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Linking and Morse Theory

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LINKING AND MORSE THEORY

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Abstract. In this paper we use Morse theory and the gradient flow of a Morse-Smale function to compute the linking number of a two-component link L in S^3, by counting the signed number of gradient flow lines passing through each component of L. We will also use three Morse-Smale functions and their gradient flows, to compute Milnor's triple linking number of three-component links by counting flow trees with sign.

Contents

1. Introduction

A real valued smooth function on a smooth Riemannian manifold (M, g) is called a *Morse function* if the Hessian matrix of f is non-singular at every critical point, that is $\mathrm{d}f \pitchfork 0_M$. To a Morse function we can associate gradient flow lines, which are solution curves $\gamma_x(t)$ of the boundary value problem $\nabla(f(\gamma)) = -\dot{\gamma}(t)$, $\gamma(0) = x$, where ∇ denotes the g-gradient. It can be shown that each gradient flow line starts and ends at a critical point. If a gradient flow line happens to pass one or several critical points before its end, it is called a *broken flow line*. Such flow lines can be seen as a concatenation of gradient flow lines. For each critical point x of f we can define the stable or unstable manifold, denoted $W^s(x)$ or $W^u(x)$ respectively, by considering all points in the manifold which contain a gradient flow line that end or start at the critical point x, i.e. $\lim_{t\to\pm\infty}\gamma(t) = x$. If all intersections between the stable and unstable manifolds of a Morse function are transverse, it is called a Morse-Smale function.

In contrast to the more modern approach to Morse theory of the gradient flow, one can also study the topology of level sets $M^a = f^{-1}((-\infty, a])$ of a Morse function f, where a is a regular value [13]. Suppose that we have two level sets M^b and M^a, and a critical point p of index λ, such that $f(p) = c \in (a, b)$. Then M^b can be shown to be diffeomorphic to M^a with a λ-handle glued to its boundary [9, 18]. This is a powerful result which enables us to essentially build the manifold once we know the critical points and their indices of a Morse function. A more modern construction related to Morse functions is a homology theory called *Morse homology*. Let $C_k = \mathbb{Z}^{\mathrm{crit}_k(f)}$ be the free Abelian group generated by the set of critical points of index k. We then define a differential $\partial_k \colon C^k \longrightarrow C^{k-1}$ that counts the number of gradient flow lines between critical points of index k and $k-1$ with a certain sign, and define the homology of $(C^*(f), \partial_*)$ as the *Morse homology* [1, 8, 17].

In classical knot theory, a *knot* is an embedding of S^1 into S^3, and one of the main objectives in knot theory is to find a complete knot invariant. A link is a collection of disjoint knots, and an invariant of two-component links is the *linking number* [14, thm 4.5.1]. A 2-manifold S with a link as its boundary is called a *Seifert surface* of the link. The linking number of a two-component link $L = K_1 \sqcup K_2$ can be defined

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as the intersection number between K_1 and any Seifert surface of K_2. In this paper we will consider a two-component link $L = K_1 \sqcup K_2$ in S^3 together with a Morse-Smale function f. We will equip each gradient flow line that goes between two knots with a certain sign which will be discussed in section 3. We will denote by n the signed number of gradient flow lines in the stable and unstable manifolds of critical points of index 0 and 1 that go between K_2 and K_1. We will then denote by m the signed number of gradient flow lines in the unstable manifold of critical points of index 2 that go between the knot K_1 and the standard (homological) cycle of 1-handles of f. We will show that the linking number satisfies the following formula

$$\mathrm{lk}(K_1, K_2) = n + m\,. \tag{1.1}$$

One of the main results needed to prove (1.1) is theorem 1.1.

Theorem 1.1. *Denote by Σ the set of gradient flow lines (including broken flow lines) that pass a point on a knot K_2 in S^3. Suppose that K_2 is null-homologous in the stable and unstable manifolds of critical points of index 0 and 1. Then Σ is an oriented 2-chain, with $\partial\Sigma = K_2$.*

If the requirement that K_2 is null-homologous in the stable and unstable manifolds of critical points of index 0 and 1 is relaxed, we can find ourselves in a situation as in fig. 4, where the bounding chain of K_2 does not lie within the stable and unstable manifolds of critical points of index 0 and 1. We will then have to use the differential in Morse homology in order to argue why we can "repair the holes". Futhermore, by realizing that gradient flow lines between knots are in one-to-one correspondence with transversal intersections between a knot and a Seifert surface, as shown in lemma 3.3, we can establish a bridge between Morse theory and the linking number of two-component links.

In section 2 we will recall the relevant notions of Morse theory. We will focus on gradient flow lines and stable/unstable manifolds, and also develop the intuition for the natural handle decomposition a Morse function gives rise to. We will briefly discuss Morse homology and define all involved signs for the differential. We end the section with a general discussion on transversality.

In section 3 we will consider a two-component link $L = K_1 \sqcup K_2$ in S^3, characterize all possible critical points of a Morse-Smale function f in S^3 and then discuss the sign of a gradient flow lines that pass through both of the knots K_1 and K_2. The rest of the section consists of proving theorem 1.1 among other results as lemma 3.3, which establishes a one-to-one correspondence between such signed gradient flow lines and the signed intersections between one of the knots and a Seifert surface of the other. We end the section with a proof of the formula (1.1).

In section 4 we use the geometric interpretation of Milnor's triple linking number discussed in [4, 11] and the results from section 3 in order to compute Milnor's triple linking number using Morse theory. Apart from the results in section 3 we need to define *flow trees* and equip them with a sign, which we prove to be in one-to-one correspondence with signed triple points of three transversal Seifert surfaces in lemma 4.2.

Throughout this paper, we will assume that every manifold is finite dimensional, compact and smooth (i.e. $\mathcal{C}^\infty$).

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2. Background notions

In this section we recall necessary notions from Morse theory and the Morse lemma in particular. As far as stable/unstable manifolds are concerned we will state the stable/unstable manifold theorem. We will furthermore prove lemma 2.11 which is a result we need together with the stable/unstable manifold theorem to prove that the intersection of stable and unstable manifolds is an embedded manifold. We will then discuss handle decompositions of manifolds that a Morse function naturally gives rise to, in section 2.1.A. We will finish this section by discussing transversality, which ends with the proof of theorem 2.25. This result essentially enables us to always arrange things so that they intersect transversally.

2.1. **Morse theory.** A real valued smooth function $f\colon\ M \longrightarrow \mathbb{R}$ on a manifold M is called a *Morse function* if the Hessian matrix of f is non-singular at every critical point of f. By Sard's theorem, the critical points are isolated, which means that for every critical point $x \in M$ there is an open ball $x \in B \subset M$ which does not contain any other critical points. In particular, if M is compact, a Morse function has only a finite number of critical points. If $x \in M$ is a critical point of f, we define the index, $\operatorname{ind}(x)$ to be the number of negative eigenvalues of the Hessian matrix at $x \in M$. The index is also the dimension of the maximal subspace of T_xM in which the Hessian matrix is negative definite. Before we can prove a powerful tool in Morse theory, called the Morse lemma, we will need lemmas 2.1 and 2.2.

Lemma 2.1 (Lemma 2.1 in [13])**.** *Suppose that $f \in \mathcal{C}^\infty(V)$ where $0 \in V \subset \mathbb{R}^n$ is convex, and where $f(0) = 0$. Then there are smooth functions g_i such that*

$$f(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \dots, x_n)\,,$$

and $g_i(0) = \frac{\partial f}{\partial x_i}(0)$ for all $i \in \{1, \dots, n\}$.

Proof. First of all, because $f(0) = 0$ we can use a small trick and write

$$f(x_1, \dots, x_n) = \int_0^1 \frac{\mathrm{d}f(tx_1, \dots, tx_n)}{\mathrm{d}t}\,\mathrm{d}t\,.$$

This means that

$$f(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n)\,\mathrm{d}t\,.$$

If we let $g_i := \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n)\,\mathrm{d}t$ we have

$$g_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(0)\,\mathrm{d}t = \frac{\partial f}{\partial x_i}(0)\,.$$

□

Lemma 2.2 (Lemma at p. 145 of [7])**.** *Let A be a diagonal matrix with diagonal elements ± 1. Then in the vector space of symmetric $n \times n$ matrices, there is a neighborhood N of A and a smooth map*

$$P\colon\ N \longrightarrow \mathrm{GL}_n(\mathbb{R})\,,$$

such that $P(A) = I$, and if $P(B) = Q$ then $Q^T BQ = A$.

