The GHP formalism, with applications to Petrov type III spacetimes

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Abstract

We give a review of the construction and application of spinor fields in general relativity and an account of the spinor-based Geroch-Held-Penrose (GHP) [1] formalism. Specifically, we discuss using the GHP formalism to integrate Einstein's equations as suggested by Held [2, 3] and developed by Edgar and Ludwig [4–7] and discuss the similaritites with the Cartan-Karlhede classification [8] of spacetimes. We use this integration method to find a one-parameter subclass and a degenerate case, for which the Cartan-Karlhede algorithm terminates at second order, of the Petrov type III, vacuum Robinson-Trautman metrics [9, 10, Ch. 28]. We use the GHP formalism to find the Killing vectors, using theorems by Edgar and Ludwig [7]. The one-parameter family admits exactly two Killing fields, whereas the degenerate case admits three and is Bianchi type VI. Finally we use the Cartan-Karlhede algorithm to show that our class, including the degenerate case, is equivalent to a subclass found by Collinson and French [11]. Our degenerate case corresponds to an example metric given by Robinson and Trautman [9] and is known [12] to be the unique algebraically special vacuum spacetime with diverging rays and a three-dimensional isometry group.

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1 Introduction

In an introductory course on general relativity the student will learn about vectors and tensors in curved spacetime, as these are the concepts needed to state the fundamental field equation

$$R_{\mu\nu} + (\Lambda - \frac{1}{2}R)g_{\mu\nu} = 8\pi T_{\mu\nu},$$

that the matter content of spacetime determines its curvature, and describe its solutions, and to formulate physics, for example elctromagnetism, in curved spacetime. Vectors and tensors are defined by their transformation properties. However, there are both experimental and theoretical reasons to not be satisfied with only tensorial transformation laws. Experimentally we know that particles with half-integer spin cannot be described by a field with a tensorial transformation law. To accomodate such fields the formalism must be generalized [13] to include objects called *spinors*.

On the theoretical side, using spinors allows one to write Einstein's field equations and its consistency constraint, the Bianchi identity, as a simpler system of differential equations [1–3, 14]. The equations provided by these spinor formalisms are significantly more compact than the tensor form of Einstein's equations[1], but still "sufficiently frightening in appearance to quail even the stoutest of hearts" [3].

Given the large and varying catalog of known solutions [10] it seems unlikely that a single form for the most general solution can be found. One can circumvent this by using *Ansätze* to find special classes of solutions. Here the spinor formalisms provide both clear candidates for *Ansätze* and physical interpretations of them. As a striking example, Kinnsersly [15] found *all* vacuum metrics of Petrov type D.

The spinor formalism are also useful for stating and proving general theorems. For instance Newman and Penrose [14] found a much shorter proof of the theorem by Goldberg and Sachs [16] and also presented a general theorem on the asymptotic properties of gravitational radiation. The existence of the Petrov classification is most easily established using spinors. Finally, spinors can be used in the Cartan-Karlhede algorithm for classifying and determining equivalence of metrics [8].

In this work, we will first present the mathematics of spinor fields and their covariant derivatives, formalizing the idea of a transformation law using representations of groups and fiber bundles. Having these tools, we present

[1] A single component of the Riemann tensor, written out in terms of the components of the metric and their derivatives, fills close to one page. Einstein's equations involve sums of many components.

the Geroch-Held-Penrose [1] spinor formalism and an integration method within this formalism due to Held [2, 3] and elaborated on by Edgar and Ludwig [4–6], and a method of treating spacetimes with symmetries, also due to Edgar and Ludwig [7].

The GHP integration procedure is quite coordinate independent, allowing one to construct coordinates from curvature invariants. Thus we relate this method to the Cartan-Karlhede algorithm and cite an example [17] where the Cartan-Karlhede classification was made simple by the coordiantes having been constructed from invariants.

We then apply the GHP integration procedure to find a subclass of the Robinson-Trautman metrics [originally published in 9; see also 10, Ch. 28; 11] of Petrov type III. Our class is a one-parameter family of metrics and remaining in the GHP formalism, we find that it admits exactly two, commuting, Killing vectors.

Finally, we use the Cartan-Karlhede algorithm and computer algebra packages to relate our class to the known Robinson-Trautman solutions. In learning how to use the packages Sheep and Classi, the author has found the lecture notes by Skea [18], bundled with the software, very useful.

2 The Lorentz Group and Physical Quantities in Relativistic Theories

2.1 Representations of the Lie Algebra $\mathfrak{so}(1,3)$

The representations of the Lorentz group $SO(1,3)$ can be found by studying those of its Lie algebra, $\mathfrak{so}(1,3)$. For simply connected Lie groups G with Lie algebra $\mathfrak{g}$, representations of $\mathfrak{g}$ are in one-to-one- correspondence with representations of G. The Lorentz group, however, is not simply connected (for it contains the rotation group $SO(3)$ as a Lie subgroup, and $SO(3)$ contains loops which cannot be shrunk to a point). What we will obtain, then, is actually the representations of a simply connected Lie group with a Lie algebra isomorphic to that of the Lorentz group.

This group is the universal cover of the Lorentz group, and it can be shown that it is $SL(2,\mathbb{C})$ [19] and is a double cover, where both I and $-I$ in $SL(2,\mathbb{C})$ map to I in $SO(1,3)$. Some, but not all, of the representations of the covering group will be projectable to the Lorentz group, they are the ones where both $I, -I \in SL(2,\mathbb{C})$ are represented by an identity transformation. The other representations have the property that a path of transformations from I to $-I$ changes the sign of the object it acts on.

The argument to find the representations of $\mathfrak{sl}(2,\mathbb{C})$, as presented in for example Ref. [20], is to form a basis for $\mathfrak{so}(1,3)$, A_i, B_i where $i = 1,2,3$ such that the A_i and B_i separately have the commutation relations of $\mathfrak{su}(2)$ and commute with each other. The relations are $J_i = A_i + B_i, K_i = i(B_i - A_i)$ where J_i are the generators of rotations and K_i of boosts. Any physicist is familiar with the representations of $\mathfrak{su}(2)$: there is one for every half-integer spin, the j representation having dimension $2j+1$; the argument can be found in any textbook on quantum mechanics, for example [21]. Therefore every representation of $\mathfrak{sl}(2,\mathbb{C})$ can be labelled by a pair of half-integers (A, B).

The $j = \frac{1}{2}$ representation of $\mathfrak{su}(2)$ can be realized by the Pauli matrices

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus the $(\frac{1}{2}, 0)$ representation is such that J_i is represented by σ_i and K_i by $-i\sigma_i$, while the $(0, \frac{1}{2})$ representation has K_i represented by $i\sigma_i$.

We call objects in these representations spinors. They have 2 complex components and we will use capital letters $A, B, \ldots$ for $(\frac{1}{2}, 0)$ indices and dotted letters $\dot{A}, \dot{B}, \ldots$ for $(0, \frac{1}{2})$ indices. We can build any representation of $\mathfrak{sl}(2,\mathbb{C})$ from these two, including of course the representations that project to representations of $\mathfrak{so}(1,3)$, which are the familiar tensor representations. We will present this correspondence explicitly in 2.4.

2.2 Relation to Weyl and Dirac Spinors

It is also possible to construct the covering group explicitly. This method works for metrics of any signature, in any dimension. This is done by using the Clifford algebra. Let a and b be 1-forms. Associate linearly to every 1-form a (complex) matrix, such that

$$\gamma(a)\gamma(b) + \gamma(b)\gamma(a) = g(a,b)I \tag{2.1}$$

where $g(\cdot,\cdot)$ is the metric. The function γ can be extended to take k-forms as argument, through

$$\gamma(a \wedge b) := \gamma(a)\gamma(b).$$

If there is an orientation and vol is a positive volume element we can define $\Gamma = \gamma(\text{vol})$. We call this the chirality element. It is a theorem that for odd-dimensional spaces $\Gamma = I$, but for even-dimensional spaces, Γ has two eigenspaces of equal dimensions corresponding to eigenvalues $\pm\lambda$ (which can be set to ± 1 by a rescaling).

The set of all $\gamma(v)$ where $g(v,v) = \pm 1$ generates a subgroup called Pin. The subgroup where all elements are products of an even number of $\gamma(v)$:s is called Spin. If $T \in$ Pin, then

$$T\gamma(v)T^{-1} = \gamma(w) \tag{2.2}$$

for a unique w such that $g(w,w) = g(v,v)$, that is, we obtain an orthogonal transformation $v \mapsto w$. Clearly I and $-I$ belong to Spin and give the same transformation. If $T \in$ Spin, the transformation has determinant 1. One thus realizes that Spin is a double cover of SO with a covering map η. However, such elements do not necessarily preserve time orientation. We denote by $\text{Spin}^{\uparrow}$ the subgroup that does preserve time orientation, which is the identity component. It is this group that for $1+3$ dimensions is isomorphic to $SL(2,\mathbb{C})$.

If a basis e^a_μ is chosen and we let $\gamma^a = \gamma(e^a_\mu)$, the Lie algebra $\mathfrak{spin}$ is spanned by matrices of the form $\gamma^a\gamma^b$, where $a \neq b$. Indeed, if ϵ_{ab} is an element of the Lie algebra $\mathfrak{so}$, the Lie algebra isomorphism onto $\mathfrak{spin}$ given by the covering map is $\epsilon_{ab} \mapsto \frac{1}{4}\epsilon_{ab}\gamma^a\gamma^b$. We can also view it as the generators $J^{\mu\nu}$ mapping to $\frac{1}{2}\gamma^\mu\gamma^\nu$.

Let us specialize to $1+3$ dimensions. Then one set of γ matrices is

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The spinors these matrices act on are called Dirac spinors, they are the spinors appearing in the Dirac equation.

Calculating the various $\gamma^\mu\gamma^\nu$, one sees that they are block-diagonal, meaning that this representation of the $\mathfrak{spin}$ algebra is reducible. For example,

$$\gamma^0\gamma^3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -i \begin{pmatrix} K_{3,(\frac{1}{2},0)} & 0 \\ 0 & K_{3,(0,\frac{1}{2})} \end{pmatrix}$$

$$\gamma^2\gamma^3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} = -i \begin{pmatrix} J_{1,(\frac{1}{2},0)} & 0 \\ 0 & J_{1,(0,\frac{1}{2})} \end{pmatrix}$$

so that in fact, the representation is equivalent to $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$.

If e^0,e^1,e^2,e^3 is positively oriented, $\mathrm{vol}=e^0\wedge e^1\wedge e^2\wedge e^3$ and

$$\Gamma=\gamma(\mathrm{vol})=\gamma^0\gamma^1\gamma^2\gamma^3=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

so the top two components correspond to the chirality -1 eigenspace and the bottom two to the $+1$ eigenspace. Let us call these eigenspaces left-handed and right-handed, so $W=W_L\oplus W_R$. But since the $\gamma^\mu\gamma^\nu$ matrices are block-diagonal in the same way in the same basis, $W_L=(\frac{1}{2},0),W_R=(0,\frac{1}{2})$. Thus we call $(\frac{1}{2},0)$ spinors left-handed and $(0,\frac{1}{2})$ spinors right-handed.

Space and time inversion Finally we note for completeness that the Weyl spinors form representations only of the proper Spin group. The Pin group includes a parity-reversing element P (actually, two). From the characterisation of left- and right-handed spinors, which depended on the orientation, it is physically obvious that P swaps eigenvalues of Γ, but let us check: take $\Gamma=\gamma^0\gamma^1\gamma^2\gamma^3$. From (2.2) and (2.1), we see that $\gamma^0\gamma^i\gamma^0=-\gamma^i,\gamma^0\gamma^0\gamma^0=\gamma^0$, so $\pm\gamma^0$ project to P. But γ^0 anticommutes with Γ, so if $\Gamma\psi=\lambda\psi$,

$$\Gamma P\psi=\Gamma\gamma^0\psi=-\gamma^0\Gamma\psi=-\lambda P\psi$$

so $P\psi$ is an eigenspinor with eigenvalue $-\lambda$. Since in the handed representations Γ has only one eigenvalue, they cannot be extended to representations of the Lorentz group including space inversion in this way. A similar argument for the time inversion $\pm\gamma^1\gamma^2\gamma^3$ shows that time inversion swaps the eigenvalue of Γ, too. However, the combination PT can be included.

2.3 Conjugate Representations and Spinors

We follow initially the account in [22]. If W is a complex vector space, then the set of *anti*linear $\mathbb{C}$-valued functions on W is also a vector space, called the conjugate dual space $\overline{W}^*$. The space of linear $\mathbb{C}$-valued functions on $\overline{W}^*$ is called the conjugate space to W and denoted $\overline{W}$.

If $u\in W,\psi\in\overline{W}^*$, then we can define $\overline{u}=f_u$ where f_u is the function $f_u(\psi)=\psi(u)$ (“the hunted becomes the hunter” [23, p. 641]). Clearly $\overline{u}$ is antilinear. If the spaces are finite-dimensional, as they will be for us, $\bar{\cdot}$ is an anti-isomorphism (then $W,\overline{W}^*$ and $\overline{W}$ have the same dimension, and $\bar{\cdot}$ is easily seen to have trivial kernel). Further, if $A:W\to W$ is a linear transformation, we have

$$\overline{Au} = f_{Au} = f_w = \overline{v}$$

for some $f_w = \overline{v} \in \overline{W}$. We get an induced transformation $\overline{A} : f_u \mapsto f_{Au}$. It is seen that $\overline{A}(\overline{u} + \overline{v}) = \overline{A}\overline{u} + \overline{A}\overline{v}$ and we have

$$\overline{A}(af_u) = \overline{A}(f_{\overline{a}u}) = f_{\overline{a}Au} = af_{Au}$$

so $\overline{A}$ is a linear transformation. From finite-dimensionality, $A \mapsto \overline{A}$ is again an anti-isomorphism $\operatorname{End} W \to \operatorname{End}(\overline{W})$ on the linear transformations $W \to W$. The conjugation operation defines a *functor* in the category of complex vector spaces.

If the vectors $\mathcal{B} = \{b_i\}$ form a basis for W, $\overline{b}_i$ form a basis $\overline{\mathcal{B}}$ for $\overline{W}$. The components of $\overline{u}$ with respect to $\overline{\mathcal{B}}$ are the complex conjugates of u's components with respect to $\overline{B}$. Similarly, if A is represented as a matrix with respect to B, the matrix representation of $\overline{A}$ with respect to $\overline{\mathcal{B}}$ is given by taking the complex conjugate elementwise. While viewed in this way the bar operation is trivial, we have defined it intrinsically, and in a way that shows that it is a *natural transformation*.

Now if $\rho : \mathfrak{g} \to \operatorname{End}(W)$ is a complex representation of the *real* Lie algebra $\mathfrak{g}$, then $\overline{\rho} : \mathfrak{g} \to \operatorname{End}(W)$ defined in the obvious way, is also a represenation of $\mathfrak{g}$. This is because for a real Lie algebra, the structure constants are all real, thus $[\overline{\xi_a}, \overline{\xi_b}] = \overline{[\xi_a, \xi_b]} = \overline{f^c_{ab}\xi_c} = f^c_{ab}\overline{\xi_c}$ (here we of course use that in a representation the Lie bracket is the commutator). So for real Lie algebras, we obtain a new representation, the conjugate representation. For the complexified Lie algebra we extend the bar anti-linearly, and then obtain

$$[\overline{i\xi_a}, \overline{i\xi_b}] = -[\overline{\xi_a}, \overline{\xi_b}] = -\overline{[\xi_a, \xi_b]} = -f^c_{ab}\overline{\xi_c} = \overline{i}f^c_{ab}\overline{i\xi_c} = \overline{if^c_{ab}i\xi_c}$$

so the bar is still a Lie algebra homomorphism. Note it is absolutely necessary to extend *anti*-linearly!

Now coming back to the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations, direct calculation using the Pauli matrices shows that the $\frac{1}{2}$ representation is equivalent to its conjugate. Therefore, $\overline{(\frac{1}{2}, 0)}$ has J_i represented by σ_i and K_i represented by $i\sigma_i$ (antilinearity!). But that is precisely the $(0, \frac{1}{2})$ representation. We conclude that $(\frac{1}{2}, 0)$ spinors are left-handed and $(0, \frac{1}{2})$ spinors are right-handed, and conjugation swaps handedness.

2.4 More Spinor Algebra

Terminology Let us first recap the terminology and notation. Spinors in the $(\frac{1}{2}, 0)$ representation are called left-handed or simply spinors and have undotted indices. Spinors in the $(0, \frac{1}{2})$ representation are called right-handed or conjugate spinors, and have dotted indices.

Any functorial construct on vector spaces can be used to construct new representations. Some basic functorial constructions are familiar: they include the dual; the direct sum; as we saw in the previous subsection, complex conjugation, and the tensor product. Two other useful functors are the symmetric tensor product and the alternating tensor product, and one can also construct tensors with symmetries such as $R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab}$ (the symmetries of the Riemann tensor). These latter functors are examples of Schur functors, which formalize the idea of applying a combination of commutators and anticommutators to the indices of a tensor. Such constructions may of course lead to reducible representations; we will see below that a tensor with Riemann symmetry is reducible in a physically meaningful way.

Vectors and tensors We can take the direct product of the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations. Since 0 is the trivial representation, we have

$$(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2}).$$

This is a 4-dimensional representation, in which we have

$$J_{(\frac{1}{2},\frac{1}{2})} = J_{(\frac{1}{2},0)} \otimes I + I \otimes J_{(\frac{1}{2},0)}$$

which is familiar to us as the addition of two spins $\frac{1}{2}$, which is the direct sum of spin 0 and spin 1. This means that the $(\frac{1}{2}, \frac{1}{2})$ representation, restricted to rotations, is reducible to the sum of a trivial representation and a three-dimensional one. That is, if v transforms in the $(\frac{1}{2}, \frac{1}{2})$ representation, $v = v^0 \oplus v^i$ where v^i transforms like a 3-vector under rotations. The notation is suggestive, $(\frac{1}{2}, \frac{1}{2})$ is the 4-vector representation.

