The function concept and university mathematics teaching

Olov Viirman
The function concept and university mathematics teaching

Olov Viirman
**Table of contents**

Acknowledgements ................................................................................................................................. 3

List of papers ........................................................................................................................................ 6

1 Introduction .......................................................................................................................................... 7
  1.1 Aims of the thesis ....................................................................................................................... 10
  1.2 How to read this thesis .............................................................................................................. 11

2 Literature review ................................................................................................................................. 13
  2.1 The mathematical concept of function ................................................................................... 13
    2.1.1 The historical development of the function concept.................................................... 13
    2.1.2 Research on the teaching and learning of the function concept ................................. 17
  2.2 Research on mathematics teaching in higher education ....................................................... 21
    2.2.1 Self-reports and theoretical analyses of teaching practice ............................................ 23
    2.2.2 Empirical research on university mathematics teaching ............................................... 25

3 Theoretical considerations ................................................................................................................ 30
  3.1 Basic theoretical assumptions ................................................................................................... 30
  3.2 Discursive approaches to mathematics education research .................................................. 32
  3.3 Sfard's commognitive framework ............................................................................................ 36

4 Methodology ....................................................................................................................................... 42
  4.1 Research questions ..................................................................................................................... 42
  4.2 Research design ........................................................................................................................... 43
    4.2.1 Preliminary study ................................................................................................................. 43
    4.2.2 Main study ............................................................................................................................ 44
  4.3 Further methodological considerations................................................................................... 49

5 Overview of the papers ..................................................................................................................... 53
  5.1 Paper I .......................................................................................................................................... 53
  5.2 Paper II ......................................................................................................................................... 54
  5.3 Paper III ....................................................................................................................................... 56
  5.4 Paper IV ....................................................................................................................................... 57
  5.5 Paper V ......................................................................................................................................... 59

6 Conclusions ......................................................................................................................................... 61
  6.1 Teaching practices regarding the function concept............................................................... 61
    6.1.1 Mathematical discourse ................................................................................................ 61
    6.1.2 Discourse of mathematics teaching ................................................................................. 65
    6.1.3 General features of the teaching practices ...................................................................... 68
6.2 The discursive constitution of the function concept ............................................................ 71

7 Discussion ............................................................................................................................................ 75

7.1 Contributions of the thesis ........................................................................................................ 75

7.1.1 Contributions to research on university mathematics teaching practice .................... 75

7.1.2 Methodological contributions ........................................................................................... 76

7.1.3 Contributions to university mathematics teaching practice ......................................... 77

7.2 Reflections on the quality of the study .................................................................................... 78

7.3 Further research .......................................................................................................................... 80

8 References ............................................................................................................................................ 83
Acknowledgements

In a sense the first seeds of this thesis were sown almost fifteen years ago, when, as a doctoral student in mathematics, I realized that I wasn’t that interested in doing research in mathematics per se. Instead my interests lay more towards the teaching of mathematics, and what Dörfler (2003) somewhat tentatively calls mathematicology, that is the meta-study of mathematics – the study of mathematics “as a given human phenomenon and activity” (ibid, p. 149). However, at the time I didn’t realize that you could conduct research in mathematics education, at least not in university mathematics education. Having discontinued my doctoral studies in mathematics, and realizing that getting a permanent position as a university mathematics teacher without a doctoral degree was almost impossible, I decided to turn to teaching mathematics in upper secondary school, and enrolled in a teacher education program. To get my degree as a teacher I needed to write a bachelor’s thesis in mathematics education, and I knew that I wanted to study conceptual development in mathematics in the context of teacher education. When planning the bachelor’s thesis I met Iiris Attorps, who was to become my advisor on the present thesis. She was very enthusiastic about my idea for the project, and already in the first meeting she and I had sketched an outline of a research project for a doctoral thesis, leaving me happy but somewhat shell-shocked.

After finishing my bachelor’s thesis (which in fact forms part of the basis for the first paper of this thesis), instead of applying for a position as a mathematics teacher, I thus found myself deciding to embark on a second round of doctoral studies. I applied for a position as a doctoral student at the University of Gävle1, as part of the Graduate school in educational sciences with emphasis on didactics - UVD2. I am very grateful towards the University of Gävle for providing me with the opportunity to write this thesis by choosing me out of over 40 applicants for their one position within the UVD graduate school. I'm sorry that my thesis turned out to have almost nothing to do with teacher education…

A lot of people have been involved in various ways in my doctoral studies, and I would like to take this opportunity to express my profound gratitude. First and foremost I would like to thank my supervisors Iiris Attorps and Lars-Erik Persson. Their unwavering support, both scientifically and otherwise, during

1 Since the University of Gävle doesn’t have a doctoral program in mathematics education, this dissertation is presented at Karlstad University, but my position and funding has been at the University of Gävle.
2 In Swedish “Utbildningsvetenskap med inriktning mot didaktik”. UVD is a collaboration between a number of Swedish universities, but it is based at Örebro University.
the years I’ve spent working on this thesis, has been crucial to me. In particular I would like to thank Lars-Erik for his help in arranging my transfer from Örebro University to Karlstad University, and Iiris for not even flinching when I decided to scrap our original research plan and change the direction of my research.

As suggested above, I have been connected to several different scientific communities during my doctoral studies. I would like to thank my colleagues at the University of Gävle for providing me with such a friendly and stimulating workplace. In particular, I would like to thank Yukiko Asami-Johansson for great teaching collaboration and for many interesting conversations on both scientific and other matters. At Karlstad University, I would like to thank everyone at the Department of mathematics, and particularly Jorryt van Bommel and Yvonne Liljekvist, for their support, and for always making me feel welcome on my visits to Karlstad. A special thank you to Jorryt for her hospitality, and for many valuable suggestions regarding various aspects of the writing of this thesis. I would also like to thank my fellow doctoral students in the UVD graduate school, as well as the senior researchers involved in UVD. The research projects of the other doctoral students in UVD cover a wide range of topics within educational research, and have introduced me to many different theoretical perspectives and frameworks within a general educational setting. This has been helpful to me in developing my own theoretical starting points. But, apart from this they are also a very likeable group of people, and I have had lots of fun in their company. Of the UVD doctoral students I would like to single out Roger Olsson, for his generous hospitality during my, at times fairly frequent, visits to Karlstad.

A number of senior researchers have at different times commented on my work, and for this I am very grateful. Thanks go in particular to Barbro Grevholm, reader at my 50%-seminar, and Andreas Ryve, opponent at my 90%-seminar. Two senior researchers have had particular influence on the development of my research: Markku Hannula, by suggesting that I focus on university teachers rather than students; and Jeppe Skott, by introducing me to the writings of Anna Sfard on thinking as communicating. Thank you both!

