Master’s thesis

Lower ramification numbers of wildly ramified power series

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LOWER RAMIFICATION NUMBERS OF WILDLY RAMIFIED
POWER SERIES

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Abstract. In this thesis we study lower ramification numbers of power series tangent to the identity that are defined over fields of positive characteristics. Let $f$ be such a series, then $f$ has a fixed point at the origin and the corresponding lower ramification numbers of $f$ are then, up to a constant, the multiplicity of zero as a fixed point of iterates of $f$. In this thesis we classify power series having ‘small’ ramification numbers. The results are then used to study ramification numbers of polynomials not tangent to the identity. We also state a few conjectures motivated by computer experiments that we performed.

Keywords: Lower ramification numbers, Minimal ramification, Ramified polynomials, Ramified power series, Difference equations, Recurrence relations, Optimal cycles, Periodic points
1. Introduction

In this thesis we study lower ramification numbers for polynomials and power series over fields of prime characteristic. Let $K$ be a field and

$$g(z) = z + \ldots$$

be a power series in $K[[z]]$. For integers $n \geq 1$ let $g^n(z)$ denote the $n$-fold composition of $g$ with itself. Given an integer $n \geq 1$ the lower ramification number $r_n$ of $g$ is then defined as the multiplicity of $z = 0$ as a zero of $(g^n(z) - z)/z$.

Note that if $Q(z) = z + a_i z^i$ is a polynomial in $K[z]$, then we have that $Q^n(z) = z + na_i z^i + \ldots$. Hence if $K = \mathbb{C}$, then the lower ramification number is constant under iterations of $Q$. However, in the case that $K$ is of positive characteristic $p$ then the ramification numbers are nontrivial. In fact the lower ramification numbers are only known in a few special cases.

Since a famous theorem of Sen [21] there has been an increasing interest in this field of research. Main contributions are Laubie and Saïne [12,13], Keating [10], Lubin [16], Laubie, Movahhedi and Saliner [11] and Wintenberger [24]. These contributions concern the possible sequences of ramification numbers as discussed in more details in §1.2 below. Further research closely related to this field is also done by Sarkis [19,20].

Lindahl and Rivera-Letelier [15] has shown that there exist a connection between the lower ramification numbers and the geometric location of periodic points of power series in non-Archimedean dynamical systems, which is a very active area of research, see [1,2,22,25] and references therein.\(^1\) The study of dynamical systems in algebraic structures is often referred to as algebraic dynamics. For more information on the subject of algebraic dynamics we refer to Anashin and Khrennikov [1].

Let $p$ be a prime and $k$ a field of positive characteristic $p$. In this paper we classify power series of the form (1.1) that have ramification numbers of the form

$$r_p^m = b(1 + p + \cdots + p^m)$$

for $b = 1$ and $b = 2$. These results are stated as Theorem A and Theorem B in section 3.2 and 3.3 respectively. In proving these results we find the lowest degree term of $p^r(\zeta) - \zeta$ for all $p$ and then apply a result of Laubie and Saïne [12, Corollary 1]. Proving these theorems involves solving first order linear nonhomogeneous difference equations with nonconstant coefficients, which is a key part in the proofs. In solving these equations explicitly we show some sum identities needed in the solutions that is of independent interest. Our results are then applied to a problem posed by Lindahl and Rivera-Letelier [15, Problem 1.4, § 1.5]. The problem posed by Lindahl and Rivera-Letelier relates to finding the ramification numbers $r_n$ for a class of polynomials in fields of prime characteristic of the form $\gamma \zeta + \zeta^2$, where $\gamma$ is a root of unity.

We now give a brief outline of the thesis. In the §1.1 we give the preliminaries needed for the thesis, and in §1.2 we discuss the notion of lower ramification numbers, and give examples of possible sequences for the ramification numbers. We also introduce key

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\(^1\)A brief discussion of non-Archimedean fields follows in the preliminaries
notions of the thesis: \textit{b-ramification} and \textit{minimal ramification}. Minimally ramified power series do in some sense have the smallest possible lower ramification numbers. We also give examples and references to important related results. In §2 we introduce a method that will be used frequently throughout the thesis.

In §3 we prove our main results Theorem A and Theorem B. In §4 we study consequences of our results from §3 and relate them to the problem posed by Lindahl and Rivera-Letelier [15, Problem 1.4]. In §5 we briefly discuss the connection between minimally ramified power series and dynamical systems. In §6 we discuss the results and their implications. We also state a couple of conjectures based on results obtained from computer experiments that we performed using SAGE.

1.1. Preliminaries. Let $R$ be a ring. Given an element $a \in R$ we let $\langle a \rangle$ denote the ideal of $R$ generated by $a$. Let $R[z]$ and $R[[z]]$ denote the ring of polynomials and the ring of power series with coefficients in $R$ respectively. We denote by char($R$) the characteristic of $R$. The characteristic of a ring $R$ is the smallest integer $n \geq 1$ such that $n \cdot 1_R = 1_R + \cdots + 1_R = 0_R$, where $1_R$ and $0_R$ denotes the multiplicative unity, and the additive identity of $R$ respectively. Note that if $R$ is a ring of characteristic $b$, then $R[z]$ and $R[[z]]$ are rings of characteristic $b$, i.e. the characteristics is inherited from the coefficient ring.

Moreover, in this thesis we will be working in fields. Recall that a field is a commutative ring where every nonzero element has a multiplicative inverse. The most common examples of fields are $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. These are all of characteristic 0.

In this thesis we will mostly work with fields and rings of prime characteristic. Let $p$ be a prime number and $k$ a field of characteristic $p$. The \textit{order} of a nonzero power series $g \in k[[z]]$ is the lowest degree non-zero term of $g$ and we denote this by ord($\cdot$). For the power series 0 put ord(0) := $+ \infty$. Defined in this way ord($\cdot$) is a valuation in $k$.

There are many different fields of characteristic $p$ where the most commonly known contains exactly $p$ elements $\{0, 1, \ldots, p-1\}$ (or is isomorphic to this field) [3, §6.5]. We will denote this field by $\mathbb{F}_p$. For further information regarding fields we refer to [4, 23]. Below we will present some examples of standard constructions of fields of characteristic $p$ that contain more elements than $\mathbb{F}_p$.

\textbf{Example 1.} An example of a field of characteristic $p$ is the field of all rational functions with coefficients in $\mathbb{F}_p$, 

\[ \mathbb{F}_p(t) := \left\{ \frac{f}{g} : f, g \in \mathbb{F}_p[t], g \neq 0 \right\}. \]

$\mathbb{F}_p(t)$ is a field since we have multiplicative inverses for all elements $h \in \mathbb{F}_p(t)$ except for $h = 0$. Since all coefficients of $\mathbb{F}_p(t)$ comes from $\mathbb{F}_p$ we can see that the characteristic of $\mathbb{F}_p(t)$ is $p$. 

Example 2. Let \( k \) be a field of prime characteristic \( p \). Then, the field of formal Laurent series \( k((t)) \) is defined by

\[
k((t)) := \left\{ \sum_{i=-\infty}^{\infty} a_i t^i : a_i \in k, j \in \mathbb{Z}, a_j \neq 0 \right\}.
\]

Clearly \( k((t)) \) is of characteristic \( p \). The fact that this is actually a field is not trivial. Though it is not hard to see that it fulfills the demands to be a commutative ring with unity, it is more difficult to see that every nonzero element has a multiplicative inverse. Let \( f \) be a series in \( k((t)) \) of the form

\[
f(t) = a_j t^j + a_{j+1} t^{j+1} + \cdots + a_0 + a_1 t + \cdots = a_j t^j + \sum_{i=j+1}^{\infty} a_i t^i,
\]

then there exists a series \( g(t) \in k((t)) \) such that \( f(t) = t^j g(t) \), with \( g(t) = \sum_{i=0}^{\infty} a_i t^i \). This means that \( g \) is a formal power series. From [17, Theorem 1] we have that formal power series are invertible iff \( a_0 \) is invertible, which is true for this case since \( k \) is a field. Then we can construct our inverse for \( f(t) \) as the series \( f^{-1}(t) = t^{-j} g^{-1}(t) \), which yields \( f(t)f^{-1}(t) = t^j g(t)t^{-j} g^{-1}(t) = 1 \) as required.

Example 3. Given a field \( K \) we can endow \( K \) with an absolute value \( | \cdot | \). That means that we in some sense can measure how “close” elements in \( K \) are. Such a field is called a valued field and is usually denoted \((K, | \cdot |)\). An absolute value on a field \( K \) is a function which maps elements from \( K \) onto the positive real line, i.e. \( | \cdot | : K \rightarrow \mathbb{R}_+ \). It also has to fulfill the following requirements

i. \(|z| = 0\), iff \( z = 0 \)
ii. \(|z_1 z_2| = |z_1||z_2|\) for all \( z_1, z_2 \in K \)
iii. \(|z_1 + z_2| \leq |z_1| + |z_2|\) for all \( z_1, z_2 \in K \)

If also the following holds then we say that the absolute value is non-Archimedean

iv. \(|z_1 + z_2| \leq \max\{|z_1|, |z_2|\}\) for all \( z_1, z_2 \in K \).

Note that if (iv) holds then (iii) also holds because (iv) is a stronger statement.

Consider the field of formal Laurent series with coefficients in \( k \) mentioned in the previous example. Given an element \( f(t) = \sum_{i=-\infty}^{\infty} a_i t^i \in k((t)) \) we can define an absolute value \( | \cdot | \) in the following way. Choose a real number \( \epsilon \) such that \( 0 < \epsilon < 1 \) and let \( \text{ord}(f) \) denote the order of the Laurent series \( f \), then we put \( |f| := \epsilon^{\text{ord}(f)} \). In this way the \( \text{ord}(\cdot) \) function naturally induces an absolute value on \( k((t)) \).

We can see that all of the above mentioned statements hold for this absolute value. If we consider the \( 0 \) power series we have from the definition that \( \text{ord}(0) = +\infty \) which yields \(|0| = \epsilon^{+\infty} = 0 \) since \( |\epsilon| < 1 \). For all other \( f \) we have that \(|f| > 0\).

To show that (ii) holds we assume that we have two different Laurent series \( f \) and \( g \) with orders \( n \) and \( m \) respectively, which means that \(|f| = \epsilon^n \) and \(|g| = \epsilon^m \), ergo
\[ |f| \cdot |g| = \varepsilon^n \cdot \varepsilon^m = \varepsilon^{n+m} \text{.} \] We also have that

\[ f \cdot g = \sum_{i=n}^{\infty} a_i t^i \sum_{j=m}^{\infty} a_j t^j = \left( a_n t^n + \sum_{i=n+1}^{\infty} a_i t^i \right) \left( a_m t^m + \sum_{j=m+1}^{\infty} a_j t^j \right) = a_n a_m t^{n+m} + \ldots, \]

since \( k \) is a field this means that \( a_n a_m \neq 0 \), and this implies that \( |f \cdot g| = \varepsilon^{n+m} \) as required.

