Wave Modelling Techniques for Medium and High Frequency Vibroacoustic Analysis Including Porous Materials

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To Zixuan
Abstract

Numerical methods based on wave modelling are explored for the vibroacoustic analysis of wave propagation, sound transmission and interior noise in vehicles and buildings at medium and high frequencies. The presence of sound absorbing porous materials in practical engineering structures is also considered. The wave modelling techniques provide computational efficiency and physical insight, and two such methods having these advantages are developed in this thesis namely: the semi-analytical finite element method and the wave expansion method.

The semi-analytical finite element method is applicable to structures which have constant properties in one direction, and it uses a finite element discretization of the cross-section and analytical functions in the third direction. Equations of motion are derived from this method to study wave propagation characteristics, which help understand the vibroacoustic behavior of structures. These characteristics may also be used by high frequency techniques, such as statistical energy analysis. The wave propagation in sandwich panels with a poroelastic core, which is modeled with Biot’s theory, is investigated thoroughly.

The semi-analytical finite element method retains the flexibility of the finite element method on geometry and also dramatically increases the computational speed thanks to the orthogonality of the analytical functions when used to calculate forced response. The calculated response of partitions is integrated into diffuse field sound transmission loss calculations of, for example, built-up train floor partitions and multilayer panels lined with porous materials. The calculations are computationally efficient and show good agreement with measurements, thus it is interesting for industrial optimizations which often need many calculation iterations.

The wave expansion method uses \textit{a priori} defined plane wave solutions to the Helmholtz equation for approximation of the sound field in geometrically complex enclosures. It reduces the requirements regarding the number of degrees of freedom compared to the finite element method, which, furthermore, is polluted by dispersion errors. Therefore, the wave expansion method is particularly appealing for high frequency (or large wavenumber) calculations. Its application in interior sound field predictions is assessed within the automobile context.
Sammanfattning

I denna avhandling studeras två numeriska metoder baserade på vibro-akustiska vågbeskrivningar för beräkning av vibrationer, ljudtransmission och buller i fordon och byggnader. Inverkan av ljudabsorberande porösa material i realistiska strukturer är av speciellt intresse. De vibro-akustiska vågbeskrivningarna ger beräkningseffektiva modeller och fysikalisk insikt, vilket är av stort värde i praktiskt ingenjörsarbete. De två metoderna är semi-analytiska finita element metoden och vågexpansionsmetoden.


Vågexpansionsmetoden använder en uppsättning lokala plana våglösningar till Helmholtz ekvation för att beskriva ljudfältet i geometriskt komplicerade kaviteter och rum. Denna approximation minskar kravet på antalet frihetsgrader jämfört med finita element metoden, som dessutom lider av "dispersionsfel" dvs att våglängen beräknas lite fel, vilket leder till stora fel på avstånd som är stora i förhållande till ljudvåglängen. Detta gör vågexpansionsmetoden särskilt lockande för beräkningar av ljudfältet för höga frekvenser. I avhandlingen analyseras metoden konvergensegenskaper och illustreras genom studier av ljudfältet i en bilkup.
I would like to express my sincere gratitude to my main supervisor Svante Finnveden for the continuous support during my PhD. His guidance helped me in research and writing all the time. I really enjoyed the often held discussions and was always amazed by his sharp ideas and superb intelligence. I would also like to thank my supervisor Ines Lopez Arteaga for her kindness of providing help at different aspects, for example organize work and stimulate excellent ideas in writing papers. She is the person I could turn to whether at my ups or downs, since I know she will always raise me up.

I gratefully acknowledge the funding from the Marie Curie initial training network Mid-Frequency for the first two years of my PhD. I felt very lucky to join this network and meet so many nice fellows all around Europe. I greatly appreciate the smooth coordination done by Bert Pluymers, Onur Atak and Wim Desmet at KU Leuven. I wish all the best to my fellow-mates on their way to pursuit excellence in research and happiness in life.

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And to my beloved Zixuan, with whom I find beauty in life.
Papers and manuscripts included in the doctoral thesis


Hao Liu investigated the subject and wrote the paper under the supervision of Svante Finnveden. Ines Lopez Arteaga and Mathias Barbagallo gave advice on the content, structure and writing of the article.

B. Hao Liu, Svante Finnveden and Ines Lopez Arteaga, *Prediction of sound transmission through elastic porous material lined multilayer panels using a semi-analytical finite element method*.

Svante Finnveden proposed the method. Hao Liu developed the elements, performed the calculations and wrote the paper under the supervision of Svante Finnveden and Ines Lopez Arteaga.


Three methods involving wave modeling are discussed in the paper. Hao Liu provided the calculation examples and wrote the theory, together with Svante Finnveden, of the waveguide finite element method based on a Rayleigh-Ritz procedure.

D. Hao Liu, Ciarán O’Reilly, Svante Finnveden and Ines Lopez Arteaga, *Prediction of sound field in geometrically complex enclosures with the wave expansion method*.

Hao Liu did the calculations and wrote the paper under the supervision of Svante Finnveden and Ines Lopez Arteaga. Ciarán O’Reilly provided the computer codes.

Publications NOT included


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Part I

Overview and Summary
Chapter 1

Introduction

The sensation of sound perceived by human ears is frequency dependent, and the loudness is usually measured with the A-weighting which shows a positive gain within the frequency range between 1000 Hz to 6000 Hz [1]. This range of particular high hearing sensitivity happens to cover the medium and high frequencies of acoustic designs for vehicles such as automobiles and railway cars. The need for validated prediction tools at the design stage of the engineering process becomes gradually more pronounced in order to avoid major cost and time losses. A preferable prediction tool first has to produce reliable approximate solutions. It is also important that the tool has a reasonable computational cost, especially when running optimization that needs many iterations of the calculation with varied parameters, which is practical in the industrial design process. The finite element method (FEM) is a well developed and widely used tool, and many commercial software packages are on the market. These packages are usually suitable for low frequency analysis, because the equations that need to be solved become too many at high frequency due to the spatial resolution required to resolve the waves propagating in the structure. The computational cost is even higher if the structure includes acoustic porous materials for sound absorption and insulation. The number of equations describing the porous materials is usually larger than for other media [2]. The parameters of porous materials are also frequency dependent which can only be directly solved instead of using a more computational affordable modal technique. The high computational expense of the FEM at medium and high frequencies to control the accuracy and resolve acoustic porous materials provides us the motivation to explore more computational affordable methods.

In many situations a continuous problem is adequately represented with a finite number of components behaving in a simplified manner such as the FEM. The FEM discretizes the full computational domain with meshes that consist of a finite number of elements and nodes. The variables at the nodes, called degrees of
freedom (DOF), are used to interpolate the field variables within the element by multiplication of the shape functions [3]. The Lagrangian polynomials are usually used for the shape functions. When the structure has constant properties in one direction, e.g. a prism bar, analytical functions such as Fourier series may be used to interpolate the field variables, and this is the basis of the semi-analytical finite element (SAFE) method. The computational domain is only discretized in the dimensions where analytical shape functions are not used. The SAFE method increases the computational efficiency by reducing the dimension of the problem.

The SAFE method is also used to investigate wave propagation in structures of indefinite length and constant property in one direction, which gives special insight into the vibroacoustic behavior of the structure. The wave propagation characteristics may provide parameters needed for modeling tools such as statistical energy analysis (SEA) [4, 5], which is a method for predicting transmission through complex structures particularly well suited at high frequency.