I have also had the opportunity to attend a number of conferences and graduate courses during the work on my thesis, and I would like to thank all the friendly and interesting people I have met there, in particular my “conference buddies” Hanna Palmér, Erika Stadler, Birgit Gustafsson and Håkan Sollervall for good company and lots of laughs.
This thesis wouldn’t exist if it wasn’t for the seven university mathematics teachers who kindly agreed to let me intrude on their teaching with my video camera, and then writing about and publishing what I found. My deepest gratitude to you all! I would also like to thank the two best mathematics teachers I have had the pleasure of being taught by, Gunnar Berg and Ryszard Rubinsztein, for sparking my interest in mathematics teaching. They both taught in the traditional way, chalk in hand at the blackboard, but still their teaching styles were very different, showing the complexity of good teaching. These acknowledgements are becoming very long, but it is still very likely that I have forgotten someone. So if you are reading this, and feel that you have contributed in some way to the work on this thesis, then you are probably right. Thank you! I would also like to thank my family and friends, for helping me lead a good life outside of mathematics education. My deepest, most heartfelt gratitude goes out to my wife, Frida, for her love and unwavering support during all the ups and downs of this long process, and to my children Hilding and Birger for helping me stay sane by not being in the least interested in mathematics education research. Tack så mycket, Hilding och Birger! Nu är pappa äntligen klar med sin bok, och får mer tid att spela Wii och bygga lego.
List of papers


1 Introduction

University mathematics education is undergoing a process of change. More students than ever enrol in study programs containing mathematics, while at the same time fewer students choose to make mathematics their main subject of study. Also, students tend to have a weaker background in mathematics when entering university (Hillel, 2001). This development can be seen also in Sweden, as evidenced for instance by figures measuring the performance of beginning students at the Royal Institute of Technology\(^3\) (Brandell, 2012), as well as by the increasingly poor performance of Swedish students in international tests such as TIMSS. Particularly, these changes place new demands on the character and quality of university mathematics teaching. Consequently, the last decade has seen an increase in research on the teaching and learning of mathematics in higher education. However, in contrast to the large body of research on teaching practices at the K-12 level, very little research on the actual teaching practices of university mathematics teachers exists (Speer, Smith & Horvath, 2010). As Speer and her colleagues point out, this means that

“the design of most existing programs and resources has not benefited from analyses of the practices of college mathematics teachers or examinations of the influences on and development of those practices. In short, the community’s efforts to support instructors as they learn to teach college mathematics is often not informed by data and research on what is involved in teaching college mathematics” (ibid, p. 100).

However, the request for more empirical research on university mathematics teaching practice put forward by Speer and her colleagues in their paper has been heeded by a number of researchers (e.g. Artemeva & Fox, 2011; Hora & Ferrare, 2013; Güçler, 2013; for a more thorough overview of research on university mathematics teaching, see section 2.2), and the present thesis is intended as a contribution to this small but growing body of research.

In this thesis I have chosen to focus on one particular form of teaching practice, namely lectures. By “lecture” I will mean a teaching mode where the teacher presents new material to the students in a traditional lecture style – the teacher talking and writing on the board, and the students listening and taking notes. The communication is thus mainly directed from the teacher to the students, although the students are of course allowed to ask questions, for instance. Lectures are seldom one-off performances. Rather, a course is typically made up of a number of lectures, together with other teaching activities such as problem solving sessions, etc. The choice to focus on lectures

\(^3\) Kungliga tekniska högskolan, KTH.
was made for two main reasons. First, lectures are a central feature of most undergraduate mathematics teaching in Sweden and elsewhere, and if we want to improve mathematics teaching at universities, any such attempts ought to be informed by knowledge of the practice of university mathematics teaching as it is actually performed. Second, and related to the first, lectures have been the subject of fairly heated debate among scholars of university teaching. On the one hand, some writers (e.g. Bligh, 2000; Biggs & Tang, 2011) have been highly critical of the lecture as an effective method of teaching; on the other hand, some writers, both within mathematics education (Pritchard, 2010; Rodd, 2003) and within general university education (Burgan, 2006; Friesen, 2011; Jones, 2007), have argued the continued relevance and usefulness of the lecture format. However, while much of the criticism of lectures fails to take into account the major differences between academic disciplines, as argued by Neumann (2001; Neumann, Parry, & Becher, 2002) and, as Pritchard (2010) remarks, thus tends to be less relevant to mathematics lectures, still the arguments for the relevance of the lecture format are seldom informed by empirical research. Furthermore, as pointed out for instance by Hora and Ferrare, much of the existing research on university mathematics teaching practice tends to reduce teaching to its form, and this is particularly true of research on lectures: “(T)he technique commonly called ‘lecture’ is particularly subject to this reductionist approach, as what is generally meant by the term – a discourse given before an audience – actually masks a myriad of specific pedagogical behaviours such as distinct rhetorical strategies and the use of different instructional technologies” (Hora & Ferrare, 2013, p. 217-218). Increased knowledge of these “myriad of specific pedagogical behaviours” is crucial to the debate on the value of lectures. Without empirically based knowledge of what is actually contained within the umbrella term of “lectures”, any such debate is bound to be flawed. It is my hope that this thesis will provide such knowledge.

Having presented the motives for my choice of institutional context, I will now discuss my choice of the function concept as the mathematical focus of the thesis. Most importantly, it is a concept central to university mathematics, but at the same time important also in secondary school mathematics, and thus highly relevant also to teacher education. In fact, it has been argued that it is a unifying concept for mathematics as a whole (see section 2.1 for a more on this issue and other aspects of the concept). Moreover, functions appear in most mathematics courses, but to my experience as a student and teacher of university mathematics it is often not very comprehensively treated as an object
of study in its own right. If this is the case, much of both the teaching and learning of functions might be expected to occur implicitly, through the way functions are talked about and used in different contexts, making them an interesting topic of study. Also, it is a concept much studied within mathematics education, providing me with a lot of background material and tools for analysing the teaching practices of the teachers in the study.

At this point, I believe it can be useful to provide some insight into the development of the thesis project. When I embarked on the project, my intention was to contribute to the growing body of research on university students’, particularly pre-service teachers’, learning of the function concept. In fact, the first paper of this thesis is a study of some university students’ concept images of the function concept in relation to the historical development of the concept, partly based on the study I did for my bachelor’s thesis. However, my growing familiarity with the field, as well as the publication of a number of dissertations in Sweden dealing with related topics (Hansson, 2006; Juter, 2006; Pettersson, 2008), led me to leave the increasingly crowded area of research on students’ conceptions of function, and also to move my focus away from teacher education, instead positioning myself within university mathematics education research more generally. Furthermore, while my interest in the function concept remained intact, I began to focus more on the teaching of the concept.

This change of focus, as well as my own development towards a view of learning, teaching and doing mathematics as a socio-historical, discursive activity prompted a change also in theoretical perspective. I chose to use the commognitive framework (Sfard, 2008; for a more thorough treatment of this theory, see section 3.3). Since this theory is relatively recent, and still under development, not that many studies have been published, although research using the framework has been the topic of a special issue of the International Journal of Educational Research (Sfard, 2012). The studies that do exist tend to focus on the mathematical learning of younger children (e.g. Sfard & Lavie, 2005; Sinclair & Moss, 2012) or on elementary mathematics, like arithmetic (Ben-Yehuda, Lavy, Linchevski, & Sfard, 2005; Caspi & Sfard, 2012). Relatively little has been published on university mathematics learning from a commognitive standpoint, although the framework is gaining increasing interest within the university mathematics education community, with examples including Berger (2013), using commognitive theory to examine student

---

4 I have since learnt that this differs quite a lot between universities, and also has changed over the last decade or so, perhaps due to the influence of research results such as those presented in section 2.1.2.
interaction in the context of work with GeoGebra; Ioannou (2012), studying undergraduate students’ learning of group theory in terms of, for instance, object-level and meta-level learning; Kim, Ferrini-Mundy and Sfard (2012), investigating the impact of language on English and Korean speaking university students’ discourses of infinity; Kjeldsen and Blomhøj (2012), arguing the role of the history of mathematics in university mathematics education from a commognitive perspective; Ryve (2006), which makes use of, but also critiques, an early version of Sfard's theory, as presented in for instance Sfard (2001), in order to investigate student interaction in problem-solving; and Ryve, Nilsson and Pettersson (2012), investigating the role of visual mediators and technical terms in establishing effective mathematical communication. So far, research using the commognitive framework has mainly been aimed at studying learning. However, it can be fruitfully applied also in research on teaching, as argued by Sinclair and Yurita (2008, p. 137). Still, as far as I know, Güçler’s case study (2013) investigating the discourse of teacher and students in an undergraduate lecture on limits, is the only published study apart from my own work using commognitive theory to study university mathematics teaching. (Güçler’s study will be discussed at greater length in section 2.2.2 and elsewhere.)