To show that (iv) holds we have \( f \) and \( g \) being formal Laurent series with orders \( n \) and \( m \) defined as before, where we can assume, without loss of generality, that \( n > m \). This means that \( |f| < |g| \), i.e. \( \max\{|f|, |g|\} = |g| = \varepsilon^m \). If we then study the valuation of the sum of \( f \) and \( g \) then we have that

\[ |f + g| = \left| \sum_{i=n}^{\infty} a_i t^i + \sum_{j=m}^{\infty} a_j t^j \right| = \left| \sum_{k=m}^{\infty} b_k t^k \right| = \varepsilon^m. \]

However if we have equality with \( n = m \) it might be the case that their sum of the coefficients for the lowest degree term cancels out, which implies that the valuation of the sum would be less than \( \varepsilon^n = \varepsilon^m \).

For more information on non-Archimedean fields we refer to [8, § 2].

1.2. Lower ramification numbers. In the introduction we briefly touched upon the concept of ramification numbers. In this section we will discuss examples and important results on the theory of ramification numbers. At the end of this section we introduce the, for this thesis, central concept of \( b \)-ramified as well as minimally ramified power series.

We first note the following lemma which shows that for fields of prime characteristic the interesting ramification numbers are those that corresponds to iterations that are powers of the characteristic \( p \).

**Lemma 1.** Let \( p \) be a prime, and \( k \) be a field of characteristic \( p \). Let \( m \) and \( n \) be integers such that \( m \in \{1, \ldots, p - 1\} \) and \( n \geq 0 \), and let \( g \) be a power series of the form \( g(\zeta) = \zeta + \sum_{i=1}^{\infty} a_i \zeta^{i+1} \in k[[\zeta]] \), then we have that

\[
\text{ord}(g^{p^n}(\zeta) - \zeta) = \text{ord}(g^{m p^n}(\zeta) - \zeta). \tag{1.2}
\]

**Proof.** We assume that the first non-zero term of the power series \( g^n(\zeta) - \zeta \) is \( a_i \zeta^i \), this means that \( \text{ord}(g^n(\zeta) - \zeta) = i \). We want to show that \( g^{m p^n}(\zeta) - \zeta \) has the same order if \( m < p \).

We start by studying \( g^{2p^n}(\zeta) \mod (\zeta^{i+1}) \). Since higher order terms do not affect the order those are neglected. Then we have that

\[
g^{2p^n}(\zeta) = (\zeta + a_i \zeta^i + \ldots) + a_i (\zeta + a_i \zeta^i + \ldots)^i + \cdots \equiv \zeta + 2 a_i \zeta^i \mod (\zeta^{i+1}).
\]
We proceed by induction in \( m \), then \( g^{mp^n}(\zeta) = \zeta + ma_i\zeta^i \) mod \( \langle \zeta^{i+1} \rangle \) is assumed to hold for some \( m \geq 1 \), then
\[
g^{(m+1)p^n}(\zeta) = g^{p^n}(g^{mp^n}(\zeta)) = g^{p^n}(\zeta + ma_i\zeta^i \ldots) = \zeta + ma_i\zeta^i + \cdots + a_i(\zeta + ma_i\zeta^i + \ldots)^i + \cdots \\
\equiv \zeta + ma_i\zeta^i + a_i\zeta^i \\
\equiv \zeta + (m+1)a_i\zeta^i \mod \langle \zeta^{i+1} \rangle.
\]
Since we are in characteristic \( p \), the coefficient for \( \zeta^i \) will be non-zero until we iterate \( p \) times, which means that (1.2) holds if \( m < p \).

The given lemma motivates the following definition.

**Definition 1.** Let \( p \) be a prime and \( k \) a field of characteristic \( p \), and let \( g \in k[[\zeta]] \) be a power series of the form
\[
g(\zeta) = \zeta + \sum_{i=2}^{\infty} a_i\zeta^i.
\]
Then we let
\[
i_n(g) := \text{ord} \left( \frac{g^{p^n}(\zeta) - \zeta}{\zeta} \right).
\]

Note that this means that \( i_n(g) < \infty \) if and only if \( g^{p^n}(\zeta) \neq \zeta \). Now we proceed with an example where we can find \( i_n(g) \) for an arbitrarily chosen \( n \).

**Example 4.** Let \( p \) be a prime and \( k \) a field of characteristic \( p \). Let \( P(\zeta) = \zeta + \zeta^p \in k[\zeta] \).
Then \( i_n(P) = p^{p^n} - 1 \).

**Proof.** We start by computing \( P^2(\zeta) \). This yields
\[
P^2(\zeta) = \zeta + \zeta^p + (\zeta + \zeta^p)^p = \zeta + 2\zeta^p + \zeta^{p^2}.
\]
Note that for a field \( k \) with positive characteristic \( p \) with elements \( a, b \) we have that \( (a + b)^p = ap + bp \), for all \( a, b \in k \). For the next iteration we have that
\[
P^3(\zeta) = \zeta + \zeta^p + 2(\zeta + \zeta^p)^p + (\zeta + \zeta^p)^{p^2} = \zeta + 3\zeta^p + 3\zeta^{p^2} + \zeta^{p^3}.
\]
In fact, by induction we obtain that for integers \( m \geq 1 \) we have \( P^m(\zeta) = \sum_{i=0}^{m} \binom{m}{i} \zeta^{p^i} \). If we let \( m = p^n \), then all coefficients except for \( i = 0 \) and \( i = p^n \) will vanish due to the characteristic \( p \), and we have that \( P^{p^n}(\zeta) - \zeta = \zeta^{p^{p^n}} \). Therefore \( i_n(P) = p^{p^n} - 1 \), for an arbitrarily chosen \( n \geq 0 \).

Note that \( i_n(P) - i_{n-1}(P) = p^{p^n} - p^{p^{n-1}} \). In particular \( p^n \) divides \( i_n(P) - i_{n-1}(P) \) in this case. This is not a coincidence, in fact, by a well-known theorem of Sen [21] often referred to as Sen’s theorem for every power series \( g \), such that \( g^n \neq \text{Id} \) for all \( n \geq 1 \), we have that
\[
i_n(g) \equiv i_{n-1}(g) \pmod{p^n}.
\]
Remark 1. The assumption that \( g^n \neq \text{Id} \) for all \( n \geq 1 \) is crucial. For example for the power series \( g(\zeta) = \frac{\zeta}{1-\zeta} = \zeta + \zeta^2 + \zeta^3 + \ldots \), we have \( g^2(\zeta) = \zeta \), i.e. \( \text{ord}(g^2(\zeta)) = +\infty \).

Remark 2. Power series of the form
\[
g(\zeta) = \zeta + \ldots \text{ in } k[[\zeta]]
\]
such that \( g^n(\zeta) \neq \text{Id} \) for all integers \( n \geq 1 \) are often referred to as wildly ramified power series. Since by Sen’s theorem the \( p \)th iterate of such \( g \) are related to wildly ramified extensions of \( k((t)) \), see e.g. [9].

By Example 4 we can see that for some cases finding the ramification number for all possible prime powers is easy, but as we will see this does not hold for a generic power series of the form \( g(\zeta) = \zeta + \ldots \text{ in } k[[\zeta]] \). However, important information about the possible sequences of lower ramification numbers is given by the following theorem of Laubie and Saïne [12].

**Theorem 1.** [12, Corollary 1] Let \( p \) be a prime and let \( k \) be a field and \( \text{char}(k) = p \). Furthermore let \( g \in k[[\zeta]] \), be a power series of the form
\[
g(\zeta) = \zeta + \sum_{i=1}^{\infty} a_i \zeta^{i+1},
\]
and let \( i_0(g) \) and \( i_1(g) \) be integers defined as in (1.3). Then the following statements hold

i. If \( p \mid i_0(g) \) then \( i_n(g) = p^n i_0 \) for all integers \( n \geq 0 \).

ii. However if \( p \nmid i_0(g) \) and \( i_1(g) < (p^2 - p + 1)i_0(g) \) then
\[
i_n(g) = i_0 + \frac{p^n - 1}{p - 1} (i_1(g) - i_0(g)),
\]
for all integers \( n \geq 0 \).

This result is a generalization of a theorem by Keating [10, Theorem 7]. By statement 1 of Theorem 1, if we know \( i_0(g) \) and that \( p \mid i_0(g) \), then we automatically have the ramification numbers for arbitrary \( n \geq 1 \).

**Example 5.** Let \( p \) be a prime and \( k \) a field of characteristic \( p \). Let \( f(\zeta) = \zeta + \zeta^{p+1} + \ldots \in k[[\zeta]] \). Then \( i_n(f) = p^{n+1} \).

**Proof.** Note that,
\[
i_0(f) = \text{ord} \left( \frac{f(\zeta) - \zeta}{\zeta} \right) = p,
\]
and from statement 1 of Theorem 1 we have \( i_n(f) = p^n i_0 \) which for this case yields \( i_n(f) = p^{n+1} \).

In this thesis we will focus on series that satisfy statement 2 of Theorem 1. In this case \( p \nmid i_0 \) and the problem is more complex.

For example, if we fix \( p = 5 \) and study two very similar polynomials \( \zeta + \zeta^2 \) and \( \zeta + \zeta^2 + \zeta^3 \), we can see that \( i_0(\zeta + \zeta^2) = i_0(\zeta + \zeta^2 + \zeta^3) = 1 \). However, after \( p \) iterations
we get $i_1(\zeta + \zeta^2) = 6$ and $i_1(\zeta + \zeta^2 + \zeta^3) = 11$. Differences can also arise for the same polynomial for different primes. This is the case for $\zeta + \zeta^3$, we have $i_1(\zeta + \zeta^3) = 26$ for $p = 3$ and $i_1(\zeta + \zeta^3) = 12$ for $p = 5$.

Of course, given a specific prime $p$ and power series $f$, the ramification number $i_1(f)$ can be computed explicitly by a computer program. However, to find $i_1$ for even a simple polynomial as $\zeta + \zeta^3$, we have $i_1(\zeta + \zeta^3) = 26$ for $p = 3$ and $i_1(\zeta + \zeta^3) = 12$ for $p = 5$.