Another way to increase the computational efficiency is by using a coarser mesh to achieve the same accuracy as using fine meshes. The main factor that requires the FEM to have fine spatial resolution at high frequency is the dispersion error [6]. The dispersion error is due to that the calculated wavenumber, which is given by $2\pi$ divided by the wavelength, is different from the exact one. An effective way to alleviate the dispersion error is to use \textit{a priori} wave solutions to interpolate the field variables, and many wave based computational methods are established on this principle, e.g. the wave based method [7], the discontinuous Galerkin method [8] and the wave expansion method (WEM) [9]. The WEM is a discretization scheme for the field variables at a local computational stencil using a number of hypothetical plane waves. The amplitudes of the plane waves are determined by minimizing the $L^2$ norm which are further used to define the shape functions. The method’s accuracy does not break down until the spatial resolution decreases towards two nodes per wavelength [10], and is thus interesting for efficient numerical calculations.

In this thesis, the SAFE and WEM are developed and applied for medium and high frequency vibroacoustic analyses in three topics: investigation of wave propagation in porous material lined multilayer structures [11], prediction of sound transmission through build-up and multilayer partitions [12, 13], and development of the WEM for Helmholtz problems [14]. The thesis consists of two parts: The first part is an overview and summary that have twofold purposes of summarizing the PhD project and documenting additional information and work in progress which are not included in the second part. The second part are appended papers that either have been published in peer reviewed journals or are prepared with journal quality scientific contents in mind, which are going to be submitted.
The first part of the thesis is organized as follows. The concept of the SAFE method is illustrated in Chap. 2 in terms of a general weak form derived from differential equations. Two applications of the SAFE are also presented: investigation of wave propagation and calculation of forced response. In Chap. 3, the computational template and boundary conditions used in the WEM is recapitulated; a model that resolves the point source in the inhomogeneous Helmholtz equations is also developed. A summary of the contribution of the appended papers is given in Chap. 4. In Chap. 5, concluding remarks are drawn, and future work that may worth considering are also mentioned.
Chapter 2

The semi-analytical finite element method

The SAFE method is a technique to combine conventional FEM discretizations and analytical shape functions and is a powerful tool to study wave propagation [15] and calculate forced response [12]. The convenience of using analytical shape functions is only allowed if the structure has constant properties in at least one direction such as a waveguide, the SAFE is therefore also referred as waveguide FEM [16–18].

2.1 Partially discretized weak form

As a starting point of the formulations, the weak form of the equilibrium equations of a structure under a harmonic traction force $F$ with time dependence $e^{i\omega t}$ is given by [3, Chap. 3]

$$\int_{\Omega} C (\delta u)^T D (u) \, d\Omega - \int_{\Gamma} \delta u^T F \, d\Gamma = 0, \quad (2.1)$$

where $C$ and $D$ are linear differential operators, $u$ contains the field variables, $\Omega$ is the computational domain with boundary $\Gamma$, $\delta u$ is an arbitrary function that satisfies the essential boundary conditions (BCs), and the superscript $^T$ denotes the transpose of a vector or a matrix.

The constant properties allow a separation of the field variables into one function of the coordinate $y$ along the waveguide and one function of the cross sectional coordinates ($x - z$ plane) which is discretized by conventional finite element shape functions, given by

$$u = N (x, z) v (y), \quad (2.2)$$

where $N (x, z)$ is a matrix that contains the shape functions defined on the cross section in the $x - z$ plane, and $v (y)$ is a column vector that contains the nodal
field variables on the nodes of the cross section mesh as functions of \( y \). The Galerkin method uses the same shape functions as the nodal field variables for the arbitrary functions \( \delta u \) introduced to formulate the weak form. The partial discretized arbitrary functions are thus given by \( \delta u = N(x, z)\delta v(y) \), and \( \delta v(y) \) remains as arbitrary.

The differential operators of \( C(\delta u) \) and \( D(u) \) are expanded according to their order of derivative with respect to \( y \) for the convenience of implementation, given by

\[
C(N\delta v) = C_0\delta v + C_1\delta v_y + \cdots + C_p\delta v_{y^p}
\]

\[
D(Nv) = D_0v + D_1v_y + \cdots + D_qv_{y^q},
\]

(2.3)

where the subscripts \( y^p \) and \( y^q \) denote the \( p \)'th and \( q \)'th order partial derivatives with respect to \( y \), respectively. Substitution of Eq. (2.3) into Eq. (2.1) yields

\[
\int_y \sum_p \sum_q \delta v^{T}_{y^p} \int_x \int_z C_p^T D_q d^xdv_{y^p} dy - \int_{\Gamma} \delta v^T N^T F d\Gamma = 0.
\]

(2.4)

The integration of \( C_p^T D_q \) in the \( x-z \) plane uses Gauss-Legendre quadrature, and the corresponding results are assembled into the generalized dynamic stiffness matrix \( A_{pq} \). For example, if the first order of derivative is used, Eq. (2.4) is written as

\[
\int_y \delta v^T A_{00} v + \delta v^T_{y} A_{10} v + \delta v^T A_{01} v + \delta v^T_{y} A_{11} v_{y} dy - \int_{\Gamma} \delta v^T N^T F d\Gamma = 0.
\]

(2.5)

The partially discretized equations in Eq. (2.5) are used in two manners: to investigate the wave propagation in structures, see Sec. 2.2, and to calculate the forced response response of structures, see Sec. 2.3.

### 2.2 SAFE for wave propagation investigation

#### 2.2.1 Wave equations

The Euler differential equations [19, Chap. 10] that follow from Eq. (2.5), without external forcing, are given by

\[
A_{00} v + (A_{01} - A_{10}) v_y + A_{11} v_{yy} = 0.
\]

(2.6)

The entries of the generalized dynamic stiffness matrices are constants at a specified frequency. The solutions to a set of ordinary differential equations with
constant coefficients have the form of
\[ \mathbf{v}(y) = \hat{\mathbf{v}} e^{i\kappa y}. \] (2.7)

Substitution of Eq. (2.7) into Eq. (2.6) yields
\[ \left[ A_{00} + i\kappa (A_{01} - A_{10}) + \kappa^2 A_{11} \right] \hat{\mathbf{v}} = 0, \] (2.8)
which is a quadratic eigenvalue problem (QEP) [20] that defines the dispersion relation giving the wavenumber as a function of the frequency. The eigenvalue \( \kappa \) is the wavenumber, and the eigenvector \( \hat{\mathbf{v}} \) is the mode shape vector. For a QEP, the number of eigenvalues are twice as many as the DOFs of the system, and the eigenvalues of Eq. (2.8) are in pairs as \((\kappa, -\kappa)\). If damping is included, the wavenumbers are always complex [21].

2.2.2 Wave sorting in a complex wavenumber–frequency space

The SAFE method solves the QEP to get the wavenumbers at different frequencies, and the solved wavenumbers are not categorized according to their wave types. The categorization brings much convenience for the study of wave propagations, for example to calculate the group velocity. However, there has always been a challenge to automatically sort the waves. The discrete wavenumbers calculated by the SAFE method are usually identified by comparing to analytical solutions, or simply by connecting the dots that seems to belong to one curve. Mencik and Ichchou [22] proposed a technique to track the waves by comparing the modes at two adjoining frequencies. The comparison in this technique may take time due to the large number of modes, and the technique might also break down when veering happens. Veering is a wave coupling phenomenon in which two dispersion curves first approach each other and then veer apart instead of a crossing over. Mace and Manconi [23] showed that the mode shapes of the two waves switch after veering for a construction of spring coupled Euler beam and string. This switch makes it difficult for the tracking technique based on mode comparison. An alternative technique based on the curvature of the dispersion curves is thus developed.

The dispersion curves can be plotted in many different ways. One way is plotting the real part or the imaginary part of the wavenumber against the frequency in a two-dimensional (2D) plane. Another way is plotting the dispersion curves in a complex wavenumber–frequency \((\kappa - \omega)\) space as shown in Fig. 2.1.
The wavenumbers are divided into two groups with respect to the imaginary part. Those with negative imaginary parts are plotted as dots, and those with positive imaginary parts are plotted as circles.

Figure 2.1: Dispersion curves of wave propagation in a porous media calculated by SAFE in a $\kappa - \omega$ space. Wavenumbers with positive imaginary part are plotted as dots \( \bullet \), and wavenumbers with negative imaginary part are plotted as circles \( \circ \).