Proof. See [7, p. 145]. □

Lemma 2.3 (Morse lemma)**.** *Suppose that $f\colon\ M \longrightarrow \mathbb{R}$ is a Morse function, and $p \in M$ a critical point of f with $\operatorname{ind}(p) = \lambda$. Then there is a chart $x\colon\ \mathbb{R}^n \longrightarrow U \subset M$, where U is a neighborhood of $p \in M$, such that $x^{-1}(p) = 0$. Moreover if (x_i) are the local coordinates of x in U, then*

$$(f \circ x)(x_1, \dots, x_n) = f(p) - x_1^2 - x_2^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2\,. \tag{2.1}$$

Proof. First we will prove that if (2.1) holds, then $\lambda = \operatorname{ind}(p)$. Suppose that there is some chart

$$z\colon\ \mathbb{R}^n \longrightarrow U \subset M\,,$$

such that $z(z_1, \dots, z_n) = q$ for some coordinates z_i. Then if

$$f(q) = f(p) - z_1^2(q) - \ldots - z_\lambda^2(q) + z_{\lambda+1}^2(q) + \cdots z_n^2(q)\,,$$

we have

$$\frac{\partial^2 f}{\partial z_i \partial z_j}(p) = \begin{cases} -2, & \text{if } i = j \leq \lambda \\ 2, & \text{if } i = i > \lambda \\ 0, & \text{otherwise} \end{cases}\,.$$

With respect to the standard basis of T_pM we have the Hessian matrix

$$\mathcal{H}(f,p) = \begin{pmatrix} -2 & & & & & \\ & \ddots & & & & \\ & & -2 & & & \\ & & & 2 & & \\ & & & & \ddots & \\ & & & & & 2 \end{pmatrix} .$$

Thus there is a subspace of T_pM of dimension λ where $\mathcal{H}$ is negative definite, and a subspace V of dimension $n-\lambda$ where $\mathcal{H}$ is positive definite. If there were a subspace of T_pM with dimension higher than λ on which $\mathcal{H}$ would be negative definite, then this subspace would intersect V, which is clearly impossible. Hence $\lambda = \operatorname{ind}(p)$.

We will now show that there is a suitable chart $y\colon\ \mathbb{R}^n \longrightarrow U \subset M$ such that (2.1) holds. Without loss of generality we can assume that M is a convex set in $\mathbb{R}^n$ with $p = 0$ and $f(p) = f(0) = 0$. By lemma 2.1 we can write

$$f = \sum_{j=1}^{n} x_j g_j \,,$$

in some neighborhood of 0, and since 0 is a critical point we have

$$g_j(0) = \frac{\partial f}{\partial x_j}(0) = 0 \,,$$

and this allows us to use lemma 2.1 once again to get

$$f = \sum_{i,j=1}^{n} x_i x_j h_{ij} \,.$$

We can assume $h_{ij} = h_{ji}$ because if not, then $\tilde{h}_{ij} = \frac{1}{2}\left(h_{ij} + h_{ji}\right)$ and we would have $\tilde{h}_{ij} = \tilde{h}_{ji}$ and

$$f = \sum_{i,j=1}^{n} x_i x_j \tilde{h}_{ij} \,.$$

Moreover, the matrix $\left(h_{ij}(0)\right) = \left(\frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ is non-singular and can be made diagonal by a change of variables. The rest of the proof is just application of lemma 2.2 and the inverse function theorem. □

Remark 2.4. The basic intuition and statement of this lemma is basically that every Morse function locally around a critical point can be (after some appropriate coordinate change) expressed as a quadratic function with the same number of negative terms as the index of the critical point. This is really just the spectral theorem used in a manifold situation.

Now follows two results which show what Morse theory can do. If M is a compact n-manifold and if f is a Morse function on M with exactly two critical points, then M is homeomorphic to S^n. This is even true if we do not require the critical points to be non-degenerate. M may be homeomorphic to S^n, but is not necessarily diffeomorphic to S^n [13, §4]. There is also a result regarding closed (meaning compact and with no boundary) n-manifolds which says that if f is a Morse function on M we have

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \cdot \#\text{critical points of Morse index } i \,,$$

where χ denotes the Euler characteristic. There is furthermore an existence result for Morse functions, stating that almost every function $f\colon\ M \longrightarrow \mathbb{R}$ is a Morse function in the sense of the Lebesgue measure. More precisely it states that the set of n-tuples $(a_0, \ldots, a_n)$ for which the function $f = \sum_{\nu=0}^{n} a_\nu x^\nu$ is *not* a Morse function has Lebesgue measure zero. The proof uses Sard's theorem, and the fact that the critical points are non-degenerate is equivalent to the fact that $\mathrm{d}f \pitchfork 0_M$, and transversality is a generic property, meaning that "almost all" intersections are transverse. A discussion of transversality and proof of the fact that it is a generic property is found in section 2.4.

2.1.A. *Handle decompositions and CW complexes.* One efficient way of studying manifolds is by studying how they decompose into smaller parts, so called *handles*. In the more general setting of a topological space, one can study how the topological space decomposes into n*-cells*, which is the $\mathcal{C}^0$-analog of a handle decomposition. These are naturally called a *cell decomposition*, and when the cells in the decomposition are neatly attached together, they form a CW complex. The CW in CW complex stands for closure finiteness and weak topology, and plays an important role in the general case when the space is not necessarily finite dimensional. Such CW complexes were first studied by J. H. C. Whitehead [19]. We will need to briefly introduce and develop the intuition for CW complex and handle decompositions and discuss how they naturally appear when dealing with Morse functions. This will mainly be used later in section 3.

A n-cell is a space that is homeomorphic to the open n-disk, D^n. If X is a topological space, a cell decomposition is a family $\{e_\alpha\}_{\alpha\in I}$ of cells such that $X = \coprod_{\alpha\in I} e_\alpha$. Since we at this point only have a collection of tuples $\{(x, e_\alpha) \mid x \in X\}$, we will consider *attaching maps*, in order to attach the cells together to form X. We first denote by X^0 a discrete set of points, which are called 0-cells. We then inductively define the set of n-cells, X^n (called the n-skeleton), by attaching cells e^n_α via maps

$$\varphi_\alpha \colon \partial e^n_\alpha = S^{n-1} \longrightarrow X^{n-1}\,.$$

That is, we can write $X^n = \left(X^{n-1} \coprod_\alpha D^n_\alpha\right) / \sim$, where $x \sim \varphi_\alpha(x)$, if and only if $x \in \partial D^n_\alpha$ [6]. Since X is is finite dimensional, we will have $X = X^n$ for some n. A pair (X, E) of a topological space X and a cell decomposition E is called a CW complex if for each n-cell $e^n_\alpha \in E$ there exists a *characteristic map* Φ_α defined by the composition

$$D^n_\alpha \hookrightarrow X^{n-1} \coprod_\alpha D^n_\alpha \longrightarrow X^n \hookrightarrow X\,.$$

For further reading, see e.g. [2, 6]. As an example, we can consider a sphere as a CW complex $(S^2, \{e^0, e^2\})$ as shown in fig. 1. We start by a 0-cell, which is homotopy equivalent to a disk. We then glue a 2-cell (disk) along the boundary of the 0-cell to form a sphere.

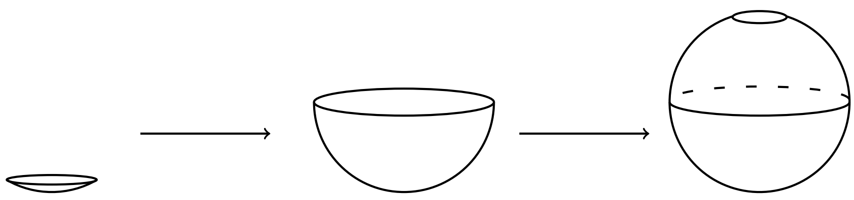


FIGURE 1. The figure shows how S^2 has the structure of a CW complex $e^0 \sqcup e^2$.

Following [9] we will now give a description of the smooth analog of a CW complex, which is the attachment of handles to smooth manifolds. We consider a *handle* to be an n-manifold H^λ with $\partial H^\lambda \cong D^\lambda \times D^{n-\lambda}$, where λ is called the *index* of the handle. If M denotes a manifold with boundary, we define $M \cup H^\lambda$ as $M \cup_h (D^\lambda \times D^{n-\lambda})$, where $h \colon \partial D^\lambda \times D^{n-\lambda} \longrightarrow \partial M$ is a smooth embedding. The operation $M \cup H^\lambda$ is referred to as attaching a λ-handle to M. In general, if $\partial M = \varnothing$ we have to "drill" a hole in M first, in order to create a boundary we can attach a handle to. Much like the CW complex, this gives a handle decomposition,

$$M_1 = (M_0 \sqcup D^\lambda \times D^{n-\lambda}) / \sim\,,$$

where $x \sim h(x)$ if and only if $x \in \partial D^\lambda \times D^{n-\lambda}$. In general, this manifold do not have to be smooth. For a more detailed discussion regarding smoothness and handle decompositions, see e.g. [9].