That v transforms properly under boosts can be argued as follows: let $|0, 0\rangle$ be the ray transforming in the trivial representation (the 0 direction) and $|1, 1\rangle, |1, 0\rangle, |1, -1\rangle$ eigenvectors of J_3 in the subspace that transforms in the 3-vector representation. These form a basis B. $|1, 0\rangle$ is annihilated by J_3, thus invariant under rotations around the 3 axis, so the $|1, 0\rangle$ component of v must be the 3-component. Write the matrix for K_3 in the basis B. It is seen to mix v^0 and v^3 in the form of a boost along the 3-axis. The argument can be repeated for the other axes.

We see that since the 4-vector representation is the product of two spinor representations, a vector v^μ can be given as a spinor tensor, $v^\mu = \sigma^\mu_{A\dot{A}} \psi^{A\dot{A}}$ for some quantity $\sigma^\mu_{A\dot{A}}$. The form of the $\sigma^\mu_{A\dot{A}}$ can be inferred from the previous argument: the 0 component is given by the projection onto the eigenspinor corresponding to $J^2 = 0$. The z component by projection onto the eigenspinor $J^2 = 1, J_z = 0$ and similarly for x and y.[2]

Having recovered the familiar 4-vector representation, any tensor representation can be formed by a suitable direct product and symmetrizations. Because these representations are equivalent in a canonical way, one can omit the $\sigma^\mu_{A\dot{A}}$ and have a clearer notation by equating pairs of dotted and undotted indices and vector indices, that is, writing for example, $F^{\mu\nu} = \psi^{A\dot{A}B\dot{B}}$. Penrose and Rindler [19, Chapter 3] discuss in detail the correspondence between spinors and what they call world-vectors and world-tensors.

A vector $v^\mu = \psi^{A\dot{A}}$ is said to be real if $\overline{v}^\mu = \overline{\psi}^{A\dot{A}} = v^\mu$. The notion extends to tensors in the obvious way. Since conjugation swaps the type of index, $\overline{\psi}$ and ψ belong to the same space only if the numbers of dotted and undotted indices are equal, and so this is the only case where we can have a notion of a spinor being real.

An important spinor tensor We know that in every representation of $\mathfrak{su}(2)$, the eigenvalues of J_i are symmetric around 0, and therefore, the trace of J_i is 0. But then in any representation of the Lie algebra of the spin group, $\mathrm{Tr}(K_i) = \mathrm{Tr}(J_i) = 0$. From the formula $\det(\exp(A)) = \exp(\mathrm{Tr}(A))$, $\det \equiv 1$ in any representation of the spin group. Consider 2-forms over $(\frac{1}{2}, 0)$ spinors. Since the spinor space is 2-dimensional, spinor 2-forms are the same thing as orientations and the definition of $\det\Lambda$ is that

$$\Lambda \cdot \epsilon_{AB} = \epsilon_{CD}(L^{-1})^C{}_A(L^{-1})^D{}_B = \det(\Lambda)\epsilon_{AB}$$

where ϵ_{AB} is a spinor 2-form, that is, an anti-symmetric tensor with two spinor down indices. Therefore, not only does the spin group preserve orientation, the volume element is invariant.

Pick any such volume element ϵ_{AB}. Then $\epsilon_{AB}\xi^B$ is a spinor 1-form and non-zero, unless $\xi^B = 0$. But then we can lower indices by $\xi^A \mapsto \xi_A = \xi^B\epsilon_{BA}$. Since ϵ_{AB} is antisymmetric, it is important to keep track of index order. The convention [22, 24] is as we have done to contract with the first index. Table 1 summarizes identites and convention. Since complex conjugation is a natural transformation, we can repeat this for conjugate spinors. A useful mnemonic is that spinor indices are raised and lowered with a contraction joining indices with an arrow like $\searrow$.

[2] That these are 3 linearly independent spinors can be checked explicitly using the matrix form of the J operators for spin-1 which can be worked out by hand or found in [21].

Table 1: Conventions and identites for raising, lowering and contracting spinor indices. W denotes the spinor space and id the identity operator. (2.3a) defines ϵ^{AB}. Each identity also has a conjugate version.

$$\epsilon^{AB}\epsilon_{BC} = -\delta^{A}{}_{C} = -\operatorname{id}_W \tag{2.3a}$$
$$\xi_A = \xi^B \epsilon_{BA} = -\xi^B \epsilon_{AB} \tag{2.3b}$$
$$\xi^A = \epsilon^{AB}\xi_B = -\xi_B \epsilon^{BA} \tag{2.3c}$$
$$\xi_A \phi^A = -\xi^A \phi_A \tag{2.3d}$$
$$\xi_A \xi^A = 0 \tag{2.3e}$$
$$\epsilon_C{}^A = \operatorname{id}_W \cong \operatorname{id}_{W^*} \tag{2.3f}$$

Tetrads and dyads Now let o^A, ι^A be such that

$$J_3 o^A = o^A \qquad J_3 \iota^A = -\iota^A \tag{2.4}$$
$$\iota_A o^A = 1 \tag{2.5}$$

One can then see that if we define

$$t^{A\dot{A}} = \frac{1}{\sqrt{2}}(o^A \overline{o}^{\dot{A}} + \iota^A \overline{\iota}^{\dot{A}}) \tag{2.6a}$$
$$x^{A\dot{A}} = \frac{1}{\sqrt{2}}(o^A \overline{\iota}^{\dot{A}} + \iota^A \overline{o}^{\dot{A}}) \tag{2.6b}$$
$$y^{A\dot{A}} = \frac{i}{\sqrt{2}}(o^A \overline{\iota}^{\dot{A}} - \iota^A \overline{o}^{\dot{A}}) \tag{2.6c}$$
$$z^{A\dot{A}} = \frac{1}{\sqrt{2}}(o^A \overline{o}^{\dot{A}} - \iota^A \overline{\iota}^{\dot{A}}) \tag{2.6d}$$

we have

$$-t^{A\dot{A}} t_{A\dot{A}} = x^{A\dot{A}} x_{A\dot{A}} = y^{A\dot{A}} y_{A\dot{A}} = z^{A\dot{A}} z_{A\dot{A}} = -1.$$

The vectors t^a, x^a, y^a and z^a are all real, and one can verify the physicality of labeling them in this way by checking that

$$J_3^{(0,\frac{1}{2})} \overline{o}^{\dot{A}} = -\overline{o}^{\dot{A}} \qquad J_3^{(0,\frac{1}{2})} \overline{\iota}^{\dot{A}} = \overline{\iota}^{\dot{A}},$$

writing out $J_i^{(\frac{1}{2},\frac{1}{2})} = J_i^{(\frac{1}{2},0)} \otimes I + I \otimes J_i^{(0,\frac{1}{2})}$ in the basis $o^A\overline{o}^{\dot{A}}, o^A\overline{\iota}^{\dot{A}}, \iota^A\overline{o}^{\dot{A}}, \iota^A\overline{\iota}^{\dot{A}}$ and checking that $x^{A\dot{A}}, y^{A\dot{A}}, z^{A\dot{A}}$ are invariant under rotations around the

$1, 2, 3$ axis respectively, and $t^{A\dot{A}}$ under all rotations.[3]

This is not the only way to form a tetrad from a dyad. One can also take

$$l^{A\dot{A}} = o^A \overline{o}^A \tag{2.7a}$$

$$m^{A\dot{A}} = o^A \overline{\iota}^A \tag{2.7b}$$

$$\overline{m}^{A\dot{A}} = \iota^A \overline{o}^A \tag{2.7c}$$

$$n^{A\dot{A}} = \iota^A \overline{\iota}^A \tag{2.7d}$$

$$\tag{2.7e}$$

which satisfies the orthogonality relations

$$l^a n_a = -1 \qquad m^a \overline{m}_a = -1$$

with the other inner products vanishing. This is called a null tetrad, as it is composed of null vectors. The vector m^a is complex. Equations (2.6a)–(2.6d) compared with (2.7) show that it is easy to construct a null tetrad from an orthonormal tetrad or vice versa.

Reduction of spinors and tensors The (A, B) classification gives all the irreducible representations of the spin group. The 4-vector representation is one of them. As remarked, this does not imply that tensors with 2 indices form an irreducible representation. In fact, the space of tensors with 2 indices is 16-dimensional, but the dimension of (A, B) is $(2A + 1)(2B + 1)$, which is odd. Working out the product $(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2}) = (0, 0) \oplus (1, 0) \oplus (0, 1) \oplus (1, 1)$ shows that the space of 2-tensors is built from one scalar, two 3-dimensional representations that are conjugates and one 9-dimensional representation. From number of components, we identify in turn these with the trace, the antisymmetric part and the traceless symmetric part, respectively.

The central idea is this: trace (contraction), symmetrization and antisymmetrization are all canonical operations, so they commute with Lorentz transformations, therefore they can be used to split tensor representations into smaller ones.

[3]The careful reader notes the similarity to the linear combinations in the decomposition of two spin 1/2 particles into a singlet and a triplet (for example [21, p. 166]), except that some signs seem to be wrong. This is because the bar changes matrix elements, we are only guaranteed to end up in a representation *equivalent to* $(0, \frac{1}{2})$ as we defined it, and then matrix elements may change signs or phases or even mix. In short, the expression is similar because it is the same two spin 1/2 into singlet and triplet decomposition, but different because it is in a different basis. It is the same decomposition because we obtain one ray that is invariant under all rotations and three rays that are invariant under one family of rotations each.

Since the spinor space is two-dimensional, every antisymmetrization gives something proportional to ϵ_{AB}. Further, confusingly, spinor traces are taken with the antisymmetric ϵ_{AB}, so if a spinor tensor is symmetric in a pair of indices, the trace over them vanishes. And

$$\epsilon^{AB}\Phi_{[AB]C\cdots} = \epsilon^{AB}\kappa\epsilon_{AB}\Upsilon_{C\cdots} = 2\kappa\Upsilon_{C\cdots} \tag{2.8}$$

so taking the trace of an antisymmetric pair of indices is just "cancelling ϵ_{AB}". We can summarize this as "only symmetric spinors matter" [24].

There is a convenient notation for the dyad components of symmetric spinors. We exemplify with the a spinor Ψ_{ABCD} which is totally symmetric. If $\{o^A, \iota^A\}$ is a dyad, we define

$$\Psi_i = \Psi_{ABCD}s_1^A s_2^B s_3^C s_4^D \tag{2.9}$$

where i of the s_k^A are ι^A, and the other are o^A. Thus for example

$$\Psi_2 = \Psi_{ABCD}\iota^A\iota^B o^C o^D = \Psi_{1100}. \tag{2.10}$$

Since Ψ_{ABCD} is totally symmetric, the order of the spinors s_k^A does not matter, or more concretely, only the number of indices with the value 1 matter. If we were to consider a a symmetric spinor with both dotted and undotted indices, we also have to specify the number of dotted indices that take the value 1. For example, if $\Phi_{\dot{A}\dot{B}CD}$ is a symmetric spinor

$$\Phi_{10'} = \Phi_{\dot{A}\dot{B}CD}\bar{o}^{\dot{A}}\bar{o}^{\dot{B}}o^C\iota^D.$$

We will use this notation from now on.

Useful identities Direct calculation shows that if $T_{ab} = T_{A\dot{A}B\dot{B}}$ is a tensor, then

$$\frac{1}{2}T_{[a,b]} = T_{(AB)[\dot{A}\dot{B}]} + T_{[AB](\dot{A}\dot{B})} \tag{2.11}$$

2.5 Fields in Curved Spacetime

If M is a spacetime and p a point in spacetime, we can consider the set of all ordered orthonormal ($x_a^\mu x_{b\mu} = \eta_{ab}$), oriented and time-oriented bases for tangent vectors at p. Let $F(M)$ be the union of all such sets, labeled by p, can be given a manifold structure: on a coordinate patch, we can take the coordinates of p together with the coordinates of the frame with respect to the coordinate vector fields $\frac{\partial}{\partial x^\mu}$ as coordinates on $F(M)$. The points in $F(M)$ are frames at points in M, so there is an obvious projection to M.

Each fiber is diffeomorphic to the Lorentz group $SO(1,3)^{\uparrow}$: pick any frame to be the identity, any other frame is related to that one by a unique proper and ortochronous Lorentz transformation. This means that $F(M)$ is a fiber bundle with fiber $SO(1,3)^{\uparrow}$ over M. The obvious right action of $SO(1,3)^{\uparrow}$ on $F(M)$ shows that $F(M)$ is a principal $SO(1,3)^{\uparrow}$-bundle.

The $\text{Spin}^{\uparrow}$ group covers the $SO(1,3)^{\uparrow}$ group. Is there a principal $\text{Spin}^{\uparrow}$-bundle, say $\tilde{F}(M)$ that covers $F(M)$ in a suitable sense? If there is one, we say that M admits spin structure. Not every manifold does, but three papers by Geroch [25–27] show that the general relativist need not lose sleep over this issue: any spacetime which is a solution of an initial value problem for Einstein's equation admits a spin structure.

The idea that fields are quantities with transformation laws can be formalized by considering pairs $[u, \xi]$ where $u \in \tilde{F}(M)$ and ξ belongs to a representation of $\text{Spin}^{\uparrow}$. Identify $[u, \xi] \sim [ug, g^{-1} \cdot \xi]$. That is, if we think of ξ as components, the components transform opposite to the frame. It can be shown [23, 28] that the set of equivalence classes has a fiber bundle structure with fiber V. The bundle is said to be associated to $\tilde{F}(M)$. We already know the tangent bundle, cotangent bundle and tensor bundles of various types. They are associated to $F(M)$ in the familiar way, since any representation of $SO(1,3)^{\uparrow}$ lifts to one of $\text{Spin}^{\uparrow}$, they are also associated to $\tilde{F}(M)$. But with $\tilde{F}(M)$ we gain also spinor bundles, that *cannot* come from any representation of $SO(1,3)$.

Let the spinor bundle $S(M)$ be the bundle associated through the $(\frac{1}{2}, 0)$ representation. We will call sections of this bundle spinor fields. Since the bar is a (smooth) functor there is a conjugate spinor bundle $\overline{S}(M)$, and it is equivalent with the bundle constructed with the $(0, \frac{1}{2})$ representation. Since $o^A \mapsto \overline{o}^A$ is a natural transformation, spinor fields can be antilinearly mapped to conjugate spinor fields.

Let J_3 be any generator of rotations. If o^A, ι^A are spinor fields such that (2.5) holds then the linear combinations in (2.6a)–(2.6d) define vector fields. The fields are everywhere orthonormal and have the same invariance properties as before, by the same calculations, so from the spin frame we obtain a tetrad. It is not surprising that we obtain a tetrad from a spin frame: the spin frame bundle covers the frame (tetrad) bundle. Likewise, we see that any tetrad defines a spin frame up to a sign: the covering is double.

We must also have that

$$g_{\mu\nu} = \epsilon_{AB} \overline{\epsilon}_{\dot{A}\dot{B}}. \tag{2.12}$$

This could actually have been guessed from the start. $g_{\mu\nu}$ is the (up to scale) unique invariant 2-tensor of $SO(1,3)$ and is symmetric.[4] $\epsilon_{AB}\overline{\epsilon}_{\dot{A}\dot{B}}$ is a symmetric 2-tensor and invariant, so it must be at most proportional to $g_{\mu\nu}$, and they can be set equal by rescaling ϵ_{AB}.

Further, the spinor tensor

$$\varepsilon_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = i(\epsilon_{AB}\epsilon_{CD}\overline{\epsilon}_{\dot{A}\dot{C}}\overline{\epsilon}_{\dot{B}\dot{D}} - \overline{\epsilon}_{\dot{A}\dot{B}}\overline{\epsilon}_{\dot{C}\dot{D}}\epsilon_{AC}\epsilon_{BD}) \tag{2.13}$$

corresponds to a real, totally antisymmetric, invariant 4-index object that transforms as a tensor under orientation-preserving transformation. There is (up to scale) only one such object: the Levi-Civita pseduo-tensor. It can be checked with (2.6a)–(2.6d) that the scale in (2.13) is $\varepsilon_{0123} = 1$.

2.6 Covariant Derivatives and Curvature

Using a principal bundle it is possible to define the covariant derivative uniformly for all associated bundles. The construction is somewhat lengthy but can be found in [28, Ch. II–III, 29, Ch. 10]. The basic idea is to pick out a subbundle of the tangent bundle of $\tilde{F}(M)$ in a way that allows parallel transport to be defined. This turns out [30, pp. 129–131] to be equivalent to associating to every local frame a $\mathfrak{so}(1,3)$-valued 1-form A, called the connection form, in a way that a certain transformation law (that of the connection coefficients, see [31, p. 262]) is satisfied. This transformation law says that under a change of frame with Λ,

$$A_\mu \mapsto \mathrm{Ad}_\Lambda(\Lambda_\mu{}^\nu A_\nu) + \text{other terms}$$

where Ad is the adjoint representation and the "other terms" prevent it from being a proper field (this is like the perhaps more familiar Christoffel symbols, which do not form a tensor).

Then in a local frame, the covariant derivative operator is

$$\nabla_\mu\varphi = \partial_\mu\varphi - A_\mu \cdot \varphi \tag{2.14}$$

where the dot means action according to the representation φ transforms in. It is well known that there is a unique way to choose A_μ so that 4-vector lengths are preserved and the torsion vanishes.

[4]Proof: Suppose $g_{\mu\nu}$ and $h_{\mu\nu}$ are two invariant 2-tensors, both not zero. Since they are invariant they can both be used to raise and lower indices, by Schur's lemma they are then invertible, write the inverse of $h_{\mu\nu}$ with up indices. But then $h^{\mu\nu}g_{\nu\sigma}$ takes 4-vectors to 4-vectors and commutes with all Lorentz transformations. Since the 4-vectors form an irreducible representation, $h^{\mu\nu}g_{\nu\sigma}$ must be proportional to δ^μ_σ, again by Schur's lemma. But then $h_{\mu\nu}$ must be proportional to $g_{\mu\nu}$.

This has the nice property that if we have the representation $\rho = \rho_1 \otimes \rho_2$, the induced Lie algebra representation is

$$d\rho = d\rho_1 \otimes I + I \otimes d\rho_2$$

and so we automatically obtain that covariant derivative has the Leibniz property with respect to tensor products.

Now from the connection 1-form one can form the curvature 2-form Ω according to Cartan's structure equation

$$\Omega_{\mu\nu} x^\mu y^\nu = (dA)_{\mu\nu} x^\mu y^\nu + [A_\mu x^\mu, A_\nu y^\nu] \tag{2.15}$$

where the brackets are the bracket in $\mathfrak{so}(1,3)$. Ω transforms in the product of the 2-form and the adjoint representations.