Having described some of the background and development of the thesis, I will now present the overall aims of the thesis project.

1.1 Aims of the thesis

The overall aims of the thesis are describing and analysing the teaching practices of university mathematics teachers regarding the function concept, and how this concept is constituted through these practices. More specifically, one aim is to describe the discourses of function, and of teaching functions, as they appear through the discursive practices of the teachers. This is the topic of papers II-IV of the thesis. Paper II compares the discursive practices of two teachers on the topic of linear transformations, focusing on word use. Papers III and IV describe specific mathematical and didactical aspects of the teaching practices of seven mathematics teachers. (A more detailed summary of the papers is given in chapter 5.) A second aim of the thesis is to investigate how the function concept is characterized through the discursive practices of the teachers. This is the topic of Paper V of the thesis.
1.2 How to read this thesis

As indicated above, this thesis consists of five papers, together with what we in Sweden call a “kappa”, a longer text written in order to introduce the papers and place them in context, but also to deepen and extend the conclusions of the papers, presenting aspects of the study which did not fit onto the separate papers. Following the example of Ryve (2006), I have chosen to focus on certain aspects of the papers, rather than attempting to present an exhaustive account. In this way I hope to be able to present the main results and contributions of the thesis, while at the same time producing a text which functions as a self-contained whole. I will now briefly describe the disposition of the thesis.

In chapter 2 I present an overview of previous research relevant to the thesis. This chapter is divided into sections serving different purposes in the context of the thesis. The first section focuses on research on the historical development of the function concept, and on the teaching and learning of it. This section both serves to provide a mathematical and didactical context for the thesis as a whole, and to introduce notions, results and analytical constructs used in the empirical analyses, particularly in Paper I, but also in the analyses of mathematical discourse in Papers II-V. The second section, on research on university mathematics teaching, serves to outline the research context in which the thesis is positioned.

The theoretical framework used in the thesis, the commognitive framework of Sfard (2008), is presented in chapter 3, together with a discussion of some more general theoretical assumptions underlying the research presented in this thesis. In this context I also clarify the use of some terms central to the thesis. The chapter also aims at placing the commognitive framework in a larger context of discursive approaches to research in mathematics education.

Chapter 4 is devoted to methodological issues. Here I present the design of the studies in relation to their aims and to the theoretical frameworks employed. I also describe the educational context in which the studies were conducted and the methods used for data collection and analysis, including considerations of research ethics. Furthermore, I discuss the methodology of the main study of the thesis in relation to two previous studies closely related to the present one.

5 The direct translation of “kappa” is “coat”, but I am not aware of any proper English term, except perhaps “thesis”. Indeed, in what follows I will be using this term to denote both the “kappa” and the work as a whole, including the five papers.
In chapter 5, extended summaries of the five papers are presented, together with some brief considerations regarding the relation between, and the progression of, the papers.

In chapter 6 I return to the aims of the thesis. I discuss the results of the papers in relation to these aims, describing the main characteristics of the discourses of mathematics and mathematics teaching, as well as the discursive constitution of the function concept. I also present some general characteristics of the teaching practices of the teachers not discussed in the individual papers. In this way I highlight the main results of the thesis, relating them to previous research.

In the seventh and final chapter I discuss the results of the thesis, highlighting its theoretical and methodological contributions in the context of research on university mathematics teaching. I also discuss possible contributions to university mathematics teaching practice. Chapter 7 also contains some reflections on the quality and relevance of the study, as well as suggestions for further research.
2 Literature review

This chapter is devoted to a discussion of previous research relevant to the thesis. The chapter consists of two sections – one on the teaching and learning of the function concept, and one on mathematics teaching at the university level. These two sections serve different purposes in the thesis. The first section serves partly to provide mathematical and didactical background to the thesis, and partly to introduce notions and results which will be useful when conducting the empirical analyses of the mathematical discourse of the teachers. The second section serves to present the research context within which the thesis is positioned.

2.1 The mathematical concept of function

This section contains relevant research concerning the function concept. It consists of two subsections – one on the historical development of the function concept, with some didactical reflections, and one on research on the function concept within mathematics education.

2.1.1 The historical development of the function concept

The central mathematical notion in this thesis is the concept of function. It therefore seems fitting to present a brief outline of the historical development of this concept, from a mathematical perspective. At least since the early 20th century, the function concept has been considered as one of the fundamental concepts of mathematics (Sierpinska, 1992, p. 32). Hence, it is perhaps somewhat surprising that it has been studied systematically only for about 300 years. As Kleiner (1989, p. 282) puts it: “The evolution of the concept of function goes back 4000 years; 3700 of these consist of anticipations.” In fact, it has been claimed that “the concept of function is one of the distinguishing features of ‘modern’ as against ‘classical’ mathematics” (ibid, p. 282). Sierpinska (1992, p. 31) claims that “the notion of function can be regarded as a result of the human endeavour to come to terms with changes observed and experienced in the surrounding world.” Of course, functional ideas of co-variation and relations between magnitudes were used even in ancient times, for instance to construct astronomical charts and tables. However, this was not considered part of mathematics (Sierpinska, 1992, p. 31). Also, ancient mathematics lacked the necessary algebraic prerequisites needed to develop a concept of function (Kleiner, 1989, p. 283). Eventually this was to change. In the two hundred years leading up to the work of Newton and Leibniz in the 1680’s, there were a
number of mathematical developments paving the way for calculus and the beginnings of the function concept as it is used today. For example the work in astronomy by the likes of Kepler and Galileo led to the establishment of the study of motion as a central problem of science (Kleiner, 1989, p. 283). This time also saw the development of symbolic algebra, and, through the work of Descartes and others, the establishment of a firm link between algebra and geometry. However, the calculus of Newton and Leibniz was still not a calculus of functions, but of quantities related to geometric entities, and the continuum in which they performed their limit processes was geometrical, not arithmetical (Guicciardini, 2003, pp. 73-74). When Leibniz introduced the term ‘function’, it was “to designate a geometric object associated with a curve” (Kleiner, 1989, p. 283). It was not until the work of Euler in the mid-1700s that calculus began to be seen as a study of functions, not curves.