Remark 3. Note that if $g$ is defined as in the previous definition and if $i_0(g) = b$ and $i_1(g) = b(1 + p)$, then by Theorem 1 $g$ is $b$-ramified. As $p$ then of course does not divide $i_0(g)$ and $i_1(g) < (p^2 - p + 1)i_0(g)$. This means that we have

$$i_n(g) = i_0(g) + \frac{p^n - 1}{p - 1} (i_1(g) - i_0(g)) = b + bp^n - 1 - b \frac{p^n - 1}{p - 1},$$

which implies that $g$ is $b$-ramified.

By letting $f \in k[[\zeta]]$ be a power series of the form

$$f(\zeta) = \gamma \zeta + \ldots,$$

The following proposition is given by Lindahl and Riveria-Letelier [15], which motivate an important notion related to $b$-ramification.

**Proposition 1.** [15, Proposition 3.2] Let $p$ be a prime and $k$ be a field of characteristic $p$ and $\gamma$ in $k$, with $q$ such that $\gamma^q = 1$, then for every power series $f(\zeta) = \gamma \zeta + \ldots$ in $k[[\zeta]]$ and every integer $n \geq 0$, we have

$$(1.5) \quad i_n(f^q) \geq q \frac{p^{n+1} - 1}{p - 1}.$$  

If $p$ is odd and equality holds for some $n \geq 1$ then it holds for every $n \geq 0$.

This proposition motivates the following definition.

**Definition 3.** Let $p$ be a prime and $k$ a field of characteristic $p$. Furthermore let $\gamma \in k$ be a root of unity with order $q$ and $f$ be a power series of the form

$$f(\zeta) = \gamma \zeta + \ldots$$

If we have equality in (1.5) for every $n \geq 0$ we say that $f$ is minimally ramified.\[\text{ii}\]

\[\text{ii}\]The notion of minimal ramification was introduced by Laubie et. al. in [11]
The notion of minimally ramified power series is a key aspect of this thesis and will be used throughout this work. Note that minimal ramification for power series with $\gamma = 1$, corresponds to 1-ramification. In the first part of this thesis we will consider power series with $\gamma = 1$ and later we extend our results of such power series to quadratic polynomials with arbitrarily chosen $\gamma$.

Remark 4. Note that for a field $k$ of prime characteristic $p$ and a root of unity $\gamma \in k$ of order $q$ and power series of the form

$$f(\zeta) = \gamma \zeta + \ldots,$$

the notion of minimally ramified relates to the notion of $b$-ramified for the power series mentioned in Definition 2 in the sense that $b$-ramification for power series with $\gamma = 1$ and minimal ramification of $f$ might have the same appearance if $q = b$. In fact, $f$ of the form (1.6) is minimally ramified if and only if $f^q$ is $b$-ramified. This is an important remark for this study in the sense that it motivates the study of $b$-ramified power series, when our main problems relates to power series of the form (1.6).

2. The $\Delta$-method

In this section we will give two lemmas that we will use frequently in our investigation of lower ramification numbers of power series.

A main ingredient in the proving of the theorems and lemmas in the upcoming sections is the $\Delta$-recursion used in [15] and in [18, Exemple 3.19]. Instead of studying the power series $g^p(\zeta) - \zeta$ for each integer $m \geq 1$ we define the power series $\Delta_m(\zeta)$ inductively by

$$\Delta_1(\zeta) := g(\zeta) - \zeta$$

and for $m \geq 2$ we let

$$\Delta_m(\zeta) := \Delta_{m-1}(g(\zeta)) - \Delta_{m-1}(\zeta).$$

The next lemma shows why we are interested in this recurrence relation.

Lemma 2. Let $p$ be a prime, and $k$ a field of characteristic $p$. Let $g = \zeta + \sum_{i=1}^{\infty} a_i \zeta^{i+1} \in k[[\zeta]]$. Consider the $\Delta$-recursion defined in (2.1), then we have the following

$$\Delta_p(\zeta) = g^p(\zeta) - \zeta.$$

By proving this lemma we can then look at the coefficients for $\Delta_p(\zeta)$ instead of $g^p(\zeta) - \zeta$, which turns out to be easier.

Proof of Lemma 2. We will use proof by induction to show this lemma. We will start by looking at $\Delta_2(\zeta)$,

$$\Delta_2(\zeta) = \Delta_1(g(\zeta)) - \Delta_1(\zeta)$$

$$= g(g(\zeta)) - g(\zeta) - (g(\zeta) - \zeta)$$

$$= g^2(\zeta) - 2g(\zeta) + \zeta$$

This is in fact the binomial expansion so we assume that for some $m \geq 1$ we have that

$$\Delta_m(\zeta) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} g^{m-i}(\zeta),$$
where we define $g^0(\zeta) := \zeta$, then we have that

$$\Delta_{m+1}(\zeta) = \Delta_m(g(\zeta)) - \Delta_m(\zeta)$$

$$= \sum_{i=0}^{m} (-1)^i \binom{m}{i} \sum_{j=0}^{m} (-1)^j \binom{m}{j} g^{m-j}(\zeta)$$

$$= \sum_{i=0}^{m} (-1)^i \binom{m}{i} g^{m-i+1}(\zeta) - \sum_{j=0}^{m} (-1)^j \binom{m}{j} g^{m-j}(\zeta)$$

$$= \sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} g^{m-i+1}(\zeta),$$

the last step comes from that we match every $i - 1 = j$ yielding for some $i$

$$\left((-1)^i \binom{m}{i}\right) g^{m-i+1}(\zeta) - \left((-1)^{i-1} \binom{m}{i-1}\right) g^{m-i+1}(\zeta) = (-1)^i g^{m-i+1}(\zeta) \left((\frac{m}{i}) - (-1) \binom{m}{i - 1}\right)$$

$$= (-1)^i g^{m-i+1}(\zeta) \left((\frac{m}{i}) + \binom{m}{i - 1}\right)$$

$$= (-1)^i g^{m-i+1}(\zeta) \left(\frac{m+1}{i}\right).$$

This covers all the cases except for $i = 0$ and $j = m$, summing up all these cases yields

$$\sum_{i=1}^{m} (-1)^i \binom{m+1}{i} g^{m-i+1}(\zeta) + g^{m+1}(\zeta) - (-1)^m = \sum_{i=1}^{m} (-1)^i \binom{m+1}{i} g^{m-i+1}(\zeta)$$

$$+ g^{m+1}(\zeta) + (-1)^m$$

$$= \sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} g^{m-i+1}(\zeta),$$

which proves that our formula holds. This means that if $m = p$, we have that $\Delta_p(\zeta) = g^p(\zeta) - \zeta$, since all other terms will be 0 due to the characteristic $p$. \qed

Since we did not have any restriction for $m$ in this lemma, we can extend our result and we have the following corollary.

**Corollary 1.** Let $p$ be a prime, and $k$ a field of characteristic $p$. Let $g(\zeta) = \zeta + \sum_{i=1}^{\infty} a_i \zeta^{i+1} \in k[[\zeta]]$. Consider the $\Delta$-recurrence relation defined in (2.1), then we have the following

$$\Delta_p^n(\zeta) = g^n p(\zeta) - \zeta.$$

**Proof.** By studying the proof from Lemma 2, we can see that since $m$ was chosen arbitrarily we can also choose it as $p^n$. This yields that

$$\Delta_p^n(\zeta) = \sum_{i=0}^{p^n} (-1)^i \binom{p^n}{i} g^{p^n-i}(\zeta),$$
for characteristic $p$ all terms will be 0 except for $i = 0$ and $i = p^n$. We can see this if we look at the $j$th term in the sum

$$(-1)^j \binom{p^n}{j} g^{p^n-j}(\zeta) = (-1)^j p^{p^n-1}(p^n-1)! \frac{1}{j!(p^n-j)!} g^{p^n-j}(\zeta),$$

and if $\rho_j = \frac{p^n-1!(p^n-1)!}{j!(p^n-j)!}$ is an integer, then we are done since all $j$ terms will be congruent 0 modulo $p$.

We need just to motivate why $\rho_j$ is an integer. If we study the number of $p$ in the numerator and denominator, denoted as $P_n$ and $P_d$ respectively. Note that $P_n < P_d$ implies that $\rho_j$ is not an integer. The number of $p$ in the numerator is $P_n = \frac{p^n-1}{p^n-1} - p$. For the denominator we have that $P_d \leq \frac{p^n-1}{p^n-1} - p$, and since $P_d$ will be largest when $j = 1$ and $j = p^n - 1$, and then we get equality otherwise we will have more $p$ in the numerator and therefore $\rho_j$ must be an integer.

So we can focus on looking at $\Delta_p(\zeta)$ and the coefficients of that. We are interested in the coefficients of $i_1(g)$, which means that we have to find the coefficients of $\Delta_p(\zeta)$. We show some computations using an example, with a polynomial $P_1(\zeta) = \zeta + \zeta^2$.

**Example 6.** Let $P_1$ be the polynomial, $P_1(\zeta) = \zeta + \zeta^2$. Using the $\Delta$-recursion from (2.1) we want to compute $\Delta_3(\zeta)$.

**Solution.** We use the recurrence relation to compute $\Delta_2$ and then we proceed with $\Delta_3$. Note that $\Delta_1(\zeta) = P_1(\zeta) - \zeta = \zeta^2$. We get the following

$$\Delta_2(\zeta) = (\zeta + \zeta^2)^2 - \zeta^2 = 2\zeta^3 + \zeta^4.$$

We use this to compute $\Delta_3$, which yields

$$\Delta_3(\zeta) = 2(\zeta + \zeta^2)^3 + (\zeta + \zeta^2)^4 - (2\zeta^3 + \zeta^4)$$

$$= 6\zeta^4 + 10\zeta^5 + 8\zeta^6 + 4\zeta^7 + \zeta^8$$

$$\equiv 6\zeta^4 + 10\zeta^5 \mod (\zeta^6),$$

and this shows the method.

The reason why only two coefficients is kept is due to the fact that if and only if the second term of these two is nonzero $P_1$ will be minimally ramified. This will be explained in the next section.

3. **Classification of lower ramification numbers of power series**

In this section we state and prove our main theorems. We will discuss certain classes of polynomials and power series where we can show that these mappings are $b$-ramified. We give a classification for both 1- and 2-ramified power series, of the form $q(\zeta) = \zeta + \ldots$. However, for the case of 2-ramified power series we need to add some assumptions for $g$ in our proof.
3.1. Ramification numbers of the quadratic polynomial $\zeta + \zeta^2$. In this section we investigate the polynomial $P_1(\zeta) = \zeta + \zeta^2 \in k[\zeta]$. We examine the order of this polynomial after $p$ iterations. It has been shown in [15] that this polynomial is in fact minimally ramified, but to see the method from the previous section at work we will give a full proof of this result. The technique that we use will be used to prove our main theorems as well.