Ω deserves to be called the curvature form, because one can show [28, p. 133] that if X and Y are vector fields and ψ is *any* field,

$$\Omega(X,Y) \cdot \psi = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})\psi \tag{2.16}$$

where the expression on the left is the familiar definition of the Riemann curvature tensor [31, p. 271], if ψ is a vector field. This equation shows that the failure of covariant derivatives to commute is measured by a single object, which is uniquely determined by the metric.

For matrix groups, $\operatorname{Ad}_g v = gvg^{-1}$, which is the transformation law for $(1,1)$-tensors. Now if $v \in \mathfrak{so}(1,3)$ and ψ belongs to some representation ρ, we have that $(\operatorname{Ad}_g v) \cdot \psi = \operatorname{Ad}_{\rho(g)}(v \cdot \psi)$. Therefore, to any one field, we can associate a tensor which has 2-form whose value is a $(1,1)$-tensor over ψ. More concretely, this object has two 4-vector down indices in which it is anti-symmetric, one set of up indices like the field's indices, and one corresponding set of down indices. For instance, for vector fields X, Y, and left-handed spinors and right-handed spinors respectively,

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})\psi^A = \chi_{\mu\nu}{}^A{}_B \psi^B \tag{2.17a}$$

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})\psi^{\dot A} = \chi'_{\mu\nu}{}^{\dot A}{}_{\dot B} X^\mu Y^\nu \psi^{\dot B} \tag{2.17b}$$

and since 4-vectors are products of spinors,

$$\begin{aligned}(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})v^\tau &= (\chi_{\mu\nu}{}^A{}_B \bar\epsilon^{\dot A}{}_{\dot B} + \chi'_{\mu\nu}{}^{\dot A}{}_{\dot B} \epsilon^A_B) X^\mu Y^\nu v^{B\dot B} \\ &= R_{\mu\nu\sigma}{}^\tau X^\mu Y^\nu v^\sigma.\end{aligned} \tag{2.18}$$

with $R_{\mu\nu\sigma}{}^\tau$ being the standard Riemann tensor (up to an insignificant permutation of what each index means). Now since $\bar{\cdot}$ is continuous and anti-linear,

$\nabla\overline{\psi} = \overline{\nabla\psi}$, therefore

$$\chi'_{\mu\nu}{}^{\dot{A}}{}_{\dot{B}}\overline{\psi}^{\dot{B}} = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})\overline{\psi}^{\dot{B}} \tag{2.19}$$
$$\tag{2.20}$$

In analogy with $R_{\mu\nu\sigma}{}^{\tau}$ being called the curvature tensor, let us call $\chi_{A\dot{A}B\dot{B}}{}^{C}{}_{D}$ the curvature spinor.

Irreducible components of the curvature spinor Now we shall use the convention that the 4-vector index a corresponds to the pair of spinor indices $A\dot{A}$, b to $B\dot{B}$ and so on. Lowering indices with $g_{ab} = \epsilon_{AB}\overline{\epsilon}_{\dot{A}\dot{B}}$, we obtain

$$R_{abcd} = \chi_{A\dot{A}B\dot{B}CD}\overline{\epsilon}_{\dot{C}\dot{D}} + \overline{\chi}_{A\dot{A}B\dot{B}\dot{C}\dot{D}}\epsilon_{CD}. \tag{2.21}$$

Since R_{abcd} is antisymmetric in the cd pair, we can apply the identity (2.11):

$$R_{abcd} = 2\chi_{A\dot{A}B\dot{B}(CD)}\overline{\epsilon}_{\dot{C}\dot{D}} + 2\overline{\chi}_{A\dot{A}B\dot{B}(\dot{C}\dot{D})}\epsilon_{CD}$$

so $\chi_{A\dot{A}B\dot{B}CD}$ is symmetric in the CD pair. Then since R_{abcd} is antisymmetric in the ab pair, it must be that $\chi_{A\dot{A}B\dot{B}CD}$ can be written

$$\chi_{A\dot{A}B\dot{B}CD} = \Lambda_{ABCD}\overline{\epsilon}_{\dot{A}\dot{B}} + \Phi_{\dot{A}\dot{B}CD}\epsilon_{AB} \tag{2.22}$$

where

$$\Lambda_{ABCD} = \Lambda_{(AB)(CD)} \qquad \Phi_{\dot{A}\dot{B}CD} = \Phi_{(\dot{A}\dot{B})(CD)}$$

(verification: the right-hand side has the required symmetry and the same number of independent components (18 complex) as the left.)

Then (2.21) and the symmetry $R_{abcd} = R_{cdab}$ implies that Λ_{ABCD} also has exchange symmetry, $\Lambda_{ABCD} = \Lambda_{CDAB}$ and $\Phi_{\dot{A}\dot{B}CD}$ is real. We are done with $\Phi_{\dot{A}\dot{B}CD}$ since it is symmetric in all possible index pairs.

Because of exchange symmetry, $\epsilon^{AC}\Lambda_{ABCD}$ is antisymmetric in BD, therefore proportional to ϵ_{BD}. In fact,

$$\epsilon^{BD}\epsilon^{AC}\Lambda_{ABCD} = \epsilon^{BD}\kappa\epsilon_{BD} = \epsilon^{B}_{B}\kappa = 2\kappa \tag{2.23}$$

so if we define

$$\Psi_{ABCD} := \Lambda_{ABCD} - \Lambda(\epsilon_{AB}\epsilon_{CD} + \epsilon_{AC}\epsilon_{BD}) \tag{2.24}$$

where

$$\Lambda = \frac{1}{6}\epsilon^{BD}\epsilon^{AC}\Lambda_{ABCD} \tag{2.25}$$

it is seen to have the symmetries of Λ_{ABCD}, but Λ is chosen precisely so that the second term is the BC-antisymmetric part, so Ψ_{ABCD} is totally symmetric and cannot be reduced further. Therefore,

$$\chi_{A\dot{A}B\dot{B}CD} = \Psi_{ABCD}\bar{\epsilon}_{\dot{A}\dot{B}} + \Phi_{\dot{A}\dot{B}CD}\epsilon_{AB} + \Lambda(\epsilon_{AB}\epsilon_{CD} + \epsilon_{AC}\epsilon_{BD})\bar{\epsilon}_{\dot{A}\dot{B}}. \quad (2.26)$$

We now have 5 complex components from Ψ_{ABCD}, 1 complex scalar from Λ and 9 real components from $\Phi_{\dot{A}\dot{B}CD}$, which is the same as 21 real components. The Riemann tensor has only 20 components, but we have not used the last symmetry, the first Bianchi identity, $R_{a[bcd]} = 0 \Leftrightarrow R_{[abcd]} = 0$. This symmetry is one equation for the components, the only possible way to remove 1 real component from our decomposition is if Λ is real. This is in fact the case: use (2.13) to form the antisymmetrization $R_{[abcd]}$ using (2.21). Every part except the one with Λ is symmetric in some pair of indices, and so vanishes when contracted with the Levi-Civita. What remains is

$$\begin{aligned} 0 = i(&\epsilon_{AB}\epsilon_{CD}\bar{\epsilon}_{\dot{A}\dot{C}}\bar{\epsilon}_{\dot{B}\dot{D}} - \bar{\epsilon}_{\dot{A}\dot{B}}\bar{\epsilon}_{\dot{C}\dot{D}}\epsilon_{AC}\epsilon_{BD})\cdot \\ &\big(\Lambda(\epsilon^{AB}\epsilon^{CD} + \epsilon^{AC}\epsilon^{BD})\bar{\epsilon}^{\dot{A}\dot{B}}\bar{\epsilon}^{\dot{C}\dot{D}} + \overline{\Lambda}(\bar{\epsilon}^{\dot{A}\dot{B}}\bar{\epsilon}^{\dot{C}\dot{D}} + \bar{\epsilon}^{\dot{A}\dot{C}}\bar{\epsilon}^{\dot{B}\dot{D}})\epsilon^{AB}\epsilon^{CD}\big) \end{aligned}$$

where we used that the Λ term is symmetric in the index pair AC and the $\overline{\Lambda}$ term is symmetric in $\dot{A}\dot{C}$. They therefore vanish when contracted with the first and second part of the Levi-Civita, respectively.

$$\begin{aligned} 0 = \quad & \Lambda\epsilon_{AB}\epsilon_{CD}\bar{\epsilon}_{\dot{A}\dot{C}}\bar{\epsilon}_{\dot{B}\dot{D}}(\epsilon^{AB}\epsilon^{CD} + \epsilon^{AC}\epsilon^{BD})\bar{\epsilon}^{\dot{A}\dot{B}}\bar{\epsilon}^{\dot{C}\dot{D}} \\ & - \overline{\Lambda}\bar{\epsilon}_{\dot{A}\dot{B}}\bar{\epsilon}_{\dot{C}\dot{D}}\epsilon_{AC}\epsilon_{BD}(\bar{\epsilon}^{\dot{A}\dot{B}}\bar{\epsilon}^{\dot{C}\dot{D}} + \bar{\epsilon}^{\dot{A}\dot{C}}\bar{\epsilon}^{\dot{B}\dot{D}})\epsilon^{AB}\epsilon^{CD} \\ & = k\Lambda - \overline{k}\,\overline{\Lambda} \end{aligned}$$

but contractions of ϵ_{AB} can only give real integers, so Λ must be real.

From number of components, it is now clear that Λ is proportional to the Ricci scalar R, $\Phi_{\dot{A}\dot{B}CD}$ corresponds to the traceless Ricci tensor and Ψ_{ABCD} corresponds to the Weyl tensor. From (2.21) and (2.26) we see that

$$C_{abcd} = \Psi_{ABCD}\bar{\epsilon}_{\dot{A}\dot{B}}\bar{\epsilon}_{\dot{C}\dot{D}} + \overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}\epsilon_{AB}\epsilon_{CD}. \quad (2.27)$$

The traceless Ricci tensor is given by contracting the Φ part over b and d

$$\begin{aligned} R_{ac} &= \sum_{\dot{B}=\dot{D},B=D} 2\Phi_{\dot{A}\dot{B}C}{}^{D}\epsilon_{AB}\bar{\epsilon}_{\dot{C}}{}^{\dot{D}} \\ &= 2\Phi_{\dot{A}\dot{B}C}{}^{B}\epsilon_{AB}\bar{\epsilon}_{\dot{C}}{}^{\dot{B}} \\ &= -2\Phi_{\dot{A}\dot{C}CA} = -2\Phi_{AC\dot{A}\dot{C}}. \end{aligned}$$

The Ricci scalar is the full contraction of the Λ part, which can be found to be 24Λ and we conclude

$$\Lambda = \frac{R}{24} \tag{2.28}$$

Principal null directions Suppose that $\Psi_{ABCD} \neq 0$. Then there is a spinor ι^A such that

$$\Psi_{ABCD}\iota^A\iota^B\iota^C\iota^D = 1$$

and also a spinor o^A such that $o_A\iota^A = 1$. If we set $\alpha^A = z\iota^A + o^A$, then

$$f(z) = \Psi_{ABCD}\alpha^A\alpha^B\alpha^C\alpha^D$$

is a polynomial in z and can be factored

$$f(z) = (z - c_1)(z - c_2)(z - c_3)(z - c_4).$$

But for $i = 1, 2, 3, 4$ we can set$(\kappa_i)^A = o^A + c_i\iota^A$, or

$$z - c_i = (\kappa_i)_A\alpha^A$$

and therefore for all spinors α^A,

$$\Psi_{ABCD}\alpha^A\alpha^B\alpha^C\alpha^D = (\kappa_1)_A(\kappa_2)_B(\kappa_3)_C(\kappa_4)_D\alpha^A\alpha^B\alpha^C\alpha^D$$
$$\Psi_{(ABCD)}\alpha^A\alpha^B\alpha^C\alpha^D = (\kappa_1)_{(A}(\kappa_2)_B(\kappa_3)_C(\kappa_4)_{D)}\alpha^A\alpha^B\alpha^C\alpha^D.$$

But the action of a totally symmetric tensor on $\alpha^A\alpha^B\alpha^C\alpha^D$ is sufficient to determine the tensor: let ι^A, o^A be a basis so that $\iota^A = \delta^A_0, o^A = \delta^A_1$. Then letting α^A have the five values $\iota^A, o^A, \iota^A + o^A, \iota^A - o^A, \iota^A + io^A$ yields five linearly indepent equations for the five independent components of Ψ_{ABCD}. We conlude that in fact

$$\Psi_{ABCD} = (\kappa_1)_{(A}(\kappa_2)_B(\kappa_3)_C(\kappa_4)_{D)}. \tag{2.29}$$

and refer to κ_i as the *principal spinors*. Since a spinor determines a null direction through $\kappa_i^A\overline{\kappa}_i^{\dot{A}}$, we obtain (up to) four null directions for the Weyl tensor:

$$\Psi_{ABCD} = (\kappa_1)_{(A}(\kappa_2)_B(\kappa_3)_C(\kappa_4)_{D)} \Rightarrow \Psi_{ABCD}(\kappa_1)^A = 0 \Rightarrow C_{abcd}\kappa^A\overline{\kappa}^{\dot{A}} = 0.$$

If some of the null directions coincide, the spacetime is said to be algebraically special. The statement can be further refined in the so-called Petrov classification, where each Petrov type corrseponds to an integer partition of 4 according to how the principal null directions coincide. Table 2 shows the Petrov types.

Table 2: The Petrov classification. In this table, α_A is not proportional to β_A, and so on. Type I is the algebraically general case.

Petrov type	Ψ_{ABCD}	Partition
I	$\alpha_{(A}\beta_B\gamma_C\delta_{D)}$	$1+1+1+1$
II	$\alpha_{(A}\alpha_B\beta_C\gamma_{D)}$	$2+1+1$
D	$\alpha_{(A}\alpha_B\beta_C\beta_{D)}$	$2+2$
III	$\alpha_{(A}\alpha_B\alpha_C\beta_{D)}$	$3+1$
N	$\alpha_{(A}\alpha_B\alpha_C\alpha_{D)}$	4
O	0	

For each Petrov type except O, one can effect a reduction of the structure group by considering dyads where the basis spinors are parallel with the principal spinors. For type N, one basis spinor can be fixed completely, the structure group is reduced to the subgroup of $SL(2,\mathbb{C})$ that leaves this spinor invariant, that is the group of matrices of the form $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ for $z \in \mathbb{C}$, which is $\mathbb{C}$ with addition. For type D, the dyad (α_A, β_A) is fixed up to rescaling $\alpha_A \mapsto z\alpha_A, \beta \mapsto z^{-1}\beta_A$ and swapping α_A and β_A, which is $(\mathbb{C}, *) \times \mathbb{Z}_2$. For types I, II and III, the dyad can be fixed up to a discrete group [32].

A physical interpretation of the Petrov types is provided by a theorem given by Newman and Penrose [14]. The Weyl spinor is the part of the gravitational field that is not related to matter (since Einstein's equations couple only the Ricci tensor and scalar to matter) so it should be characteristic of gravitational radiation, the Newtonian gravitational field and frame-dragging effects. Newman and Penrose show that the most general type I corresponds to the near field of gravitational radiation, while type N corresponds to the radiation zone, with types II and III corresponding to transition zones.

Segré classification The Ricci spinor can be classified as belonging to one of 14 different algebraic types [10, Ch. 5, 32] in a scheme called the Segré classification. Like with the Petrov types, for each Segré type one can choose a standard dyad, at least up to some subgroup, that is, effect a reduction of the structure group. Several cases common in physics, for example when the matter content of spacetime is a perfect fluid or a radiation field, coincide with a Segré type, and so a standard form for the Ricci spinor can be chosen in these cases. In general the Ricci spinor and the Weyl spinor cannot both be put in standard form.

Table 3: The spin coefficients.

$A\dot{A}$ \ B^C	$_0{}^0$	$_0{}^1$	$_1{}^0$	1^1
$0\dot{0}$	ε	$-\kappa$	$-\tau'$	$-\varepsilon$
$0\dot{1}$	$-\beta'$	$-\rho$	$-\sigma'$	β'
$1\dot{0}$	β	$-\sigma$	$-\rho'$	$-\beta$
$1\dot{1}$	$-\varepsilon'$	$-\tau$	$-\kappa'$	ε'

2.7 Ricci rotation coefficients and spin coefficients

Covariant differentiation is encoded in a $\mathfrak{so}(1,3)$-valued 1-form, the connection form. Using the Lie algebra isomorphism provided by the covering map $SL(2,\mathbb{C}) \to SO(1,3)^{\uparrow}$, we can regard it instead as a $\mathfrak{sl}(2,\mathbb{C})$-valued form. It is easy to realize that the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is the space of complex traceless 2×2 matrices. It is 6-dimensional as a real space, so 3-dimensional as complex space. Thus in the first case we have $4 \cdot 6$ real components, and in the latter $4 \cdot 3$ complex components.

We can use the expression for the covariant derivative in a frame (2.14) applied to the frame fields to extract the components of the connection form. Thus, the covariant derivative of a 1-form becomes

$$\begin{aligned} e_a^\mu (e_b)_\nu (\nabla_\mu e_c)_\nu &= e_a^\mu (e_b)_\nu (\partial_\mu (e_c)_\nu - (A_\mu \cdot e_c)_\nu) \\ &= 0 - e_a^\mu (e_b)_\nu (\gamma_{abc})^\sigma e_{\sigma c} \\ &= \gamma_{abc}. \end{aligned}$$

The use of a different kernel letter is justified since what we are really extracting is the coefficients of a representation of A, the components in the 1-form representation. We call the γ_{abc} the Ricci rotation coefficients.

One can do the same thing with spinors, of course. Let $o^A, \iota^A, o^A \iota_A = 1$ be a dyad and use dyad indices $\phi_0^A = o^A, \phi_1^A = \iota^A$. Then

$$\phi_A^X (\overline{\phi}_{\dot{A}})^{\dot{X}} \phi^B{}_Y (\nabla_{X\dot{X}} \phi_C)^Y = -\gamma_{A\dot{A}}{}^B{}_C.$$

The $\gamma_{A\dot{A}}{}^B{}_C$ are called the spin coefficients. At face value there are 16 of them, but we have the tracelessness condition $\gamma_{A\dot{A}}{}^B{}_B = 0$ that eliminates 4 of them. We are left with 12 complex quantites, or 24 real.