In 1718 Johann Bernoulli gave the first definition of the function concept: “One calls here Function of a variable a quantity composed in any manner whatever of this variable and constants” (Rüthing, 1984, p. 72). This definition of course was very imprecise and was somewhat improved by Euler. Common to these early definitions was that they required the function to be given by an analytic expression. A lively debate concerning the problem of describing the motion of a vibrating string, centred on the interpretation of a function as on the one hand the solution to a physical problem, and on the other hand an analytic expression (for a more detailed discussion, see Kleiner, 1989, p. 285-288) caused Euler to change his view on the concept of function. In 1755, he gave the following definition of the function concept: “a quantity should be called a function only if it depends on another quantity in such a way that if the latter is changed, the former undergoes change itself” (Sfard, 1992, pp. 62-63). Instead of relying on analytic expressions, this definition uses the idea of co-variation of quantities (Thompson, 1994, pp. 28-29). Using Sfard’s (1991) terminology of process-object duality (see section 2.1.2), Euler’s definition is clearly process-oriented.

The 18th and early 19th centuries saw rapid development of calculus, but the central concepts were handled in a less than rigorous manner. When Fourier published his work on trigonometric series in 1822, inspired by the problem of heat conduction in a rod, it challenged the old ideas about the concepts of calculus. In passing, it could be mentioned that Malik (1980) makes the observation that the heat conduction problem is mathematically similar to the problem of the vibrating string, but that the obvious geometrical interpretation of the string problem made it difficult, and even unnecessary, for the 18th
century mathematicians to make the conceptual leap made by Fourier and others a hundred years later. However, it was obvious that Fourier’s proofs were lacking conceptual precision. Also, mathematicians teaching analysis at universities found this a difficult task without a solid conceptual foundation (Lützen, 2003, p. 155). A rigorous re-treatment of the concepts of calculus was needed, and this endeavour was undertaken in the first half of the 19th century by such mathematicians as Cauchy, Dirichlet and Weierstrass. Regarding the concept of function, we have seen that already Euler in his 1755 definition attempted to avoid the close conceptual link between functions and analytic expressions. This was the case also with the definitions given by Cauchy and Fourier. However, in practice they both still relied on the idea of function as analytic expression, either in the way they talked about functions, or in the way they presented proofs (Kleiner, 1989, pp. 290-291; Lützen, 2003, pp. 156-158). In fact, “it was quite usual for early 19th-century mathematicians to define functions in a general way and then implicitly or explicitly ascribe various additional properties to them in the course of the arguments” (Lützen, 2003, p. 158). The first mathematician to consistently refrain from this was Dirichlet (Lützen, 2003, p. 158), and it was he who gave what can be seen as the first modern definition of the function concept: “If a variable $y$ is so related to a variable $x$ that whenever a numerical value is assigned to $x$ there is a rule according to which a unique value of $y$ is determined, then $y$ is said to be a function of the independent variable $x$. ” (Sierpinska, 1992, p. 46). In one sense this definition could be viewed as less general than the one by Euler quoted above, speaking of "rule" rather than "dependence", but at the same time the idea of co-variation is less pronounced, making it a precursor to modern definitions. This is made even more explicit by the fact that it is the first definition to mention one-valuedness. According to Even (1990), one-valuedness and arbitrariness are the two essential features of the concept of function in the modern sense.

One-valuedness simply means that for each element in the domain there is a unique element in the range. This feature might seem obvious to the modern reader, but in the early development of calculus it was not required. However, as the analysis of functions got more advanced, the use of multi-valued symbols became unwieldy. Keeping track of the different values of a multi-valued function when for instance taking higher order derivatives soon becomes

---

6 Even uses the term univalence, but since this is a term used in a different sense in other mathematical contexts, it is perhaps less appropriate.
unnecessarily difficult. The added requirement of one-valuedness was needed to make analysis of functions manageable (Even, 1990).

Arbitrariness means that the value of a function at any point is independent of the value at other points but also that the domain and range can be arbitrary sets; specifically they need not be number sets. The idea of functions operating on objects other than numbers was still in the future at the time of Dirichlet. However, through how he speaks of relation and rule, as opposed to Euler’s idea of change in the independent variable causing change in the dependent, Dirichlet approaches the independence aspect of arbitrariness. In fact, the famous Dirichlet function, defined as having the value 1 for all rational \( x \) and 0 for all irrational \( x \), is one of the first illustrations of this idea. In the wake of the example of the Dirichlet function, a whole new field of study opened up, aiming at constructing “pathological” functions, in order to test the robustness of the results of calculus. The need was seen for defining more restrictive classes of functions, such as continuous or differentiable functions, for which the results were valid. For instance, Weierstrass’ example of a function which was everywhere continuous but nowhere differentiable showed the need for separating these two classes, whereas the earlier belief was that continuity implied differentiability (Kleiner, 1989, p. 293).

In the late 19th and early 20th century, development in different areas of mathematics, such as analysis, topology, set theory and algebra helped generalize the function concept even further, freeing it from its connection to numbers. Group homomorphisms, linear transformations of vector spaces, and functionals (functions whose domain is a set of functions and whose range is the real or complex numbers) were all examples of functions not necessarily defined on sets of numbers. The time was ripe for a thoroughly abstract, set-theoretical definition of the function concept, as exemplified by the following one, given by Bourbaki in 1939:

Let \( E \) and \( F \) be two sets, which may or may not be distinct. A relation between a variable element \( x \) of \( E \) and a variable element \( y \) of \( F \) is called a functional relation in \( y \) if, for all \( x \) in \( E \), there exists a unique \( y \) in \( F \) which is in the given relation with \( x \).

We give the name of function to the operation which in this way associates with every element \( x \) in \( E \) the element \( y \) in \( F \) which is in the given relation with \( x \); \( y \) is said to be the value of the function at the element \( x \), and the function is said to be determined by the given relation. Two equivalent functional relations determine the same function. (Rüthing, 1984, p. 77)

Here is made explicit reference to domain and range, something which wasn’t needed earlier, when functions were implicitly assumed to be defined on real or
complex numbers. In this definition there is no reference at all to number sets. Moreover, the definition is totally static, or, in Sfard’s (1991) terminology, structural. The concept of function has become reified, and is now seen as an object in its own right, not just as a process operating on objects. Again using the language of Sfard, the historical development of the function concept could be described as going from an operational to a structural view of the concept. However, it is worth noting that the function concept is one example of a mathematical concept where alternative formulations of the definition are still in parallel use. For example, more operationally oriented definitions, like the one given by Dirichlet above, are still often used in calculus.

2.1.2 Research on the teaching and learning of the function concept

The function concept has received quite a lot of attention within the mathematics education research community, with conferences and books (Harel & Dubinsky, 1992; Romberg, Fennema, & Carpenter, 1993) devoted to the subject. A major reason for this is the role which the function concept is often seen to play as a unifying concept in mathematics. This view was put forth already by Felix Klein in 1908 (Sierpńska, 1992, p. 32), and has been discussed by many researchers since (e.g. Akkoc & Tall, 2005; Carlson, 1998; Hansson, 2006; Leinhardt, Zaslavsky, & Stein, 1990). According to this view, “functional thinking should pervade all of mathematics and, at school, students should be brought up to functional thinking” (Sierpńska, 1992, p. 32). Eisenberg speaks of a “sense for functions”, and states that developing such a sense in students “should be one of the main goals of the school and collegiate curriculum” (Eisenberg, 1992, p. 153).