Morse functions give rise to a handle decomposition of a manifold in a natural way. Consider the space $M^a = f^{-1}((-\infty, a])$, where f is a Morse function on a manifold M. If p is a critical point of f with $f(p) = c$, and if $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact for some $\varepsilon > 0$, then $M^{c-\varepsilon}$ has the same homotopy type as $M^{c+\varepsilon} \sqcup e^\lambda$, where $\operatorname{ind}(p) = \lambda$ [13]. In [9, 18] it is proven that $M^{c-\varepsilon}$ is diffeomorphic to $M^{c+\varepsilon} \sqcup H^\lambda$, where H^λ is a λ-handle. Now, if we assume that M is a closed manifold and $\{p_i\}$ are the critical points of f with corresponding indices $\{\lambda_i\}$, then $M = \coprod_i H^{\lambda_i}$ is a natural handle decomposition.

2.2. **Stable and unstable manifolds.** From now on we will consider (M, g) to be a compact Riemannian n-manifold, and ∇ will denote the g-gradient. A particularly interesting feature of Morse functions is the solution curves $\gamma_x = \gamma_x(t)$ to the differential equation

$$\dot{\gamma}_{x_0}(t) = -\nabla f(\gamma_{x_0}), \qquad \gamma_{x_0}(0) = x_0\,. \tag{2.2}$$

Any solution curve $\gamma_{x_0}(t)$ satisfying (2.2), is called the *negative gradient flow line* of f that passes through x_0. Throughout this paper we will drop "negative", and just say "gradient flow line" when the sign in the right hand side of (2.2) does not matter. If the point $\gamma_x(0) = x_0$ is not relevant, we write $\gamma(t)$ to denote an arbitrary flow line of f.

We will later need lemma 2.5.

Lemma 2.5 (Lemma 7.4.1 in [8]). *There is a constant c_0 with the property that for any gradient flow line $\gamma(t)$ satisfying* (2.2), *we have* $\|\dot{\gamma}\| \leq c_0$. *In particular, $\dot{\gamma}$ is uniformly Lipschitz continuous.*

Proposition 2.6. *Any smooth function $f\colon M \longrightarrow \mathbb{R}$ decreases along its negative gradient flow lines. In particular there are no non-constant orbits with*

$$\lim_{t\to-\infty} f(\gamma(t)) = \lim_{t\to\infty} f(\gamma(t))\,.$$

Proof. By considering the tangent to the graph of f along negative gradient flow lines we have

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) = \mathrm{d}f(\gamma(t))\dot{\gamma}(t) = \langle \nabla f(\gamma(t)), \dot{\gamma}(t)\rangle = -\|\dot{\gamma}(t)\|^2 < 0\,,$$

and the result follows. □

Proposition 2.7. *For any Morse function on M, the gradient flow lines start and end at critical points.*

Proof. A consequence of proposition 2.6 is that

$$f(\gamma(t_1)) - f(\gamma(t_2)) = \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))\,\mathrm{d}t = \int_{t_1}^{t_2} \|\dot{\gamma}(t)\|^2\,\mathrm{d}t = \int_{t_1}^{t_2} \|\nabla f(\gamma(t))\|^2\,\mathrm{d}t\,. \tag{2.3}$$

If e.g. $f_\infty := \lim_{t\to\infty} f(\gamma(t)) > -\infty$ we have for $t \in [0, \infty)$ that

$$f_0 := f(\gamma(0)) \geq f(\gamma(t)) \geq f_\infty\,,$$

since f decreases along negative gradient flow lines. This means that (2.3) gives

$$\int_0^\infty \|\dot{\gamma}(t)\|^2\,\mathrm{d}t = f_\infty - f_0 < \infty\,. \tag{2.4}$$

Due to $\dot{\gamma}(t) = -\nabla f(\gamma(t))$ and lemma 2.5, equation (2.4) implies

$$\lim_{t\to\infty} \nabla f(\gamma(t)) = \lim_{t\to\infty} \dot{\gamma}(t) = 0\,.$$

An analogous argument shows the same result in the case when $t \to -\infty$. □

Definition 2.8 (Stable and unstable manifolds). *If $f\colon M \longrightarrow \mathbb{R}$ is a Morse function, and x a critical point of f, then we define the* stable manifold *of $x \in M$ as*

$$W^s(x) := \left\{ y \in M \mid \lim_{t\to\infty} \gamma_y(t) = x \right\}\,,$$

and the unstable manifold *of $x \in M$ as*

$$W^u(x) := \left\{ y \in M \mid \lim_{t\to-\infty} \gamma_y(t) = x \right\}\,.$$

Theorem 2.9 (Corollary 7.3.1 of [8]). *The stable and unstable manifolds, $W^s(x)$ and $W^u(x)$ of a gradient flow γ for a Morse function f are injectively immersed manifolds.*

Proof. See [8, cor 7.3.1]. □

Theorem 2.10 (Stable/unstable manifold theorem). *Given a Morse function on M, and a critical point $x \in M$ of f, the tangent space of M at x can be decomposed as follows*

$$T_x M = T_x^u M \oplus T_x^s M\,,$$

where $T_x^u := T_x W^u(x)$ and $T_x^s := T_x W^s(x)$. Moreover, the spaces $W^s(x)$ and $W^u(x)$ are in fact embedded manifolds with $\dim(W^u(x)) = \mathrm{ind}(x)$ *and* $\dim(W^s(x)) = \dim(M) - \mathrm{ind}(x)$.

Proof. See e.g. [8, cor 7.4.1] or [1, thm 4.15]. □

Lemma 2.11 (Lemma 4.20 of [1]). *If $f\colon M \longrightarrow \mathbb{R}$ is a Morse function and p a critical point of f, then there are maps*

$$E^s\colon T_p^s(M) \longrightarrow W^s(p)$$
$$E^u\colon T_p^u(M) \longrightarrow W^u(p)\,,$$

which are homeomorphisms onto their images.

Proof. Without loss of generality it is enough to consider only E^s since the stable manifold of f is the unstable manifold of $-f$. In the proof of theorem 4.15 in [1] it is shown that E^s is smooth, so what is left to show is that $(E^s)^{-1}$ is continuous.

We will choose a neighborhood $p \in U \subset M$ such that p is the only critical point in U. This is possible since non-degenerate critical points are isolated. We then choose a neighborhood $V \subset T_p^s M$ around $0 \in T_p^s(M)$ such that $E^s(V) \subset U$, and such that $E|_V^s$ is a homeomorphism onto its image. Next consider a sequence (x_j) in $W^s(p)$ such that $x_j \to x \in W^s(p)$ as $j \to \infty$. Then define

$$v_j := (E^s)^{-1}(x_j), \quad v := (E^s)^{-1}(x)\,.$$

We will now assume that $v_j \not\to v$, and try to arrive at a contradiction, which will prove the theorem since that would mean that if $x_j \to x$ then $v_j \to v$ (which is continuity of $(E^s)^{-1}$). For each $t \in \mathbb{R}$, define $(x_j^t) := (\varphi_t(x_j))$, where φ_t is a gradient flow line. Then define $x^t := \varphi_t(x)$ so that $x_j^t \to x^t$ as $j \to \infty$. Then we have a corresponding sequence in the preimage defined by the following.

$$v_j^t := (E^s)^{-1}(x_j^t), \quad v^t := (E^s)^{-1}(x^t)\,.$$

Since $v_j \not\to v$ we must have $v_j^t \not\to v^t$, which in turn means that $||v_j^t|| \to \infty$ as $j \to \infty$. Otherwise there would be some $t > 0$ such that $v_j^t \in V$ for a sufficiently large j. We would then have $v_j^t \to v^t$. So if $||v_j^t|| \to \infty$ then there is a subsequence of (x_j^t) that converges to some critical point q, due to proposition 2.7 and the fact that f has a finite number of critical points if M is assumed to be compact. The terms x_j^t still lie in the open set U for a sufficiently large j, and since p was the only critical point in U by assumption, we must have $p = q$. Because of this, there would be a non-constant gradient flow line from p to itself, but this is a contradiction by proposition 2.6. Hence $v_j^t \to v^t$ and thus $v_j \to v$, and $(E^s)^{-1}$ is therefore continuous. □

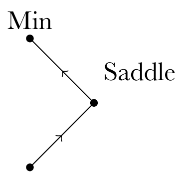
Definition 2.12 (Smale condition). *A Morse function f is said to satisfy the* Smale condition *if $W^u(x) \pitchfork W^s(y)$ (i.e. they intersect transversally) for any pair of critical points x and y, of f. If the Smale condition is satisfied, f is called a* Morse-Smale *function.*

We will from now on assume that every Morse function also is Morse-Smale if nothing else is stated. See section 2.4 for a definition of transverse intersection.