2.8 Writing Einstein's Equations in Spinor Form, or the NP Formalism

The application to general relativity is now rather straightforward. The relativistic description of the gravitational field consists of the relation between the connection form and the curvature form, Einstein's field equations as equations for the Ricci tensor, and the Bianchi identities as consistency constraints. They are most familiarly written in the world-tensor form, but since the spin coefficents and the curvature spinor are equivalent with the Christoffel symbols and the Riemann world-tensor, the spinor form can also be used.

In the Newman-Penrose (NP) formalism [14] one starts by choosing a null tetrad or dyad, as in (2.7) and introducing the NP operators $D := l^a\nabla_a, \Delta := n^a\nabla_a, \delta := m^a\nabla_a, \overline{\delta} := \overline{m}^a\nabla_a$. One labels each spin coefficient with respect to the chosen dyad with a Greek letter according to Table 3. The equations of general relativity then take the form of (equation numbers refer to [14])

- 18 equations (4.2) relating the spin coefficients and the NP operators acting on them to the dyad components of the Weyl spinor and the Ricci spinor, and the curvature scalar. (The Ricci equations or identities.)
- 8 equations (4.5) expressing the Bianchi identities using the components of the Riemann spinor, the NP operators acting on them, and the spin coefficients.
- 4 equations (4.4) for the commutators of the NP operators. Two of these have complex conjugates that give new, independent equations.

From Einstein's equations in world-tensor form

$$R_{\mu\nu} + (\Lambda - \frac{1}{2}R)g_{\mu\nu} = 8\pi T_{\mu\nu}$$

the matter content ($T_{\mu\nu}$) of spacetime determines the curvature scalar, and the components of the traceless Ricci tensor. From the latter, the dyad components of the Ricci spinor can be found, and this is how Einstein's equations enter the NP formalism.

3 The GHP Formalism

We give an account of the Geroch-Held-Penrose formalism, originally presented in [1]. The idea is similar to the orthonormal tetrad formalism. In general, a vector field X on a manifold can be written as $X^\mu = f^a(x)e^\mu_a(x)$[5]

[5]Here we use an upper index $^\mu$ to indicate that the objects are vectors, it is not to be taken as a reference to some (coordinate) basis – the e^μ_a are coordinate independent objects and form a basis in their own right. Compare with the introduction in [1] or the discussion in [19].

for some functions $f^a(x)$ and basis vector fields e^μ_a, that are linearly independent at every point in some region. If $e^\mu_{\hat{a}}$ is another set of basis vector fields, we in general have

$$e^\mu_{\hat{a}} = M(x)^a_{\hat{a}} e^\mu_a \qquad f^{\hat{a}} = (M^{-1}(x))^{\hat{a}}_a f^a(x) \tag{3.1}$$

for some $n \times n$ matrix-valued function. In mathematical terms, the structure group of the tangent bundle is $GL(\mathbb{R}^n)$, the group of invertible linear operators on $\mathbb{R}^n$.

However, if there is further structure available, we can achieve a reduction of the structure group. If our manifold is equipped with a metric, as is of course the case for spacetimes in general relativity, we can, given any set of e^μ_a perform the Gram-Schmidt process at each point simultaneously and and obtain e'^μ_a that are such that $e'^\mu_a e'_{\mu b} = \mathrm{diag}(1, -1, -1, -1)$. Then the matrices in (3.1) are constrained to lie in $O(1, 3)$, the Lorentz group. If the spacetime is oriented (and time-oriented), the e'^a_μ can be made to respect this and we achieve a further reduction to $SO(1, 3)^{(\uparrow)}$, the proper (and orthochronous) Lorentz group. This is the method of orthonormal tetrads.

The GHP formalism is a more sophisticated reduction of the structure group, that can be made to work if there are given two null vector fields, l^a and n^a, everywhere nonzero. They can be arranged to be future pointing and normalized so that $l^a n_a = 1$. For a complete basis, we need also X^a, Y^a that can be taken to be spacelike, $X^a X_a = Y^a Y_a = -1$ and orthogonal, $X^a Y_a = 0$. If primed fields represent some other choice fulfilling the same requirements, we must have

$$l'^a = r l^a \qquad n'^a = r^{-1} n^a \tag{3.2a}$$

$$\begin{pmatrix} X'^a \\ Y'^a \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X^a \\ Y^a \end{pmatrix} \tag{3.2b}$$

for some $r > 0$ and some angle θ, that both vary in spacetime. The structure group is then $\mathbb{R}^+ \times U(1) \cong \{\mathbb{C}^*, *\}$, that is, non-zero complex numbers with multiplication.

The transformation (3.2b) can be "diagonalized" by complexifying. We introduce the complex vector fields

$$m^a = \frac{1}{\sqrt{2}}(X^a + iY^a) \qquad \overline{m}^a = \frac{1}{\sqrt{2}}(X^a - iY^a) \tag{3.3}$$

which satisfy $m^a \overline{m}_a = -1, m^a m_a = \overline{m}^a \overline{m}_a = 0$. If $\lambda^2 = re^{i\theta}$, the transformations can be written

$$l'^a = \lambda\overline{\lambda}l^a \tag{3.4a}$$

$$n'^a = \lambda^{-1}\overline{\lambda}^{-1}n^a \tag{3.4b}$$

$$m^a = \lambda\overline{\lambda}^{-1}m^a \tag{3.4c}$$

$$\overline{m}^a = \overline{\lambda}\lambda^{-1}m^a \tag{3.4d}$$

(the last transformation is, of course, the complex conjugate of the penultimate). Thus we have a null tetrad and there are two corresponding dyads, according to (2.7), which we repeat here:

$$l^{A\dot{A}} = o^A\overline{o}^A \tag{3.5a}$$

$$m^{A\dot{A}} = o^A\overline{\iota}^A \tag{3.5b}$$

$$\overline{m}^{A\dot{A}} = \iota^A\overline{o}^A \tag{3.5c}$$

$$n^{A\dot{A}} = \iota^A\overline{\iota}^A. \tag{3.5d}$$

We see that transformations may be labeled by the two integers (p, q) ("is of type (p, q)") which we take to mean that the factor involved is $\lambda^p\overline{\lambda}^q$. $\frac{1}{2}(p-q)$ is called the spin-weight and $\frac{1}{2}(q-p)$ is called the boost-weight.

3.1 Spin- and Boost-Weighted Quantities

Just like (r, s) label representations of $SO(1, 3)$, (p, q) label representations of $\mathbb{C}^*$, on tensor products of $\mathbb{C}$, that is, on $\mathbb{C}$. If η and ζ are of types (p, q) and (u, v), $\eta \otimes \zeta$ is of type $(p+u, q+v)$. We might as well omit the $\otimes$.

Complex conjugation Since complex conjugation is a natural transformation, there is a natural bundle map between quantities of type (p, q) and of type (q, p). Denote the corresponding bundles by $E(p, q)$ and $E(q, p)$. Then if $\eta = [u, z] \in E(p, q)$, define the map $\bar{\cdot} : E(p, q) \to E(q, p)$ by $\overline{\eta} = [u, \overline{z}]$. $\bar{\cdot}$ is well-defined because for any other representative of η,

$$\overline{[u\lambda, \lambda^p\overline{\lambda}^q z]} = [u\lambda, \overline{\lambda}^p\lambda^q\overline{z}] = [u, \overline{z}] = \overline{\eta}.$$

Clearly $\bar{\cdot}$ preserves fibers and is conjugate-linear.

Vectors and spinors of type (p, q) Now consider the bundle $TM_{(p,q)} := TM \otimes E(p, q)$. As a vector bundle, $E(p, q)$ is the trivial complex line bundle, so as a vector bundle $TM_{(p,q)}$ is just the complexification of the tangent bundle. Therefore, it makes sense to talk about vectors of type (p,q). The transformations (3.4a)–(3.4d) show that l^μ is a vector of type $(1,1)$, n^μ of type $(-1,-1)$, m^μ of type $(1,-1)$ and $\overline{m}$ is of type $(-1,1)$. Note that the last two are consistent with the bar operation from the previous paragraph. Spinors are naturally complex, so the concept of a spinor of type (p,q) makes sense immediately. Naturally, if o^A is a spinor of type (p,q), $\overline{o}^{\dot{A}}$ is a spinor of type (q,p).

The GHP prime operation Suppose that a tensor $T_{\mu\nu}$ of type $(0,0)$ (an ordinary tensor) is given. Its tetrad components are $T_{\mu\nu}m^\mu l^\nu$ and so on. These have weights, because the tetrad vectors do (a tensor with down indices eats vectors and gives scalars; if it eats weighted vectors, it gives weighted scalars). The discrete transformation, denoted by a prime, $'$, given by

$$o^A \mapsto i\iota^A \qquad \iota^A \mapsto io^A \qquad \overline{o}^{\dot{A}} \mapsto -i\overline{\iota}^{\dot{A}} \qquad \overline{\iota}^{\dot{A}} \mapsto -i\overline{o}^{\dot{A}}$$

preserves the dyad normalisation. On the null tetrad it acts like

$$l^a \mapsto n^a \quad n^a \mapsto l^a \quad m^a \mapsto \overline{m}^a \quad \overline{m}^a \mapsto m^a$$

and preserves the tetrad normalisation. There are other discrete transformations like this, they have been studied by Ludwig [33].

Now if $\alpha = T(l^a, m^a)$, then we can define

$$\alpha' := ('^*T)(l^a, m^a) = T((l^a)', (m^a)').$$

Of course the operation is extended naturally to the general case. Since $'$ negates the weights of the dyad spinors, we have that if α is of type (p,q), α' is of type $(-p,-q)$.

It is important to note that $'$ acting on quantities is linear in the sense that $(a\alpha+b\beta)' = a\alpha'+b\beta'$, but $\alpha = 0$ does not imply $\alpha' = 0$. This is because $'$ acts on functions that when evaluated for certain vectors give α and β, not on α and β. Keeping this in mind, the prime is a convenient notational tool.

3.2 The spin coefficients

Now that we have the dyad and the prime operation, we can restate Table 3. The spin coefficients are

$$\kappa = o^A\bar{o}^{\dot{A}}o^B\nabla_{A\dot{A}}o_B \tag{3.6a}$$
$$\sigma = o^A\bar{\iota}^{\dot{A}}o^B\nabla_{A\dot{A}}o_B \tag{3.6b}$$
$$\rho = \iota^A\bar{o}^{\dot{A}}o^B\nabla_{A\dot{A}}o_B \tag{3.6c}$$
$$\tau = \iota^A\bar{\iota}^{\dot{A}}o^B\nabla_{A\dot{A}}o_B \tag{3.6d}$$
$$\kappa' = -\iota^A\bar{\iota}^{\dot{A}}\iota^B\nabla_{A\dot{A}}\iota_B \tag{3.6e}$$
$$\sigma' = -\iota^A\bar{o}^{\dot{A}}\iota^B\nabla_{A\dot{A}}\iota_B \tag{3.6f}$$
$$\rho' = -o^A\bar{\iota}^{\dot{A}}\iota^B\nabla_{A\dot{A}}\iota_B \tag{3.6g}$$
$$\tau' = -o^A\bar{o}^{\dot{A}}\iota^B\nabla_{A\dot{A}}\iota_B \tag{3.6h}$$

and

$$\beta = o^A\bar{\iota}^{\dot{A}}\nabla_{A\dot{A}}o_B \tag{3.7a}$$
$$\varepsilon = o^A\bar{o}^{\dot{A}}\nabla_{A\dot{A}}\iota_B \tag{3.7b}$$
$$\beta' = -\iota^A\bar{o}^{\dot{A}}\nabla_{A\dot{A}}\iota_B \tag{3.7c}$$
$$\varepsilon' = -\iota^A\bar{\iota}^{\dot{A}}\nabla_{A\dot{A}}o_B. \tag{3.7d}$$

Under a transformation $o^A \mapsto \lambda o^A, \iota^A \mapsto \lambda^{-1}\iota^A$, the coefficients in (3.6) transform like weighted scalars; the terms with derivatives on λ have no derivatives on o_B or ι_B, which is then contracted with a matching spinor, giving 0. However, the spin coefficients in (3.7) do not transform like weighted scalars. The weights are

$$\kappa : (3,1) \qquad \sigma : (3,-1) \qquad \rho : (1,1) \qquad \tau : (1,-1)$$
$$\kappa' : (-3,-1) \qquad \sigma : (-3,1) \qquad \rho : (-1,1) \qquad \tau : (-1,1).$$

Transformation of the spin coefficients The Lorentz transformation

$$\begin{aligned} l^a &\mapsto l^a \\ m^a &\mapsto m^a + \overline{Z}l^a \\ \overline{m}^a &\mapsto \overline{m}^a Z l^a \\ n^a &\mapsto n^a + Zm^a + \overline{Z}\overline{m}^a + Z\overline{Z}l^a \end{aligned}$$

with Z a complex scalar field of weight $(-2, 0)$, clearly preserves the null vector l^a and is seen to give a null tetrad with the same normalization as $(l^a, n^a, m^a, \overline{m}^a)$. The spinor equivalent of this transformation is

$$\begin{aligned} o^A &\mapsto o^A \\ \iota^A &\mapsto \iota^A + Z o^A. \end{aligned}$$

The transformation thus deserves the name *null rotation around l^a*.

Such a null rotation is not in the group of GHP transformations, so the spin coefficients transform differently under it. It is simple but tedious to calculate the coefficients in the new tetrad. Letting the subscript 1 denote quantities with respect to the transformed tetrad and the subscript 0 quantities with respect to the transformed tetrad, we have

$$\kappa_1 = \kappa_0 \tag{3.8a}$$
$$\sigma_1 = \sigma_0 \tag{3.8b}$$
$$\tau_1 = \tau_0 + Z\sigma_0 + Z\overline{Z}\kappa_0 + \overline{Z}\rho_0 \tag{3.8c}$$
$$\rho_1 = \rho_0 + Z\kappa_0 \tag{3.8d}$$
$$\kappa'_1 = \kappa'_0 - Þ'Z + Z(\rho'_0 - ðZ) + \overline{Z}(\sigma'_0 - ð'Z) + Z\overline{Z}(\tau'_0 - ÞZ) - Z^2\tau_0 \tag{3.8e}$$
$$- Z^3\sigma_0 - Z^2\overline{Z}\rho_0 - Z^3\overline{Z}\kappa_0 \tag{3.8f}$$
$$\sigma'_1 = \sigma'_0 - ð'Z + Z(\tau'_0 - ÞZ) - Z^3\kappa_0 - Z^2\rho_0 \tag{3.8g}$$
$$\tau'_1 = \tau'_0 - ÞZ + Z^2\kappa \tag{3.8h}$$
$$\rho'_1 = \rho'_0 + \overline{Z}(\tau'_0 - ÞZ) - ðZ - Z^2\sigma_0 - Z^2\overline{Z}\kappa_0. \tag{3.8i}$$

The transformation laws under the analogous null rotation that preserves n^a are given by priming and letting $Z \to \overline{Z}$. Several of the transformations are considerably simpler in the case that $\kappa_0 = \sigma_0 = 0$.

Geometrical interpretations The spin coefficients have a fairly simple geometrical interpretation. The coefficent κ is proportional to the covariant derivative of l^a along itself. Hence $\kappa = 0$ if the integral curves of l^a are geodesics. In fact, $\kappa = 0$ is equivalent with the integral curves of l^a being geodesics [10].

If $\kappa = 0$, one can show [10] that

$$\rho := m^b \overline{m}^a \nabla_a l_b = -(\Theta + i\omega)$$

where

$$\Theta := \frac{1}{2}\nabla_a l^a \qquad \omega^2 := \frac{1}{2}\nabla_{[b} l_{a]} \nabla^b l^a.$$

One can think of a (geodesic) null vector field as the 4-velocity field of some null fluid. Since $\nabla_{[a}l_{b]} = (dl)_{ab}$, the quantity ω is essentially the magnitude of the vorticity or twist. Θ is the 4-divergence of the velocity field of the fluid, and so called the expansion. One also finds

$$\sigma = -m^a\overline{m}^a\nabla_b l_a$$

where

$$\sigma\overline{\sigma} = \frac{1}{2}\nabla_{(b}k_{a)}\nabla^b k^a - \Theta^2$$

so σ is essentially the magnitude of the traceless symmetric part of $\nabla_b k_a$, which is the shear.

The interpretations of the primed coefficients are the same, with l^a and n^a swapped.

We can now state a famous and useful theorem in a physically clear way

Theorem 1 (Goldberg-Sachs [16])**.** *A vacuum metric is algebraically special if and only if there is a geodesic shearfree null field l^a. Furthermore, l^a is a repeated principal null direction.*

We refer to [14] for the proof but comment that it is much shorter than the original proof, an illustration that especially for algebraically special metrics, spinor methods can be powerful.

3.3 Differential Operators

Just like the partial derivatives of the components of a vector do not form a vector themselves, one cannot expect that partial derivatives of (p, q)-weighted scalars are properly (p, q)-weighted. However, it is not too hard to convince oneself that the operator

$$O_{A\dot{A}}\eta = (\nabla_{A\dot{A}} - p\iota^B\nabla_{A\dot{A}}o_B - q\bar{\iota}^{\dot{B}}\nabla_{A\dot{A}}\bar{o}_B)\eta \tag{3.9}$$

gives a (p, q)-weighted quantity when acting on one. We have said quantity instead of saying scalar, because using $\nabla_{A\dot{A}}$ instead of $\partial_{A\dot{A}}$ means $O_{A\dot{A}}$ works on weighted spinors, vectors and tensors, and it adds a covariant spacetime index. The last property means that the operator $O_{A\dot{A}}$ is equivalent to having the four scalar operators

$$Þ = o^A\bar{o}^{\dot{A}}O_{A\dot{A}} \qquad (1,1) \tag{3.10a}$$

$$Þ' = \iota^A\bar{\iota}^{\dot{A}}O_{A\dot{A}} \qquad (-1,-1) \tag{3.10b}$$

$$ð = -o^A\bar{\iota}^{\dot{A}}O_{A\dot{A}} \qquad (1,-1) \tag{3.10c}$$

$$ð' = -\iota^A\bar{o}^{\dot{A}}O_{A\dot{A}} \qquad (-1,1) \tag{3.10d}$$

where we have indicated the type of each operator. Now one realizes that, using (3.7)

$$Þ\eta = o^A \bar{o}^{\dot{A}} (\nabla_{A\dot{A}} - p\iota^B \nabla_{A\dot{A}} o_B - q\bar{\iota}^{\dot{B}} \nabla_{A\dot{A}} \bar{o}_B)\eta = (\nabla_{l_a}\eta - p\varepsilon - q\bar{\varepsilon})\eta$$

and a similar relation for the other operators. Collecting, we have

$$Þ = l^a \nabla_a - p\varepsilon - q\bar{\varepsilon} \tag{3.11a}$$

$$ð = \overline{m}^a \nabla_a - p\beta + q\overline{\beta} \tag{3.11b}$$

$$Þ' = n^a \nabla_a + p\varepsilon' + q\bar{\varepsilon}' \tag{3.11c}$$

$$ð' = m^a \nabla_a + p\beta' - q\overline{\beta}'. \tag{3.11d}$$

The idea is now to write the equations of general relativity using the differential operators (3.11) and the well-behaved spin coefficients (3.6). It is to be expected that the formalism has quantities that do not transform "properly". The usual world-vector formulation of general relativity has the Christoffel symbols, which are not a tensor, and hides them in covariant derivatives and the Riemann tensor, which are tensors. We hide the spin coefficients that do not transform like weighted quantities in the differential operators, which do transform properly.