Unlike in a 2D plot, the dispersion curves in Fig. 2.1 do not cross each other, and each wave tend to have a sophisticated curvature of their own in the $\kappa - \omega$ space. This property of the dispersion curves can be used to sort the waves, bringing convenience to the study of each individual wave type.

A simple algorithm is built to sort the calculated wavenumbers. After solving the QEP of a $K$ DOFs system for $L$ specified frequencies, a $2K$-by-$L$ matrix of wavenumbers is obtained. The specified frequency is noted as $f_l \ (l = 1, 2, 3 \cdots)$. Firstly, the $2K$ wavenumbers at $f_1$ are used as starting points. For each chosen wavenumber at $f_1$, a second wavenumber, which is closest to it, is selected at $f_2$. Up till this step, there are two wavenumbers selected for each wave. With an additional wavenumber, a curve in the $\kappa - \omega$ space can be defined through the three wavenumber points. The one that gives the curve a minimum curvature is chosen at $f_3$. The curvature $r$ of a dispersion curve for equal parted frequency $\Delta f = f_l - f_{l-1}$ is evaluated by

$$ r = \frac{\kappa_l - 2\kappa_{l-1} + \kappa_{l-2}}{\Delta f^2}. \quad (2.9) $$

A wavenumber at $f_l$ is selected by evaluating the curvature with wavenumbers at $f_{l-1}$ and $f_{l-2}$. In this way, all the wavenumbers can be sorted into individual waves. This algorithm has been effective and robust when used to sort the waves for the cases considered in this thesis. If the dispersion curves do not look reasonable or converged, finer frequency resolution $\Delta f$ should solve most problems; otherwise,
a higher order of curvature evaluation formula might be helpful.

### 2.2.3 Wave propagation in elastic sandwich panels

An example of the wave propagation in an elastic sandwich panel is given to demonstrate the SAFE method. The panel consists of two aluminum skin plates and a foam core, and the parameters are given in Tab. 2.1. In this section, the dispersion curves and wave motions are studied. The wave propagation characteristics of this sandwich panel are similar to those of waves dominated by the skin plates and solid frame of the sandwich panel with a poroelastic core studied by Liu et al. [11].

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<tr>
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<th>Thickness (mm)</th>
<th>( \rho ) (kg/m(^3))</th>
<th>( E ) (Pa)</th>
<th>( \eta ) [-]</th>
<th>( \nu ) [-]</th>
</tr>
</thead>
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<tr>
<td>Skin plates (aluminum)</td>
<td>0.5</td>
<td>2815</td>
<td>( 6.9 \times 10^{10} )</td>
<td>0.01</td>
<td>0.3</td>
</tr>
<tr>
<td>Core (foam)</td>
<td>20</td>
<td>57</td>
<td>( 4.14 \times 10^{5} )</td>
<td>0.19</td>
<td>0.24</td>
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**2.2.3.1 Dispersion curves**

In the studied frequency range of 10 to 2000 Hz, most of the computed wavenumbers have very large imaginary part, i.e. they are quickly evanescing waves. Based on this observation, the dispersion curves of such waves are discarded, and only six branches are selected. The wavenumbers of these six waves have negative imaginary part and positive real part. The real part of the selected dispersion curves is shown in Fig. 2.2, and the absolute value of the imaginary part is shown in Fig. 2.3.

Often it is the propagating waves that are of most interest. These waves may evolve from an evanescent wave at the cut-off frequency, where the group velocity is zero [21], but this kind of pure cut-off frequency does not exist if damping is attributed. Alternatively, the wave is highly attenuated below cut-off frequency, and lightly attenuated above the cut-off frequency, which is mainly due to the damping described here by the loss factor [24]. Therefore, an intuitive way to distinguish the propagating waves from the evanescent waves is to study how the imaginary part of the wavenumbers changes by varying the loss factors. For a propagating wave, the imaginary part of the wavenumbers is often proportional to the loss factor; while for an evanescent wave, the imaginary part of the wavenumbers is often little affected, as could be observed from the analytical dispersion relation of an Euler beam.
Figure 2.2: Dispersion curves (real part) of the elastic sandwich panel. Solid lines, propagating waves; dash dotted lines, evanescent waves; Roman number I - VI, annotation of the branches.

Figure 2.3: Dispersion curves (imaginary part) of the elastic sandwich panel. Solid lines, with the original loss factors; dashed lines, with the loss factors reduced by 10 times; Roman number I - VI, annotation of the branches.
Fig. 2.3 shows the imaginary part of the wavenumbers for the panel with the original loss factors (solid lines) and with 10 times smaller loss factors (dashed lines). These branches are identified as: one propagating wave at all frequencies, branch III; one evanescent wave at all frequencies, branch V; three branches which become propagating waves above certain frequencies, branches I, IV and VI; and one propagating wave which has a complex cut-off between 1200 and 1700 Hz, branch II. At the complex cut-off of branch II, the real part of the wavenumbers decreases as the frequency increases, thus the group velocity is negative [25]. The propagating and evanescent waves, sorted in this way, are in Fig. 2.2 shown as solid and dash dotted lines. The modification of loss factor helps to distinguish the propagating and evanescent waves, but it may change the dispersion phenomenon like veering, which may be avoided if a smaller modification of loss factor is used.

2.2.3.2 Wave motions

To visualize the wave motions of an individual branch, its wavenumber and eigenvector are used. Fig. 2.4 shows the wave motions for two and a half wavelengths in the $y-z$ plane. When plotting the wave motions, the imaginary part of the wavenumber is neglected, and therefore no spatial decay is seen. Displacements in the $y$ and $z$ directions are not according to the same scale.

![Wave motions](image)

Figure 2.4: Wave motions (in the $y-z$ plane) of the elastic sandwich panel at frequencies of interest, and the scale in the $y$ and $z$ directions are different. (a) branch I at 1000 Hz; (b) branch II at 100 Hz; (c) branch III at 20 Hz; (d) branch IV at 800 Hz; (e) branch V at 2000 Hz; (f) branch VI at 2000 Hz.

As shown in Figs. 2.4(a) and 2.4(b), branches I and II mainly have compressional motions of the skin plates in the waveguide direction. For branch I, the
compressional motions of the skin plates are in antiphase, i.e. the upper skin plate is compressed while the lower skin plate is extended, and vice versa. Since the core is attached to the skin plates, its upper and lower halves also move in antiphase. This antiphase motion generates a displacement gradient in the thickness direction, which induces a shear force in the core that couples the skin plates. Therefore, branch I is here named as the antiphase coupled compressional wave.

For branch II, the compressional motions of the skin plates and the core are all in phase, thus it may be regarded as the quasi-longitudinal motion of the sandwich composite. It is named here as the in-phase composite compressional wave. Due to the Poisson contraction, motions are also seen in the thickness direction.

Branches III and IV are mainly related to the bending motions of the skin plates, see Figs. 2.4(c) and 2.4(d). For branch III, the skin plate bending motions are antisymmetric, and it is usually called the antisymmetric bending wave. This wave has been studied in the literature [26–28], and it has a typical behavior that: at low frequencies, this wave behaves as if the sandwich structure was a single layer plate. As frequency increases, the shear motion in the core softens the bending rigidity of the panel, which gradually decouples the sandwich composite. At even higher frequencies, the skin plates behave like decoupled ones, and branch III approaches the bending wavenumber of the skin plate.

For branch IV, the bending motions of the skin plates are symmetric, and thus it is called the symmetric bending wave. For this wave, the core acts as springs that it is extended when the plates move away from each other and compressed when the plates move towards each other. In many cases, this wave is also taken into account, besides the antisymmetric bending wave, to predict the sound transmission loss through sandwich panels [27, 28].

For the remaining two waves, branches V and VI, the core plays a more important role, as is further discussed in Paper A in this thesis [11].

