would be unlikely to appear in the eventual unavoidable set. Heesch was initially approving of this idea, but since it relied on creating an unavoidable set of “likely reducible configurations” he felt that it would not work in the end.

Haken had considered setting the problem aside for a few years, since he had little knowledge of computers and waiting would mean that there would be better computers available. He had talked with “computer experts”, who claimed that his ideas could not be programmed. He was giving a lecture on the horseshoe episode, he said:

“The computer experts have told me that it is not possible to go on like that. But right now I’m quitting. I consider this to be the point beyond which one cannot go without a computer.”

Present at this lecture was Kenneth Appel, who thought that the computer “experts” were talking nonsense, and that their opinion reflected the fact that they were unwilling to invest a great deal of time in something without a certain outcome. Haken accepted Appel’s offer to manage the computing side of things, and together they started working towards a proof.

A rule of thumb they used to identify suitable configurations for inclusion into the unavoidable set was the m-and-n rule: if $n$ is the ring-size of a configuration without a few obstacles, configurations that had been shown to be complicated to evaluate, and $m$ is the number of faces inside the surrounding ring, then the likelihood of reducibility depends on the relative sizes of $n$ and $m$; in particular, if $m$ is larger than $\frac{3}{2}n - 6$ then the configuration is almost always reducible.

For example, the reducible Birkhoff diamond has ring-size 6 and 4 interior faces, and 4 is larger than $(\frac{3}{2} \cdot 6) - 6 = 3$; on the other hand the Shimamoto horseshoe, which is not reducible, has ring-size 14 and 10 interior faces, and 10 is smaller than $(\frac{3}{2} \cdot 14) - 6 = 15$.

When Appel and Haken started working on their proof in 1972, the first problem that they faced was being swarmed with repetitive data. Appel recalls, “We started with certain ideas and kept discovering that we had to become more sophisticated to avoid being swamped by useless or repetitive data”. Since they were looking for an unavoidable set, rather than reducible configurations, their computer runs only took a few hours, and after changing the program, they ran a second
set of computer calculations a month later, and found that the output had been reduced greatly in size.

From then on, every two weeks or so, they modified the computer program so that the program grew, and the output shrank. This dialogue with the computer continued for about six months, where they solved every problem that appeared. After these six months, their experiments showed them that their method for producing a finite unavoidable set, without a few complicated special cases, in a reasonable time, was feasible.

It took them a year to prove theoretically that their method would provide an unavoidable set, without these complicated cases. Following this, they experimented with a special case, maps that do not contain two adjacent pentagons, a much easier case than the general one, and they produced an unavoidable set, containing only 47 configurations of ring-size 16 or less, without these special cases. They estimated that the general problem would contain about 50 times as many configurations as this set, which turned out to be an optimistic estimation. In 1975, they introduced one of the special cases mentioned before, the so called hanging 5-5 pair (shown below), a pair of adjacent pentagons that adjoin a single face inside the surrounding ring, a procedure that doubled the size of the unavoidable set.

They soon realised that they would need more help with the computer programming, so they got a student who was willing to write his thesis paper on the computing part of the proof. John Koch was interested, and he agreed to help them out with the programming. His thesis project was set up so that its completion would not depend on the outcome of Appel and Haken’s proof of the four color theorem.

Koch was put to work on C-reducibility of configurations of ring-size 11; recall that a C-reducible configuration is one that can be adapted so that reducibility arguments go through more easily, but it’s not always clear how this adaption can be carried out. Koch devised a very efficient method to show reducibility in about 90%
of configurations of ring-size 11, a method that was extended to ring-sizes 12, 13 and 14 by Appel.

After a while the discharging program ran into trouble. The problem was that while trying to disperse the positive charge on each pentagon to its neighbours, they regularly came up against barriers of hexagons. Haken constructed the idea that the positive charge on pentagons could be able to “jump over” the hexagons. This would make the discharging process a lot more efficient, but it would take an incredible amount of time to rewrite the program. In the end they decided to implement the final version of the discharging process by hand, this would also require a lot of work, but would give them some flexibility to make minor changes as was needed. This led to so many improvements that they were able to restrict all their configurations to ring-size 14 and below.

During the first half of 1976, Appel and Haken worked on the final details of the discharging procedure that would give them an unavoidable set of reducible configurations. The final process used 487 discharging rules, requiring the investigation by hand of about ten thousand neighbourhoods of faces with positive charge, and reducibility testing by computer of about two thousand configurations. They set a time limit for the computer on checking any configuration; if it took more time than this, they abandoned the configuration and replaced it by other ones, since thoroughly checking an awkward configuration might take years. This greatly reduced the computer time required.

They drafted the help of Haken’s daughter, Dorothea, and began working through the two thousand or so configurations that would eventually form the unavoidable set. One person wrote a section, and then the other two checked it for mistakes independently, a procedure that took about three months. Suddenly, in June, they had completed the construction of the unavoidable set, and within two days Appel was able to test the final configuration for reducibility. All that remained, was a detailed checking of the proof.

Appel celebrated this achievement by writing, the now famed words, “Modulo careful checking, it appears that four colors suffice” on the blackboard of the mathematics departement of the University of Illinois.

Since other people were also making progress on proving the theorem, they drafted the help of five of their children to help them check the proof for errors. They found about one error per page in the 800 pages of the proof, all but 50 of these were immediately corrected. Appel managed to recompute the 50 cases so that only 12 of them
failed to work, these 12 were replaced by 20 new ones, and so the process continued.

At this point, they felt safe. Even if a few configurations turned out not to be reducible after all, there was more than enough self-correction in the whole system for these configurations to be replaceable fairly easily. Since they could adapt their list to produce hundreds of unavoidable sets of reducible configurations, they didn’t have one proof, they had several hundred. If the one they submitted would be faulty in some way, they could just take a new one without that specific configuration.

On June 23rd 1976, The Times of London wrote the following:

“Two American mathematicians have just announced that they have solved a proposition that has been puzzling their kind for more than 100 years . . . Their proof, published today, runs to 100 pages of summary, 100 pages of detail and a further 700 pages of back-up work. It took each of them about 40 hours of research a week and 1000 hours of computer time. Their proof contains 10,000 diagrams, and the computer printout stands four feet high on the floor.”

Appel and Haken had reached their goal: the four color theorem was proved.
REFERENCES