The GHP equations (Appendix A) thus consist of (equation numbers refer to [1])

- 6 equations (2.21–2.26)and their primed versions relating the spin coefficients and the GHP operators acting on them to the dyad components of the curvature spinor. (Ricci equations or identities)

- 4 equations (2.33–2.36) and their primed versions expressing the Bianchi identities using the dyad components of the curvature spinor, the GHP operators on them and the spin coefficients.

- 2 contracted Bianchi identities (2.37–2.38) and their primed versions.

- 3 equations (2.30–2.32) for the commutators of the GHP operators. For one of these, complex conjugation, the prime operation, and both give three new independent equations.

In the GHP formalism, there are thus 12 Ricci equations, whereas the NP formalism has 18. The missing Ricci equations involve the badly behaved spin coefficents; this information is instead contained in the GHP commutators, which are more complicated than their NP counterparts.

3.4 Integration in the GHP Formalism

The problem of integrating Einstein's equations is different from other problems in physics. Quoting Wald [22, p. 225], "[i]n other theories of classical physics we are given the spacetime background and our task is to determine the time evolution of quantities in the background from their initial values and time derivatives. However, in general relativity we are solving for the spacetime itself."

Consider solving a problem in electrodynamics in flat spacetime. Then the geometry is given, it is flat, but the problem may suggest suitable coordinates. In general relativity, the geometry is what we are looking for, but we can take the stance that matter tells space how to curve [31] and make assumptions about the geometry and see if the problem suggests coordinates. We formalize it in the following.

Theorem 2. *Suppose that M is a spacetime of dimension n and n scalar functions $f_1, \dots, f_n$ are given, such that at every $x \in M$, $df_i(x)$ form a linearly independent set. Then the f_i can be taken as coordinates, at least locally.*

Proof. Collect the functions in an $\mathbb{R}^n$-valued function $\mathbf{f}$. Then $d\mathbf{f}(x) = \begin{pmatrix} df_1(x) & \dots & df_n(x) \end{pmatrix}^T$ is a linear map $T_x M \to \mathbb{R}^n$ and has rank n. By Theorem 2.25 in [23], $\mathbf{f}$ is a local diffeomorphism with $\mathbb{R}^n$, which is the same thing as the f_i providing coordinate functions locally. □

The idea is now to use the GHP equations to find 4 functionally independent scalars $f^1, \dots, f^n$, of weight $(0, 0)$. Use them locally as coordinates indexed by μ. Then the components l^μ of the tetrad vector l^a are found from

$$l^\mu = l \cdot f^\mu = l^i \nabla_i f^\mu = \text{þ} f^\mu$$

where the first equality is the vector field l acting as a differential operator on the coordinate function f^μ, the second equality is covariant differentiation agreeing with Lie derivative on scalars, and the third is the definition of the þ operator, for $(0, 0)$ weighted scalars. Similarly, we find the components of the other tetrad vectors $n^\mu, m^\mu, \overline{m}^\mu$.

3.5 Dealing with the Non-Optimal Situation

It may be the case that 4 functionally independent scalars cannot be constructed from the GHP scalars. This seems to have been the general case when integration in the GHP formalism has been employed, as in Ref. [6, and references therein]. One intuitive situation where this happens is when the spacetime possesses a continuous symmetry, that is, a Killing field. In the integration procedure, one aims to find 4 coordinates defined by curvature invariants; if there are symmetries the curvature invariants cannot be sufficient. In the most concrete example, an observer in a static universe with time coordinate t would not be able to tell the passage of time by measuring curvature invariants.

3.5.1 Killing Vectors

We first recall some general facts about continuous symmetries. Let ξ^μ be a Killing field. Recall that this means that

$$(\mathcal{L}_\xi g)_{\mu\nu} = 0$$

and that in general any quantity T such that $\mathcal{L}_\xi T = 0$ is said to be Lie derived by ξ. Killing fields are of interest because they are the generators of symmetries.

By a general theorem, the "straightening theorem" [23, Theorem 2.101], given any vector field ξ^μ, it is possible to pick coordinates $x^0, \ldots, x^3$ so that $\xi^\mu = \frac{\partial}{\partial x^0}$. However, given two vector fields, ξ^μ and ζ^μ, it is not always possible to pick coordinates such that $\xi^\mu = \frac{\partial}{\partial x^0}$ and $\zeta^\mu = \frac{\partial}{\partial x^1}$. In fact this is possible if and only if $[\xi, \zeta]^\mu = 0$, that is, the vector fields commute.

Let ξ be a Killing field and straighten it, choosing coordiantes so $\xi = \partial_0$. If one writes out (3.14), below, in index notation in the coordinate frame $\{\partial_\mu\}$, the condition that ξ is a Killing field becomes

$$0 = (\mathcal{L}_\xi g)_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^0}$$

that is, x^0 is a cyclic coordinate. In general, a tensor T is Lie derived with respect to ∂_{x^0} if and only if its components in the coordinate frame are independent of x^0. Since the Christoffel symbols and the components of the Riemann tensor can be found from first and second partial derivatives of the components of the metric, the Riemann tensor all its covariant derivatives, and contractions formed form them must also be Lie derived. This conclusion is independent of coordinate system, so it must hold for all Killing vectors, even though we may be unable to straighten more than one at a time.

A weaker condition holds for invariantly defined directions. An invariant direction is defined by

$$0 = f^{abc\ldots}(D^p(R), k^\rho) \tag{3.12}$$

where $D^p(R)$ means the Riemann tensor and its covariant derivatives up to order p, and if k^a is a solution, so is λk^a for any function λ. Taking the Lie derivative with respect to ξ, we obtain

$$0 = \frac{df^{abc\dots}}{dl^\rho}[\xi, \lambda l]^\rho \Rightarrow 0 = (\xi \cdot \lambda) l^\rho + \lambda[\xi, l]^\rho$$

so that

$$(\mathcal{L}_\xi l)^\rho \propto l^\rho.$$

As an example, the equation

$$0 = \Psi_{ABCD}\iota^A\iota^B$$

will be satisfied for Petrov type III if and only if $\iota^A\bar{\iota}^{\dot{A}}$ is the repeated principal null direction. One easily obtains the corresponding equations for Petrov types other than O and I [24].[6]

Edgar and Ludwig have introduced a "GHP Lie derivative" useful for treating Killing vector within the GHP formalism. We give a summary of the main results in their paper [7]. This operator, defined by

$$\text{Ł}_\xi := \mathcal{L}_\xi + \left(\frac{p}{2} + \frac{q}{2}\right) n_\mu(\mathcal{L}_\xi l)^\mu + \left(\frac{p}{2} - \frac{q}{2}\right) \bar{m}_\mu(\mathcal{L}_\xi m)^\mu \tag{3.13}$$

for quantites of weight (p, q), where $\mathcal{L}$ is the usual Lie derivative, respects weights. This is seen by using the formula

$$\begin{aligned}(\mathcal{L}_\xi T)(X_1, \dots, X_n) = \quad & (\nabla_\xi T)(X_1, \dots, X_n) + T(\nabla_\xi X_1, \dots, X_n) \\ & + \dots + T(X_1, X_2, \dots, \nabla_\xi X_n)\end{aligned} \tag{3.14}$$

and noting that the extra terms in (3.13) allow us to replace ∇_ξ with Θ_ξ when acting on T. When acting on the X_i, this replacement has no effect if X_i are zero-weighted, but if they are not, the result is still properly weighted.

We can now state the main theorem of [7].

Theorem 3. *Suppose that a null tetrad*

$$l_\mu n^\mu = 1 \qquad m_\mu \overline{m}^\mu = -1$$

can be formed from intrinsic directions. Then ξ^a is a Killing field if and only if the null tetrad is GHP Lie derived.

Proof. Assume that ξ^a is a Killing field. From the orthogonality relations, we find

$$[\xi, n]_\mu = An_\mu \qquad\qquad [\xi, l]_\mu = -Al_\mu$$
$$[\xi, m]_\mu = iBm_\mu \qquad\qquad [\xi, \overline{m}]_\mu = -iB\overline{m}_\mu$$

[6]For type I and O, one can of course not expect to fix a direction using the Weyl tensor.

where A, B are real. But if R solves the differential equation $\xi \cdot \ln R = A$ and S solves $\xi \cdot S = B$,[7] a GHP gauge transformation with $\lambda^2 = Re^{-iS}$ gives us a Lie derived null tetrad. In this gauge, (3.13) shows that

$$(\text{Ł}_\xi Z_i^a)_{\text{gauge}} = (\mathcal{L}_\xi Z_i^a)_{\text{gauge}} = 0$$

for all tetrad fields Z_i^a. But then it must be that

$$\text{Ł}_\xi Z_i^a = 0 \tag{3.15}$$

in any gauge.

Conversely, since the zero-weighted metric is given by

$$g_{\mu\nu} = l_\mu n_\nu + l_\nu n_\mu - m_\mu \overline{m}_\nu - m_\nu \overline{m}_\mu$$

it is Lie derived if the tetrad vectors are GHP Lie derived. □

We state without proof the useful

Theorem 4. *If and only if ξ^a is a Killing field, then for any intrinsically defined null tetrad and $(0,0)$ weighted scalar η*

$$[\text{Ł}_\xi, \Theta_\mu]\eta = 0 \tag{3.16}$$

where Θ_μ is the GHP derivative operator. If η is (p,q) weighted, ξ being a Killing vector is sufficient but not necessary.

3.5.2 Auxiliary Scalars

In the case that we cannot form 4 functionally independent zero-weighted scalars from the GHP scalars, we can introduce new scalars by specifying the action of the GHP operators on them. For zero-weighted scalars η, $\Theta_\mu \eta = (d\eta)_\mu$, so we are really writing the equation $(d\eta)_\mu = \alpha_\mu$ in tetrad components. For this equation to have a solution, it is necessary that $(d\alpha)_{\mu\nu} = 0$. This condition is also sufficient for a solution to exist locally. In the GHP formalism the integrability condition is that the GHP commutators are satisfied.

In the search for such auxiliary scalars, one is led to make *Ansätze* that lead to differential equations, the solutions of which may involve integration constants. This raises the question if, by making an *Ansatz*, the class of solutions found is restricted and if the integration constants are important. This is not discussed in Refs. [6, 7, 34], but the authors, giving tables for auxiliary scalars without describing the metrics they find as a *sub*class, seem to be implicitly using the next theorem.

[7]Such functions exist because applying the straightening theorem and taking $R = R(x^0), S = S(x^0)$ the conditions are ODEs.

Theorem 5. *Suppose that a set of completely involutive tables for 3 or fewer zero-weighted real scalars A_i and one weighted scalar η of weight (p, q) with $p \neq \pm q$ has been found, and that these scalars are intrinsic and are a maximal functionally independent set. Then all geometric information is contained in these tables.*

Proof. By the Cartan-Karlhede algorithm [8] a full description of the geometry is given by the tetrad components of the Riemann tensor and its covariant derivatives (up to order 7 in the worst case). Fix the gauge such that in this gauge, $\eta_{\text{gauge}} \equiv 1$. Then from (3.11) we obtain a system of linear equations for $\beta, \varepsilon, \beta'$ and ε'. One realizes that for the system to have a unique solution $p \neq \pm q$ is necessary and sufficient.

Since A_i, η are a maximal functionally independent set of intrinsic scalars, every tetrad component of the Riemann tensor is a function of A_i, η. Knowing the badly behaved spin coefficients is then sufficient to solve for, in this gauge, $\nabla_\mu \Psi_i$, $\nabla_\mu \Phi_{ij}$ and $\nabla_\mu \Lambda$ from the tables, and all higher covariant derivatives, that is, all Karlhede scalars.[8] Since the tables were involutive, the Karlhede scalars are functions only of the A_i. □

Thus no geometric information is found in auxiliary scalars, so making an *Ansatz* for their tables does not restrict the class of solutions, and any integration constants appearing are superflous and can be set to whatever is most convenient.

3.6 Integration in the GHP Formalism and the Karlhede Classification

Specifying the Petrov type and matter content of spacetime fixes the tetrad components of the Riemann tensor, if the tetrad is suitably chosen. These components are the zeroth order Cartan-Karlhede scalars, $\mathcal{R}^0$. Using a calculation like that in Ref. 35, Appendix 1, one becomes convinced that $\mathcal{R}^0$, the GHP derivatives of $\mathcal{R}^0$, along with the spin coefficients determines the tetrad components of the first covariant derivative of the Riemann tensor, that is, the set $\mathcal{R}^1$ is determined by $\mathcal{R}^0 \cup \Theta_m \mathcal{R}^0 \cup \mathcal{S}$ where $\mathcal{S} = \{\rho, \sigma, \kappa, \tau, \rho', \sigma', \kappa', \tau'\}$ is the set of spin coefficients.

At the next order, the set $\mathcal{R}^2$ is determined by $\Theta_m R^1$, and so on. The Karlhede algorithm terminates when the dimension of the group of tetrad

[8] Since the tetrad components of the Riemann tensor and its covariant derivatives are weighted scalars, the result can be written in gauge-covariant form at the end by introducing suitable factors of η and $\overline{\eta}$. It may also be possible to perform the calculation without fixing a gauge, as for instance in ref. 35.

transformation is unchanged, *and* no new functionally independent scalar is provided.

Lemma 1. *Suppose that a spacetime is not conformally flat. Then it is possible to fix a tetrad up to discrete transformations at at most first order in the Cartan-Karlhede algorithm, except for type N and $\rho = 0$.*

Proof. In Ref. [36], it was shown that for Petrov type D the tetrad can be fixed completely at first order. For Petrov types I, II and III the tetrad can be fixed up to discrete transformations at zeroth order [8]. For Petrov type N and $\rho \neq 0$, the tetrad can be fixed at first order by, at zeroth order, taking the standard tetrad, and at first order performing the null rotation described in [2, 3] and fixing the gauge to $\Psi_4 \equiv 1$, where Ψ_4 is a component of the Weyl spinor, according to (2.9) and (2.10). If $\rho = 0$, only the first step can be performed and a one complex-parameter freedom remains. □

In the conformally flat case, for non-vacuum solutions it may be possible to fix the tetrad at first order, as was done in [6]. The conformally flat vacuum metrics are known to be de Sitter, anti-de Sitter and Minkowski.

Thus except in the conformally flat case, when a fully involutive set of tables for $\mathcal{R}^n$ is found, the Cartan-Karlhede algorithm terminates at order $n+1$. This is of course the case if four functionally independent, zero-weighted, real quantities can be constructed. The tables must then be involutive, even though the exact functional form of all entries may not be known. The $(n+1)$:th step may then provide additional information, such as differential equations for the unknown functions.

As an example, in Ref. [6] Edgar and Ludwig found a set of tables for four real functionally independent quantities at third order; thus guaranteed to be involutive. These tables had three unknown functions and in the fourth step, these functions were found to be functions only of one coordinate. The subsequent classification by Skea [37] indeed terminated at fourth order. A detailed analysis [17] showed that in the special case that all the unknown functions were constants, the Karlhede classification algorithm terminates at third order, but the general case needs four derivatives of the Riemann tensor.

4 Finding a Class of Type III Robinson-Trautman solutions

In this section we integrate the GHP equations for vacuum spacetimes in the case $\rho = \overline{\rho} \neq 0$, with the *Ansatz* that the metric is of Petrov type III. One can pick the standard tetrad for type III metrics, where $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0$, where Ψ_i are the components of the Weyl spinor. By the Goldberg-Sachs theorem, the spin coefficients $\sigma = \kappa = 0$. However, the problem is easier if one performs a null rotation (compare [2, 3]) around the repeated principal null direction.

Letting the subscripts 0 and 1 stand for quantities in the standard respectively the roatated tetrad, it is possible to find a null rotation such that $\tau_1 = 0$, if ρ_0 is everywhere non-zero. Comparing with the transformation laws (3.8), one sees that the rotation parameter to achieve this is $\overline{Z} = -\frac{\tau_0}{\rho_0}$. Then it follows from either the transformation law (3.8h) and the Ricci equations in the original tetrad, or from the Ricci equations in the new tetrad that we have $\tau_1' = 0$. This null rotation comes at the cost that in general $\Psi_{4,1} = Z\Psi_{3,0} \neq 0$.

For analysis of Killing vectors later, we note here that the rotated tetrad is what Edgar and Ludwig [7] call intrinsic, that is, it is defined by invariants of the Riemann tensor and its covariant derivatives.