2.2.4 SEA parameters from wave propagation characteristics

Parameters used in an SEA model include modal density, loss factor and coupling loss factor etc. The loss factor is calculated in this section as an example of what may be extracted from the wave propagation characteristics.
2.2.4.1 Loss factor

The loss factor of a damped structure is defined from an energy point of view as [29]

$$\eta = \frac{\langle \tilde{D} \rangle}{\omega (\langle \tilde{U} \rangle + \langle \tilde{T} \rangle)},$$  \hspace{1cm} (2.10)

where $\langle \tilde{D} \rangle$, $\langle \tilde{U} \rangle$ and $\langle \tilde{T} \rangle$ are the time and space averaged dissipated power, strain energy and kinetic energy, respectively, and they are calculated by

$$\left\{ \begin{array}{l}
\langle \tilde{D} \rangle = \frac{1}{2} \omega \hat{v}^H \text{Im} [\tilde{A}_{00} + i\kappa \tilde{A}_{01} - i\kappa^* \tilde{A}_{10} + |\kappa|^2 \tilde{A}_{11}] \hat{v} \\
\langle \tilde{U} \rangle = \frac{1}{4} \hat{v}^H \text{Re} [\tilde{A}_{00} + i\kappa \tilde{A}_{01} - i\kappa^* \tilde{A}_{10} + |\kappa|^2 \tilde{A}_{11}] \hat{v} \\
\langle \tilde{T} \rangle = \frac{1}{4} \omega^2 \hat{v}^H \tilde{M} \hat{v}, \end{array} \right.$$ \hspace{1cm} (2.11)

where $^*$ denotes the conjugate of a complex number, and $M$ is the mass matrix from $A_{00} = \tilde{A}_{00} + \omega^2 M$.

The calculated loss factors are shown in Fig. 2.5. The loss factors of the evanescent waves, branch V and parts of branches IV and VI, are even larger than the loss factor of the core material of 0.19. Therefore, it shows again that the damping in an evanescent wave is controlled beyond the material loss factor itself. Branch VI is a wave whose motions in mainly in the core, and therefore its loss factor is around 0.19 of the core material. On the other hand, branch I is the skin plate dominated in-phase longitudinal wave. At low frequencies, the composite behaves like a quasi-longitudinal wave, and its loss factor is closer to that of the skin plates of 0.01; at high frequencies, the core does not move as in a plan wave anymore and the loss factor increases to that of the core. The loss factor of branch III, the antiphase bending wave, decreases as the two skin plates gradually decouple. This decoupling leads to more motion and energy in the skin plates, the damping is therefore reduced because of the low loss factor of the skin plates.

2.3 SAFE for forced response calculation

From the partially discretized weak form in Eq. (2.5), the wave equations are derived in Eq. (2.8) to study the wave propagation in the previous section. The partially discretized weak form may also be used to calculate the forced response following an approximation of the field variables by Fourier series and minimization of the residual by the Ritz method.
2.3.1 Fourier series expansion of the field variables

The continuous field variables of a finite structure with length $L_y$ are represented by a Fourier series, as discussed in details by Zienkiewicz et al. [30]. Without loss of generality, the series is sorted into two sets so that $v(y)$ is approximated by

$$v(y) = \sum_n T_n^{(1)}(y) \hat{v}_n^{(1)} + T_n^{(2)}(y) \hat{v}_n^{(2)}. \quad (2.12)$$

If an entry of the diagonal matrix $T_n^{(1)}$ is $\sin \kappa_n y$ where $\kappa_n = n\pi/L_y$, the corresponding entry of the diagonal matrix $T_n^{(2)}$ is $\cos \kappa_n y$, and vice versa.

We presume one of the $T_n^{(1)}$ and $T_n^{(2)}$ in Eq. (2.12) is enough to define the field variables and the trigonometric functions are orthogonal with respect to Eq. (2.5), the integration in the $y$ direction is transformed into a summation, and the residual is given as

$$r = \sum_n \frac{L_y}{2} \delta \hat{v}_n^T (A_{00} + A_{01}\kappa_n + \kappa_n A_{10} + \kappa_n A_{11}\kappa_n) \hat{v}_n - \delta \hat{v}_n^T \hat{F}_n, \quad (2.13)$$

where

$$\hat{F}_n = \int\Gamma T_n^T N^T F d\Gamma. \quad (2.14)$$

The elements of the diagonal matrix $\kappa_n$ are $\kappa_n$, if the corresponding element in $T_n$ is $\sin \kappa_n y$; and $-\kappa_n$ for $\cos \kappa_n y$. 

Figure 2.5: Loss factors of the waves traveling in the elastic sandwich panel.
2.3. SAFE FORCED RESPONSE CALCULATION

2.3.2 Coefficients of the Fourier series by the Ritz method

The coefficients of the Fourier series are determined by minimizing the residual following the Ritz procedure [31], given by

\[
\frac{\partial r}{\partial \delta \hat{v}_n} = \frac{L_y}{2} (A_{00} + A_{01} \kappa_n + \kappa_n A_{10} + \kappa_n A_{11} \kappa_n) \hat{v}_n - \hat{F}_n = 0. \tag{2.15}
\]

The nodal field variables are recovered as a continuous function in the \( y \) direction after solving Eq. (2.15) for the coefficients of \( \hat{v}_n \), given by

\[
u(y) = \sum_n T_n(y) \hat{v}_n \tag{2.16}\]

This SAFE method described here solves a number of small 2D problems instead of a larger three-dimensional problem, which dramatically decreases the computational time and reduces the memory usage.

2.3.3 Formulation of the semi-analytical porous element

An important step in the SAFE formulation is the choice of the orthogonal harmonic functions as described in Sec. 2.3.1. Examples of orthogonal trigonometric functions are given below for the porous materials, which is used in the later calculations for the forced response.

Biot’s theory describes the porous material as a two-phase homogenized medium and employs the frame and fluid displacements as field variables. Atalla et al. instead proposed a mixed frame displacement - fluid pressure formulation, and the weak form with time dependence \( e^{i\omega t} \) is given as [32]

\[
\begin{align*}
\int_{\Omega} \sigma (\delta u)^T \varepsilon(u) \, d\Omega - \int_{\Omega} \omega^2 \rho \delta u^T u \, d\Omega & - \int_{\Omega} \phi \delta u^T \nabla p + \phi \left( 1 + \frac{\tilde{Q}}{R} \right) \nabla \cdot \delta u p \, d\Omega \\
+ \int_{\Omega} \frac{\phi^2}{R} (\nabla \delta p)^T \nabla p d\Omega & - \int_{\Omega} \frac{\phi^2}{R} \delta p d\Omega - \int_{\Omega} \phi \frac{(\nabla \delta p)^T u + \phi \left( 1 + \frac{\tilde{Q}}{R} \right) \delta p \nabla \cdot u \, d\Omega \\
- \int_{\Gamma} \delta u^T \sigma_n^T d\Gamma & - \int_{\Gamma} \phi \delta p (U - u)^T n \, d\Gamma = 0
\end{align*}
\]

where \( \delta u \) and \( \delta p \) denote admissible variations of the frame displacement \( u \) and fluid pressure \( p \), respectively, \( U \) is the fluid displacement, \( \phi \) is the porosity, \( \nabla \) is the
CHAPTER 2. THE SEMI-ANALYTICAL FINITE ELEMENT METHOD

gradient operator, \( \nabla \cdot \) is the divergence operator, \( i \) is the imaginary unit, \( \omega \) is the angular frequency, \( \sigma^t \) is the total stress tensor, \( n \) is the normal of the boundary \( \Gamma \), \( \tilde{Q} \) is a coupling coefficient between the dilatation and stress of the two phases, \( \tilde{R} \) is the bulk modulus of the fluid, and \( \tilde{\alpha} \) is the dynamic tortuosity. \( \tilde{\rho} = \tilde{\rho}_{11} - (\tilde{\rho}_{12})^2 / \tilde{\rho}_{22} \) is introduced for convenience, with the modified effective densities \( \tilde{\rho}_{11} \), \( \tilde{\rho}_{22} \), and \( \tilde{\rho}_{12} \). The engineering strain vector \( \epsilon = [\epsilon_x \epsilon_y \gamma_{xy} \gamma_{yz} \gamma_{xz}]^T \) is related to the engineering stress vector of the frame \textit{in-vacuo} \( \tilde{\sigma} \) with the constitutive matrix \( D \) as \( \tilde{\sigma} = D \epsilon \).