Proposition 2.13. *If f is a Morse function on M and if $p, q \in M$ are critical points such that $W^u(p) \cap W^s(q) \neq \varnothing$, then $W^u(p) \cap W^s(q)$ is an embedded manifold of dimension* $\operatorname{ind}(p) - \operatorname{ind}(q)$.

We will prove this in section 2.4, when we have discussed transversality in a little more depth and proven corollary 2.23.

Definition 2.14 (Broken flow lines). *A* broken flow line *of the gradient flow of a Morse function $f\colon M \longrightarrow \mathbb{R}$, is a flow line $\gamma(t)$ between two critical points $p \in M$ and $q \in M$ such that γ can be written as a concatenation of two (or more) gradient flow lines. That is, $\gamma = \mu \star \nu$, where $\mu \subset W^u(r) \cap W^s(p)$ and $\nu \subset W^u(q) \cap W^s(r)$ for a critical point r. If $\gamma = \nu_1 \star \nu_2 \star \cdots \star \nu_m$, then γ is a* broken flow line of order m.



It is worth mentioning that lemma 2.11 does not hold if we allow the use of broken flow lines. We will deal with this later in section 3. Let f be a Morse function and define $\mathcal{M}^f_{p,q} := W^s(p) \cap W^s(y)$. We will now define what it means for a Morse function to satisfy the Palais-Smale condition. Morse functions that satisfy this condition possess the property that the set of flow lines can be compactified by adding broken flow lines. We use the same definition as in [17, def. 2.34].

Definition 2.15 (Palais-Smale condition)**.** *Consider a function* $f\colon\, M \longrightarrow \mathbb{R}$. f *is said to fulfill the* Palais-Smale condition *if any sequence* $(x_n)_{n\in\mathbb{N}} \subset M$*, such that* $\Big(\big|f(x_n)\big|\Big)_{n\in\mathbb{N}}$ *is bounded, and* $\big|\nabla f(x_n)\big| \to 0$*, has a convergent subsequence.*

Note that f automatically satisfies the Palais-Smale condition since M is compact.

Theorem 2.16 (Theorem 7.4.1 in [8])**.** *Consider a function f that satisfies the Palais-Smale condition. For any sequence $(x_n(t))_{n\in\mathbb{N}} \subset \mathcal{M}^f_{p,q}$ we can choose a subsequence. Then there exist critical points*

$$p = p_1, \dots, p_n = q\,,$$

flow lines $y_i \in \mathcal{M}^f_{p_i,p_{i+1}}$ and $t_{n,i} \in \mathbb{R}$ where $n \in \mathbb{N}$ and $i \in \{1, \dots, k-1\}$, such that the flow lines $x_n(t + t_{n,i})$ converge to y_i for $n \to \infty$. In this case we say that $x_n(t)$ converges to the broken flow line $y_1 \star y_2 \star \cdots \star y_{n-1}$.

Proof. See e.g. [8, thm. 7.4.1.] or [17, prp. 2.35]. □

Before we discuss transversality we will introduce Morse homology in order to prove lemma 3.2.

2.3. **Morse homology.** Morse homology is a homology theory that was discovered by R. Thom, S. Smale and J. Milnor, and is defined for any smooth manifold. Morse homology is the finite dimensional analog of Floer homology.

Define $\widehat{\mathcal{M}}^f_{p,q} := \mathcal{M}^f_{p,q}/\mathbb{R}$ as the space of unparametrized flow lines. Note that proposition 2.13 implies that $\widehat{\mathcal{M}}^f_{p,q}$ is a manifold of dimension $\operatorname{ind}(p) - \operatorname{ind}(q) - 1$. In the spirit of theorem 2.16 we have the following result.

Theorem 2.17 (Proposition 2.35 in [17])**.** *If f is a Morse-Smale function that satisfies the Palais-Smale condition, the manifold $\widehat{\mathcal{M}}^f_{p,q}$ is compact up to broken flow lines of order* $\operatorname{ind}(p) - \operatorname{ind}(q)$.

Proof. See [17, p. 61] □

The content of theorem 2.17 is that by adding broken flow lines of order $\operatorname{ind}(p) - \operatorname{ind}(q)$, the space $\widehat{\mathcal{M}}^f_{p,q}$ becomes compact.

Corollary 2.18 (Corollary 2.36 in [17])**.** *If* $\operatorname{ind}(p) = \operatorname{ind}(q) + 1$, *then $\widehat{\mathcal{M}}^f_{p,q}$ is a finite set of unparametrized flow lines.*

Proof. Follows immediately from proposition 2.13. □

Let $f\colon\, M \longrightarrow \mathbb{R}$ be a Morse function on M. Denote by $\operatorname{crit}_k(f)$ the set of critical points of index k. Define $C_*(f) = \oplus_{k=0}^n C_k(f)$, where $C_k(f) = \mathbb{Z}^{\operatorname{crit}_k(f)}$ is the free Abelian group generated by the set of critical points of index k. Now define a homomorphism $\partial\colon\, C_k(f) \longrightarrow C_{k-1}(f)$ as

$$\partial(p) = \sum_{q\in\operatorname{crit}_{k-1}(f)} \#\widehat{\mathcal{M}}^f_{p,q} \cdot q\,,$$

where $\#\widehat{\mathcal{M}}^f_{p,q}$ is the number of elements in $\widehat{\mathcal{M}}^f_{p,q}$ counted with a certain sign. We will follow the convention used in [1] to define this sign. Let $\gamma_{p,q} \in \widehat{\mathcal{M}}^f_{p,q}$, and choose a basis of $T_pW^u(p)$. Then choose a basis $\hat{B}^u_p$ so that $\{\gamma'_{p,q}, \hat{B}^u_p\}$ becomes a positive basis in $T_pW^u(p)$. Choose an orientation on $W^u(q)$ and use the induced orientation on $W^s(q)$, according to

$$T_pM = T_pW^u(q) \oplus T_pW^s(q)\,.$$

Denote by B^s_q a positive basis in $T_pW^s(q)$, and parallel transport $\hat{B}^u_p$ along $\gamma_{p,q}$ to q. Now we assign the sign $+1$ to $\gamma_{p,q}$ if $\{\hat{B}^u_p, B^s_q\}$ is a positive basis in T_qM and the sign -1 otherwise. We now claim that $(C_*(f), \partial)$ is a chain complex, and define the *Morse homology* as the homology groups $H_*(C_*(f), \mathbb{Z})$. The proof of this claim, and a proof that the Morse homology is isomorphic to singular homology, is found in e.g. [1, 8, 17].

2.4. **Transversality.** The main result of this subsection is to establish the fact that transversality is a generic property. This means that we can always perturb manifolds so that their intersections become transverse, which will be desirable later, in section 3. Recall that if $S, R \subset M$ are two submanifolds of M, then we say that S and R intersect *transversally* if

$$T_x S \oplus T_x R = T_x M \,,$$

for all $x \in S \cap R$. In that case we write $S \pitchfork R$.

Definition 2.19 (Transversality of maps). *If M, N and Z are manifolds, and if $f\colon\ M \longrightarrow N$ and $g\colon\ Z \longrightarrow N$ are two smooth maps of manifolds, then we say that $f \pitchfork g$ if for every x and z such that $f(x) = g(z) = y \in N$ we have*

$$\mathrm{d}f_x\,(T_x M) \oplus \mathrm{d}g_z\,(T_z Z) = T_y N \,.$$

Note that, from this definition we can say that $f \pitchfork Z$ if for every $f(x) = z \in Z$ we have

$$\mathrm{d}f_x\,(T_x M) \oplus T_z Z = T_z N \,,$$

by choosing $g\colon\ Z \hookrightarrow N$ and using definition 2.19.

Theorem 2.20 (Local immersion theorem). *If $f\colon\ M \longrightarrow N$ is an immersion at $x \in M$ and $y = f(x) \in N$, then there exist local coordinates around x and y such that*

$$f(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0) \,,$$

i.e. so that f is a local inclusion.