However, much of the research on the function concept from an educational perspective has tended to reveal difficulties related to the learning of the concept. There are numerous studies of students' conceptions of the function concept, showing inconsistencies both within conceptions and between conceptions and definitions. Such studies have been conducted at the upper secondary\(^7\) and tertiary\(^8\) levels, as well as within teacher education\(^9\). For example, in a widely cited study, Vinner and Dreyfus (1989) used the theoretical construct of concept definition/concept image (Tall & Vinner, 1981, see below) to investigate the conceptions of functions held by a number of university

\(^7\) e.g. Akkoc & Tall, 2002; Eisenberg & Dreyfus, 1994; Elia, Panaoura, Eracleous, & Gagatsis, 2007; Gerson, 2008; Ronda, 2009; Saika, 2003; Slavit, 1997; Tall & Bakar, 1991; Vinner, 1992

\(^8\) e.g. Carlson, 1998; Dubinsky & Harel, 1992; Evangelidou, Spyrou, Elia, & Gagatsis, 2004; Tall & Bakar, 1991; Vinner, 1992; Vinner & Dreyfus, 1989; Williams, 1998

\(^9\) e.g. Bayazıt, 2011; Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Chinnappan & Thomas, 2001; Dede & Soybas, 2011; Even, 1993; Hansson, 2006; Meel, 2000; Thomas, 2003; Wilson, 1993
students and junior high school teachers. Comparing the definitions given by the participants to their use of the concept, they found several examples of compartmentalization, meaning that different parts of the concept image, invoked in different situations, are mutually inconsistent. Moreover, the concept images of the function concept held by many of the students were undeveloped. In a similar vein, Vinner (1992) and Dubinsky and Harel (1992) found common restrictions in students’ function conceptions, for instance the belief that all functions need to have numbers as input and output, and the belief that all graphs of functions need be continuous. From a slightly different perspective, working within the framework of pedagogical content knowledge and subject-matter knowledge for teaching (Even, 1990; Shulman, 1986), Even (1993) studied pre-service secondary teachers’ conceptions of function, and found that many of the pre-service teachers did not hold modern conceptions. Also, their expectations of the behaviour of functions led them to disregard the actual definition of the concept. There have also been a number of studies (e.g. Hitt, 1998; Norman, 1992; Vinner & Dreyfus, 1989) investigating the function conceptions of practicing teachers, and even within this group numerous examples of inadequate conceptions have been found. Approaching the problem from a different angle, Mesa (2004) looks at the mathematical practices associated with the function concept in textbook problems on functions, in order to investigate what conceptions of functions may be stimulated through working with these problems.

Research on students’ conceptions of the function concept and related concepts has also been conducted in the Nordic countries. Of relevance to this study is for instance the work of Attorps (2006), Hansson (2006), Juter (2006), Pettersson (2008) and Viholainen (2008). Specifically, Hansson (2006) conducted a study using concept maps to assess pre-service teachers’ conceptions of function starting from simple, algebraic representations of functions (Hansson & Grevholm, 2003; Hansson, 2005). One finding of the study is that the function concept is not so well integrated into the general knowledge structure of the students, casting some doubt on the idea of the function concept as a unifying concept (Hansson, 2006, p. 37).

The results of this line of research are aptly summarized by Breidenbach and colleagues, who write: “college students, even those who have taken a fair number of mathematics courses, do not have much of an understanding of the function concept” (Breidenbach et al, 1992, p. 247). So, despite its seemingly straightforward nature, the function concept obviously is a complex one for students, and their concept development “appears to evolve over a number of
years and appears to require an effort of ‘sense-making’ to understand and orchestrate individual function components to work in concert” (Carlson, 1998, p. 143). In the light of this, it is not surprising that researchers have investigated the character of the function concept and the teaching of it, trying to understand the difficulties related to students’ learning of functions. Many of the studies cited above indicate that the structural nature of the set-theoretical definition is problematic for learners. This has led a number of researchers to suggest that more operational descriptions ought to be used when students are first introduced to the function concept. For instance, Thompson (1994; Oehrtman, Carlson, & Thompson, 2008), as well as Smith (2003), Slavit (1997) and others, has argued forcefully for using a covariance approach, as opposed to the view of function as correspondence. On the other hand, a covariance approach makes the distinction between independent and dependent variables less clear, something which Sierpinska (1992, p. 38) speaks of as an epistemological obstacle in the learning of the function concept.

A related body of research concerns the use of multiple representations in the teaching and learning of the function concept. The main idea behind this research is that abstract mathematical objects can be accessed through different representations, the principal ones for the function concept being algebraic, graphical and tabular. Understanding of the concept then requires the grasp of these different representations and of the interplay between them, and these studies typically concern students’ abilities to work with these representations, and to move from one representational system to another. A related example from within a Nordic context is (Hähköniemi, 2006). In this context one may also include studies investigating the use of technological tools designed to facilitate the use of different representational systems. However, the idea of representations as a central object of study has been criticized, for instance by Thompson (1994), Dörfler (2000b, 2006) and Sfard (2008). Thompson’s criticism is primarily pedagogical, questioning whether, to students, the different representations actually represent any abstract mathematical concept, and not rather something else, such as aspects of a specific situation (Thompson, 1994, p. 39). The criticism put forward by Dörfler and Sfard is more ontological in nature, questioning the whole idea that there is something

---

10 e.g. Leinhardt, Zaslavsky, & Stein, 1990; Malik, 1980; Sfard, 1992; Sierpinska, 1992; Smith, 2003; Tall, 1996; Thompson, 1994.
12 e.g. Bardini, Pierce, & Stacey, 2004; Bloch, 2003; Borba & Confrey, 1996; Očak, 2008; Schwarz & Dreyfus, 1995.
beyond the representations; some abstract object. As Dörfler writes: “I do not consider prototypic objects as representing the relevant abstract concept or abstract object. Rather, for me, an abstract concept or object is a specific way of talking about the prototypic objects based on experiences with them” (Dörfler, 2000b, pp. 105-106). Abstract mathematical concepts are discursive objects. “Formal definition should not be considered as creating an abstract entity but, rather, as an explicit rule for a correct discourse” (ibid, p. 106). For a related discussion, see sections 3.1-3.2.

The function concept has also played a significant role in theory development within mathematics education research. A number of theories have been formulated attempting to describe concept development, and more often than not the function concept has been used as a central example. Highly influential such theories include the APOS theory of Dubinsky and colleagues (e.g. Asiala et al, 1996; Breidenbach et al, 1992; Dubinsky & Harel, 1992) and Sfard’s (1991, 1992) theory of process-object duality. Sfard claims that mathematical concepts can be viewed in two complementary ways – as processes and objects - and that being able to see a mathematical concept both as a process and an object is necessary for a deep understanding of the concept (Sfard, 1991, p. 5). However, Sfard claims, operational conceptions generally precede structural, both historically and in individual learning. Concerning individual conceptual development, Sfard proposes a three-stage model for concept formation, culminating in reification: the ability to view the concept as an object in its own right (ibid, p. 18-20). Another influential theory of concept development already mentioned is the concept definition/concept image theory (Tall & Vinner, 1981; Vinner, 1983), which distinguishes between the concept definition, being the formal mathematical definition as accepted by the larger mathematical community, and the concept image, which is seen as something much wider - an individual mental construction representing “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (Tall & Vinner, 1981, p. 152).