The field variables of a poroelastic material that have constant properties in the \( y \) direction are partially discretized as

\[
\mathbf{u} (x, y, z) = \sum_n \mathbf{T}_n^u (y) \begin{bmatrix} N_u (x, z) \\ N_v (x, z) \\ N_w (x, z) \end{bmatrix} \mathbf{u}_n \text{ and } \mathbf{p} (x, y, z) = \sum_n \mathbf{T}_n^p (y) \mathbf{N}_p (x, z) \mathbf{p}_n
\]

(2.18)

where the row vectors \( \mathbf{N}_u \), \( \mathbf{N}_v \), and \( \mathbf{N}_w \) contain the finite element shape functions [3] that interpolate the frame displacements \( u, v, \) and \( w \) in the \( x, y \) and \( z \) directions, respectively. The row vector \( \mathbf{N}_p \) contains the shape functions of the fluid pressure. \( \mathbf{T}_n^u \) and \( \mathbf{T}_n^p \) could be one of either the orthogonal combinations given by

\[
\mathbf{T}_n^u = \begin{bmatrix} \sin \kappa_n y & 0 & 0 \\ 0 & \cos \kappa_n y & 0 \\ 0 & 0 & \sin \kappa_n y \end{bmatrix} \quad \text{or} \quad \mathbf{T}_n^u = \begin{bmatrix} \cos \kappa_n y & 0 & 0 \\ 0 & \sin \kappa_n y & 0 \\ 0 & 0 & \cos \kappa_n y \end{bmatrix}
\]

(2.19)

The engineering strain vector is rewritten with Eqs. (2.18) and (2.19) as \( \epsilon = \sum (\mathbf{B}_0 \mathbf{T}_n^u + \mathbf{B}_1 \partial \mathbf{T}_n^u / \partial y) \mathbf{u}_n \), where

\[
\mathbf{B}_0 = \begin{bmatrix} \partial \mathbf{N}_u / \partial x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \partial \mathbf{N}_v / \partial x & 0 \\ 0 & 0 & \partial \mathbf{N}_w / \partial x \\ \partial \mathbf{N}_u / \partial z & 0 & \partial \mathbf{N}_w / \partial x \end{bmatrix} \quad \text{and} \quad \mathbf{B}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbf{N}_v & 0 \\ 0 & 0 & \mathbf{N}_w \\ \mathbf{N}_u & 0 & 0 \\ 0 & 0 & \mathbf{N}_w \end{bmatrix}
\]

(2.20)

The generalized frame stiffness matrices \( \mathbf{K} \) are given by

\[
\mathbf{K} = \frac{\tilde{\rho}_{11}}{\tilde{\rho}} \sum_n \int_\Gamma \mathbf{B}_0^T \mathbf{K}_0 \mathbf{B}_0 + \kappa_n^u \mathbf{B}_0^T \mathbf{K}_1 \mathbf{B}_0 + \kappa_n^u \mathbf{B}_1^T \mathbf{K}_0 \mathbf{B}_0 + \kappa_n^u \mathbf{B}_1^T \mathbf{K}_1 + \kappa_n^u \mathbf{B}_1^T \mathbf{K}_0 \mathbf{B}_0 + \kappa_n^u \mathbf{B}_1^T \mathbf{K}_1 \mathbf{B}_0 \mathbf{d}x \mathbf{d}z,
\]

(2.21)
where
\[
\kappa_n^u = \begin{bmatrix}
\kappa_n & 0 & 0 \\
0 & -\kappa_n & 0 \\
0 & 0 & \kappa_n
\end{bmatrix}
\] (2.22)

if the first set in Eq. (2.19) is used. There is no multiplication between \(\sin \kappa_n y\) and \(\cos \kappa_m y\), and the terms in the Fourier series matrix given in Eq. (2.19) are orthogonal in the well-known form that
\[
\int_0^{L_y} \sin \kappa_m y \sin \kappa_n y dy = \int_0^{L_y} \cos \kappa_m y \cos \kappa_n y dy = \begin{cases} 
\frac{L_y}{2} & (m = n) \\
0 & (m \neq n)
\end{cases}
\] (2.23)

for a constitutive matrix having a format of
\[
D = \begin{bmatrix}
D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\
D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\
D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & D_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & D_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & D_{66}
\end{bmatrix},
\] (2.24)

which can represent isotropic, orthotropic and transversely isotropic materials, covering many types of porous medium. However, the orthogonality is not valid for the fully anisotropic constitutive matrix. The stiffness matrix of the fluid is given by
\[
Q_{00} = \int_x \int_z \frac{\phi^2}{R} N_p^T N_p dx dz,
\] (2.25)

and the mass matrix of the frame is given by
\[
M_{00} = \int_x \int_z \tilde{\rho} (N^u)^T N^u dx dz,
\] (2.26)

where \(N^u = [N_u, N_v, N_w]^T\) also leads to orthogonal relations of the Fourier series matrix, and so do the remaining domain integral terms in Eq. (2.17), as can be verified by direct inspection. The gradient of the fluid pressure is given as
\[
\nabla p = \sum (G_0 T_p + G_1 \partial T_p / \partial y) \tilde{p}_n,
\]

where
\[
G_0 = \begin{bmatrix}
\partial N_p / \partial x \\
0 \\
\partial N_p / \partial z
\end{bmatrix}
\] and \(G_1 = \begin{bmatrix} 0 \\ N_p \\ 0 \end{bmatrix}.
\] (2.27)
Therefore, the mass matrix of the fluid is given by

\[
H_{00} = \int_x \int_z \frac{\phi^2}{\omega_p^2} G_0^T G_0 \, dx \, dz \quad H_{01} = 0 \quad H_{11} = \int_x \int_z \frac{\phi^2}{\omega_p^2} G_1^T G_1 \, dx \, dz.
\] (2.28)

The coupling terms are given by

\[
C_{00} = \int_x \int_z \frac{\phi}{\alpha} (N_u)^T G_0 + \frac{\phi}{\alpha} \left(1 + \frac{\tilde{Q}}{R}\right) R_0^T N_p \, dx \, dz
\]

\[
C_{01} = \int_x \int_z \frac{\phi}{\alpha} (N_u)^T G_1 \, dx \, dz
\]

\[
C_{10} = \int_x \int_z \phi \left(1 + \frac{\tilde{Q}}{R}\right) R_1^T N_p \, dx \, dz
\]

(2.29)

where \(\nabla \cdot u = \sum (R_0 T_n^u + R_1 \partial T_n^u / \partial y) \tilde{u}_n\), with \(R_0 = \partial N_u / \partial x + \partial N_w / \partial z\) and \(R_1 = N_v\). The generalized dynamic stiffness matrices for poroelastic materials are thus given as

\[
A_{00} = \begin{bmatrix}
K_{00} - \omega^2 M_{00} & -C_{00} \\
-C_{00}^T & \frac{1}{\omega^2} H_{00} - Q_{00}
\end{bmatrix}, \quad A_{01} = \begin{bmatrix}
K_{01} & -C_{01} \\
-C_{01}^T & 0
\end{bmatrix}, \quad A_{11} = \begin{bmatrix}
K_{11} & 0 \\
0 & \frac{1}{\omega^2} H_{11}
\end{bmatrix}.
\] (2.30)