4.1 Preliminary Setup for Integration

The full system of the general GHP equations is found in Appendix A. We state them in our case, in the null rotated tetrad, using $\rho = \overline{\rho}$ and$\tau' = \tau = 0$. The GHP equations consist of the Ricci equations

$$\eth\rho = 0 \quad (4.1a) \qquad \eth'\rho' = -\Psi_3 \quad (4.1a')$$

$$þ\rho = \rho^2 \quad (4.1b) \qquad þ'\rho' - \eth\kappa' = \rho'^2 \quad (4.1b')$$

$$0 = 0 \quad (4.1c) \qquad -\eth'\kappa' = \Psi_4 \quad (4.1c')$$

$$0 = 0 \quad (4.1d) \qquad -þ\kappa' = \Psi_3 \quad (4.1d')$$

$$0 = -\overline{\sigma}'\rho \quad (4.1e) \qquad 0 = 0 \quad (4.1e')$$

$$þ'\rho = \rho\overline{\rho'} \quad (4.1f) \qquad þ\rho' = \rho\rho' \quad (4.1f')$$

the Bianchi equations

$$0 = 0 \qquad (4.2a) \qquad\qquad þ'\Psi_3 - ð\Psi_4 = 4\rho'\Psi_3 \qquad (4.2a')$$
$$0 = 0 \qquad (4.2b) \qquad\qquad ð\Psi_3 = 0 \qquad (4.2b')$$
$$þ\Psi_3 = 2\rho\Psi_3 \qquad (4.2c) \qquad\qquad 0 = 0 \qquad (4.2c')$$
$$þ\Psi_4 - ð'\Psi_3 = \rho\Psi_4 \qquad (4.2d) \qquad\qquad 0 = 0 \qquad (4.2d')$$

and the commutators

$$[þ, þ'] = 0 \tag{4.3a}$$
$$[ð, ð'] = (\overline{\rho}' - \rho')þ + p\rho\rho' - q\rho\overline{\rho}' \tag{4.3b}$$
$$[þ, ð] = \rho ð \tag{4.3c}$$
$$[þ, ð'] = \rho ð' \tag{4.3d}$$
$$[þ', ð'] = \overline{\rho}' ð' - \kappa' þ + p(\rho\kappa' + \Psi_3) \tag{4.3e}$$
$$[þ', ð] = \rho' ð - \overline{\kappa}' þ + q(\rho\overline{\kappa}' + \overline{\Psi}_3). \tag{4.3f}$$

We have written out also those equations which are trivial in the current application, for correspondence with the original paper [1], where the Ricci equations are (2.21) through (2.26) and the Bianchi equations are (2.33) through (2.36). Our equations are in the same order and the equations with primed labels are the primed equations. The contracted Bianchi identities are all trivial for the vacuum case.

Some immediate simplifications can be made. (4.1e) gives $\sigma' = 0$. The left-hand side of (4.1f) is real, so it must be that ρ' is also real. This simplifies the commutator (4.3b) to

$$[ð, ð'] = (p - q)\rho\rho'. \tag{4.3b}$$

Since ρ and ρ' are real, (4.1a) and (4.1a') give us the actions of both ð and $ð'$ on both ρ and ρ'.

4.2 Making an *Ansatz*

The full system is intractable in the GHP formalism. Therefore, we assume that

$$\Psi_3 = -\rho\kappa'. \tag{4.4}$$

With this *Ansatz*, two more commutators become simpler,

$$[þ', ð'] = \rho' ð' - \kappa' þ \tag{4.3e}$$
$$[þ', ð] = \rho' ð - \overline{\kappa}' þ. \tag{4.3f}$$

Table 4: The GHP differential operators on the GHP quantities. S_0 is a so far unknown function of appropriate weight.

	ρ	ρ'	κ'	Ψ_3	Ψ_4
þ	ρ^2	$\rho\rho'$	$-\Psi_3$	$2\rho\Psi_3$	$2\rho\Psi_4$
$\text{þ}'$	$\rho\rho'$	ρ'^2	$-\rho'\Psi_3/\rho$	$2\rho'\Psi_3$	$2\rho'\Psi_4$
$\eth$	0	$-\overline{\Psi}_3$	0	0	$-2\rho'\Psi_3$
$\eth'$	0	$-\Psi_3$	$-\Psi_4$	$\rho\Psi_4$	S_0

Using (4.4), (4.1a) and (4.1c'), one finds

$$\eth'\Psi_3 = \rho\Psi_4 \tag{4.5}$$

so that (4.2d) implies

$$\text{Þ}\Psi_4 = 2\rho\Psi_4. \tag{4.6}$$

Further, (4.2b') implies

$$\eth\kappa' = 0. \tag{4.7}$$

This result, the commutator (4.3b) and (4.1c'), can be used to establish that

$$\eth\Psi_4 = -2\rho'\Psi_3 \tag{4.8}$$

which in (4.2a') implies

$$\text{þ}'\Psi_3 = 2\rho'\Psi_3. \tag{4.9}$$

But then

$$2\rho'\Psi_3 = \text{þ}'\Psi_3 = -\text{Þ}'(\rho\kappa') = -\kappa'\rho\rho' - \rho\text{Þ}'\kappa' = \rho'\Psi_3 - \rho\text{Þ}'\kappa'$$

and so

$$\text{Þ}'\kappa' = -\frac{\rho'}{\rho}\Psi_3. \tag{4.10}$$

Now with the commutator (4.3e) and the previous results, one finds

$$\text{þ}'\Psi_4 = 2\rho'\Psi_4 \tag{4.11}$$

The results so far are summarized in Table 4. Strictly speaking, (4.4) and the Leibniz rule makes one of the columns for Ψ_3, ρ, κ' superfluous, but it is useful to have all three.

4.3 Finding Coordinate Candidates

One obvious real zero-weighted quantity is

$$A := \rho\rho'. \tag{4.12}$$

Another simple combination is

$$B := \rho\sqrt{\Psi_3\overline{\Psi}_3}. \tag{4.13}$$

We require also a complex quantity, which we take as $\Psi_3 = PQ$ where

$$P := \sqrt{\frac{\Psi_3}{\overline{\Psi}_3}},\ (-1, 1) \qquad\qquad Q := \sqrt{\Psi_3\overline{\Psi}_3},\ (-1, -1).$$

with indicated weights. Note that P has only spin-weight and Q has only boost-weight, or intuitively, the magnitude of P and the phase of Q are gauge invariant. In fact, the following relations are obvious but useful:

$$P\overline{P} = 1 \tag{4.14a}$$

$$\overline{Q} = Q. \tag{4.14b}$$

Finally to bring Ψ_4 into the coordinate candidates we introduce the zero-weighted

$$C := \frac{\Psi_4}{\Psi_3^2}. \tag{4.15}$$

Note that C is complex. But since it is zero-weighted, we can of course form $C_R = \frac{C+\overline{C}}{2}, C_I = \frac{C-\overline{C}}{2i}$ to extract the real and imaginary parts.

It is now a matter of applying the Leibniz rule to rewrite Table 4 as tables for A, B, C, P and Q. We give an example to illustrate one possible procedure:

$$\begin{aligned}
þB^2 &= 2BþB = þ(\rho^2\Psi_3\overline{\Psi}_3) \\
&= 2\rho^3\Psi_3\overline{\Psi}_3 + \Psi_3\rho^2 \cdot 2PB + \rho^2\Psi_3 \cdot 2\overline{P}B \\
&= Q^2 \cdot 2\frac{B^3}{Q^3} + 4\frac{B^3}{Q} = 6\frac{B^3}{Q} \\
\therefore þB &= 3\frac{B^2}{Q}.
\end{aligned}$$

Carrying this through results in the tables

$$\begin{aligned} þA =& \ 2\tfrac{AB}{Q} \\ þ'A =& \ 2\tfrac{A^2Q}{B} \\ ðA =& \ -B\overline{P} \\ ð'A =& \ -BP \end{aligned} \qquad (4.16.A) \qquad \begin{aligned} þP &= 0 \\ þ'P &= 0 \\ ðP &= -\tfrac{B\overline{C}}{2} \\ ð'P &= \tfrac{BCP^2}{2} \end{aligned} \qquad (4.16.P)$$

$$\begin{aligned} þB =& \ 3\tfrac{B^2}{Q} \\ þ'B =& \ 3QA \\ ðB =& \ \tfrac{B^2}{2}\overline{PC} \\ ð'B =& \ \tfrac{B^2}{2}PC \end{aligned} \qquad (4.16.B) \qquad \begin{aligned} þQ &= 2B \\ þ'Q &= 2\tfrac{Q^2A}{B} \\ ðQ &= \tfrac{B\overline{C}PQ}{2} \\ ð'Q &= \tfrac{BCPQ}{2} \end{aligned} \qquad (4.16.Q)$$

$$\begin{aligned} þC &= -2\tfrac{BC}{Q} \\ þ'C &= -2\tfrac{AQC}{B} \\ ðC &= -2\tfrac{A}{B}\overline{P} \\ ð'C &= P(S - 2C^2B) \end{aligned} \qquad (4.16.C)$$

where S is a complex zero-weighted function, as yet undetermined. It is related to S_0 of Table 4 through

$$S_0 = P^3Q^2S. \qquad (4.17)$$

Since $B \neq 0$, neither A nor B can be constant. We have that $dA \neq \lambda dB$, so A and B are functionally independent, except if $CA = -3$, which implies $A^3/B^2 =$ constant. Consistency with the table for C then implies that this constant is

$$\frac{A^3}{B^2} = \frac{3}{2}. \qquad (4.18)$$

We will refer to this case as the *singular case*. In the following, unless stated otherwise, we shall be working with the assumption that A and B are functionally independent.

Applying the commutators to A and B shows that they are satisfied, but yields no new information. The same is true for the commutators on $PQ = \Psi_3$. This is to be expected since the only unknown function is S, which only appears in $ð'C$, and C only appears in $ð'B$ and $ð'P$. As such no information about S can be found from these commutators.

Looking at the þ and þ′ rows in each table reveals that they are proportional with constant of proportionality $\frac{AQ^2}{B^2}$. Since þ and þ′ are real operators, $\{A, B, \mathrm{Re}(C), \mathrm{Im}(C)\}$ is a functionally dependent set. It could be the case, however, that $\{A, B, \mathrm{Re}(C)\}$ or $\{A, B, \mathrm{Im}(C)\}$ (or both) is a functionally independent sets. If the real (imaginary) part is a function of A and B, then the imaginary part is too, since it is a function of A, B and the real (imaginary) part. Therefore we have that in these cases $C = C(A, B)$. Can such a function exist? It must satisfy

$$Q\text{þ}C = Q\frac{\partial C}{\partial A}\text{þ}A + Q\frac{\partial C}{\partial B}\text{þ}B = -2BC \tag{4.19}$$

$$P\eth C = P\frac{\partial C}{\partial A}\eth A + P\frac{\partial C}{\partial B}\eth B = -2\frac{A}{B} \tag{4.20}$$

The first equation written out and divided by B is

$$2A\frac{\partial C}{\partial A} + 3B\frac{\partial C}{\partial B} = -2C \tag{4.21}$$

and the general solution of this is

$$C(A, B) = \frac{F\left(\frac{B^2}{A^3}\right)}{A} \tag{4.22}$$

where F is an arbitrary, complex, function of one variable. Inserting this into the second equation gives

$$\frac{B}{A^2}F(B^2/A^3) + 3F'(B^2/A^3)\frac{B^3}{A^5} + F'(B^2/A^3)\frac{B^3}{A^4}\frac{\overline{F}}{A} = -2\frac{A}{B} \tag{4.23}$$

which on putting $t = B^2/A^3$ is equivalent to

$$tF(t) + t^2F'(t)(3 + \overline{F}(t)) = -2 \tag{4.24}$$

which is a system of ODE for the real and imaginary part of F. By the fundamental theorem of ODE:s, a solution exists. Of course $\text{þ}C$ and $\eth' C$ have to be correct too, but since $\text{þ}'(A, B, C)$ is proportional to $\text{þ}(A, B, C)$ this is the case, and since S was unknown,

$$\eth' C = P(S - 2C^2B)$$

is really an equation for S. One can check that with this form of C, $[\eth, \eth']B = 0$ still holds.

It is important to note that $F \neq -3$.

It is convenient to change the coordinate candidates at this point to $X := B/A^{3/2}$ and $Y := A^{1/2}$. Then the tables are

$$\begin{aligned} \text{þ}X &= 0 \\ \text{þ}'X &= 0 \\ \eth X &= \tfrac{1}{2}X^2Y(\overline{F}+3)\overline{P} \\ \eth' X &= \tfrac{1}{2}X^2Y(F+3)P \end{aligned} \tag{4.25.X}$$

$$\begin{aligned} \text{þ}Y &= \tfrac{XY^4}{Q} \\ \text{þ}'Y &= \tfrac{Q}{X} \\ \eth Y &= -\tfrac{XY^2}{2}\overline{P} \\ \eth' Y &= -\tfrac{XY^2}{2}P \end{aligned} \tag{4.25.Y}$$

and the function $F = F(X)$ satisfies the equation

$$2t^2F(t) + t^3F'(t)(3 + \overline{F}(t)) = -4. \tag{4.26}$$

One can realize that there are two cases for the function F

Lemma 2. *Let F be a solution to* (4.26)*. On any interval I such that F is defined on all of I, either F is always real, or never real.*

Proof. The fundamental theorem of ODE:s assures that given an initial value $F(t_0) = F_0$, a unique solution to

$$F'(t) = \phi(t, F(t))$$

exists on an interval I containing t_0. If F has a real value on an interval I, we can pick that point as t_0 and the real value as F_0 and by uniqueness the solution to this initial value problem agrees with the function we started with on I. It is then intuitively obvious that F must be real on all of I. A formal argument is given by noting that the proof of the fundamental theorem [38, p. 317] shows that the F is the limit of the sequence

$$F_{n+1}(t) = F_0 + \int_{t_0}^{t} \phi(s, F_n(s))ds$$

where since F_0 is real, $F_1(t)$ is real, and so on, and hence $F(t)$ must be real for all $t \in I$. □

4.3.1 Introducing Auxiliary Scalars

A zero-weighted scalar V can be found with the *Ansatz* $ðV = iVXY^3\overline{P}$ The commutator equations then give the following table,

$$\begin{aligned} þV =& \; 2V(\ln V + k_1)\tfrac{XY^3}{Q} \\ þ'V =& \; 2V(\ln V + k_1)\tfrac{Q}{XY} \\ ðV =& \; iVXY^3\overline{P} \\ ð'V =& \; -iVXY^3P \end{aligned} \qquad (4.27)$$

where k_1 is an integration constant. If we put $W = \ln V + k_1$, the table for W is

$$\begin{aligned} þW =& \; 2W\tfrac{XY^3}{Q} \\ þ'W =& \; 2W\tfrac{Q}{XY} \\ ðW =& \; iXY^3\overline{P} \\ ð'W =& \; -iXY^3P \end{aligned} \; . \qquad (4.28)$$

Thus the integration constant is superfluous, in accordance with Theorem 5.

In the case that $F = \overline{F}$, a fourth scalar U can be introduced with the table

$$\begin{aligned} þU =& \; 0 \\ þ'U =& \; \alpha(X)\tfrac{Q}{Y^2} \\ ðU =& \; 0 \\ ð'U =& \; 0 \end{aligned} \qquad (4.29)$$

which will be consistent with the commutator equations if $\alpha(X)$ is a solution to the equation

$$X(F+3)\alpha'(X)+(F+2)\alpha(X)=0. \tag{4.30}$$

This equation can be solved up to a real constant k and a quadrature,

$$\alpha(X)=k\exp\left(\int_1^X \frac{F(t)+2}{tF(t)+3t}dt\right)$$

which makes it clear that unless $k=0$, $\alpha(X)\neq 0$. However, it is also clear that if we adopt the coordinate candidate U as a coordinate u, changing k only has the effect of a rescaling $u\to\frac{k}{k'}u$ and so produces an equivalent metric, again in accordance with Theorem 5.

It is easy to check using the determinant on the tables for X,Y,U,W that these are functionally independent. The problem is then essentially solved and we can write down the metric. In the coordinates x,y,w,u and the corresponding coordinate basis,

$$g^{\mu\nu}=\begin{pmatrix} -\frac{1}{2}(F(x)+3)^2x^4y^2 & \frac{1}{2}(F(x)+3)x^3y^3 & 0 & 4wy^3 \\ \frac{1}{2}(F(x)+3)x^3y^3 & -\frac{1}{2}x^2y^4+2y^4 & xy^2\alpha(x) & 0 \\ 0 & xy^2\alpha(x) & 0 & 2wxy\alpha(x) \\ 4wy^3 & 0 & 2wxy\alpha(x) & 8w^2y^2-2x^2y^6 \end{pmatrix}. \tag{4.31}$$

4.3.2 Killing Vectors

The coordinate u is cyclic and so these spacetimes admit at least one Killing vector. We can confirm this with the GHP Lie derivative Ł. Supppose that

$$\xi=\xi^x\frac{\partial}{\partial x}+\xi^t\frac{\partial}{\partial y}+\xi^u\frac{\partial}{\partial u}+\xi^w\frac{\partial}{\partial w}$$

is a Killing vector. Since x and y are intrinsic, and $\xi\cdot x=\xi^x$, it must be that $\xi^x=\xi^y=0$. Since the right-hand side of the tables for X and Y involves only intrinsic quantities, the GHP-Lie commutators are satisfied for x and y.

The first two GHP-Lie commutators applied to u are

$$[\text{Ł}_\xi,\text{þ}]u=0=0-\text{þ}\xi^u=-\frac{xy^4}{Q}\frac{\partial\xi^u}{\partial y}-2w\frac{xy^3}{Q}\frac{\partial\xi^u}{\partial w} \tag{4.32a}$$

$$[\text{Ł}_\xi,\text{þ}']u=0=0-\text{þ}'\xi^u=-\frac{Q}{x}\frac{\partial\xi^u}{\partial y}-\frac{2wQ}{xy}\frac{\partial\xi^u}{\partial w}-\frac{\alpha(x)Q}{y^2}\frac{\partial\xi^u}{\partial u}. \tag{4.32b}$$

Using the first equation in the second, we find $\frac{\partial \xi^u}{\partial u} = 0$. Either equation then gives

$$\xi^u = \xi^u(x, \frac{w}{y^2}).$$

Now the commutator $[\mathcal{L}_\xi, \eth]u = 0$ gives

$$0 = \frac{\partial \xi^u}{\partial x} P \eth x + P \xi^u_{,2} \eth \frac{w}{y^2} \tag{4.32c}$$

where $\xi^u_{,2}$ means the partial derivative with respect to the second argument. Now $P\eth x$ is real, but $P\eth \frac{w}{y^2}$ has a non-zero imaginary part. Therefore we must have $\xi = k_1$ where k_1 is a constant. The fourth commutator, being the complex conjugate of the one just considered, gives nothing new.