Proof. Since the differential of f has constant (max) rank in a neighborhood of x, this is a direct consequence of the rank theorem, c.f. [16, thm 9.32]. ☐

Lemma 2.21. *If $f\colon\ M \longrightarrow N$ is a smooth map of manifolds, then if $L \subset N$ is an embedded submanifold we have $f \pitchfork L$ for all regular values $l \in \mathrm{im}(f)$.*

Proof. If $f(k) = l$ is a regular value then $\mathrm{d}f_k(T_k M) = T_l N$, so in particular we have $\mathrm{d}f_k(T_k M) \oplus T_l L = T_l N$, which means that $f \pitchfork L$ by definition. ☐

Theorem 2.22 (Inverse image theorem). *Let $Z \subset N$ be an immersed submanifold, and $f\colon\ M \longrightarrow N$ a smooth map. If $f \pitchfork Z$ then $f^{-1}(Z) \subset M$ is a submanifold with*

$$\mathrm{codim}\left(f^{-1}(Z)\right) = \mathrm{codim}(Z) \,.$$

Proof. By theorem 2.20 it suffices to prove this theorem locally. So this means that we can choose a chart on N such that (N, Z) is represented by $(U \times V, U \times \{0\})$ where $U \times V \subset \mathbb{R}^z \times \mathbb{R}^{n-z}$ is an open neighborhood of $(0,0)$. The map $f\colon\ f^{-1}(U \times V) \longrightarrow U \times V$ is transverse to $U \times \{0\}$ if and only if the map

$$g\colon\ f^{-1}(U \times V) \xrightarrow{f} U \times V \xrightarrow{\pi} V \,,$$

has 0 as a regular value due to lemma 2.21. This can be seen since if 0 is a regular value of g, then the map

$$\mathrm{d}g_0\colon\ T_{g^{-1}(0)} f^{-1}(U \times V) \longrightarrow T_0 V \,,$$

is surjective, and by definition $f \pitchfork Z = U \times \{0\}$ is equivalent with the fact that

$$\mathrm{im}(\mathrm{d}f_x) + T_z Z = T_z(U \times V) \,,$$

for $z = f(x)$. This means that since $f^{-1}(U \times \{0\}) = g^{-1}(0)$, $f^{-1}(Z)$ is a submanifold with the same codimension as Z. ☐

Corollary 2.23 (Corollary 5.12 of [1]). *If M^m and Z^z are immersed submanifolds of N^n, and $M^m \pitchfork Z^z$, then $M^m \cap Z^z$ is an immersed submanifold of N^n of dimension $m + z - n$.*

Proof. We apply theorem 2.22 to the inclusion $i\colon\ M \hookrightarrow N$, and note that

$$m - \dim(M \cap Z) = n - z \,.$$

☐

Equipped with corollary 2.23 we can prove proposition 2.13.

Proof of proposition 2.13. By theorem 2.10 $W^u(p)$ and $W^s(q)$ are embedded manifolds of M, and by definition of a Morse-Smale function they intersect transversally. So by lemma 2.11 and corollary 2.23, $W^u(p) \cap W^s(q)$ is a smooth embedded submanifold of dimension

$$\dim(W^u(p)) + \dim(W^s(q)) - m = \mathrm{ind}(q) + (m - \mathrm{ind}(p)) - m = \mathrm{ind}(q) - \mathrm{ind}(p)\,.$$

□

We will now state theorem 2.24, which is a transversality result for sections of smooth maps.

Theorem 2.24 (Theorem 14.6 in [3])**.** *Let $f\colon\ E \longrightarrow M$ be a smooth map of manifolds, and let $s\colon\ M \longrightarrow E$ be a smooth section of f, i.e. $f \circ s = \mathrm{id}_M$. Let $N \subset E$ be a submanifold, then arbitrarily close to s there exists a section $t\colon\ M \longrightarrow E$ such that $t \pitchfork N$. If the transversality condition on s is already satisfied for all points of a closed subset $A \subset M$ then one can choose the section t so that $t|_A = s|_A$.*

Proof. See [3, thm 14.6]. □

Theorem 2.25 (Transversality theorem for maps)**.** *Let $f\colon\ M \longrightarrow N$ be a smooth map and let $L \subset N$ be a smooth submanifold. Then, arbitrarily close to f there exist maps $g\colon\ M \longrightarrow N$ transverse to L. If the transversality condition on f is already satisfied at the points of a closed subset $A \subset M$, then one can choose g so that $f|_A = g|_A$.*

Proof. Consider the composition

$$M \xrightarrow{s} M \times N \xrightarrow{\pi} N\,,$$

where $s = (\mathrm{id}, f)$, and π is the projection onto the second coordinate. Then $f = \pi \circ s$ and π is a submersion, which means that every point in the image of π is a regular value. Thus $f \pitchfork L$ due to lemma 2.21, and $\pi^{-1}(L) = M \times L \subset M \times N$. We may then use theorem 2.24 to approximate the sections s of the projection π by a section t to $M \times L$. Hence, by lemma 2.21 $\pi \circ t$ is transverse to $\pi(M \times L) = L$. □

3. Linking number

Consider a two-component link $L = K_1 \sqcup K_2$ in S^3. The main idea of this section is to prove that the signed number of gradient flow lines, γ, with the property that $\gamma(t) \in K_2$ and $\gamma(s) \in K_1$ for some $s > t$, is equal to the linking number $\mathrm{lk}(K_1, K_2)$. It turns out that this only holds if one of the knots, or both, are null-homologous in the stable and unstable manifolds of critical points with index 0 and 1. In the case when this does not hold, we have have to use lemma 3.2 and take the linking with the standard cycle of 1-handles into account.

We now consider a Morse function $f\colon\ S^3 \longrightarrow \mathbb{R}$. From now on, we will assume that no knots contain any critical point. A critical point of a Morse function on S^3 has an index less than, or equal to 3 and an index 0 critical point is easily realized to be a minimum, since the interior of any small ball around this critical point lies in the stable manifold of the critical point. Analogously, a critical point of index 3 is a maximum. The critical points with index index 2 and 1 are both saddles. It is worth nothing that we consider S^3 to be $\mathbb{R}^3 \cup \{\infty\}$ (through the use of stereographic projection and identifying the north pole with a point at ∞). We will furthermore equip S^3 with the induced metric from $\mathbb{R}^4$.

Critical point of index 3: Consider a critical point $p \in S^3$ with $\mathrm{ind}(p) = 3$. From lemma 2.3 we know there is a neighborhood U (and a coordinate system) of p such that $f = f(p) - x^2 - y^2 - z^2$ throughout of $U \subset S^3$. Now considering the negative gradient flow we have an ordinary differential equation we want to solve, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = -\nabla(f(\gamma(t))), \qquad \gamma(0) = p\,. \tag{3.1}$$

If we let $\gamma(t) = (x_1, x_2, x_3)$, (3.1) becomes

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) = (2x_1, 2x_2, 2x_3)\,,$$

which gives $\gamma(t) = \left(p_1 e^{2t}, p_2 e^{2t}, p_3 e^{2t}\right)$, where $\gamma(0) = p = (p_1, p_2, p_3)$. These flow lines are just lines through the origin, pointing out from the origin. This is not very surprising since the critical point is a maximum. The surface in S^3 to which the flow lines $\gamma(t)$ are orthogonal is a 2-sphere.

Critical point of index 2: The same construction can be made as above, but instead locally around a critical point $q \in S^3$ with $\operatorname{ind}(q) = 2$ we have $f = f(q) - x^2 - y^2 + z^2$. The differential equation (3.1) then gives

$$\gamma(t) = \left(q_1 e^{2t}, q_2 e^{2t}, q_3 e^{-2t}\right) ,$$

which is not radial, but instead the surface to which the flow lines $\gamma(t)$ are orthogonal to is a hyperboloid of one sheet, whose axis of rotation is the z-axis.

Critical point of index 1: If $r \in S^3$ now is a critical point with $\operatorname{ind}(r) = 1$ we can write $f = f(r) - x^2 + y^2 + z^2$, to get

$$\gamma(t) = \left(r_1 e^{2t}, r_2 e^{-2t}, r_3 e^{-2t}\right) .$$

Analogous to the previous case, the surface to which the flow lines are orthogonal is a hyperboloid of one sheet, whose axis of rotation is the x-axis.

Critical point of index 0: This is analogous to the first case, and here we have

$$\gamma(t) = \left(s_1 e^{-2t}, s_2 e^{-2t}, s_3 e^{-2t}\right) .$$

It is again a 2-sphere that the flow lines are orthogonal to, but now the flow is inward, and we have a minimum.