**Proposition 2.** Let $p$ be a prime, $k$ a field of characteristic $p$. Let $P_1 \in k[\zeta]$ be the polynomial

$$P_1(\zeta) = \zeta + \zeta^2.$$  

Then, $P_1$ is minimally ramified for all primes $p$.

**Proof.** Note that in view of Theorem 1 $i_1(P_1) = 1 + p$ would imply minimal ramification.

The $\Delta$-relation in (2.1) is a key ingredient here and for $m \geq 1$ we let $\Delta_1(\zeta) := P_1(\zeta) - \zeta$, and for $m \geq 2$ we put $\Delta_m(\zeta) = \Delta_{m-1}(P_1(\zeta)) - \Delta_{m-1}(\zeta)$. We prove that for any $m \in \{1, \ldots, p\}$ we have that

$$\Delta_m(\zeta) = mL_m^m + C_{\Delta_m} \zeta^{m+2} + \ldots,$$

where

$$C_{\Delta_m} = (m-1)! \left(\frac{m}{2}\right) + (m+1)C_{\Delta_{m-1}}, \quad C_{\Delta_1} = 1.$$

By insertion we can see by comparison of Example 6 that it holds for $m = 1, 2, 3$. We continue by induction in $m$ and assume that it holds for some integer $m \geq 1$. Then

$$\Delta_{m+1}(\zeta) = \Delta_m(\zeta) - \Delta_{m}(\zeta)$$

$$= mL_m^m + C_{\Delta_m}(\zeta) + C_{\Delta_m}m^2 + \ldots - (mL_m^m + C_{\Delta_m}m^2)$$

$$= mL_m^m + C_{\Delta_m}m^2 + \ldots$$

$$= (m+1)!L_m^{m+2} + \ldots$$

This concludes the induction step.

This gives us a way to describe the coefficient $C_{\Delta_m}$ using the recurrence relation in (3.2). Now we will look at the case where $m = p$. By insertion in (3.2) we have that

$$C_{\Delta_p} = (p-1)! \left(\frac{p}{2}\right) + (p+1)C_{\Delta_{p-1}} = C_{\Delta_{p-1}}.$$
Therefore

\[ C_{\Delta_p} = C_{\Delta_{p-1}} = (p-2)! \frac{(p-1)(p-2)}{2} + pC_{\Delta_{p-2}} = (p-2)! \frac{(-1)(-2)}{2} = (p-2)! = 1, \]

which shows that the coefficient of the term with degree \( m + 2 \) in (3.1) is non-zero for all \( p \), which means that

\[ \frac{P^p_1(\zeta) - \zeta}{\zeta} = \zeta^{p+1} + \sum_{i=p+2}^{\infty} a_i z^i. \]

This means that \( i_1(P_1) = 1 + p \) and by Theorem 1 and Remark 3 this shows that \( P_1(\zeta) \) is minimally ramified.

If we study the polynomial in the previous example we can see that a small perturbation \( \tilde{P}_1(\zeta) = \zeta + a\zeta^2 \), wouldn’t change the order of the polynomial, if \( a \neq 0 \). We have that \( \tilde{P}_1^p(\zeta) - \zeta = a^{p+1} \zeta^{p+2} + \ldots \), and of course \( a^n \neq 0 \), for any integer \( n \), because no nilpotent elements exist in fields.

3.2. Classification of 1-ramified power series. In this section we classify all 1-ramified power series of \( g \) of the form

\[ g(\zeta) = \zeta + \cdots \in k[[\zeta]]. \]

More precisely we prove the following theorem which is a generalization of Proposition 2.

**Theorem A.** Let \( p \) be a prime, \( k \) a field of characteristic \( p \), and \( h \) be a power series of the form

\[ h(\zeta) = \zeta(1 + a_1 \zeta + a_2 \zeta^2) + \sum_{i=3}^{\infty} a_i \zeta^{i+1} \in k[[\zeta]]. \]

The power series \( h \) is minimally ramified if and only if \( a_1 \neq 0 \) and \( a_1^2 \neq a_2 \).

This result is known by Keating [10] for the special case that \( p = 3 \). Theorem A was also proven by Rivera-Letelier [18, Example 3.19] for the case \( k = \mathbb{F}_p \) and \( a_1 = 1 \). Theorem A can be proven using [18, Example 3.19]. However, for completeness we here give a full proof.

To prove Theorem A as in the proof of Proposition 2, we first use the \( \Delta \)-relation from section 2 to find \( i_1(h) \) and then apply Theorem 1.

**Proof of Theorem A.** In view of Remark 3, it is sufficient to prove that \( i_1(h) = 1 + p \). Analogous to Proposition 2 for each integer \( m \geq 1 \) we define the power series \( \Delta_m(\zeta) \) in \( k[[\zeta]] \) inductively by \( \Delta_1(\zeta) = h(\zeta) - \zeta = a_1 \zeta^2 + a_2 \zeta^3 \) and for \( m \geq 2 \) by

\[ \Delta_m(\zeta) := \Delta_{m-1}(h(\zeta)) - \Delta_{m-1}(\zeta). \]
Note that Lemma 2 holds and \( \Delta_p(\zeta) = h^p(\zeta) - \zeta \). We will prove that for any \( m \in \{1, \ldots, p\} \) we have that
\[
\Delta_m(\zeta) = a_1^{m-1}m!\zeta^{m+1} + \tilde{C}_{\Delta_m}\zeta^{m+2} + \ldots,
\]
where
\[
\tilde{C}_{\Delta_m} = a_1^{m-1}(m-1)!a_1^2\left(\frac{m}{2}\right) + a_1m(m+1)\tilde{C}_{\Delta_{m-1}}, \quad \tilde{C}_{\Delta_1} = a_2.
\]

We proceed by induction. For \( m = 1 \) (3.3) and (3.4) holds by definition. Assume that it holds for some integer \( m \geq 1 \). Now we show that it holds for \( m + 1 \), this yields
\[
\Delta_{m+1}(\zeta) = \Delta_m(h(\zeta)) - \Delta_m(\zeta) \equiv a_1^{m+1}(m+1)!\zeta^{m+2} + a_1m+1\left(\frac{m+1}{2}\right)\zeta^{m+3} + a_1m+1a_2\zeta^{m+2} + \ldots - \Delta_m(\zeta) \equiv a_1^{m+1}(m+1)!\zeta^{m+2} + a_1m+1\left(\frac{m+1}{2}\right)\zeta^{m+3} + a_1m+1a_2\zeta^{m+2} + \ldots.
\]

this shows that the coefficient of the term of degree \( m + 2 \) in (3.3) is given by the recurrence relation (3.4). Hence we want to find the coefficient \( \tilde{C}_{\Delta_p} \) (note that the coefficient \( a_1^m m! \) is zero after \( p \) iterations). One way to find the solution would be to solve the non-homogenous difference equation (3.4), but this is not necessary for this case. By insertion in (3.4) we get the following
\[
\tilde{C}_{\Delta_p} = a_1^{p-1}(p-1)!\left(a_1^p\left(\frac{p}{2}\right) + a_2p\right) + a_1(p + 1)\tilde{C}_{\Delta_{p-1}} = a_1\tilde{C}_{\Delta_{p-1}}.
\]

Consequently,
\[
\tilde{C}_{\Delta_p} = a_1\tilde{C}_{\Delta_{p-1}} = a_1\left(a_1^{p-2}(p-2)!\left(a_1^p\left(\frac{p-1}{2}\right) + (p-1)a_2\right) + a_1(p - 1 + 1)\tilde{C}_{\Delta_{p-2}}\right) = a_1^{p-1}(p-2)!\left(a_1^p\left(\frac{p-1}{2}\right) + (p-1)a_2\right).
\]

Clearly for \( \tilde{C}_{\Delta_p} \) to be non-zero it is necessary that \( a_1 \neq 0 \). Furthermore, by Wilson’s theorem [3, § 6.5, Theorem 6.5.1] we have \( (p-1)! = -1 \), in a field of characteristic \( p \). Together with the fact that
\[
\left(\frac{p-1}{2}\right) = \frac{(p-1)(p-2)}{2} = \frac{(-1)(-2)}{2} = 1,
\]
we obtain that
\[
\tilde{C}_{\Delta_p} = a_1^{p-1}(a_1^2 - a_2).
\]
We conclude that $C_{\Delta_p} \neq 0$ if and only if $a_1 \neq 0$ and $a_1^2 \neq a_2$. Together with (3.3) and Lemma 2 this proves our theorem. \hfill $\square$

3.3. Classification of 2-ramified power series. In this section we will discuss a new class of power series with no quadratic term. We give a classification of all 2-ramified maps within this class. It turns out there is a connection between the quartic polynomial $Q(\zeta) = \zeta + a_3\zeta^3 + b\zeta^4$ and minimally ramified maps of the form $P_\gamma = \gamma\zeta + \zeta^2$. In the next section we use this classification for the power series studied here to study the polynomial $P_{\gamma}$. Now we prove the following main result.

**Theorem B.** Let $p$ be a prime and let $k$ be a field of characteristic $p$. Let $q$ be a power series of the form

$$q(\zeta) = \zeta + a_3\zeta^3 + a_4\zeta^4 + a_6\zeta^6 + \sum_{i=7}^{\infty} a_i\zeta^i \in k[[\zeta]].$$

Then $q$ is 2-ramified if and only if $a_3 \neq 0$ and $\frac{3a_3^2 + 2a_4^2}{2} \neq 0$.

The proof of this theorem comes in the end of this section. Apart from the theorem of Laubie and Saïne [12], the main ingredient of the proof is the following proposition.

**Proposition 3.** Let $p$ be a prime and $k$ field of characteristic $p$. Let $q$ be a power series of the form

$$q(\zeta) = \zeta + a_3\zeta^3 + a_4\zeta^4 + a_6\zeta^6 + \sum_{i=7}^{\infty} a_i\zeta^i \in k[[\zeta]],$$

then

$$q^p(\zeta) - \zeta \equiv a_3^{p-2} \left( \frac{3a_3^2 + 2a_4^2}{2} \right) \zeta^{3+2p} \pmod{\zeta^{4+2p}}.$$

The proof of Proposition 3 comes after the following three lemmas.