These theories take a broad view of concept development in mathematics, but there are also a number of theories dealing more specifically with the development of the function concept. One way of going about this is through the idea of multiple representations (Schwarz & Dreyfus, 1995), sometimes in combination with some variant of process-object duality, like in the “facets” and “layers” of DeMarois and Tall (1996) or the “horizontal and vertical growth” of Schwingendorf, Hawks and Beineke (1992). Another is through
focusing more on the specific characteristics of functions, whether speaking of prototypes (Schwarz & Herschkowitz, 1999; Tall & Bakar, 1991) or a property-oriented view (Slavit, 1997) of functions. A recent development, combining a more traditional process-object approach to concept development with a property-oriented view is (Ronda, 2009), speaking of “growth points”, describing the big ideas in students’ understanding of the function concept. Another recent development is the appearance of studies concerning the learning of the function concept in more advanced mathematics, for instance the work of Hamdan (2006) on functions in the context of set theory, and of Trigueros and Martinez-Planell (2010) and Kabael (2011) on two-variable functions.

Although the bulk of research on the teaching and learning of the function concept is of the cognitive type presented above, there are some studies looking at the learning of functions from a more sociocultural perspective. Moschkovich, for instance, investigates “the impact that interaction with a tutor had on a learner, how the learner appropriated goals, actions, perspectives, and meanings that are part of mathematical practices, and how the learner was active in transforming several of the goals that she appropriated.” (Moschkovich, 2004, p. 49). This study is of particular interest also because the analysis of the observational data is done on a very detailed, line-by-line level, something which is, perhaps somewhat surprisingly, not very common in research on students’ development of the function concept. Using Sfard’s commognitive framework, Nachlieli and Tabach (2012) look at how students taking their first steps in the discourse of function are able to participate in this discourse despite the fact that they are still only in the beginning of their objectification of function. Indeed, Nachlieli and Tabach claim that this participation is in fact a prerequisite for the objectification of function (ibid, p. 17).

2.2 Research on mathematics teaching in higher education

As mentioned in the introduction, university mathematics education is a highly active research field, and has become more so over the last decade. Traditionally, however, research in this field has focused on student learning from a cognitive perspective, often focusing on the learning of specific mathematical topics, in the vein of for instance (Tall, 1991). Many of the studies referenced in the previous section fit into this strand of research, which is still very much a going concern. Restricting oneself just to dissertations from the

---

13 There are of course exceptions, for instance Sajka (2003).
Scandinavian countries over the last 10 years, one still finds a large number of works. The first paper of this thesis also belongs to this tradition. In recent years, however, this focus has begun to shift, to include a greater amount of studies, for instance, of institutional aspects of university mathematics education, as well as studies of university mathematics teaching and teachers. Holton (2001), an ICMI study on the teaching and learning of university mathematics, while still containing much cognitive research, also contains a fair amount of research on other aspects of university mathematics education. Another early example is the work of Burton (e.g. 1999a, 1999b, 2004), looking at for instance the learning processes and views of mathematics of professional mathematicians, as well as gender issues. An indication of the shift is the fact that the working group on “Advanced mathematical thinking” at the CERME conferences changed its name to “University mathematics education”, beginning with CERME 7 in 2011. In fact, the leader of this working group, Elena Nardi, is one of the most influential researchers behind the shift. In her work (e.g. Iannone & Nardi, 2005; Nardi, 1999, 2008; Nardi, Jaworski, & Hegedus, 2005) she uses data from interviews with university mathematicians to investigate their views on various issues relating to university mathematics teaching, and the pedagogical awareness and teaching knowledge of university mathematics teachers. In the remainder of this section, I will give an overview of previous research on university mathematics teaching. Where they are relevant to my aims, I will also include works on university mathematics education more generally, as well as on university teaching more generally. I should perhaps mention here that of course the huge field of research on mathematics teacher education also belongs to university mathematics education research, much of it related to teaching, but since this research mostly deals with the knowledge of elementary mathematics needed for prospective teachers, it falls outside the scope of this overview.

Speer and colleagues note that, in contrast to the case of K-12 mathematics teaching, at the collegiate level “very little empirical research has yet described and analysed the practices of teachers of mathematics” (Speer et al, 2010, p. 99). Instead, they claim, most research on university mathematics teaching consists either of analyses of the impact of various “innovative” or “reform” teaching practices (in this strand I also include research on the use of ICT in university mathematics teaching), or of reflections on teaching practice, often informed by

---

14 e.g. Hansson, 2006; Juter, 2006; Pettersson, 2008; Viholainen, 2008.
15 e.g. Attorps, Björk, Radic, & Tossavainen, 2013; Bookman & Friedman, 1994; Darken, Wynegar, & Kuhn, 2000; Frid, 1994; Ganter & Jirottek, 2000; Larsen & Zandieh, 2008; Miller-Reilly, 2007; Rasmussen, Kwon, Allen, Marrongelle, & Burtch, 2006; Schwingendorf, McCabe, & Kuhn, 2000
the authors’ own past teaching experiences\textsuperscript{16}. This is a description consistent with for instance the one given by Harel and Trgalova (1996). Of these two strands of research, the first mainly focuses on curricular aspects, and on forms of teaching. Rarely are there any analyses of teaching practices, although for instance (Frid, 1994) does contain some consideration of data collected through observations of teaching. However, these observations are used for providing a quantitative description of the teaching, monitoring for instance the use of symbols, technical terms and justifications, without paying any attention to the more specific character of them. Furthermore, these observations are used only to a very small extent in the study, which mainly focuses on the analysis of student interviews.

\textbf{2.2.1 Self-reports and theoretical analyses of teaching practice}

This second main strand of university mathematics teaching research is of greater relevance for the present thesis, since these reflections, although not so well grounded in empirical data, often concern themselves with the qualitative aspects of actual teaching practices, and not merely with their form. For instance, the paper by Oikkonen (2009), describing and reflecting over a lecture course in calculus developed and taught by the author himself, stands in contrast to the research belonging to the first strand mentioned in the previous section, since the innovation in Oikkonen’s course concerned mainly the handling and ordering of content, and not the outer form of the teaching. The paper contains fairly detailed descriptions of the handling of content; for instance, Oikkonen describes how a careful choice of examples paves the way for the formal definition of limit. Taking a different perspective, Rodd (2003) uses results from research on theatre and public performance to argue the value of the university lecture as a site of “mathematical awe and wonder”. She claims that the “dominance of overt social interaction within the teaching paradigms currently advocated” (Rodd, 2003, p. 18) tends to ignore the fact that participation need not necessarily be explicit. The attendant of a lecture might be an active participant in the sense of the audience of a play. Also, lectures play an important part in the establishment of a community of practice (Wenger, 1998), thus providing inspiration and motivation. As Rodd puts it, far from being just means of information transmission, lectures “have the potential to develop students’ identity within a mathematical community by means of

\textsuperscript{16}\textit{e.g.} Barton, 2011; Marrongelle & Rasmussen, 2008; Mason, 2008; Mason & Watson, 2008; Oikkonen, 2009; Pritchard, 2010; Rodd, 2003; Schoenfeld, 1994, 1998; Wood & Harding, 2007

23
using theatrical experience to stimulate their imaginations and to inspire them to act” (p. 20).