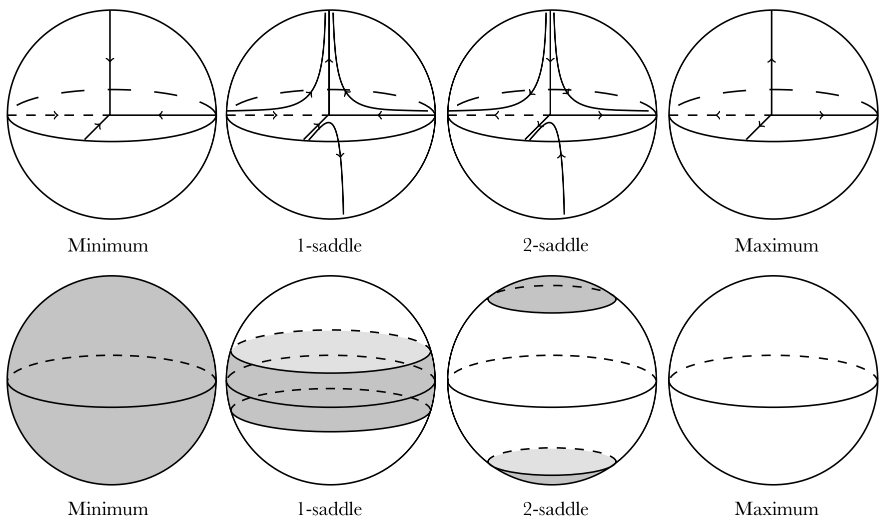


FIGURE 2. The figure shows two ways to visualize the gradient flow lines $\gamma(t)$.

In fig. 2 we can depict the local situation around a critical point of each index in S^3. Every negative gradient line roughly goes from the gray areas towards the center of the sphere, to get pushed outward towards the white areas.

In order to count the *signed* number of gradient flow lines between the knots we need to be more precise about what it means for a flow line to be signed. Let K_1 and K_2 be oriented and let γ be a gradient flow line of f such that $\gamma(t_1) \in K_1$ and $\gamma(t_2) \in K_2$ with $t_1 > t_2$. Let $\tau_1 \in T_{\gamma(t_1)}K_1$ and $\tau_2 \in T_{\gamma(t_2)}K_2$ be positive bases. We then parallel transport τ_2 and $\gamma'(t_2)$ along γ to t_1. Then we say that γ is a gradient flow line with sign $+1$, if the basis $\{\tau_1, \tau_2, \gamma'(t_1)\}$ is positive in $T_{\gamma(t_1)}S^3$ after the parallel transport has been made. Otherwise we give it a sign -1.

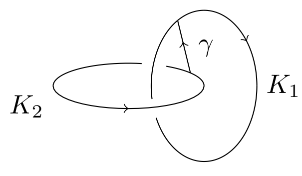


FIGURE 3. The figure illustrates a gradient flow line with sign $+1$.

All knots in S^3 are null-homologous and thus K_2 is the boundary of some 2-chain. The goal is to prove that such 2-chain can be made up from gradient flow lines of f. First off, we will consider flow lines that pass a point on K_2, and eventually ends up at a minimum. Such flow line can go directly to a minimum, or be a broken flow line that passes a critical point of index 1. Since $\dim(W^s(r)) = 1$ for $r \in \mathrm{crit}_2(f)$ we can assume that $W^s(r) \cap K_2 = \varnothing$ holds (if not, perturb K_2 slightly). It is thus sufficient to consider flow lines in the stable and unstable manifolds of critical points with indices 0 and 1. Now, it it crucial for the counting of gradient flow lines between the knots, whether the bounding chain of K_2 lies in these spaces or not. If the bounding chain of K_2 does not lie entirely in these spaces, as in fig. 4, we have to modify the formula for the linking number and count the signed number of flow lines coming from 2-saddles (see theorem 3.4).

Theorem 3.1. *Denote by Σ the set of gradient flow lines (including broken flow lines) that pass a point on a knot K_2 in S^3. Suppose that K_2 is null-homologous in the stable and unstable manifolds of critical points of index 0 and 1. Then Σ is an oriented 2-chain, with $\partial\Sigma = K_2$.*

Proof. Let $p \in \mathrm{crit}_0(f)$ and define $\delta_p := W^s(p) \cap K_2$. By theorem 2.10, $W^s(p)$ is an embedded manifold, and K_2 can be perturbed so that $W^s(p) \pitchfork K_2$ by theorem 2.25. It is thus clear that δ_p is an orientable 1-submanifold of K_2. For each $y \in \delta_p$ there is a unique flow line $\gamma^{(y)}(t)$ that passes through y. We will reparametrize each such flow line $\gamma^{(y)}(t)$ so that $\gamma^{(y)}(0) = y$, i.e. so that $\gamma^{(y)}(t) = \gamma_y(t)$. Define

$$\Delta_p(t) = \bigcup_{x \in \delta_p} \{\gamma_x(t)\} \ .$$

This means that we can write $\Delta_p([0,\infty)) = \delta_p \times [0,\infty)$, where $\Delta_p([0,\infty)) = \bigcup_{x\in\delta_p} \gamma_x([0,\infty))$. From this we can see that $\Delta_p([0,\infty))$ is an orientable 2-submanifold of $W^s(p)$, with boundary. The given orientation is the one induced from K_2 and each flow line $\{x\} \times [0,\infty) \subset \Delta_p([0,\infty))$.

We now claim that there is a finite number of points on K_2 that *do not* belong to any δ_p for any minimum p. We furthermore claim that through each such point it passes a broken flow line. Suppose that $z \in K_2$ does not belong to any δ_p. Since z is assumed not to be a critical point, there is a unique flow line $\gamma^{(z)}(t)$ that passes through z (which can be assumed to be parametrized so that $\gamma^{(z)}(t) = \gamma_z(t)$). By proposition 2.7, $\gamma_z(t)$ must end up at a saddle point q. We can assume that the index of q is 1, because if $\mathrm{ind}(q) = 2$ then $\dim(W^s(q)) = 1$ and so we can perturb K_2 so that $W^s(q) \cap K_2 = \varnothing$.

The next step is to show that the closure of $\Delta_p([0,\infty))$, $\overline{\Delta_p([0,\infty))}$, is an orientable 2-simplex. By theorem 2.16 we realize that $\overline{\Delta_p([0,\infty))}$ consists of $\Delta_p([0,\infty))$, the point p and the broken flow lines that start at points in $\partial\delta_p$ and pass a 1-saddle. Since S^3 is compact, so is $\overline{\Delta_p([0,\infty))}$. By lemma 2.3 we can find a neighborhood of any critical point $c \in U \subset \overline{\Delta_p([0,\infty))}$ such that $f = f(c) - x^2 + y^2$, for some coordinates (x, y) throughout U. By solving (3.1) in U we can construct an explicit diffeomorphism between $\overline{\Delta_p([0,\infty))}$ and an orientable 2-simplex. The orientation is the one induced by the boundary.

We will now patch the oriented simplices $\overline{\Delta_p([0,\infty))}$ together. To do this, we will use the assumption that K_2 is null-homologous in the stable and unstable manifolds of critical points with index 0 and 1. Pushing K_2 along the gradient flow lines, we end up at a 1-skeleton of 1-handles of S^3. Since K_2 is null-homologous in the stable and unstable manifolds of critical points of index 0 and 1, the 1-skeleton can be shrunk to a point. This means that a bounding chain to K_2 can be constructed as

$$\Sigma = \bigcup_{p\in\mathrm{crit}_0(f)} \overline{\Delta_p([0,\infty))}\,,$$

which can be seen as a formal sum of 2-simplices, i.e. a 2-chain. The orientation of Σ is the one given by its boundary $\partial\Sigma$. The boundary is $\bigcup_{p\in\mathrm{crit}_0(f)} \delta_p$ and the points that lie on a broken flow line, i.e. the whole of K_2. □

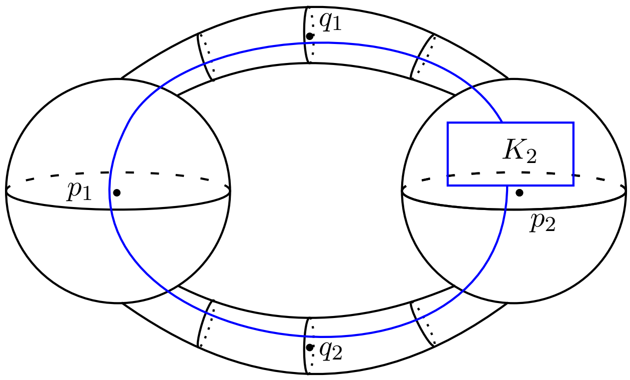


FIGURE 4. The figure shows a knot K_2, whose bounding chain does not lie entirely in the stable and unstable manifolds of $q_1, q_2 \in \mathrm{crit}_1(f)$ and $p_1, p_2 \in \mathrm{crit}_0(f)$.