Since the right-hand side of the table for w is independent of u, the GHP-Lie commutators for w will be linear equations for ξ^w; we can already now say that $\xi^w \equiv 0$ is a solution, so $\frac{\partial}{\partial u}$ is indeed a Killing vector.

The first two commutators are

$$[\mathcal{L}_\xi, \text{þ}]w = 0 = \frac{2xy^3}{Q}\xi^w - \frac{\partial \xi^w}{\partial w} 2wxy^3 - \frac{\partial \xi^w}{\partial y} xy^4 \tag{4.33a}$$

$$[\mathcal{L}_\xi, \text{þ}']w = 0 = \frac{2Q}{xy}\xi^w - \frac{\partial \xi^w}{\partial w}\frac{2w}{xy} - \frac{\partial \xi^w}{\partial y}\frac{Q}{x} - \frac{\partial \xi^w}{\partial u}\alpha(x)y^2. \tag{4.33b}$$

Again these two show that $\frac{\partial \xi^w}{\partial u} = 0$.

The third commutator is

$$P[\mathcal{L}_\xi, \eth]w = 0 = -\eth \xi^w = -i\frac{\partial \xi^w}{\partial w} xy^3 - \frac{\partial \xi^w}{\partial x}\frac{1}{2}x^2 y(F(x)+3) + \frac{\partial \xi^w}{\partial y}\frac{1}{2}xy^2. \tag{4.33c}$$

Since the first term is the only one that has an imaginary part, $\frac{\partial x^w}{\partial w} = 0$. Either of the first two commutators can be used to find

$$\frac{2}{y}\xi^w = \frac{\partial \xi^w}{\partial y},$$

which has the general solution

$$\xi^w = C(x)y^2.$$

The real part of (4.33c) then gives $C(x)$ up to a quadrature.

The final result is that

$$\xi^w = k_2 y^2 \exp\left(\int_1^x \frac{2dt}{t(F(t)+3)}\right). \tag{4.34}$$

where k_2 is an integration constant. We have thus found two obviously linearly independent and commuting Killing vectors:

$$\xi_1 = \frac{\partial}{\partial u} \tag{4.35a}$$

$$\xi_2 = y^2 \exp\left(\int_1^x \frac{2dt}{t(F(t)+3)}\right)\frac{\partial}{\partial w}. \tag{4.35b}$$

4.4 The Singular Case

In the case that A and B are functionally dependent, that is, $X^2 = B^2/A^3 = \frac{2}{3}$, we have only one functionally independent zero-weighted scalar. Making this substitution in the table for the auxiliary scalar W from the non-singular case, one finds that the commutators are still satisfied for W. The table for W is

$$Þ W = 2k\frac{WY^3}{Q} \tag{4.36}$$

$$Þ' W = 2\frac{WQ}{kY} \tag{4.37}$$

$$ð W = ikY^3\overline{P} \tag{4.38}$$

$$ð' W = -ikY^3 P \tag{4.39}$$

where $k = (2/3)^{1/2}$.

We get a hint from the table for X and consider a scalar Z with the table

$$Þ Z = 0 \tag{4.40}$$

$$Þ' Z = 0 \tag{4.41}$$

$$ð Z = g(Y,Z)\overline{P} \tag{4.42}$$

$$ð' Z = g(Y,Z)P \tag{4.43}$$

for some as yet unknown, real function g. The commutator $[Þ, ð]Z = 0$ gives $g(Y,Z) = g_0(Z)Y$. We can always transform $Z \mapsto k\int_0^Z \frac{dt}{g_0(t)}$ to set $ðZ = kY\overline{P}$. The table is then consistent with all the commutators.

Finally, following an *Anzatz* used previously, we find that the table

$$Þ U = 0 \tag{4.44}$$

$$Þ' U = e^{Z/2}\frac{Q}{Y^2} \tag{4.45}$$

$$ð U = 0 \tag{4.46}$$

$$ð' U = 0 \tag{4.47}$$

is consistent with the commutators. There are no essential integration constants, so the singular case corresponds to exactly one metric. In our y, z, u, w coordinates

$$g^{\mu\nu} = \begin{pmatrix} \frac{5}{3}y^4 & \frac{2}{3}y^3 & \frac{2^{1/2}}{3^{1/2}}y^2e^{z/2} & 4wy^3 \\ \frac{2}{3}y^3 & -\frac{4}{3}y^2 & 0 & 0 \\ \frac{2^{1/2}}{3^{1/2}}y^2e^{z/2} & 0 & 0 & \frac{2^{3/2}}{3^{1/2}}wye^{z/2} \\ 4wy^3 & 0 & \frac{2^{3/2}}{3^{1/2}}wye^{z/2} & 8w^2y^2 - \frac{4}{3}y^6 \end{pmatrix} \tag{4.48}$$

4.4.1 Killing Vectors for the Singular Case

The argument starting with (4.32a) can be carried over with some modifications. Since z is not an intrinsic scalar, we now have

$$[\mathcal{L}_\xi, \text{þ}]u = 0 = 0 - \text{þ}\xi^u = -\frac{ky^4}{Q}\frac{\partial \xi^u}{\partial y} - 2w\frac{xy^3}{Q}\frac{\partial \xi^u}{\partial w} \tag{4.49}$$

$$\frac{1}{Q}[\mathcal{L}_\xi, \text{þ}']u = 0 = \frac{\xi^z}{2}\frac{e^{z/2}}{y^2} - \text{þ}'\xi^u$$

$$= -\frac{\partial \xi^u}{\partial y}\cdot\frac{1}{k} - \frac{2w}{ky}\frac{\partial \xi^u}{\partial w} - \frac{\partial \xi^u}{\partial u}\frac{e^{z/2}}{y^2} \tag{4.50}$$

where using the first equation in the second gives

$$\frac{\xi^z}{2} = \frac{\partial \xi^u}{\partial u}. \tag{4.51}$$

Since $\xi^z \equiv 0$ is consistent with the Lie-GHP commutators applied to z, we find that $\xi_1 = \frac{\partial}{\partial u}$ is a Killing vector.

The argument starting with (4.33a) carries over to establish that $\frac{\partial \xi^w}{\partial w} = 0$, and furthermore,

$$\xi^w = \xi^w(ye^{z/2})$$

and using both these in (4.33a) gives

$$\zeta^w = k_2y^2e^z \tag{4.52}$$

with k_2 an integration constant.

Similarly using the $[\mathcal{L}_\xi, \text{þ}]z$ and $[\mathcal{L}_\xi, \text{þ}']z$ commutators we find $\frac{\partial \xi^z}{\partial u} = 0$ and the $[\mathcal{L}_\xi, \eth]z$ commutator implies $\frac{\partial \xi^z}{\partial w} = 0$ and $\xi^z = \xi^z(ye^{z/2})$. This is only consistent with the first commutator if $\zeta^z = k_3$ for a constant k_3. In this case, (4.51) implies that $\xi^u = k_3u/2$. We have thus found three linearly independent Killing vectors:

$$\xi_1 = \frac{\partial}{\partial u} \tag{4.53}$$

$$\xi_2 = y^2 e^z \frac{\partial}{\partial w} \tag{4.54}$$

$$\xi_3 = \frac{\partial}{\partial z} + \frac{u}{2}\frac{\partial}{\partial u}. \tag{4.55}$$

Clearly ξ_1 and ξ_2 commute, but the other Lie brackets are non-vanishing:

$$[\xi_1, \xi_3] = \xi_1/2, \tag{4.56}$$

$$[\xi_2, \xi_3] = -\xi_2. \tag{4.57}$$

Rescaling $\xi_1 \mapsto -\xi_1, \xi_2 \mapsto -\xi_2$ and $\xi_3 \mapsto 2\xi_3$ and comparing with [39] tells us that this Lie algebra is Bianchi type VI with $h = 2$.

Kerr and Debney [12] showed that there is a unique algebraically special vacuum metric with $\rho \neq 0$; this metric first appeared in the paper by Robinson and Trautman [9] as an example of a type III metric. In the next section, we will show that it is precisely this metric that we have found, using the Cartan-Karlhede algorihtm.

5 A Cartan-Karlhede Classification and Equivalence Problem

5.1 The General Type III Robinson-Trautman Metric

The general type III or more special Robinson-Trautman metric has the form [11], in the coordinates $u, r, \zeta, \overline{\zeta}$

$$g^{22} = -2\Delta L + 2r\partial_u P$$
$$g^{12} = 1$$
$$g^{34} = -4\frac{P^2}{r^2}$$

where $P = P(u, \zeta, \overline{\zeta}), L = \ln P$ and $\Delta = 4P^2\partial_\zeta\partial_{\overline{\zeta}}$. The metric is also found in [10] as equation (28.13), but with the coordinates rescaled, $\zeta \mapsto \zeta/2$. The field equations take the form of the constraint

$$\Delta\Delta P = 0. \tag{5.1}$$

This means

$$\partial_\zeta \partial_{\overline{\zeta}} \Delta P = 0 \Rightarrow \begin{cases} \partial_{\overline{\zeta}} \Delta P = a(u,\overline{\zeta}) \Rightarrow \Delta P = A(u,\overline{\zeta}) + C(u,\zeta) \\ \partial_\zeta \Delta P = b(u,\zeta) \Rightarrow \Delta P = B(u,\zeta) + D(u,\overline{\zeta}) \end{cases},$$

so clearly we can take $\Delta P = f(u,\zeta) + \overline{f}(u,\overline{\zeta})$. The form of the Robinson-Trautman metric is invariant under the transformation [10]

$$u \mapsto G(u) \quad r \mapsto rG' \quad \zeta \mapsto g(\zeta) \quad P \mapsto P|g'|/G'. \tag{5.2}$$

A null tetrad for this metric is given in the coordinate basis[9] by

$$l^a = (0,1,0,0) \tag{5.3a}$$
$$m^a = (0,0,2P/r,0) \tag{5.3b}$$
$$\overline{m}^a = (0,0,0,2P/r) \tag{5.3c}$$
$$n^a = (1,-U - r\partial_u P,0,0) \tag{5.3d}$$

and a calculation in this tetrad shows that

$$\tau = 0 \tag{5.4}$$
$$\Psi_3 = -\frac{2P}{r^2}\partial_{\overline{\zeta}} \Delta P \tag{5.5}$$
$$\Psi_4 = -\frac{4}{3}\partial_{\overline{\zeta}}\big(P^2 \partial_{u\overline{\zeta}} L) + \frac{4}{r^2}\partial_{\overline{\zeta}}\big(P^2 \partial_{\overline{\zeta}} \Delta L)\big) \tag{5.6}$$

Hence if $\partial_\zeta \Delta P = 0$ the spacetime is type N or conformally flat. (In the latter case, the metric is therefore flat, since it is a vacuum solution with zero cosmological constant.)

If the metric is type III, we see that the null tetrad is in the same class as the tetrad we used for integration. If we do a gauge transformation to $\Psi_3 \equiv 1$, and choose this gauge for our class also, we can apply the equivalence algorithm. Although our choice of tetrad is then different from the standard choice in the literature [8, 32], the Cartan-Karlhede algorithm will still determine equivalence correctly, since we have fixed the tetrad components of the curvature and and the spin coefficients in the same way for both metrics. The classification produced will however not be comparable to one performed with the the standard form $\Psi_i = \delta_{i3}$.

Collinson and French [11] used the NP formalism to find algebraically special vacuum metrics with more than one Killing vector, among them some

[9] Since we are using complex coordinates, m^a and $\overline{m}^a$ are complex conjugates even though their components are not; the basis vector ∂_ζ is mapped by complex conjugation to $\partial_{\overline{\zeta}}$.

Robinson-Trautman solutions. They found several cases, for each of which the function L and the field equation (5.1) could be put in a standard form using the transformations (5.2). To identify our class we search among the type III cases for which there are two Killing vectors.

Now our class is defined by the requirement $\rho\kappa' = -\Psi_3$, in the tetrad with $\Psi_i = 0, i \le 2, \tau = 0$. In the tetrad above, one finds

$$\rho = -r^{-1} \tag{5.7}$$

$$\kappa' = 2P\partial_{u\bar{\zeta}}L + 2r^{-1}P\partial_{\bar{\zeta}}\Delta L. \tag{5.8}$$

so that we must have $\partial_{u\bar{\zeta}}L = 0$. Examination of the standard forms given by Collinson and French [11] shows that this is only possible in their case (i), where, with

$$s = \zeta + \bar{\zeta} \tag{5.9}$$

L and the field equation take the forms

$$L = L(s) \tag{5.10}$$

$$s = -4e^{2L}\ddot{L}, \tag{5.11}$$

the dot being differentiation with respect to s.

At zeroth order, the invariant

$$\Psi_4 = -2r^2\dot{L} \tag{5.12}$$

is found. At first order, the invariant

$$(D\Psi)_{30'} = r^{-3}e^L \tag{5.13}$$

is found. Here $(D\Psi)_{A\dot{A}BCDE}$ means the symmetric part of the covariant derivative of Ψ_{BCDE}, and we use the notation described in (2.9) and (2.10) for its dyad components. Since the spinor is symmetric, this is enough to specify which quantity is meant. These two invariants will be functionally dependent if

$$2\dot{L}^2 + 3\ddot{L} = 0. \tag{5.14}$$

This differential equation is readily solved resulting in

$$L(s) = \frac{3}{2}\ln(s + C) + D$$

where C and D are integration constants, or $L(s) = C$. The field equation (5.11) excludes the second case and yields $C = 0, D = -\ln 6$ so that

$$L(s) = \frac{1}{2}\ln\frac{s^3}{6}. \tag{5.15}$$

This particular metric then admits three Killing vectors, and it is known as the unique algebraically special vacuum solution with exactly three Killing vectors [12]; it is Bianchi type VI [10].

The full output from CLASSI is presented in an appendix.

5.2 Classifying the Found Metrics

In 3.6 we claimed that if involutive tables were found at first order, as for our class of metrics, the Cartan-Karlhede algorithm would terminate at second order. This claim can be verified using the computer algebra packages SHEEP, CLASSI and REDUCE.

At zeroth order there is one Karlhede scalar, $\Psi_4 = F(x)/y^2$. The tetrad is completely fixed up to a null rotation by $\Psi_3 = 1$. At first order, there is one new scalar, $\mathrm{Re}(D\Psi_{30'}) = xy^3$ and we can choose the null rotation so that $\tau = 0$, fixing the tetrad completely. At second order, all Karlhede scalars are functions only of x and y, and so no new functionally independent scalar is found. Since the tetrad was fixed, the algorithm terminates.

The full output from CLASSI is presented in an appendix.

5.3 Determining Equivalence

From the invariants $(D\Psi)_{30'} \cdot (D\Psi)_{41'}$ and Ψ_4 we obtain the equations

$$4y^2 = 4s/r^2 \tag{5.16}$$

$$F/y^2 = -2r^2\dot{L} \tag{5.17}$$

where using the first equation in the second shows that $x = x(s)$ and gives

$$\frac{F(x)}{s} = -2\dot{L}(s) \tag{5.18}$$

From $(D\Psi)_{30'}$ we then obtain

$$x(s) = -2s^{-3/2}e^{L(s)}. \tag{5.19}$$

Clearly this cannot define a change of coordinates if L is given by (5.15).

One readily finds

$$\frac{dx}{ds} = e^{L(s)}(3s^{-5/2} - 2\dot{L}s^{-3/2}). \tag{5.20}$$

From (5.18) we can write the field equation,

$$
\begin{aligned}
e^{2L}\ddot{L} &= e^{2L}\left(-\frac{F'}{2s}\frac{dx}{ds} + \frac{F}{2s^2}\right) \\
&= e^{3L}\left((\dot{L}s^{-5/2} - \frac{3}{2}s^{-7/2}\right) F' + \frac{e^{2L}}{2s^2}F \\
&= \frac{x^3}{8}\left(\frac{3}{2}s - \dot{L}s^2\right) F' + \frac{x^2}{8}sF \\
&= \frac{x^3}{8}\left(\frac{3}{2} + \frac{F}{2}\right) sF' + \frac{x^2}{8}sF \\
&= -\frac{s}{4}.
\end{aligned}
$$

Multiplying through by $\frac{16}{s}$ gives precisely the ODE (4.26).

The singular case For our singular case, there is only one curvature invariant, $\Psi_4 = -3y^{-2}$. With L given by (5.15), we use (5.13) to obtain the equation

$$y^2 = \frac{s}{r^2} \tag{5.21}$$

which determines the coordinate change up to a sign. The invariants in $D\Psi$ tell us that the correct sign is

$$y = -\frac{s^{1/2}}{r}. \tag{5.22}$$

A Full Equations of the GHP formalism

All equations have primed versions.