**Lemma 3.** Let $p$ be a prime, and $k$ a field of characteristic $p$, then

$$\left( \frac{2p+1}{p} \right)! = -1.$$

**Proof.** By definition

$$\left( \frac{2p+1}{p} \right)! = \frac{(2p+1)(2p+1)(2p-3) \cdots (p+2)(p-2) \cdots 3 \cdot 1}{p}$$

$$= \frac{(p+1)(p-1)(p-3) \cdots 2(p-2) \cdots 3 \cdot 1}{p}$$

$$= (p+1)(p-1)(p-3) \cdots 2(p-2) \cdots 3 \cdot 1.$$

By rearranging the numbers in the latter product we see by Wilson’s theorem that in fact

$$\left( \frac{2p+1}{p} \right)! = (p+1)(p-1)! = (p-1)! = -1,$$

as required. \hfill $\square$
**Lemma 4.** The following identity holds for all \( n \geq 1 \)

\[
\sum_{j=1}^{n} \frac{(2j)!!}{(2j+1)!!} = \frac{(2n+2)!! - 2(2n+1)!!}{(2n)!!}.
\]

**Proof.** Direct computation shows that it holds for \( n = 1 \). On the right hand side

\[
\frac{4!! - 2 \cdot 3!!}{3!!} = \frac{8 - 6}{3} = \frac{2}{3},
\]

which is equal to left hand side of (3.6).

We proceed by induction in \( n \). Assume that the lemma holds for \( n = 1 \). Then

\[
\sum_{j=1}^{n+1} \frac{(2j)!!}{(2j+1)!!} = \frac{(2n+2)!!}{(2n+3)!!} + \sum_{j=1}^{n} \frac{(2j)!!}{(2j+1)!!}
\]

\[
\overset{\text{IA}}{=} \frac{(2n+2)!!}{(2n+3)!!} + \frac{(2n+2)!! - 2(2n+1)!!}{(2n+1)!!}
\]

\[
= \frac{(2n+3)(2n+2)!! - 2(2n+3)!!}{(2n+3)!!}
\]

\[
= \frac{(2n+4)!! - 2(2n+3)!!}{(2n+3)!!}
\]

\[
= \frac{(2n+1+2)!! - 2(2n+1+1)!!}{(2n+1+1)!!},
\]

which completes the induction step. \( \Box \)

We will also utilize the following lemma from [6].

**Lemma 5.** [6, § 1.2] Let \( f, g : \mathbb{Z}^+ \rightarrow \mathbb{R} \), and \( y_0 \in \mathbb{R} \). Given a nonhomogeneous difference equation \( y_n = f(n)y_{n-1} + g(n), y_{n_0} = y_0 \) where \( n \geq n_0 \geq 0 \). The general solution to the difference equation is given by

\[
y_n = \left[ \prod_{i=n_0}^{n-1} f(i) \right] y_0 + \sum_{r=n_0}^{n-1} \left[ \prod_{i=r+1}^{n-1} f(i) \right] g(r).
\]

**Remark 5.** Although, the previous lemma is only stated over the reals, it has an immediate generalization to any field \( k \).

No we have everything that we need in order to prove Proposition 3.

**Proof of Proposition 3.** The proof of this proposition is divided into three parts. In the first part we find the difference equations which defines the coefficients of the three lowest degree terms in \( \Delta_m \) analogous to Theorem A. In the second part we solve these difference equations, and in the last part we determine the coefficient of the lowest degree term in \( \Delta_p \) and hence of \( q^p(\zeta) - \zeta \).

**Part 1. Finding the Difference Equations.** Analogous to Theorem A, p. 12 for \( m \geq 1 \) we define the recurrence relation \( \Delta_1(\zeta) = q(\zeta) - \zeta \) and for \( m \geq 2 \)

\[
\Delta_m(\zeta) := \Delta_{m-1}(q(\zeta)) - \Delta_{m-1}(\zeta).
\]
Note that Lemma 2 on page 8 holds and \( \Delta_p(\zeta) = q^p(\zeta) - \zeta \).

E.g. concerning \( \Delta_2(\zeta) \), we have
\[
\Delta_2(\zeta) = \Delta_1(q(\zeta)) - \Delta_1(\zeta) \\
= a_3(\zeta + a_3\zeta^3 + a_4\zeta^4 + a_6\zeta^6 + \ldots)^3 + a_4(\zeta + a_3\zeta^3 + a_4\zeta^4 + a_6\zeta^6 + \ldots)^4 \\
+ a_6(\zeta + a_3\zeta^3 + a_4\zeta^4 + a_6\zeta^6 + \ldots)^6 - \Delta_1(\zeta) \\
\equiv a_3\zeta^3 + 3a_3^2\zeta^5 + 3a_4\zeta^6 + \left(\frac{3}{2}\right)a_3^3\zeta^7 + a_4\zeta^4 + 4a_3\zeta^6 + 4a_4\zeta^7 + a_6\zeta^6 - \Delta_1(\zeta) \\
\equiv 3a_3^2\zeta^5 + (a_3 + 3a_4)\zeta^6 + \left(\frac{3}{2}\right)a_3^3 + 4a_4\zeta^7 \mod (\zeta^6).
\]

In the earlier examples (e.g. Proposition 2) we could see that for \( m \in \{1, \ldots, p-1\} \) we have \( \text{ord}(\Delta_{m+1}(\zeta)) = \text{ord}(\Delta_m(\zeta)) + 1 \), and we only had to keep track of the two lowest degree terms. For \( q \) we have that \( \text{ord}(\Delta_{m+1}(\zeta)) = \text{ord}(\Delta_m(\zeta)) + 2 \) for \( m \in \{1, \ldots, p-1\} \), and we will see that it is necessary to keep track of the three lowest degree terms. Note that the \( a_6 \)-term is not contributing to those three terms.

More generally for a given \( m \in \{1, \ldots, p\} \) we have
\[
\Delta_m(\zeta) = A_m\zeta^{2m+1} + B_m\zeta^{2m+2} + C_m\zeta^{2m+3} + \ldots 
\]
The three coefficients are defined by a system of linear difference equations. The first difference equations is defined as follows
\[
(3.7) \quad A_m = a_3(2m - 1)A_{m-1} \quad A_1 = a_3.
\]
The coefficient corresponding to the term of degree \( 2m + 2 \) is defined by the difference equation
\[
(3.9) \quad B_m = a_3(2m)B_{m-1} + a_4(2m - 1)A_{m-1} \quad B_1 = a_4.
\]
The last coefficient \( C_m \) is defined as follows
\[
(3.10) \quad C_m = a_3^2(2m - 1)(m - 1)A_{m-1} + a_4(2m)B_{m-1} + a_3(2m + 1)C_{m-1} \quad C_1 = 0.
\]
To see that this actually describes the situation we study \( \Delta_{m+1}(\zeta) \) which yields
\[
\Delta_{m+1}(\zeta) = \Delta_m(q(\zeta)) - \Delta_m(\zeta) \\
= \Delta_m(\zeta + a_3\zeta^3 + b\zeta^4) - \Delta_m(\zeta) \\
= A_m(\zeta + a_3\zeta^3 + a_4\zeta^4 + a_6\zeta^6 + \ldots)^{2j+1} + B_m(\zeta + a_3\zeta^3 + a_4\zeta^4 + a_6\zeta^6 + \ldots)^{2m+2} \\
+ C_m(\zeta + a_3\zeta^3 + a_4\zeta^4 + a_6\zeta^6 + \ldots)^{2m+3} - \Delta_m(\zeta) \\
\equiv A_m a_3(2m + 1)\zeta^{2m+3} + A_m a_4(2m + 1)\zeta^{2m+4} + A_m a_3^2\left(\frac{2m + 1}{2}\right)\zeta^{2m+5} \\
+ B_m a_3(2m + 2)\zeta^{2m+4} + B_m a_4(2m + 4)\zeta^{2m+5} + C_m a_3(2m + 3)\zeta^{2m+5} \\
\equiv A_m a_3(2m + 1)\zeta^{2m+3} + (A_m a_3(2m + 1) + B_m a_3(2m + 2))\zeta^{2m+4} \\
+ (A_m a_3^2(2m + 1) + B_m a_4(2m + 2) + C_m a_3(2m + 3))\zeta^{2m+5} \mod (\zeta^{2m+6}).
\]
Eventually we need to find a closed expression of the coefficients after \( p \) iterations i.e. \( A_p, B_p \) and \( C_p \). This turns out not to be as easy as in the case in the proof of Theorem A. Now we need to find the solutions to the difference equations, and interpret the solutions of those in characteristic \( p \).

**Part 2. Solving the Difference Equations.** In this section we discuss the solutions to the difference equations (3.8), (3.9) and (3.10) given in the previous section. First note that since \( k \) is a field and equation (3.8), (3.9) and (3.10) are linear there are unique solutions \( \{ A_m \}_{m \geq 1}, \{ B_m \}_{m \geq 1} \) and \( \{ C_m \}_{m \geq 1} \) respectively. For further reference on the matter of difference equations we refer to [5] and [7]. Also note that all three difference equations are first order, and except for \( A_m \) (3.8) they are nonhomogeneous (if we can find an explicit formula for \( A_m \)).

We now apply Lemma 5 to solve the equations. We start by considering equation (3.8)

\[
A_m = a_3(2m - 1)A_{m-1}, \quad A_1 = a_3.
\]

Considering Lemma 5 we have \( f(m) = a_3(2m - 1) \) and \( g(m) = 0 \). This means that our solution is

\[
A_m = \prod_{i=1}^{m-1} a_3(2m - 1) \quad A_1 = a_3^{m-1}(2m - 1)!!a_3 = a_3^m(2m - 1)!!.
\]

Now we have the solution to our first difference equation and inserting this into \( B_m \) (equation (3.9)) we obtain

\[
B_m = a_3(2m)B_{m-1} + a_4(2m - 1)A_{m-1} = a_3(2m)B_{m-1} + a_4(2m - 1)a_3^{m-1}(2m - 3)!! = a_3(2m)B_{m-1} + a_3^{m-1}a_4(2m - 1)!!.
\]

Now we can see that \( B_m \) is in fact a nonhomogeneous difference equation. If we study the formula given in Lemma 5 we can see that it would simplify our problem if \( f(n) \) or \( g(n) \) were constants. Therefore we try to do a substitution for \( B_m \) so that our difference equations becomes easier to solve. We do the substitution

\[
B_m^* = \frac{B_m}{a_3^{m-1}a_4(2m - 1)!!}
\]

inserting (3.12) into our equation gives us

\[
a_3^{m-1}a_4(2m - 1)!!B_m^* = a_3^{m-1}a_4(2m)(2m - 3)!!B_{m-1}^* + a_3^{m-1}a_4(2m - 1)!! \quad \iff B_m^* = \frac{2m}{2m - 1}B_{m-1}^* + 1.
\]