Barton (2011), Mason (2008; Mason & Watson, 2008), and Pritchard (2010) all present more elaborate frameworks for understanding university mathematics teaching. Although these frameworks are mainly aimed at practicing university teachers, they all contain elements which are of relevance from the perspective of empirical research on teaching. Mason aims at providing a psychological structure for mathematical topics, based on the notion of concept image as well as for instance the notion of “dimensions of variation” (Marton, Runesson, & Tsui, 2004). Though the framework deals mainly with the preparation of teaching, such aspects as the role of examples in developing an understanding of a topic, and the notion of example space – “The class of examples which come to mind in association with a concept or technique” (Mason, 2008, p. 257) – could be useful in the analysis of actual teaching practices. Barton’s framework is partly grounded in discussions with colleagues around video recordings of Barton’s own lectures, and constructs undergraduate teaching as “the interplay between the mathematical essence and the learning culture” (Barton, 2011, p. 965). This interplay is conceptualized through three potential curricular contributions of teaching – the pragmatic, epistemic and heuristic contributions, referring to different ways in which teaching might affect current and future understanding. Though the framework is mainly meant as a development tool, Barton does provide some analysis of specific characteristics of university mathematics teaching, for instance the role of examples. In a typical lecture, these mainly serve as tools for providing students with worked examples similar to exam problems, and he uses the framework to attempt to understand this “tyranny of examples”. Barton thus presents a view of examples somewhat different from, and complementing, Mason’s.

Pritchard’s (2010) framework focuses specifically on university mathematics lectures, and is partly developed in response to the criticism levelled at them from general educationalists (e.g. Bligh, 2000; Biggs & Tang, 2011). Pritchard fears that a wholesale dismissal of the lecture as a form of teaching, without first examining what mathematics lectures actually achieve, runs the risk of causing harm rather than benefit to students (Pritchard, 2010, p. 609). He highlights three types of functions that lectures can perform well: communicating information; modelling problem solving strategies, forms of reasoning and thought processes; and motivating students. I find the modelling aspect particularly interesting and relevant. As Pritchard notes, the nature of mathematical reasoning is not straightforward, and needs to be learnt by
illustration as well as practice. This is a view which fits well with the model of learning as initiation into mathematical discourse given within commognitive theory. Also, as emphasized by Pritchard as well as others (e.g. Greiffenhagen, 2008; Artemeva & Fox, 2011), a proof or a solution to a problem presented in its finished written form is conceived of as a single complete object, and the process through which it was developed is obscured (Pritchard, 2010, p. 617). The lecture format provides a means for the teacher to highlight this process. This is also a main reason for why the use of transparencies or PowerPoint slides is so uncommon among mathematics lecturers (Artemeva & Fox, 2011, p. 357; see also Hora & Ferrare, 2013, p. 247). As Greiffenhagen observes:

“The perhaps most remarkable feature of mathematical lectures is the fact that the lecturer spends most of the time writing at the board. In other words, the lecturer is predominantly not talking "about" mathematics, but actually "doing" mathematics at the board (...) In other words, the lecture can be thought of as a recipient-designed demonstration of mathematical reasoning” (Greiffenhagen, 2008, par. 35).

In this context one could also mention more instructional texts on the teaching of university mathematics, like for instance (Baumslag, 2000), and the well-known book by Krantz (1999). These books are written by mathematicians, and tend to be written from the perspective of the experienced and fairly opinionated colleague sharing his experience with a beginner. They contain lots of very hands-on advice on all aspects of university mathematics teaching, but they are seldom informed by empirical research, and mainly reflect the, albeit often well-argued, views and teaching philosophies of their authors. This is said not as criticism, but to point out the limits to the usefulness of this type of texts when doing research on university mathematics teaching.

2.2.2 Empirical research on university mathematics teaching

In their survey, Speer et al (2010) find only five studies explicitly concerned with empirical analyses of university mathematics teaching practices. Of these, three (Arcavi, Meira, Kessel, & Smith, 1998; Wagner, Speer, & Rossa, 2007; Speer & Wagner, 2009) focus on teaching which was non-traditional either in terms of content or instructional approach, and which involved large amounts of group discussion. The teaching practices analysed there are thus quite different from those considered in this thesis. One (Speer, 2008) deals with the relation between a novice teachers beliefs and his teaching, also with a focus on small group work, and is also of less interest in the context of this thesis. The last one, however, is highly relevant. Weber (2004), a case study of one university mathematics professor’s lectures in a course on real analysis, is an in-
depth empirical analysis of traditional mathematics teaching in what Weber calls the “definition-theorem-proof” (DTP) format. One of the observations made in the study is that, despite the consistency in overall form, the lecture styles used by the professor were not uniform. Weber describes three different such teaching styles: logico-structural, focusing on using definitions and formal deductive inference to produce proofs, without discussing semantic meaning; procedural, focusing on techniques and heuristics used to produce proofs; and semantic, focusing on the semantic meaning of concepts, using intuitive descriptions of ideas to give motivation for the plausibility of a proof. Weber also discusses the professor’s rationales for using these lecturing styles, and how they are based on his beliefs about his students, and about mathematics. Weber’s study shows that although teaching may on the surface appear to be similar in form, closer inspection can reveal major differences in actual teaching practices. A consequence of this is that research focusing mainly on surface differences in form, and for instance treating the lecture format as essentially homogeneous, runs the risk of missing important aspects of the teaching (ibid, p. 131). In relation to Weber’s teaching styles, but from outside of mathematics education, Saroyan and Snell (1997) compare three lectures in dermatology, representative of three different lecturing styles: content-, context- and pedagogy-driven lectures, with content-driven lectures being teacher-centred and aimed at maximizing information delivery; context-driven lectures being student-centred and situating the content within a context chosen to help promote the instructional goals; and pedagogy-driven lectures being student centred and using a wide variety of pedagogical tools aimed at making learning possible. This categorisation, however, seems less distinctive than Weber’s, and it is not entirely clear to what extent the categorisation can actually be useful in describing teaching.

There are some empirical studies of traditional university mathematics teaching practices which are not covered by the survey of Speer and her colleagues. Already mentioned is the study by Greiffenhagen (2008), a sociological study containing a very detailed micro-analysis of a graduate lecture in mathematical logic, focusing on the lecturer’s handling of proof and proving techniques. The study, framed by ethnomethodology and conversation analysis, is not aimed at studying mathematics teaching per se, but rather the professional practices of working mathematicians. However, Greiffenhagen identifies the graduate lecture as one possible site where mathematical practices become visible. As he puts it:
“Although a lecture is predominantly a monologue, it is nevertheless in a certain sense an interactional event, since the lecturer is not talking for himself, but rather for the students. In other words, mathematical lectures are situations in which an experienced mathematician demonstrates mathematical expertise to novices as an important part of their progressive induction into professionally competent autonomous mathematical practice” (ibid, par. 31).

Although the study is thus not aimed at analysing mathematical practices as teaching practices it is nevertheless an excellent example of highly detailed micro-analysis of video-recorded mathematical practice. The analysis deals for instance with the role of posture and gestures in conveying aspects of mathematical arguments, and Greiffenhagen claims that this shows mathematical practice as an embodied, situated activity, far from the image of mathematics as abstract and formal (ibid, par. 63). More directly relevant to mathematics teaching practice, the analysis shows how the lecturer gives vernacular formulations of key ideas of the proof in advance of actually writing them down, thus helping students follow the construction of the proof (ibid, par. 66).