As shown above, the orthogonality relations are obtained for either of the two Fourier series sets in Eq. (2.19). These two sets of series functions also correspond to two sets of BCs at the ends, which are: generalized simply supported BCs (first set in Eq. (2.19)), both the frame and fluid displacements in the cross section plane are restricted and displacements in the \(y\) direction is allowed; and generalized sliding BCs (second set in Eq. (2.19)) that have the opposite restrictions. The BCs described by the displacements are straightforward to see for the frame but not for the fluid since the pressure is used as a field variable. The fluid displacement as functions of the frame displacement and pore pressure is given by [33]

\[
U = \frac{\phi}{\tilde{\rho}_{22} \omega^2} \nabla p - \frac{\rho_{12}}{\tilde{\rho}_{22}} \tilde{u},
\]

(2.31)

from which it gives the same BCs as the frame displacement. Boundary conditions can also be applied on the nodal field variables of the finite element mesh, e.g. a zero displacement fixes the displacement at all locations along the \(y\) direction.
2.3.4 Numerical examples

2.3.4.1 Porous layer under normal incident pressure

The porous layer has a surface area of $0.1 \times 0.1 \text{ m}^2$ and thickness of 0.05 m. The material properties are: porosity is 0.96; flow resistivity is $32000 \text{ Ns/m}^4$; tortuosity is 1.7; viscous characteristic length is 90 $\mu \text{m}$; thermal characteristic length is 165 $\mu \text{m}$; Poisson’s ratio is 0.3; damping loss factor is 0.1; and Young’s modulus is 145000 Pa.

The displacements on the bottom surface of the porous layer are zero, and the four surfaces perpendicular to the bottom surface are simply supported. The porous layer is under a normal incident pressure of 1 Pa, and the resulting real and imaginary parts of the displacement at the middle point on the upper surface of the layer are shown in Figs. 2.6 and 2.7. The SAFE calculation shows good agreement with the converged FEM results.

![Figure 2.6: Real part of the displacement at the middle point on the upper surface of the porous layer under a normal incident pressure.](image)

2.3.4.2 Thick plate under point excitation

The forced response of a single leaf plate is used to test the convergence and accuracy of the SAFE method. The flexibility at the center of a plate calculated both by the SAFE and a modal summation technique are shown in Figure 2.8.

The dimensions and material properties of the plate is the same as the one used by Sgard et al. [34]. It is a 0.001 m thick rectangular plate of area $0.35 \times$
0.22 m². The plate is made of aluminum with Young’s modulus 71 GPa, Poisson’s ratio 0.33, mass density 2814 kg/m³ and a loss factor of 0.01. The plate is simply supported around all edges, and the point excitation is in the middle of the plate. The flexibility calculated by the SAFE with a mesh of 11 nodes on the cross section has a good agreement with the modal summation results till the fourth peak. Due to the limitation of the mesh, the results start deviating. However, at around 600 Hz, the peak of the analytical solutions is again well described by the SAFE, because it does not have smaller wavelength on the cross section, but increased wavenumber in the wave propagating direction. The SAFE has a good accuracy with only a few DOFs used on the cross section of structure.

2.4 Conclusions

A general procedure to formulate the SAFE is devised, from partially discretized weak forms, for structures which have constant properties in the axial direction, and the method is used to investigate wave propagation and to calculate forced response. When the length of the structure is infinite and there is no external forcing, a QEP is derived and solved to determine the dispersion properties of the waves traveling in the axial direction. A sorting algorithm is presented to track the waves by selecting the smoothest dispersion curves in a three-dimensional (3D) complex wavenumber - frequency space. The wave propagation characteristics of this relation and the corresponding wave motions provide insight into the vibro-
2.4. CONCLUSIONS

coustic behavior of the structure, and an example of a multilayer elastic panel is given in this chapter. The loss factor of the waves traveling in the panel is also calculated, which may be used in high frequency methods such as SEA.

For structures with finite length and under harmonic excitation, Fourier series are used to represent the field variables in the axial direction. Certain combinations of the series are orthogonal with respect to the partially discretized weak form, and therefore they transform the 3D problem into independent 2D problems, which increases the computational efficiency. The semi-analytical porous and plate elements are formulated and used to calculate the forced response of a porous layer under normal incident pressure and point excited plate. The SAFE calculations show good agreement with conventional finite element and modal summation results.

The SAFE provides physical insight, through the study of the the wave propagation characteristics, and efficient calculation, from the forced response, which are important for the vibroacoustic analysis of structures in the medium and high frequencies.

Figure 2.8: Flexibility of a plate under point excitation calculated by SAFE (11 nodes) and modal summation technique.
Chapter 3

The wave expansion method

The WEM is a variant derived from the Green’s function discretization which was firstly developed by Caruthers et al. [35]. For this discretization scheme, the pressure field at a local computational stencil with the center node of interest and its neighboring nodes is approximated using the free field Green’s function of a number of hypothetical point sources located outside the stencil, which has a similar rationale as the method of fundamental solutions [36]. The amplitudes of the point sources are determined by minimizing the $L^2$ norm of the pressure at the neighboring nodes which are further used to define the shape functions [3, Chap. 15]. The existence of a single universally valid Green’s function is not always applicable in the case of position dependent coefficients, therefore Caruthers and coworkers [9] employ a more convenient local plane wave expansion which shows no sacrifice in accuracy. Actually, the accuracy of the method does not break down until the spatial resolution decreases towards two nodes per wavelength [10], and it is thus interesting for efficient numerical calculations.

In this chapter, the computational template of the WEM is first recapitulated, and a point source model is then developed.

3.1 Helmholtz equation and the wave expansion scheme

3.1.1 Helmholtz equation

The acoustic wave propagating in a domain $\Omega$ is governed by the Helmholtz equation

$$\nabla^2 p + \kappa^2 p = 0,$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian, $\kappa$ is the wavenumber defined by angular velocity $\omega$ and speed of sound $c_0$ as $\kappa = \omega/c_0$. 

3.1.2 The wave expansion discretization scheme

In the discretized computational domain of the WEM, as shown in Fig. 3.1, the center node \( a \) has 8 neighboring nodes. The pressure at each node may be approximated by a superposition of field defined by \( J \) hypothetical plane waves of amplitude \( \gamma_j \) and with their propagation directions given by unit vectors \( \alpha_j \). The pressure at node \( a \) is thus given by

\[
p_a = \sum_{j=1}^{J} \gamma_j e^{i \kappa \alpha_j \cdot x_a} = h_a \gamma,
\]

which can be written in a compact format as

\[
p_a = h_a \gamma,
\]

where \( h_a \) is a \( 1 \times J \) row vector

\[
h_a = [ e^{i \kappa \alpha_1 \cdot x_a} \ e^{i \kappa \alpha_2 \cdot x_a} \ \ldots \ e^{i \kappa \alpha_J \cdot x_a} ],
\]

\( \gamma \) is a \( J \times 1 \) column vector

\[
\gamma = \begin{bmatrix} \gamma_1 & \gamma_2 & \ldots & \gamma_J \end{bmatrix}^T,
\]

and superscript \( T \) denotes the transpose of a matrix or vector. The pressure at the neighboring nodes, \( p_{nb} \), around node \( a \) is also approximated by the same set of waves, given by

\[
p_{nb} = H \gamma,
\]

where \( p_{nb} = \begin{bmatrix} p_b & p_c & p_d & p_e & p_f & p_g & p_h & p_i \end{bmatrix}^T \), and \( H \) is a \( 8 \times J \) matrix for the stencil illustrated in Fig. 3.1, given by

\[
H = \begin{bmatrix} h_b^T & h_c^T & h_d^T & h_e^T & h_f^T & h_g^T & h_h^T & h_i^T \end{bmatrix}^T.
\]

As more plane waves are used than the number of neighboring nodes, the system in Eq. (3.4) is under determined, which is solved in a least square sense to select the best approximation. The wave amplitudes \( \gamma \) are thus calculated by premultiplicating Eq. (3.4) with \( H^+ \), which is the Moore-Penrose pseudo-inverse of \( H \), which yields

\[
\gamma = H^+ p_{nb}.
\]

Substitution of Eq. (3.5) into Eq. (3.3), the relation between the sound pressure of node \( a \) and at its neighboring nodes is established as

\[
p_a = h_a H^+ p_{nb}.
\]

This relation is used to approximate the solution of the equation of motion at node \( a \), and \( h_a H^+ \) is similar to the shape functions used in the FEM.