Note that \( B_1 = a_4 \) implies that \( B_1^* = 1 \) from (3.12). Now we can solve \( B_m^* \) using Lemma 5 which yields

\[
B_m^* = \prod_{i=1}^{m} \frac{2i}{2i - 1} B_1^* + \sum_{r=1}^{m} \prod_{i=r+1}^{m} \frac{2i}{2i - 1} = \frac{(2m)!!}{(2m - 1)!!} + \sum_{r=1}^{m} \prod_{i=r+1}^{m} \frac{2i}{2i - 1}.
\]
The solution to the sum is

\[
\sum_{r=1}^{m} \left[ \prod_{i=r+1}^{m} \frac{2i}{2i-1} \right] = (2m + 1) - \frac{(2m)!!}{(2m - 1)!!}
\]

We proceed by induction in \( m \) to see that the solution holds. For \( m = 2 \) it holds and we prove that it holds for arbitrarily chosen \( m \), but first note that

\[
\sum_{r=1}^{m} \left[ \prod_{i=r+1}^{m} \frac{2i}{2i-1} \right] = \sum_{r=1}^{m} \frac{(2m)!!(2r - 1)!!}{(2m - 1)!!(2r)!!}
\]

Now we proceed by induction in \( m \) and study \( m + 1 \)

\[
\sum_{r=1}^{m+1} \frac{(2m + 2)!!(2r - 1)!!}{(2m + 1)!!(2r)!!} = \frac{(2m + 2)!!(2m + 1)!!}{(2m + 1)!!(2m + 2)!!} + \sum_{r=1}^{m} \frac{(2m + 2)!!(2r - 1)!!}{(2m + 1)!!(2r)!!}
\]

\[
= 1 + \frac{2m + 2}{2m + 1} \sum_{r=1}^{m} \frac{(2m)!!(2r - 1)!!}{(2m - 1)!!(2r)!!}
\]

\[
= 1 + \frac{2m + 2}{2m + 1} \left( (2m + 1) - \frac{(2m)!!}{(2m - 1)!!} \right)
\]

\[
= 1 + (2m + 2) - \frac{(2m + 2)!!}{(2m + 1)!!}
\]

\[
= (2m + 3) - \frac{(2m + 2)!!}{(2m + 1)!!}
\]

as required.

Substitution of \( B_m^* \) in (3.12) for \( B_m \) yields

\[
B_m = a_3^{m-1} a_4 (2m - 1)!! \left( (2m + 1) - \frac{(2m)!!}{(2m - 1)!!} \right)
\]

\[
(3.13)
\]

So we have a solution for \( B_m \) as well which means that we can express \( C_m \) (equation (3.10)) as a nonhomogeneous difference equation only depending on \( C_{m-1} \). Inserting the solutions for \( A_m \) and \( B_m \) (3.11) and (3.13) respectively into (3.10) gives us for \( m = 1 \) we have \( C_1 = 0 \) and for \( m \geq 2 \) we have that

\[
C_m = a_3^{m+1} (2m - 1)!!(m - 1) + 2ma_3^{m-2}a_4^2 ((2m - 1)!! - (2m - 2)!!) + a_3 (2m + 1)C_{m-1}.
\]

Since this is a linear difference equation we simplify this problem by splitting this equation into two separate equations. Therefore we define for \( m \geq 1 \), \( D_m \) and \( E_m \) as

\[
D_m := a_3 (2m + 1)D_{m-1} + d(m), \quad D_1 := 0
\]

and

\[
E_m := a_3 (2m + 1)E_{m-1} + e(m), \quad E_1 := 0
\]
respectively. Choosing $d(m) = (2m - 1)!!a_3^{m+1}(m - 1)$ and $e(m) = 2ma_3^{m-2}((2m - 1)!! - (2m - 2)!!)$ satisfies our equation (3.14). This can be seen as follows

$$D_m + E_m = a_3(2m + 1)D_{m-1} + d(m) + a_3(2m + 1)E_{m-1} + e(m)$$

$$= a_3(2m + 1)(D_{m-1} + E_{m-1}) + d(m) + e(m)$$

$$= a_3(2m + 1)(C_{m-1}) + d(m) + e(m)$$

$$= C_m.$$ 

So we solve $D_m$ and $E_m$ separately and sum up the solutions to retrieve $C_m$.

We start by finding the solution for

$$(3.15)\quad D_m = a_3(2m + 1)D_{m-1} + (2m - 1)!!a_3^{m+1}(m - 1).$$

Analogous with (3.12) we use substitution to simplify the equation. We use the following substitution

$$D_m^* = \frac{D_m}{a^{m+1}(2m + 1)!!}.$$ 

Inserting the substitution into (3.15) yields

$$D_m^* = D_{m-1}^* + \frac{m - 1}{2m + 1},$$

and since $D_1^* = 0$ this means that for every iteration of the recursion the fraction term will add on. This yields a solution of the form

$$D_m^* = \sum_{j=1}^{m} \frac{j - 1}{2j + 1}.$$ 

Changing variable back to $D_m$ we get that

$$(3.16)\quad D_m = a_3^{m+1}(2m + 1)!! \sum_{j=1}^{m} \frac{j - 1}{2j + 1}.$$ 

We will later on discuss the solution for $m = p$ in $k$.

The difference equation for $E_m$ is given by

$$(3.17)\quad E_m = a_3(2m + 1)E_{m-1} + 2ma_3^{m-2}((2m - 1)!! - (2m - 2)!!).$$

As in the previous case for $D_m$ we use substitution. Our substitution is

$$E_m^* = \frac{E_m}{a_3^{m-2}(2m + 1)!!}.$$ 

Inserting this into (3.17) yields

$$E_m^* = E_{m-1}^* + 2m\left(\frac{(2m-1)!! - (2m-2)!!}{(2m+1)!!}\right) \iff E_m^* = E_{m-1}^* + \frac{2m}{2m + 1} - \frac{(2m)!!}{(2m + 1)!!}.$$ 

By the same reasoning as above, since $E_1^* = 0$, we have

$$E_m^* = \sum_{j=1}^{m} \left[ \frac{2m}{2m + 1} - \frac{(2m)!!}{(2m + 1)!!} \right].$$
which means that the solution to our original equation is

\[
E_m = a_3^{m-2}(2m+1)!! \sum_{j=1}^{m} \left[ \frac{2m}{2m+1} - \frac{(2m)!!}{(2m+1)!!} \right].
\]

Consequently,

\[
C_m = D_m + E_m
\]

\[
(3.19) \quad = a_3^{m+1}(2m+1)!! \sum_{j=1}^{m} \frac{j-1}{2j+1} + a_3^{m-2}a_3^2(2m+1)!! \sum_{j=1}^{m} \left[ \frac{2m}{2m+1} - \frac{(2m)!!}{(2m+1)!!} \right].
\]

Now we have a closed form expressions of the coefficients in (3.7) and to prove our lemma we should find their value in \(k\). Now we have what we need to determine the coefficients of (3.7) in \(k\).

**Part 3. Determining the Coefficients of \(q^p(\zeta) - \zeta\).** First we conclude that the two terms with lowest degree in (3.7), \(A_p\) and \(B_p\) are in fact zero. We have that \(A_p = a_3^p(2p-1)!! = 0\), and \(B_p = a_3^{p-1}a_4((2p+1)!! - (2p)!!) = 0\), in \(k\), due to the fact that all three double factorials contains a multiple of \(p\). For the coefficient of the term corresponding to degree \(2m+3\) we have that

\[
C_p = a_3^{p+1}(2p+1)!! \sum_{j=1}^{p} \frac{j-1}{2j+1} + a_3^{p-2}a_3^2(2p+1)!! \sum_{j=1}^{p} \left[ \frac{2j}{2j+1} - \frac{(2j)!!}{(2j+1)!!} \right].
\]

Note that the last part of the second sum is given by (3.6) in Lemma 4. Hence we need to study \((2p+1)!! \sum_{j=1}^{p} \frac{j-1}{2j+1}\) and \((2p+1)!! \sum_{j=1}^{p} \frac{2j}{2j+1}\) in detail. We start by the former of them

\[
(2p+1)!! \sum_{j=1}^{p} \frac{j-1}{2j+1} = (2p+1)!! \left( \frac{0 + 1}{3} + \frac{2}{5} + \cdots + \frac{p-2}{2p-1} + \frac{p-1}{2p+1} \right)
\]

\[
= \frac{(2p+1)!!}{3} + 2\frac{(2p+1)!!}{5} + \cdots + (p-2)\frac{(2p+1)!!}{2p-1} + (p-1)\frac{(2p+1)!!}{2p+1}.
\]

All of these terms contains a factor \(p\) except for the \(i\)th term where \(i = \frac{p-1}{2}\) using Lemma 3 this implies that we have \((\frac{p-1}{2} - 1)\frac{(2p+1)!!}{2(2p+1)!!} = (\frac{p-3}{2})\frac{(2p+1)!!}{(2p-1)!!} = (\frac{p-3}{2})(-1) = \frac{p+3}{2} = \frac{3}{2} \).

Now we study the second part of the expression

\[
(2p+1)!! \sum_{j=1}^{p} \frac{2j}{2j+1} = (2p+1)!! \left( \frac{2}{3} + \frac{4}{5} + \cdots + \frac{2p-2}{2p-1} + \frac{2p}{2p+1} \right)
\]

\[
= \frac{2(2p+1)!!}{3} + 4\frac{(2p+1)!!}{5} + \cdots + (2p-2)\frac{(2p+1)!!}{2p-1} + 2p\frac{(2p+1)!!}{2p+1}.
\]

All terms contains a factor \(p\) except for the \(i\)th where \(i = \frac{p-1}{2}\) which yields that the only nonzero term is \(2\left(\frac{p-1}{2}\right)\frac{(2p+1)!!}{p} = (p-1)(-1) = 1\).
This means that we can find an expression for $C_p$. We have that

$$C_p = a_3^{p+1}(2p + 1)!! \sum_{j=1}^{p} \frac{j - 1}{2j + 1} + a_3^{p-2}a_2^3(2p + 1)!! \sum_{j=1}^{p} \left[ \frac{2j}{2j + 1} - \frac{(2j)!!}{(2j + 1)!!} \right]$$

$$= a_3^{p+3} + a_3^{p-2}a_2^3 \left( 1 - (2p + 1)!! \sum_{j=1}^{p} \frac{(2j)!!}{(2j + 1)!!} \right)$$

$$= a_3^{p+3} + a_3^{p-2}a_2^3 \left( 1 - ((2p + 2)!! - 2(2p + 1)!!) \right)$$

$$= a_3^{p+3} + a_3^{p-2}a_2^3$$

$$= a_3^{p-2} \left( \frac{3a_2^2 + 2a_4^2}{2} \right),$$

which proves that the coefficient for the lowest degree nonzero term of $q^p(\zeta) - \zeta$ is in fact $a_3^{p-2} \left( \frac{3a_2^2 + 2a_4^2}{2} \right)$. This completes the proof of Proposition 3.