In theorem 3.1 we have only considered knots that are null-homologous in the stable and unstable manifolds of critical points with index 0 and 1. By considering the knots that do not satisfy this, we see that by pushing the knot along the flow lines of f, we are left with a cycle Γ, whose boundary consists of K_2, and $A = \bigcup_{q \in \mathrm{crit}_1(f)} W^u(q)$, which satisfies $\partial\Gamma = K_2 - A$ in homology. We call A the standard cycle of 1-handles of S^3 and f, while having section 2.1.A in mind. Such a knot is depicted in fig. 4.

Lemma 3.2. *Consider the standard cycle of 1-handles of S^3 for a given Morse function f, which is denoted by*

$$A = \bigcup_{q \in \mathrm{crit}_1(f)} W^u(q)\,.$$

Then a bounding chain for A is $C = \bigcup_{r_k \in \mathrm{crit}_2(f)} \widehat{\mathcal{M}}^f_{r_k,p_k}$, where $p_k \in A$ are minima.

Proof. First consider a knot K_2 as in fig. 4. By pushing K_2 along the gradient flow lines, we end up at the standard cycle of 1-handles, A. We know that $H_1(C_*(f), \mathbb{Z}) = 0$ for S^3, so A is null-homologous, i.e. there must exist a bounding chain of A.

By using Morse homology as described in section 2.3, in this case, we see that we need something to kill the contribution to $H_1(C_*(f), \mathbb{Z})$ coming from 1-saddles. This requires us to have 2-saddles whose unstable manifolds will contain broken flow lines of order 2, that pass the 1-saddles. Since A contains a finite number of 1-saddles and minima, we see that due to a similar argument as in the proof of theorem 3.1, $C = \bigcup_{r_k \in \mathrm{crit}_2(f)} \widehat{\mathcal{M}}^f_{r_k,p_k}$ is a bounding chain of A, where $p_k \in A$ are minima. ☐

In lemma 3.2 we essentially repair the "holes" by adding the flow lines that come from 2-saddles, whose existence is ensured by Morse homology. This situation is depicted in fig. 5.

Lemma 3.3. *Consider two knots, K_1 and K_2, in S^3. The signed number of gradient flow lines that pass a point on each knot, is the same as the intersection number $I(S, K_1)$, where S is a Seifert surface of K_2.*

Proof. Assume that B is a bounding chain of K_2 that consists of gradient flow lines of f.By theorem 2.25 it is always possible to perturb the knot K_1 so that $K_1 \pitchfork B$. Since transversal intersections are isolated, there is an open neighborhood $U_i \subset B$ for each intersection point p_i, such that U_i is homeomorphic to $\mathbb{R}^2$. We can thus find a Seifert surface S of K_2 such that $S|_{U_i}$ is equal to $B|_{U_i}$ for all i. This gives us the desired result, since there is a unique gradient flow line that passes p_i, for each i, and since it is quite easy to realize that the sign given to the flow lines is the same as the sign of each intersection in $I(S, K_1)$. ☐

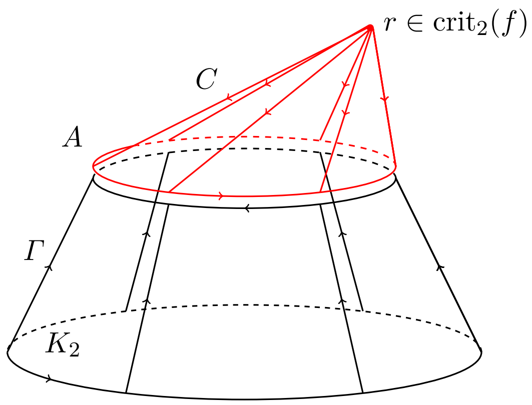


FIGURE 5. If K_2 is not null-homologous in the stable and unstable manifolds of critical points of index 0 and 1, we will have to correct the formula for the linking number by counting the flow lines coming from 2-saddles.

Theorem 3.4. *Consider a link $L = K_1 \sqcup K_2$ in S^3 together with a Morse-Smale function $f\colon\ S^3 \longrightarrow \mathbb{R}$. Let n be the signed number of gradient flow lines in the stable and unstable manifolds of critical points of index 0 and 1, γ, of f such that $\gamma(r) \in K_2$ and $\gamma(s) \in K_1$ for $s > r$. Furthermore let $r \in \mathrm{crit}_2(f)$ and $p \in \mathrm{crit}_0(f)$. Let m be the signed number of gradient flow lines $\tilde{\gamma} \in \widehat{\mathcal{M}}^f_{r,p}$ such that $\tilde{\gamma}(t) \in K_1$ and $\tilde{\gamma}(u) \in A$, for $u > t$, where*

$$A = \bigcup_{q \in \mathrm{crit}_1(f)} W^u(q)\,,$$

is the standard cycle of 1-handles of S^3 and f. Then $\mathrm{lk}(K_1, K_2) = n + m$.

Proof. First of all, the knot K_2 can either be null-homologous in the stable and unstable manifolds of critical points with index 0 and 1, or not. In the first case, theorem 3.1 tells us that $m = 0$ since the bounding chain does not intersect any unstable manifold that belongs to some 2-saddle. In lemma 3.2 we have shown that A has a bounding chain that belongs to the unstable manifolds of 2-saddles, so what is left is to construct a bounding chain of K_2 in this case. By a similar argument as in theorem 3.1, we push K_2 along the gradient flow lines until we eventually reach A. This gives a chain Γ with $\partial\Gamma = K_2 - A$ in homology. We can then glue the bounding chain C from lemma 3.2 along A, to get a bounding chain for K_2, since $\partial(\Gamma + C) = K_2$, as in fig. 5, whose orientation is the one induced by the boundary, K_2.

Having established that K_2 always can be seen as a boundary of a 2-chain that consists of gradient flow lines, we can use lemma 3.3, and the rest follows from the fact that the intersection number $I(S, K_2)$ is the linking number [15]. □

Example 3.5 (Hopf link)**.** We will now consider the Hopf link $L_{\mathrm{Hopf}} = K_1 \sqcup K_2$ and a Morse function f on S^3. As far as an explicit construction is concerned we can consider K_1 to be the curve $(\cos(\theta), \sin(\theta), 0)$ and K_2 to be the curve $(0, \sin(\theta) + 1, \cos(\theta))$, for $\theta \in [0, 2\pi]$. For simplicity we can illustrate the situation, provided that Σ does not contain a broken flow line, as in fig. 6.

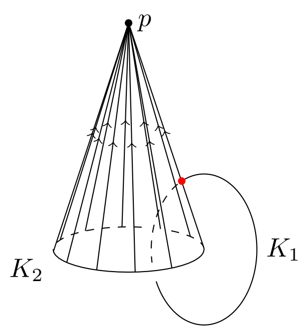


FIGURE 6. The figure illustrates the knot K_2 and the gradient flow lines all starting at K_2 and ending at a minimum, forming an oriented 2-chain.

It is clear from the geometrical point of view that there will only be one signed flow line going from K_2 to K_1. Depending on the orientations given to K_1 and K_2 we have $\mathrm{lk}(K_1, K_2) = \pm 1$.

If we instead have a situation as in fig. 5, and so that K_1 intersects C. Then it is easy to see that the sign of the intersection does not change, and the linking number remains unchanged.

4. Milnor's triple linking number

In 1957 Milnor introduced a homotopy invariant of links, which he defined purely algebraically [12]. In 2003 Mellor and Melvin disucussed a geometric interpretation of Milnor's invariant which is of our interest [4, 11]. Mellor and Melvin use Seifert surfaces, but as we have seen in lemma 3.3, transverse intersections with Seifert surfaces correspond to signed gradient flow lines between knots, which allows us to reformulate the formula for Milnor's triple linking number in terms of Morse theory. We will define signed flow trees and show that signed flow trees are in one-to-one correspondence with signed triple points of three Seifert surfaces, and we will also define the numbers m_x, m_y and m_z, defined in [4, 11] in terms of the gradient flow.

Let K_1, K_2 and K_3 be three disjoint, oriented knots in S^3 together with three generic Morse-Smale functions f_1, f_2 and f_3. Here "generic" means that all intersections between the stable and unstable manifolds among the functions are transverse. For the knot K_i we can find an oriented 2-chain Σ_i which consists of gradient flow lines of f_i, as shown in theorem 3.4. We will arrange the knots, by theorem 2.25, so that the intersections between the knots and the 2-chains are transversal.