Ricci Equations

$$\eth\rho - \eth'\sigma = (\rho - \overline{\rho})\tau + (\overline{\rho}' - \rho')\kappa - \Psi_1 + \Phi_{01} \tag{A.1a}$$
$$\text{Þ}\rho - \eth'\kappa = \rho^2 + \sigma\overline{\sigma} - \overline{\kappa}\tau - \tau'\kappa + \Phi_{00} \tag{A.1b}$$
$$\text{Þ}\sigma - \eth\kappa = \sigma(\rho + \overline{\rho}) - \kappa(\tau + \overline{\tau'}) + \Psi_0 \tag{A.1c}$$
$$\text{þ}\tau - \text{þ}'\kappa = \rho(\tau - \overline{\tau'}) + \sigma(\overline{\tau} - \tau') + \Psi_1 + \Phi_{01} \tag{A.1d}$$
$$\eth\tau - \text{þ}'\sigma = -\rho'\sigma - \overline{\sigma}'\rho + \tau^2 + \kappa\overline{\kappa}' + \Phi_{02} \tag{A.1e}$$
$$\text{þ}'\rho - \eth'\tau = \rho\overline{\rho}' + \sigma\sigma' - \tau\tau' - \kappa\kappa' - \Psi_2 - 2\Lambda \tag{A.1f}$$

Bianchi Equations

$$\begin{aligned}&\text{Þ}\Psi_1 - \eth'\Psi_0 - \text{Þ}\Phi_{01} + \eth\Phi_{00}\\ &\quad = - \tau'\Psi_0 + 4\rho\Psi_1 - 3\kappa\Psi_2 + \overline{\tau}'\Phi_{00}\\ &\qquad - 2\overline{\rho}\Phi_{01} - 2\sigma\Phi_{01} + 2\kappa\Phi_{11} + \overline{\kappa}\Phi_{02}\end{aligned} \tag{A.2a}$$
$$\begin{aligned}&\text{Þ}\Psi_2 - \eth'\Psi_1 - \eth'\Phi_{01} + \text{Þ}'\Phi_{00} + 2\text{þ}\Lambda\\ &\quad =\sigma'\Psi_0 - 2\tau'\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3\\ &\qquad + \overline{\rho}'\Phi_{00} - 2\overline{\tau}\Phi_{01} - 2\tau\Phi_{10} + 2\rho\Phi_{11} + \overline{\sigma}\Phi_{02}\end{aligned} \tag{A.2b}$$
$$\begin{aligned}&\text{Þ}\Psi_3 - \eth'\Psi_2 - \text{þ}'\Phi_{21} + \eth\Phi_{20} - 2\eth'\Lambda\\ &\quad =2\sigma'\Psi_1 - 3\tau'\Psi_2 + 2\rho\Psi_3 - \kappa\Psi_4\\ &\qquad - 2\rho'\Phi_{10} + 2\tau'\Phi_{11} + \overline{\tau}'\Phi_{20} - 2\overline{\rho}\Phi_{21} + \overline{\kappa}\Phi_{22}\end{aligned} \tag{A.2c}$$
$$\begin{aligned}&\text{Þ}\Psi_4 - \eth'\Psi_3 - \eth'\Phi_{21} + \text{þ}'\Phi_{20}\\ &\quad = + 3\sigma'\Psi_2 - 4\tau'\Psi_3 + 2\rho\Psi_4\\ &\qquad - 2\kappa'\Phi_{10} + 2\sigma'\Phi_{11} + \overline{\rho}'\Phi_{20} - 2\overline{\tau}\Phi_{21} + \overline{\sigma}\Phi_{22}\end{aligned} \tag{A.2d}$$

The contracted Bianchi identities are

$$\begin{aligned}&\text{Þ}\Phi_{11} + \text{Þ}'\Phi_{00} - \eth\Phi_{10} - \eth'\Phi_{01} + 3\text{þ}\Lambda\\ &\quad =(\rho + \overline{\rho}') + \Phi_{00} + 2(\rho + \overline{\rho})\Phi_{11} - (\tau' + 2\overline{\tau})\Phi_{01}\\ &\qquad - (2\tau + \overline{\tau}')\Phi_{10} - \overline{\kappa}\Phi_{12} - \kappa\Phi_{21} + \sigma\Phi_{20} + \overline{\sigma}\Phi_{02}\end{aligned} \tag{A.3a}$$
$$\text{þ}\Phi_{12} + \text{þ}'\Phi_{01} - \eth\Phi_{11} - \eth'\Phi_{02} + 3\eth\Lambda \tag{A.3b}$$
$$=(\rho' + 2\overline{\rho}')\Phi_{01} + (2\rho + \overline{\rho})\Phi_{12} - (\tau' + \overline{\tau})\Phi_{02} \tag{A.3c}$$
$$- 2(\tau + \overline{\tau}')\Phi_{11} - \overline{\kappa}'\Phi_{00} - \kappa\Phi_{22} + \sigma\Phi_{21} + \overline{\sigma}'\Phi_{10} \tag{A.3d}$$

Commutators η is a weighted quantity with weights (p, q). When applying prime or complex conjugation or both to $[\text{Þ}, \eth]$, one must remember that these operations change the weights of η. Quoting the GHP paper [1], "under the prime p becomes $-p$ and q becomes $-q$; under the bar, p becomes q and q becomes p,; under both bar and prime p becomes $-q$ and q becomes $-p$."

$$\begin{aligned}[\text{Þ}, \text{Þ}']\eta =& [(\overline{\tau} - \tau')\eth + (\tau - \overline{\tau}')\eth' \\ & - p(\kappa\kappa' - \tau\tau' + \Psi_2 + \Phi_{11} - \Lambda) \\ & - q(\overline{\kappa\kappa}' - \overline{\tau\tau}' + \overline{\Psi}_2 + \Phi_{11} - \Lambda)]\eta\end{aligned} \tag{A.4a}$$

$$\begin{aligned}[\text{Þ}, \eth]\eta =& [\overline{\rho}\eth + \sigma\eth' - \overline{\tau}'\text{þ} - \kappa\text{þ}' \\ & - p(\rho'\kappa - \tau'\sigma + \Psi_1) \\ & - q(\overline{\sigma}'\overline{\kappa}' - \overline{\rho}'\overline{\tau}' + \Phi_{01})]\eta\end{aligned} \tag{A.4b}$$

$$\begin{aligned}[\eth, \eth'] =& [(\overline{\rho}' - \rho')\text{þ} + (\rho - \overline{\rho})\text{þ}' \\ & + p(\rho\rho' - \sigma\sigma' + \Psi_2 - \Phi_{11} - \Lambda) \\ & - q(\overline{\rho\rho'} - \overline{\sigma\sigma'} + \Psi_2 - \Phi_{11} - \Lambda)]\eta\end{aligned} \tag{A.4c}$$

B The Cartan-Karlhede Algorithm in Brief

The *equivalence problem* in general relativity is: given the components of a metric tensor $g^{\mu\nu}$ in the coordinate basis using coordinates (x^0, x^1, x^2, x^3), and a the components of a metric tensor $\tilde{g}^{\mu\nu}$ in the coordinate basis using coordinates (y^0, y^1, y^2, y^3), does there exist a change of coordinates such that $g^{\mu\nu}$ transforms to $\tilde{g}^{\mu\nu}$? Or more plainly stated, do $g^{\mu\nu}$ and $\tilde{g}^{\mu\nu}$ really describe the same geometry?

The equivalence problem was in principle solved by Elie Cartan [40]. Cartan's solution is to view the problem as a system of linear equations for forms on the frame bundle. The consistuents of the system are the coordinate differentials pulled back to the frame bundle, and the connection form on the frame bundle. The necessary and sufficient conditions for such a system to be solvable is formulated in terms of relations between the derivatives of the forms involved. Since the exterior derivative of the connection form is related to the curvature form, for the equivalence problem the condition is therefore stated in terms of the curvature form on the frame bundle and its derivatives. This is precisely looking at the problem in terms of frame components.

The zeroth order integrability condition is naturally that the equation

$$\Omega(x^\mu, \xi^\nu) = \tilde{\Omega}(y^\mu, \eta^\nu) \tag{B.1}$$

is solvable, where (x^μ, ξ^ν) and (y^μ, η^ν) form bundle charts for the frame bundle and Ω is the curvature 2-form on the frame bundle. But already at this stage Cartan's method is intractable since in 4 dimensions, the curvature tensor has 20 independent components. Brans [41] suggested using standard frames to make Cartan's method more feasible. However, it was Karlhede [8] who gave the first practical method.

The Cartan-Karlhede algorithm is as follows.

1. Let $p = 0$ and choose any frame.
2. Calculate the frame components of $\nabla^p R$ where R is the Riemann tensor.
3. Let $\mathcal{R}^p$ be all components calculated so far and n the number of functionally independent elements in $\mathcal{R}^p$.
4. Determine a standard form for $\mathcal{R}$ and the subgroup of $SO(1,3)$ that leaves this form invariant, and its dimension d. Change to a frame where $\mathcal{R}^p$ has the standard form.
5. If neither d or n have changed since the last iteration, return $\mathcal{R}^p$, otherwise set $p := p + 1$ and go to step 2.

The set $\mathcal{R}^p$ is an invariant classification of the geometry. For two metrics to be equivalent, it is necessary and sufficient that $\mathcal{R}^p = \tilde{\mathcal{R}}^p$ is solvable as algebraic equations, treating either set of coordinates as known and the other as the unknowns. It may be possible to directly conclude that two metrics are *not* equivalent, for example if the algorithm terminates at different orders, or the number of functionally independent coordinates at each step is different, but it may also happen that a particular component vanishes identically for one metric but not for the other.

Step 4 in the algorithm is somewhat vaguely formulated. What is a "standard form"? The algorithm does not care, as long as any tensor can be put into some standard form. Since the Ricci spinor, the curvature scalar and the Weyl spinor together are equivalent to the Riemann tensor, one can calculate $\mathcal{R}^p$ using their dyad components. Then there exist known standard forms for these spinors and algorithms to determine which type a given spinor belongs to [32].

Karlhede describes the method as a "maximally generalized Petrov classification". The Petrov classification classifies metrics based on one part of the Riemann tensor. The Cartan-Karlhede classification does the same, but looks at the whole Riemann tensor and also its derivatives. The set of invariants found is sufficient to uniquely identify a metric, and so the classification is *maximally* generalized.

Finally, we comment on the number of derivatives needed in the algorithm. In n dimensions there are at most n functionally independent invariants and the semi-orthogonal group has dimension $\frac{n(n-1)}{2}$. At each step, at worst either exactly one new independent invariant is found or the dimension of the invariance group decreases by one. Therefore the algorithm terminates after at most $\frac{n(n+1)}{2}$ iterations, and this bound was known to Cartan [40].

It is known that information may be found at different stages depending on how one fixes the tetrad, but also that the order where the algorithm terminates is independent of the choice of tetrad. This can be understood from Cartan's original formulation, which does not consider fixing tetrads. Therefore it makes sense to talk about the number of derivatives needed to classify a metric as an intrinsic property.

In dimension 4 Karlhede [8] improved Cartan's bound to 7. It was unknown for a long time if there actually were any metrics that required the seventh derivative. In 1990, no metrics requiring the fourth derivative were known [35]; the first known case was a subclass of the conformally flat pure radiation metrics, classified by Koutras in 1992 [42]. In 2000, Skea [43] found a metric requiring the fifth derivative. The question was finally settled in 2008 when Milson and Pelavas [44] found a class of type N metrics requiring seven derivatives.

C Cartan-Karlhede Classification Results

One-parameter family

The Cartan-Karlhede algorithm for the metric (4.31) gives the following information. The metric is a vacuum solution, with

$$\Psi_3 = 1 \tag{C.1a}$$
$$\Psi_4 = Fy^{-2} \tag{C.1b}$$

and so is of Petrov type III. We can use $\operatorname{Re}\Psi_4 = \Psi_4$ as a curvature invariant. We have $d(\Psi_4) = y^{-2}F_{,x}\,dx - 2y^{-3}F\,dy$ which can never be 0, since the ODE F solves does not allow F and $F_{,x}$ to both vanish.

Proceeding to $D\Psi$, the non-vanishing components are

$$(D\Psi)_{3\dot{0}} = 2xy^3 \tag{C.2a}$$
$$(D\Psi)_{4\dot{0}} = 2xyF \tag{C.2b}$$
$$(D\Psi)_{4\dot{1}} = 2x^{-1}y^{-1} \tag{C.2c}$$
$$(D\Psi)_{5\dot{0}} = 2xy^{-1}F^2 - 2x^{-1}y^{-1} \tag{C.2d}$$
$$(D\Psi)_{5\dot{1}} = 2x^{-1}y^{-3}F - 4x^{-1}y^{-3} \tag{C.2e}$$

and clearly $(D\Psi)_{3\dot{0}}$ and Ψ_4 are functionally independent and form a maximal functionally independent set at this stage.

Proceeding to $D^2\Psi$, the non-vanishing components are

$$(D^2\Psi)_{3\dot{0}} = 6x^2y^6 \tag{C.3a}$$
$$(D^2\Psi)_{4\dot{0}} = 6Fx^2y^4 \tag{C.3b}$$
$$(D^2\Psi)_{4\dot{1}} = \frac{1}{6}Fx^3y^2F_{,x} + \frac{1}{3}Fx^2y^2 + \frac{1}{2}x^3y^2F_{,x} + \frac{20}{3}y^2 \tag{C.3c}$$
$$(D^2\Psi)_{4\dot{2}} = -2 \tag{C.3d}$$
$$(D^2\Psi)_{5\dot{0}} = 6F^2x^2y^2 + \frac{5}{6}Fx^3y^2F_{,x} + \frac{5}{3}Fx^2y^2 + \frac{5}{2}x^3y^2F_{,x} - \frac{8}{3}y^2 \tag{C.3e}$$
$$(D^2\Psi)_{5\dot{1}} = \frac{1}{6}F^2x^3F_{,x} + \frac{1}{3}F^2x^2 + \frac{1}{2}Fx^3F_{,x} + \frac{20}{3}F - 7 \tag{C.3f}$$
$$(D^2\Psi)_{5\dot{2}} = \frac{1}{6}Fxy^{-2}F_{,x} - \frac{5}{3}Fy^{-2} + \frac{1}{2}xy^{-2}F_{,x} + 2\frac{0}{3}x^{-2}y^{-2} \tag{C.3g}$$
$$(D^2\Psi)_{6\dot{0}} = 6F^3x^2 + 2F^2x^3F_{,x} + 4F^2x^2 + 6Fx^3F_{,x} - 4F + 2 \tag{C.3h}$$
$$(D^2\Psi)_{6\dot{1}} = 6F^2y^{-2} + \frac{1}{2}Fxy^{-2}F_{,x} - 9Fy^{-2} + \frac{3}{2}xy^{-2}F_{,x} - 4x^{-2}y^{-2} \tag{C.3i}$$
$$(D^2\Psi)_{6\dot{2}} = -2F^2y^{-4} + 6Fx^{-2}y^{-4} - 20x^{-2}y^{-4} \tag{C.3j}$$

and there cannot be any new independent scalars. The algorithm terminates.

The singular case

For the singular case with the metric (4.48) we find at zeroth order

$$\Psi_3 = 1 \tag{C.4a}$$
$$\Psi_4 = -3y^{-2} \tag{C.4b}$$

and at first order

$$(D\Psi)_{3\dot{0}} = \left(\frac{2}{3}\right)^{3/2} y^3 \tag{C.5a}$$
$$(D\Psi)_{4\dot{0}} = -2^{3/2}3^{1/2}y \tag{C.5b}$$
$$(D\Psi)_{4\dot{1}} = 6^{1/2}y^{-1} \tag{C.5c}$$
$$(D\Psi)_{5\dot{0}} = 5 \cdot 6^{1/2}y^{-1} \tag{C.5d}$$
$$(D\Psi)_{5\dot{1}} = -5 \cdot 6^{1/2}y^{-3} \tag{C.5e}$$
$$\tag{C.5f}$$

which are obviously functionally dependent.

It appears that the algorithm would terminate, but this is mistaken for a subtle reason. If this metric were classified in the standard tetrad, $\Psi_i = \delta_{i3}$, there would be no functionally independent scalars at zeroth order, so the algorithm would need two iterations: first order to find the coordinate y, second order to confirm that no more independent curvature invariants can be found.

To find this metric we used a tetrad where $\Psi_0 = \Psi_1 = \Psi_2 = 0, \Psi_3 = 1$ and Ψ_4 unspecified, and $\tau = 0$. At zeroth order in the algorithm, we can only be sure that the conditions on Ψ_i are fulfilled, and there is still the freedom of a null rotation. Only at first order do we have the possibility to guarantee $\tau = 0$. Thus the number of independent scalars has not increased, but the tetrad freedom has been decreased to the trivial group, and so we must go to second order, where the algorithm terminates.

Type III case (i) according to Collinson and French

In the coordinates $u, r, \zeta, \overline{\zeta}$ the metric is

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2(\zeta+\overline{\zeta}) & 0 & 0 \\ 0 & 0 & 0 & -4r^2e^{2L} \\ 0 & 0 & -4r^2e^{2L} & 0 \end{pmatrix} \tag{C.6}$$

where $L = L(s) = L(\zeta + \overline{\zeta})$.

Again, we find that these are vacuum metrics and of type III with

$$\Psi_3 = 1 \tag{C.7a}$$
$$\Psi_4 = -2r^2 L_{,s} \tag{C.7b}$$

and we can again take Ψ_4 as a curvature invariant.

For $D\Psi$, CLASSI finds

$$(D\Psi)_{3\dot{0}} = -4r^{-3}e^{L} \tag{C.8a}$$
$$(D\Psi)_{4\dot{0}} = 8r^{-1}e^{L}L_{,s} \tag{C.8b}$$
$$(D\Psi)_{4\dot{1}} = -rse^{-L} \tag{C.8c}$$
$$(D\Psi)_{5\dot{0}} = rse^{-L} - 16re^{L}(L_{,s})^2 \tag{C.8d}$$
$$(D\Psi)_{5\dot{1}} = 2r^3se^{-L}L_{,s} + 2r^3e^{-L} \tag{C.8e}$$

and from this set one can take $(D\Psi)_{3\dot{0}}$ as a second curvature invariant. The two invariants will be functionally independent if

$$2e^{L}(L_{,s})^2 + 3e^{L}L_{,ss} \neq 0 \tag{C.9}$$

and if they are independent, they are a maximal independent set.

Proceeding to $D^2\Psi$, we obtain

$$(D^2\Psi)_{3\dot{0}} = 24r^{-6}e^{2L} \tag{C.10a}$$
$$(D^2\Psi)_{4\dot{0}} = -48r^{-4}e^{2L}L_{,s} \tag{C.10b}$$
$$(D^2\Psi)_{4\dot{1}} = 6r^{-2}s \tag{C.10c}$$
$$(D^2\Psi)_{4\dot{2}} = -2 \tag{C.10d}$$
$$(D^2\Psi)_{5\dot{0}} = -6r^{-2}s + 96r^{-2}e^{2L}(L_{,s})^2 \tag{C.10e}$$
$$(D^2\Psi)_{5\dot{1}} = -12sL_{,s} - 7 \tag{C.10f}$$
$$(D^2\Psi)_{5\dot{2}} = \frac{3}{2}r^2s^2e^{-2L} + 4r^2L_{,s} \tag{C.10g}$$
$$(D^2\Psi)_{6\dot{0}} = 24sL_{,s} - 192e^{2L}(L_{,s})^3 + 2 \tag{C.10h}$$
$$(D^2\Psi)_{6\dot{1}} = -\frac{3}{2}r^2s^2e^{-2L} + 24r^2\zeta(L_{,s})^2 \tag{C.10i}$$
$$(D^2\Psi)_{6\dot{2}} = -3r^4s^2e^{-2L}L_{,s} - 5r^4\zeta e^{-2L} \tag{C.10j}$$

from which no new invariants can be constructed. The algorithm terminates.

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