This process is repeated for each node in the computational domain, for exam-
3.2. POINT SOURCE IN THE WEM

Figure 3.1: Illustration of the wave expansion discretization scheme in 2D. A number of hypothetical plane waves are evenly distributed around the nodes. Two computational stencils are presented as solid lines with center node $x_a$ and dashed lines with center node $x_b$.

ple node $b$ and its neighboring nodes within the stencil of dashed lines as shown in Fig. 3.1. Upon this basis, a global matrix $K$, which is unsymmetric and sparse, is assembled containing all the local vectors $h_nH^+$. Source terms $Q$ may be added to the right hand side, and therefore the system under excitation is defined by

$$Kp = Q.$$ (3.7)

The implementation of the three types of BCs, Dirichlet BC, Neumann BC and impedance BC are described in the appended Paper D.

3.2 Point source in the WEM

Point source modeling has always been challenging for wave based methods such as the WEM which uses plane waves. In this section, a free field Green’s function of a point source is used to overcome this problem.
3.2.1 The inhomogeneous Helmholtz equations

The inhomogeneous Helmholtz equations with a monopole point source at the right hand side is given by

\[ \nabla^2 p + \kappa^2 p = -4\pi \hat{S} \delta (x - x_s), \]

where \( x_s \) is the location of the point source, and \( \hat{S} \) is the monopole amplitude. For a volume flow source, it is given by

\[ \hat{S} = -\frac{i\omega \rho_0 Q_s}{4\pi}, \]

where \( \rho_0 \) is the fluid density, and \( Q_s \) is the volume velocity. Substitution of Eq. (3.9) into Eq. (3.8) gives

\[ \nabla^2 p + \kappa^2 p = i\omega \rho_0 Q_s \delta (x - x_s). \]

3.2.2 Pressure field approximation around the source stencil

The pressure field at the nodes in a stencil which contains the source point may be approximated by using the plane waves and the free field wave induced by the point source as illustrated in Fig. 3.2.

The free field pressure field of the point source is only considered at its neighboring nodes, and the center node is excluded because of the singularity. When the point source node functions as a neighboring node of the computational stencil, as the green stencil in Fig. 3.2, the pressure field at node \( x_1 \) is given by

\[ p_1 = h_1 \alpha_1 + q, \]

where \( q = i\omega \rho_0 Q_s G (x_1 | x_s) \). The pressure at the neighboring nodes are given as

\[ p_n = H_n \alpha_1 + Q_1, \]

where \( Q_1 = i\omega \rho_0 Q_s G (x_n | x_s) \) at all the nodes except for the source node where the pressure is not defined, and \( G (x_m | x_s) \) is the free field Green’s function, given by

\[ G (x_m | x_s) = \frac{1}{4} H_0^{(2)} (\kappa |x_m - x_s|) \quad 2D \]

\[ = \frac{e^{-i\kappa|x_m - x_s|}}{|x_m - x_s|} \quad 3D, \]

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Figure 3.2: Illustration of the computational stencils of the source at the center and as a neighboring point under the pressure field approximated by plane waves and free field waves induced by a point source.
where $H_0^{(2)}$ is the zero order Hankel function of the second kind. It follows, the pressure at the node that has the source as a neighboring node is given by

$$p_1 - h_1 H_n^+ p_n = q - h_1 H_n^+ Q_1. \quad (3.14)$$

By considering the neighboring nodes, the monopole point source is distributed at all the nodes in the stencil but the center node which the point source locates as given by Eq. (3.14).

### 3.2.3 Evaluation of the monopole point source model

The model of the monopole point source described previously is evaluated in two aspects: the convergence study and comparison with calculations of the finite element method.

#### 3.2.3.1 Convergence

This 2D computation example is for a square with an edge length of 1 m and rigid wall BCs at all edges. The source is located at the $(0.1 \text{ m}, 0.1 \text{ m})$, and the acoustic pressure calculated at the center of the square $(0.5 \text{ m}, 0.5 \text{ m})$. The computational domain is meshed with equally distanced nodes.

The acoustic pressure at the measured point converges to 2.418, as shown in Fig. 3.3(a). The relative error defined as the difference between the finest mesh and every other mesh normalized by the finest mesh calculation is shown in Fig. 3.3(b), and the relative error is very small, being almost $1 \times 10^{-5}$.

#### 3.2.3.2 Comparison with calculations by the finite element method

The pressure field of the square excited by a point source at $(0.1 \text{ m}, 0.1 \text{ m})$ are compared between the FEM and WEM calculations with a mesh of 240 by 240 nodes at 200 Hz and 400 Hz, respectively. As shown in Figs. 3.4 and 3.5, the contour plots are almost identical for the FEM and WEM calculations. The relative differences are small, with a mean around $1 \times 10^{-3}$, except around the source and at positions where the acoustic pressure is close to zero.

### 3.3 Conclusions

This chapter introduces the background theory and computation template of the WEM. A point source model is also developed, which uses the free field Green’s function, to overcome the limit of plane waves in modelling the singularity. The source model is compared with the finite element method for the calculation of a
Figure 3.3: The absolute value of the acoustic pressure and relative error at node (0.5 m, 0.5 m) with different degrees of freedom at 200 Hz. (a) the absolute value of the acoustic pressure (b) relative error.

2D square excited by a point source, and the pressure field comparison shows very small relative errors. The WEM is a promising method for the prediction of the pressure field in enclosures of complex geometry at medium frequencies thanks to its high computational efficiency.
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(a) FEM calculation at 200 Hz.  
(b) WEM calculation at 200 Hz.

(c) Relative difference between FEM and WEM calculations, the mean is $1 \times 10^{-3}$

Figure 3.4: Comparison of the pressure field with FEM calculation at 200 Hz.
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(a) FEM calculation at 400 Hz.

(b) WEM calculation at 400 Hz.

(c) Relative difference between FEM and WEM calculations, the mean is $7.4 \times 10^{-3}$

Figure 3.5: Comparison of the pressure field with FEM calculation at 400 Hz.
Chapter 4

Contribution of appended papers

The contribution of the appended papers includes:

- provide an in-depth understanding of wave propagation characteristics in sandwich panels with a poroelastic core (Paper A);
- develop a computational efficient method for sound transmission calculations of multilayer panels lined with porous materials (Paper B);
- apply the SAFE to calculate the sound transmission through built up partitions used in railway car (Paper C);
- formulate the wave expansion method for interior Helmholtz problems of complex geometry (Paper D).

More details of the contribution of each paper are given below.

4.1 Paper A

Wave propagation in sandwich panels with a poroelastic core

This paper examines wave propagation in sandwich panels with a poroelastic core modeled by Biot’s theory. The semi-analytical poroelastic element is developed based on the displacement-pressure weak form [32]. Dispersion curves and wave modes are computed with the SAFE. The dispersion curves are studied for varying level of damping and they are identified as propagating waves and evanescent waves. The wave characteristics of the eight branches are analyzed by investigating the wave motions, by using simplified analytical models, and by studying the energy distributions. One important contribution of this paper is the understanding of the wave propagation characteristics of such a complex multilayer construction including fluid-structure interactions within in the poroelastic core and between the core fluid and skin plates. This understanding provide a
special insight to the vibroacoustic behavior of such panels. Parameters for modeling techniques such as SEA might also be extracted to further qualitatively study sound transmission paths through the panel. Another original contribution is the development of the semi-analytical poroelastic element, which can be readily used in other cases of wave propagation study involving poroelastic materials.