Proof of Theorem B. Assuming $a_3 \neq 0$, by [12, Theorem 2] we know that it is sufficient to show that $i_1(q) = 2(1 + p)$ to show that $q$ is 2-ramified. We consider the polynomial

$$q(\zeta) = \zeta + a_3\zeta^3 + a_4\zeta^4 + a_5\zeta^6 + \sum_{i=7} a_i z^i \in k[[\zeta]].$$

Note that from Proposition 3 we know that $q^p(\zeta) - \zeta \equiv C_p\zeta^{2p+3} \mod \langle \zeta^{2p+4} \rangle$ and the coefficient $C_p$ for the lowest degree nonzero term of $q^p(\zeta) - \zeta$ is $a_3^{p-2} \left( \frac{3a_2^2 + 2a_4^2}{2} \right)$. Clearly we see that $a_3 \neq 0$ is a criterion for $q$ to be 2-ramified and moreover if $a_3 \neq 0$ then it is necessary that $3a_2^3 \neq -2a_4^2$ for $q$ to be 2-ramified, which proves our theorem.

Remark 6. The reason why we need $a_5 = 0$ in Theorem B is due to the fact that the relation between $a_3, a_4$ and $a_5$ is very complex and nested. We show that the choice of $a_5$ is not redundant with the following example. Let $k$ be a field of characteristic 3, and consider the polynomials $Q(\zeta) = \zeta + \zeta^3 + \zeta^4 + \zeta^5$ and $Q'(\zeta) = \zeta + \zeta^3 + \zeta^4 + 2\zeta^5$. Note that the criterions for $Q$ and $Q'$ to be 2-ramified according to Theorem B are fulfilled (except for $a_5 \neq 0$), though only $Q'$ is 2-ramified. Computations shows that $i_1(Q') = 8$ and $i_1(Q) = 11$.

4. Implications of Theorem B for minimally ramified series

Theorem B has the following implication for minimally ramified power series as defined in Definition 3. We show in the next corollary that we can prove minimal ramification for some polynomials of the form $\gamma \zeta + \zeta^2 \in k[\zeta]$, where $\gamma$ is a root of unity, using the results from Theorem B.

Corollary 2. Let $p$ be a prime, and $k$ a field of characteristic $p$, then the polynomial

$$P_{-1}(\zeta) = -\zeta + \zeta^2 \in k[\zeta]$$

is minimally ramified for all primes $p$, if $p \neq 11$. 
Proof. The order $q$ of $\gamma = -1$ is equal 2 for all primes, which means that $P_{-1}^2(\zeta) = P_{-1}^3(\zeta)$ for all primes. This yields that

$$P_{-1}^3(\zeta) = -(-\zeta + \zeta^2) + (-\zeta + \zeta^2)^2 = \zeta - 2\zeta^3 + \zeta^4.$$  

To determine whether $P_{-1}$ is minimally ramified or not is to check if $i_n(P_{-1}^n) = q(1 + p + \cdots + p^n)$ for some $n \geq 1$. Therefore we study $P_{-1}^2 = P_{-1}^3$. Note that $P_{-1}^3$ is on the same form as in the polynomial class in Theorem B, and since $q = 2$, minimal ramification for $P_{-1}$ is the same as 2-ramification for $Q(\zeta) = \zeta + a\zeta^3 + b\zeta^4$.

Theorem B states that $a \neq 0$ and $3a^3 + 2b^2 \neq 0$ implies 2-ramification for $Q(\zeta)$ which is minimal ramification for $P_{-1}$, and since $a = -2 \neq 0$ this holds. Furthermore

$$\frac{3a^3 + 2b^2}{2} = \frac{3(-2)^3 + 2(1)^2}{2} = \frac{-24 + 2}{2} = -11,$$

which proves that $P_{-1}$ is minimally ramified for all $p$ except for $p = 11$. \hfill $\Box$

It is interesting to notice from the above mentioned corollary is that $i_0(P_{-1}^n) = q$ for all odd $p$. This shows that it is not sufficient to study whether $i_0(P_{-1}^n) = q$ or not to claim minimal ramification. In the next lemma we discuss a matter closely related to this.

Lemma 6. Let $p$ be a prime and $k$ a field with $\text{char}(k) = p$. Furthermore let $\gamma \in k$, and $q \in \mathbb{N}$, such that $\gamma^q = 1$. Moreover we let $P_\gamma(\zeta) = \gamma \zeta + \zeta^2 \in k[\zeta]$. If $q = 3$ then $i_0(P_\gamma^q) = q$ for all $p$ and $\gamma$ except for $p = 7$ and $\gamma = 2$.

Even though we can show that $i_0(P_\gamma^q) = 3$ for almost all primes and $\gamma$ we cannot conclude that this class of polynomials is minimally ramified (with the mentioned exception). It is tempting to suggest that $i_0(P_\gamma^q) = q$ implies minimal ramification due to numerical results. But, as we saw in Lemma 2 for the case of $p = 11$ we have that $i_0(P_{-1}^2) = 2$ and $i_1(P_{-1}^2) > 24$, which means that even though minimal ramification holds for $n = 0$ it does not imply for it hold for all $n \geq 1$.

Numerical results shows that for all primes less than 100 and all $\gamma \in \mathbb{F}_p$ there is only for $p = 7$ and $\gamma = 2$, with $\gamma^3 = 1$ that is not minimally ramified.

Proof of Lemma 6. If $q = 3$ then

$$P_\gamma^3(\zeta) - \zeta = P_\gamma^3(\zeta) - \zeta \equiv (1 + \gamma^2 + \gamma^4)\zeta^2 + 2(1 + \gamma^2 + \gamma^4)\zeta^3 + (6 + \gamma + \gamma^2 + \gamma^4)\zeta^4 \mod \langle \zeta^5 \rangle.$$  

From Proposition 1 we have that $i_0(P_\gamma^3) \geq 3$. This means that the coefficient for the degree 2 (and 3) term is zero which implies that $1 + \gamma^2 + \gamma^4 = 0$.

Rearranging the coefficient for the term with degree $q + 1 = 4$ we get that

$$6 + \gamma + \gamma^2 + \gamma^4 = 5 + \gamma + (1 + \gamma^2 + \gamma^4) = 5 + \gamma.$$  

If $5 + \gamma = 0$ this means that $\gamma = -5$ which implies that $1 = -125$ since $\gamma^3 = 1$. This means that $0 = -126$, and the factoring of $-126 = (-1) \cdot 2 \cdot 3^2 \cdot 7$ implies that $p$ has to be one of these primes in the factoring, and of these primes only $p = 7$ has nontrivial (other than $\gamma = 1$) elements such that $\gamma^3 = 1$, namely $\gamma = 2$ and $\gamma = 4$, of which only
the former fulfills the criteria $\gamma + 5 = 0$. This proves that $i_0(P^3_\gamma) = 3$ for all $p$ and $\gamma$ except for $p = 7$ and $\gamma = 2$, when $q = 3$. \hfill \Box

5. Connection to dynamical systems

In this section we will briefly discuss the connection between minimally ramified power series and the geometric location of periodic points in for non-Archimedean power series. Let $K$ be a field of positive characteristic and $(K, | \cdot |)$ be a non-Archimedean field with absolute value $| \cdot |$; see Example 3. Let $P_\lambda(z) = \lambda z + z^2 \in K[z]$, we will show that periodic points of minimal period $p^n$ all lies on the same circle around $z = 0$. We also find the radius of this circle. Before proving these results we define some concepts that are used in the proofs that follow.

**Definition 4.** Let $K$ be a field and $f$ be a power series in $K[[z]]$. A point $z_0 \in K$ is said to be a periodic point of $f$ if there exists some integer $n \geq 1$ such that $f^n(z_0) = z_0$. In addition, $n$ is the smallest integer with this property, then we say that $n$ is the minimal period of $z_0$.

Let $(K, | \cdot |)$ be an non-Archimedean field. Denote $\mathcal{O}_K$ the closed unit disc of $K$, then $\mathcal{O}_K$ is a subring of $K$. Let $m_K$ be the maximal ideal of $\mathcal{O}_K$. Then $m_K$ is the open unit disk. The quotient of a ring and its maximal ideal is always a field [23]. Let $\bar{K} := \mathcal{O}_K/m_K$ denote the residue field of $K$. Furthermore, let $a \in K$, then $\bar{a}$ is the reduction of $a$ in $\bar{K}$.

In this way we define the reduction of a power series $g \in \mathcal{O}_K[[z]]$ as the power series $\bar{g}$ in $\bar{K}[[z]]$ whose coefficients is the reduction of the coefficients in $g$. For such a power series $g \in \mathcal{O}_K[[z]]$ we let $\text{wideg}(g)$ denote the Weistrass degree of $g$. The Weistrass degree of $g$ is the order of the reduction $\bar{g}$ of $g$. Suppose that $g(z) = \sum_{i=1}^\infty a_i z^i$ and $\text{wideg}(g) < +\infty$. Then $\text{wideg}(g) = \min\{i : |a_i| = 1\}$. In particular, if $\text{wideg}(g) = 0$, then by ultrametricity $g$ has no roots in the open unit disk $m_K$.

To show that all periodic points of minimal period $p^n$, in the dynamical system generated by $P_\lambda$, in fact lies on the same circle, we first show that the mapping $P_\lambda$ is an isometry from the open unit disc into itself.

**Lemma 7.** Let $K$ be an non-Archimedean field and let $\lambda \in K$, such that $|\lambda| = 1$. Furthermore, let $P_\lambda = \lambda z + z^2 \in K[z]$. Then the mapping, from and to the open unit disk,

$$P_\lambda : D_1(0) \mapsto D_1(0)$$

is an isometry. 

**Proof.** To prove this is an isometry we have to show that $|P_\lambda(y) - P_\lambda(z)| = |y - z|$. Note that the following holds for an non-Archimedean field $(K, | \cdot |)$. Let $z_1, z_2 \in K$ then $|z_1 + z_2| = \max\{|z_1|, |z_2|\}$ if $z_1 \neq z_2$. We start by looking at the following. Assuming

\[\text{An isometry is a mapping that is distance preserving}\]
that $y \neq z$ we have
\[
|P_\lambda(y) - P_\lambda(z)| = |\lambda y + y^2 - \lambda z - z^2|
= |\lambda(y - z) + (y^2 - z^2)|
= \max\{|\lambda(y - z)|, |y^2 - z^2|\}
= \max\{|\lambda||y - z|, |y^2 - z^2|\}
= \max\{|y - z|, |y^2 - z^2|\},
\]
since $z \in D_1(0)$ we have that $\max\{|y - z|, |y^2 - z^2|\} = |y - z|$ as required.