Definition 4.1 (Flow tree). *Consider three generic Morse-Smale functions f_1, f_2 and f_3. Also consider a directed and ordered tree G with three edges e_i, three 1-valent vertices v_i and one 3-valent vertex. We then define a* flow tree *as G together with a continuous function $g\colon\ G \longrightarrow S^3$ such that each e_i injectively parametrizes a flow line of f_i, and such that each v_i is mapped to a critical point of f_i for $i \in \{1, 2, 3\}$. Furthermore the 1-valent vertices should be cyclically ordered according to their index. We will refer to a flow tree as the tree G, without explicit mention of the parametrization.*

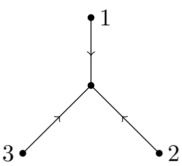


Figure 7. The figure shows a flow tree.

Lemma 4.2. *Let $f_1, f_2, f_3\colon\ S^3 \longrightarrow \mathbb{R}$ be three generic Morse-Smale functions, and let $L = K_1 \sqcup K_2 \sqcup K_3$ be a link in S^3. Then to each triple point $\tau \in \Sigma_1 \cap \Sigma_2 \cap \Sigma_3$ there corresponds exactly one flow tree up to reparametrization.*

Proof. If $\tau \in \Sigma_1 \cap \Sigma_2 \cap \Sigma_3$ then by definition of the Σ_is there are flow lines $\gamma_1 \in \Sigma_1$, $\gamma_2 \in \Sigma_2$ and $\gamma_3 \in \Sigma_3$ such that $\tau = \gamma_1(t_1) = \gamma_2(t_2) = \gamma_3(t_3)$ for some $t_1, t_2, t_3 \in \mathbb{R}$. We choose to map the 3-valent vertex to τ.

Thus we can find a directed tree G such that the edges injectively parametrizes each γ_i. It is clear that the tree G is unique up to reparametrization since we will map the 3-valent vertex to the triple point τ. □

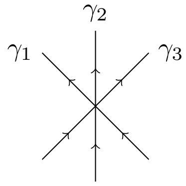


Figure 8. The figure shows how three flow lines $\gamma_1 \in \Sigma_1$, $\gamma_2 \in \Sigma_2$ and $\gamma_3 \in \Sigma_3$ intersect in the triple point τ. To obtain the flow tree we cut off the part of the flow lines that has passed τ.

We will equip a flow tree G with a sign $\operatorname{sgn}(G)$, which comes from the sign of the basis $\{v_1, v_2, v_3\}$ in $T_\tau S^3$, where v_i is a basis vector of $T_{\gamma_i(0)}K_i$ that has been parallel transported along γ_i to $\tau = \gamma_i(t_i)$. Here τ denotes the triple point of Σ_1, Σ_2 and Σ_3. If τ lies in the unstable manifold of some critical point $r_i \in \operatorname{crit}_2(f_i)$, we will consider the standard cycle of 1-handles, A_i. By lemma 3.2, A_i has a bounding chain C_i, which needs to carry the same orientation as the knot K_i, so in this case we will consider a basis vector w_i, of T_pA_i, where p is some minimum which is the end of the flow line $\gamma^{(\tau)}$. We will then parallel transport w_i backwards along the flow line $\gamma^{(\tau)}$, to τ, and consider the same kind of basis $\{z_1, z_2, z_3\}$ where $z \in \{v, w\}$. We can always perturb K_i so that the τ does not lie on a broken flow line, which means that this method works in general. We then note that the edge e_i in the flow tree G will point towards the 3-valent vertex if the corresponding basis vector z_i is v_i, and outward towards the 1-valent vertex otherwise if z_i is w_i.

Transversal intersections are isolated, so we can find three oriented Seifert surfaces S_i which coincides with Σ_i in a neighborhood U_i, of τ. We then equip τ with the sign coming from the sign of the basis $\{n_1, n_2, n_3\}$ in $T_\tau S^3$, where n_i is the normal of S_i at τ [11]. Since the orientation of the bounding chain of K_i, near K_i, is given by the tangent space of K_i and a flow line passing a point on K_i, it is quite easy to see that the sign of τ and the sign of G must in fact coincide.

To compute Milnor's triple linking number, it is not enough to count flow trees with sign. We also need to consider the pairwise linking of the knots, and the following construction. Following [4, 11], we define a product w_1 by starting at any point on K_1. We then follow it around (in the positive direction, according to the orientation) and add a factor to w_1 for each point on K_2 that lie on a signed gradient flow line coming from K_2 or K_3. If the sign of a gradient flow line γ is denoted by $\operatorname{sgn}(\gamma)$, we add the factor $y^{\operatorname{sgn}(\gamma)}$ for each flow line γ, coming from K_2 and $z^{\operatorname{sgn}(\gamma)}$ for each flow line γ, coming from K_3. We thus end up with a product containing $y^{\pm 1}$ and $z^{\pm 1}$. We can then calculate $m_x \coloneqq m_{yz}(w_1)$ as the number of times $y^{\pm 1}$ appear before $z^{\pm 1}$ minus the number of times $y^{\pm 1}$ appear before $z^{\mp 1}$ in w_1, while ignoring intervening letters. We also compute $m_y \coloneqq m_{zx}(w_2)$ and $m_z \coloneqq m_{xy}(w_3)$ in a similar way. Note that if the intersections take place outside the stable and unstable manifolds of indices of index 0 and 1, we have to adjust this method slightly, according to theorem 3.4.

Theorem 4.3. *Suppose that $f_1, f_2, f_3\colon\ S^3 \longrightarrow \mathbb{R}$ are three generic Morse-Smale functions, and $L = K_1 \sqcup K_2 \sqcup K_3$ a three-component link in S^3. Let t be the signed number of flow trees (up to reparametrization). Furthermore let m_x, m_y and m_z be the numbers obtained as discussed before, by counting gradient flow lines with sign. Then we have*

$$\overline{\mu}(L) = m_x + m_y + m_z - t \pmod{\gcd(p, q, r)},$$

where $\overline{\mu}(L)$ is Milnor's triple linking number, and

$$\begin{aligned} p &= \operatorname{lk}(K_1, K_2) \\ q &= \operatorname{lk}(K_1, K_3) \\ r &= \operatorname{lk}(K_2, K_3). \end{aligned}$$

Proof. Since f_1, f_2 and f_3 are assumed to be generic we have $\Sigma_1 \pitchfork \Sigma_2 \pitchfork \Sigma_3$. By lemma 4.2 each triple point of Σ_1, Σ_2 and Σ_3, corresponds to a unique flow tree both counted with sign. Our definition of m_x, m_y and m_z is easily seen to be equivalent with the corresponding numbers defined in [4, 11], due to lemma 3.3 and theorem 3.4. The proof that $m_x + m_y + m_z - t \pmod{\gcd(p, q, r)}$ is Milnor's triple linking number can be found in e.g. [11]. □

Example 4.4 (Borromean rings)**.** We will consider the Borromean rings which are the simplest Brunnian link, i.e. a link where the pairwise linking number is zero.

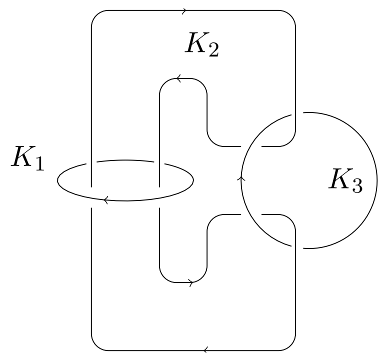


FIGURE 9. The figure shows the Borromean rings

From fig. 9 we get the following (for a given choice of starting point on each knot) values of w_1, w_2 and w_3.

$$\begin{aligned} w_1 &= 1 \\ w_2 &= xz^{-1}x^{-1}z \\ w_3 &= y^{-1}y = 1\,. \end{aligned}$$

As far as computation is concerned, it is easier to consider Seifert surfaces, rather than 2-chains of flow lines. We will consider Seifert surfaces as disks in the same plane as its boundary (the knot). We immediately get $m_x = m_z = 0$ from these values. In the second expression we have z^{-1} appearing before x^{-1} hence contributing $+1$ to m_y. The last z does not appear before any $x^{\pm 1}$, so this z does not contribute to m_y. It is also clear from the picture that if we choose Seifert surfaces to be disks in the same plane as its boundary, $t = 0$, whereas we have

$$\mu(L_{\text{Borromean}}) = 0 + 1 + 0 - 0 = 1\,.$$

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