\section*{4.2 Paper B}

\textbf{Prediction of sound transmission through poroelastic material lined multilayer panels using a semi-analytical finite element method}

In this paper, the SAFE is used in a one-way model to calculate the sound transmission loss through multilayer panels lined with elastic porous materials. The field variables in the constant property direction are represented by a Fourier series expansion. Two combinations that lead to an orthogonal relation of the weak formulations are found, which correspond to two specific sets of boundary conditions: the generalized simply supported boundary conditions that the displacements within the cross section are constrained and displacements perpendicular to the cross section are free; and the generalized sliding boundary conditions that in the opposite. The formulations of the mixed displacement-pressure poroelastic and thick plate elements are described for this kind of structures.

Predictions of the sound transmission loss through one flat and two curved panels show good results compared to measurements. Two assumptions made in the calculation are discussed: First, the radiation loss is considered which compensate for the neglect of the radiation impedance. Thus added radiation loss factors to the mass matrix of the skin plates improve the estimation around the transverse dilatational frequency. Second, boundary condition are shown mainly to affect the sound transmission loss at low frequencies. The computational efficiency is also discussed, and it is seen that the SAFE dramatically reduces computation demands compared to the full finite element model.

The SAFE is versatile for structures with a complex cross section such as the extruded train floor panel and is readily adopted into an optimization routine. The proposed method has potential to be used in more vehicle industrial applications.

\section*{4.3 Paper C}

\textbf{Wave modelling in predictive vibro-acoustics: Applications to rail vehicles and aircraft}

Three different predictive methods based on wave descriptions of the acoustic
field are presented and used to calculate transmission and radiation properties of typical rail and aerospace structures. The SAFE is applied to: (i) calculate the transmission through a double wall structure; (ii) assess the sound transmission of an extruded floor structure and also to (iii) determine the sound pressure inside a large section of a rail car excited by external sound sources. The method presented can be used to effectively support decision making in the design process of trains and aircraft. One reason is that the method is a powerful mean to provide physical insight to acoustic transmission problems, e.g. by analysing wave numbers, wave forms and wave speeds of complex sandwich structures. Such analysis can be a basis for design guidelines and to communicate acoustic design drivers to non-acousticians who are also stakeholders and decision makers in the design process. A second reason is that this model based on semi-analytical formulations is typically several orders of magnitude more computational efficient than standard FEM, for cases where it applies.

The SAFE is in view of its computational efficiency, judged to have a high potential to assess acoustical performance in multi-disciplinary optimization schemes. It is concluded that, despite certain modelling inflexibility, combined wave finite element calculations are useful whenever a transmission loss suit measurement of a section of an aircraft or rail car is. In relation to taking measurements, the main advantage of predictive modelling is associated with time savings in the product development, in particular when parametric studies are of interest, and also it can be made before the structure is built.

4.4 Paper D

Prediction of sound field in geometrically complex enclosures with the wave expansion method

In this paper, we extend the application and demonstrate the capability of the WEM to predict the sound field in geometrically complex enclosures. The convergence and dispersion error are compared between the FEM and WEM on a 2D simplified car cavity case. The convergence shows that the WEM needs 2 to 3 times less nodes per wavelength to achieve the same accuracy than the FEM. The dispersion error analysis demonstrates that the WEM calculation yields good results even when a coarse mesh used, while the FEM calculation produces unacceptable errors. A 3D industrial car cavity case with fine geometrical details is considered in three cases, for low and high damping and with impedance boundary conditions, and the WEM results all show good agreement with the FEM calculations. In a 3D calculation case the WEM needs about 8 to 27 times less
DOFs compared to the FEM, which dramatically reduces the computer memory usage and CPU time. Therefore, the WEM pushes the envelope higher for the calculation frequency range in the automobile industry.
Chapter 5

Concluding remarks and future work

5.1 Concluding remarks

The semi-analytical finite element and wave expansion methods are developed as acoustic design tools for medium and high frequency vibroacoustic analysis within a close context of automobile and railway car industries. The two methods are used in three projects as: investigation of wave propagation in porous material lined multilayer structures, prediction of sound transmission through built-up and multilayer partitions, and development of the WEM for Helmholtz problems.

The SAFE method is used to study wave propagation in sandwich panels with a poroelastic core. Dispersion curves and wave modes are computed with the SAFE method. The dispersion curves are studied for varying level of damping and they are identified as propagating waves and evanescent waves. The wave characteristics of eight branches of dispersion curves are analyzed by investigating the wave motions, by using simplified analytical models, and by studying the energy distributions. The eight branches are studied in groups according to their energy distributions and their origins in the uncoupled components. Four branches are thus related to the compressional and bending waves of the skin plates; two branches are related to the frame-borne dilatational and rotational waves in the core, and two branches are related to the airborne dilatational wave in the core. The two branches dominated by the air are an oblique wave with approximately half a wavelength between the skin plates; and a plane wave, in which the airborne wave pushes and pulls the skin plates.

The SAFE method is also used to calculate sound transmission loss of built-up train floor panel and multilayer panels lined with porous materials. The SAFE method is in view of its computational efficiency, judged to have a high potential
to assess acoustical performance in multi-disciplinary optimization schemes of structures with constant properties in at least one direction. It is concluded that, despite certain modelling inflexibility, combined wave-finite element calculations are useful whenever a transmission loss suit measurement of a section of an aircraft or rail car is useful. In relation to taking measurements, the main advantage of predictive modelling is associated with time savings in the product development, in particular when parametric studies are of interest.

The WEM increases the computational efficiency by using a coarse mesh to achieve the same accuracy as other methods using fine meshes. The applications of the WEM are extended to predict the sound field in geometrically complex enclosures. The convergence and dispersion error are compared between the FEM and WEM on a 2D simplified car cavity case. The convergence shows that the WEM needs 2 to 3 times less nodes per wavelength to achieve the same accuracy as the FEM. The dispersion error analysis demonstrates that the WEM calculation yields good results even when a coarse mesh is used, while the FEM calculation produces unacceptable errors for such meshes. A 3D industrial car cavity case with fine geometrical details is also considered in three cases, low and high damping and impedance boundary conditions, and the WEM results all show good agreement with the FEM calculations.

The developed methods and conducted projects are expected to contribute to the medium and high frequency vibroacoustic analysis.

### 5.2 Future work

Time is limited. There are a few things in this PhD work that may be worth being considered in the future.

#### 5.2.1 SEA models based on parameters from the wave characteristics calculated by the SAFE method

The wave propagation study provides an investigation of the way that the sandwich panel with a poroelastic core behaves from a wave point view, which is helpful to understand the sound transmission and radiation properties of the panel. In many cases, it is even more helpful if the transmission paths of the sound through such a panel is quantified, and the SEA is readily to be used for this task based on the wave characteristics calculated by the SAFE method. This quantification is believed to provide further understanding of the roles played by the fluid and solid phases in the poroelastic core.
5.2.2 Sound transmission loss through different types of engineering partitions with the SAFE method

The main cases considered for the sound transmission loss calculations are the extruded train floor panel and porous material lined trim panels. The application of the method will be more widely extended if more types of engineering structures can be considered. One example is taking into account of the reinforcement beams usually placed between waveguide like base plates. Another example is to formulate the elements in a cylindrical coordinates to have a better resolution of curved structures such as aircraft fuselages.

5.2.3 Extension of the WEM applications in vibroacoustic analysis

The WEM is a general method to solve linear partial differential equations and was firstly proposed almost twenty years ago. The applications of this method is however mainly limited to Helmholtz problems even with many appealing advantages such as fast convergence rate and low dispersion error. The WEM has potential to model elastic or poroelastic medium.
References


REFERENCES


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