It follows from the above lemma that for every $z$ in $D_1(0)$ we have $|P(z)| = |z|$. Accordingly, for every integer $n \geq 1$ we have $|P^n(z)| = |z|$. In particular if $z_0$ is a periodic point of $P$ with minimal period $p^n$ then all periodic points lie on the circle \[\{z \in K : |z| = |z_0|\}\].

**Lemma 8.** Let $(K, |\cdot|)$ be a non-Archimedean field, and let $K$ be algebraically closed. Moreover, let $\lambda \in K$ be such that $|\lambda| = 1$, and $|\lambda - 1| < 1$ and not a root of unity, and $\mathcal{O}_K$, be the closed unit disk in $K$, and $\mathfrak{m}_K$ its maximal ideal. Let $P_\lambda(z) = \lambda z + z^2 \in \mathcal{O}_K[z]$ and let $z_0$ be a periodic point of $P_\lambda$ with minimal period $p^n$. Then
\[
|z_0| = \left|\frac{\lambda^p - 1}{\lambda^{p^n-1} - 1}\right|^{\frac{1}{p^n}}.
\]

**Proof.** From Lemma 7 we have that all periodic points of minimal period $p^n$ of $P_\lambda$ have the same distance to $z = 0$. Put
\[
h(z) := \frac{P^{p^n}_\lambda(z) - z}{P^{p^n-1}_\lambda(z) - z}.
\]
Let $z_0$ be a periodic point of minimal period $p^n$ and let $o := \{P^{p^n}_\lambda(z)\}_{n=1}^{p^n}$ be the forward orbit of $z_0$. Then $o$ consists of $p^n$ distinct elements in $K$ since $K$ is algebraically closed. Put $\phi(z) := \prod_{i \in o} (z - z_i)$, then $\phi(z)$ divides $h(z)$, which implies that $h(z)/\phi(z) = U(z)$, for some $U(z) \in \mathcal{O}_K[[z]]$. Note that the following holds for $h$
\[
\text{wideg}(h) = \text{wideg} \left( \frac{P^{p^n}_\lambda(z) - z}{P^{p^n-1}_\lambda(z) - z} \right)
= \text{ord} \left( \frac{\widetilde{P}^{p^n}_\lambda(z) - z}{\widetilde{P}^{p^n-1}_\lambda(z) - z} \right)
= \text{ord}(\widetilde{P}^{p^n}_\lambda(z) - z) - \text{ord}(\widetilde{P}^{p^n-1}_\lambda(z) - z)
= (1 + p + \cdots + p^n) - (1 + p + \cdots + p^{n-1})
= p^n.
\]
In the last step we used that $\widetilde{P}(z) = z + z^2$, which is minimally ramified from Proposition 2, hence the order is given.
As \( \phi(z) \) consist of all the periodic points of minimal period, and \( |z_i| \) for all \( z_i \in o \), we must have that \( \text{wideg}(\phi(z)) = p^n \), since the reduction of each coefficient except for the term of degree \( p^n \) will be 0 in \( \bar{K} \). This means that

\[
\text{wideg}(U) = \text{wideg}(\frac{h(z)}{\phi(z)}) = \text{wideg}(h(z)) - \text{wideg}(\phi(z)) = p^n - p^n = 0.
\]

From this we have the implication that \( h(z) = \phi(z) \cdot U(z) \), where \( U(z) \neq 0, \forall z \in D_1(0) \).

Consequently

\[
\phi(z) = \prod_{i=1}^{p^n}(z - z_i) = c_0 + c_1z + c_2z^2 + \cdots \in O_K[z].
\]

Moreover, \( c_0 = \prod_{i=1}^{p^n} z_i \) and by Lemma 7 \( |c_0| = |z_0|p^n \).

We have on the other hand that \( Pp^n(z) = \lambda p^n z + \ldots \) and hence the constant term of

\[
U(z) = h(z)/\phi(z) \text{ is equal to}
\]

\[
(\lambda p^n - 1)/ \left( \lambda p^{n-1} - 1 \prod_{z_i \in o} z_i \right).
\]

Together with the fact that \( \text{wideg}(U) = 0 \) we obtain

\[
|z_0| = \left| \frac{\lambda p^n - 1}{\lambda p^{n-1} - 1} \right|^{\frac{1}{p^n}},
\]

as required.

This result was proven in [14] by Lindahl for the case \( K \) is a \( p \)-adic field.

Lindahl and Rivera-Letelier [15, Theorem C] show that if \( \gamma \zeta + \zeta^2 \in k[\zeta] \) is minimally ramified, then the geometric location of the cycle of periodic points of \( \lambda z + z^2 \in K[z] \) are optimal, under some conditions for \( \lambda \). Optimal meaning in this case that it “minimizes the distance to \( z = 0 \) among cycles of the same minimal period” [15, §1, page 1].

Furthermore, the implication of our study is that in view of to Corollary 2 \( P_{-1}(z) = -z + z^2 \in K[z] \) exhibit optimal cycles if and only if \( p \neq 11 \).

6. Discussion

In our work we show the complexity of the problem related to lower ramification numbers of wildly ramified power series. In Theorem A we gave a complete classification of 1-ramified power series of the form

\[
g(\zeta) = \zeta + \ldots.
\]

Also in Theorem B we gave a classification of all 2-ramified power series of the form

\[
q(\zeta) = \zeta + a_3\zeta^3 + a_4\zeta^4 + a_6\zeta^6 + \sum_{i=7}^{\infty} a_i\zeta^i \in k[[\zeta]].
\]

As noted in Remark 6 there is no straight forward generalization to the case \( a_5 \neq 0 \).
Our results show an interesting relation to minimally ramified polynomials of the form \( \gamma \zeta + \zeta^2 \). One of the main motivations of this thesis was the problem posed by Lindahl and Rivera-Lettelier [15, Problem 1.4]. The problem is the following.

**Problem.** Let \( p \) be an odd prime number, \( \mathbb{F}_p \) a field of \( p \) elements, and \( \overline{\mathbb{F}}_p \) an algebraic closure of \( \mathbb{F}_p \). Determine all those \( \gamma \in \overline{\mathbb{F}}_p^* \) for which the quadratic polynomial \( P_\gamma(\zeta) = \gamma \zeta + \zeta^2 \) in \( \mathbb{F}_p[\zeta] \) is minimally ramified.

The complexity of this problem is that minimal ramification does not only depend on \( p \) and the order \( q \) of \( \gamma \), but on \( \gamma \) itself as well. Indeed by Corollary 2 we have that for \( \gamma = -1 \), \( P_\gamma \) is minimally ramified if and only if \( p \neq 11 \). This case is particularly interesting since it emphasizes the fact that \( i_0 = q \) is not a sufficient condition for minimal ramification. Also recall Lemma 6 where we showed that when the order of \( \gamma \) is equal to 3, we have that \( i_0(P_\gamma) = q \) for all \( \gamma \) and \( p \), except for \( p = 7 \) and \( \gamma = 2 \). Numerical results also suggests that when the order of \( \gamma \) is 3, then \( P_\gamma \) is minimally ramified for all \( p \) and \( \gamma \) except for \( p = 7 \) and \( \gamma = 2 \).

In the computer experiments done for this study we have found, for all \( \gamma \in \mathbb{F}_p \) and all primes \( < 100 \), in which cases \( P_\gamma \) is minimally ramified. The computer program was written and executed in SAGE, and the code can be found in the appendix. Studying the results of these computations led us to believe that the following conjectures are true.

**Conjecture 1.** Let \( p \) be a prime and \( k \) a field of characteristic \( p \). Let \( \gamma \in k \) be a root of unity with order \( q \). Moreover, let \( P_\gamma(\zeta) = \gamma \zeta + \zeta^2 \in k[\zeta] \), and put \( \gamma = \pm 2 \). Then \( P_\gamma \) is minimally ramified if and only if \( q = p - 1 \).

We have found that for all primes \( \leq 350 \) Conjecture 1 holds. Unfortunately, we do not have the same amount of data supporting the following conjecture. However, for all primes not exceeding 100 we know that it holds.

**Conjecture 2.** Let \( p \) be a prime and \( k \) a field of characteristic \( p \). Let \( \gamma \in k \) be a root of unity with order \( q \). Moreover, let \( P_\gamma(\zeta) = \gamma \zeta + \zeta^2 \in k[\zeta] \), and put \( \gamma = 4 \). Then \( P_\gamma \) is minimally ramified if and only if \( q = \frac{p-1}{2} \).

It is interesting that for the two cases of \( \gamma = \pm 2 \) and \( \gamma = 4 \) minimal ramification seems to depend only of the order of \( \gamma \) and not on \( p \) at all.

Let \( (K, | \cdot |) \) be a non-Archimedean field. If the conjectures posed here holds then we have found a rather large class of polynomials of the form \( \gamma \zeta + \zeta^2 \in k[[\zeta]] \) exhibiting minimal ramification, and hence optimal cycles.

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\(^{iv}\)All primes not exceeding 350 might not be a strong statement, but the computational effort to find \( i_1(P_\gamma) \) is cumbersome. Within the range of this study we didn’t have the possibility to enhance the numerical computations.
References

Appendix

This program searches for which primes \( p \leq 101 \) and \( \gamma \in \mathbb{F}_p \), \( P_\gamma(\zeta) = \gamma \zeta + \zeta^2 \) is minimally ramified.

```python
F.<a>=FiniteField(3)
K.<z>=PolynomialRing(F)
P=Primes()
p=P.first()

for i in range(25):
p = P.next(p)
F.<a>=FiniteField(p)
K.<z>=PolynomialRing(F)
for g in range(2,p):
o = F(g).multiplicative_order()
Q = g*z + z^-2
q=Q
for k in range(1,o*p):
    q = q(Q).mod(z^(o*(p+1) + 2))
print "p, gamma, , order(gamma), Q, min. ramif."
    print [p , g, o, q - z, o*(p+1) + 1]
```

This program searches for when \( P_\gamma \) is minimally ramified for the 100 first primes, with the particular condition \( \gamma = -2 \)

```python
F.<a>=FiniteField(3)
K.<z>=PolynomialRing(F)
P=Primes()
p=P.first()

for i in range(100):
p = P.next(p)
F.<a>=FiniteField(p)
K.<z>=PolynomialRing(F)
g=-2
o = F(g).multiplicative_order()
Q = g*z + z^-2
q=Q
for k in range(1,o*p):
    q = q(Q).mod(z^(o*(p+1) + 2))
print "p, gamma, , order(gamma), Q, min. ramif."
    print [p , g, o, q - z, o*(p+1) + 1]
```