Wood, Joyce, Petosz and Rodd (2007) use a case study of one lecture on de Moivre’s theorem to discuss the multimodality of lectures, and the use of multiple representations of mathematical concepts in promoting student learning, building on Duval’s (2006) notion of “registers of semiotic representation”. In another paper based on the same data, Wood and Smith (2004) discuss the role of spoken and written language in mathematics lectures. They describe lecturing as a mixed-mode activity, using oral language, written language, mathematical notations and visual diagrams. Written language is characterized by a high degree of depersonalization (passives, objectification etc.), whereas the spoken language is less precise and more personal. Nickerson and Bowers (2008), another case study, investigates the teaching practices of a renowned teacher teaching a course aimed at prospective secondary teachers, focusing on the interaction between teacher and students, and on the description of interaction patterns. Two prominent such patterns were found: the ERE (elicit-respond-elaborate) and PD (proposition-discussion) patterns. The first was connected to the teacher’s introduction of new activities, while the second began to emerge as the students developed a more sophisticated understanding of the topic. Bergsten (2007), in a rare example of Swedish research on university mathematics teaching practice, uses empirical data from a case study of one university lecture on limits of functions to help develop a model for analysing quality aspects of mathematics lectures. This model includes aspects of mathematical exposition (including for instance handling of content, mathematical process, and establishment of socio-mathematical
norms), teacher immediacy (including personalization and humour), and general criteria of quality teaching (for instance orchestration of teaching and activation of students). Another study in this vein is (Movshovitz-Hadar & Hazzan, 2004), presenting six criteria for good lecturing, based on the analysis of one lecture given by an award-winning mathematics teacher.

Using data from tutorials rather than lectures, (Jaworski, 2002) is an attempt at analysing university mathematics teaching using a theoretical tool, the teaching triad, previously developed in the context of secondary school teaching. This triad comprises three elements – management of learning, sensitivity to students, and mathematical challenge – and Jaworski compares two teaching episodes, showing how the triad can be used to characterize the teaching. Analysing data from the same project, but focusing more on the teachers’ own reflections on their teaching, Nardi, Jaworski and Hegedus (2005) present a “spectrum of pedagogical awareness”, ranging from “naïve & dismissive” to “confident & articulate” (p. 293). In another study on university mathematics teachers’ reflections on their practice, Iannone and Nardi (2005) present teachers’ reflections on the role of communication and interaction within mathematics teaching, and of teaching as an initiation into the “genre speech” of mathematics.

After the appearance of the paper by Speer et al (2010), with its call for empirical research on university mathematics teaching practice, a number of such papers have been published. For instance, Artemeva and Fox (2011) approach tertiary mathematics teaching from a linguistic perspective. They identify a central pedagogical genre of the university mathematics classroom, namely chalk talk: “writing out a mathematical narrative on the board while talking aloud” (ibid, p. 345). They describe a number of characteristic features of this pedagogical genre, for instance the use of running commentary and meta-commentary, the use of movement and gestures to indicate relationships and highlight key issues, and the use of rhetorical questions to signal transitions, pause for reflections and check student understanding. The paper contains in-depth analyses of observational data but, perhaps inevitably given the linguistic viewpoint, lacks detail on the handling of the mathematical content. Fukawa-Connelly (2012), a case study of a traditionally-taught abstract algebra course, focusing on the teaching of proof, finds the range of teaching practices to be greater than indicated by earlier research. The results of the study show how the teacher models some aspects of proof and some modes of thinking about proof, but not others. Also, there was a greater amount of student participation than expected. Despite this, however, little of the responsibility for the
reasoning was delegated to the students. Rather, the teaching provided a model of reasoning about proof. Hora and Ferrare (2013) look at mathematics (and science) teaching and course planning from a systems-of-practice perspective. Their results identify a great deal of disciplinary specificity in teaching practices, meaning that mathematics teachers tend to use the same methods and combinations of techniques. Also, mathematics teachers tended to use a more limited repertoire of teaching practices than, for instance, physics and chemistry teachers. This repertoire included for instance lecturing and problem solving activities, and the dominant technological medium of instruction was the chalkboard. Hora and Ferrare uses the term lecturing to refer to the teacher providing verbal discourse to an audience, focusing on the presentation of facts and concepts (ibid, p. 213; p. 247), and note that while this is a central part of the instructional practices documented in the study, it is still just one of a number of teaching practices used by the teachers. The observational data in the study were collected using an observational instrument designed to capture more of this variety of pedagogical techniques. However, as Hora and Ferrare themselves acknowledge, this instrument is still a relatively blunt tool for capturing teaching activity, looking mainly at the surface form of the activities. Still, the classification of questions used in the study (rhetorical, conceptual, algorithmic and comprehension questions) provides a useful categorization in the context of this thesis.

Finally, the already mentioned study by Güçler (2013) is of particular interest here, since it uses commognitive theory to investigate university mathematics teaching\(^\text{17}\). It is a case study of the discourse of limits in a beginning-level undergraduate calculus classroom, focusing on the teachers’ and the students’ discourses. The results show that there were distinct differences in these discourses. The teacher used mainly an objectified discourse of limits, particularly in his written discourse, with operational language used in verbal discourse. He endorsed two main meta-level narratives\(^\text{18}\) – “limit as number” and “limit as process” – and used different meta-rules to endorse these different narratives. The students, on the other hand, used mainly an operational discourse, and they also utilized different meta-rules when endorsing narratives.

\(^{17}\) A comparison of Güçler’s study with the present thesis, from a methodological perspective, can be found in section 4.3.

\(^{18}\) For the precise meaning of these technical terms, see section 3.3.
3 Theoretical considerations

This chapter contains a discussion of theoretical questions relevant to the thesis. I first present two theoretical assumptions that form a foundation for the research presented in the thesis. I then present the theoretical framework used, in relation to other relevant frameworks.

3.1 Basic theoretical assumptions

In this section I wish to discuss two basic theoretical assumptions of a philosophical nature underlying this thesis. The first assumption concerns the way I view learning. Learning is a central notion in research on education, and also a term often burdened with implicit assumptions. In an oft-cited paper (Sfard, 1998) and elsewhere (e.g. Sfard, 2003; 2008), Sfard has attempted to address this issue by writing of two metaphors underlying most, if not all, theories of learning: the acquisition and participation metaphors. Through the acquisition metaphor learning is described in terms bringing to mind the accumulation of material goods, while through the participation metaphor learning is seen as the process of becoming a member of a certain community. The mode of acquisition is of less importance in this context, so both theories viewing learning as passive reception and as active construction are counted as relying on the acquisition metaphor (Sfard, 2007, p. 567). Piagetian constructivism could be considered as the paradigmatic example of an acquisitionist framework (Sfard, 2003, p. 356), while the theory of “legitimate peripheral participation” (Lave & Wenger, 1991) is often cited as an example of a framework making extensive use of the participation metaphor (Sfard, 1998, 2003). Sfard’s recent “commognitive” framework, which will be discussed in greater detail in section 3.3, is an attempt at developing a participationist theory of thinking in general, and mathematical thinking and learning in particular. Two points need to be made here. Firstly, it should be noted that both metaphors are most often found to be underlying any given theory. What can be seen is which metaphor that takes precedence (Sfard, 1998, pp. 6-7; 2003, pp. 355-356). In the present thesis, the first paper is written in an acquisitionist mode, while the rest of the papers rely mainly on the participation metaphor. Secondly, the metaphors should be seen as theoretical constructs describing different ways of reasoning about learning. They should not be taken as descriptive of actual teaching, an interpretation which is sometimes seen in the literature (e.g. Emanuelsson & Sahlström, 2008). As I see it, it doesn’t make sense to speak of acquisitionist or participationist teaching. On the contrary,
teaching practices not involving much in the way of student participation, such as those of the teachers in my study, can still be analysed fruitfully within a participationist framework.

The second assumption concerns the nature of mathematics. Traditionally, philosophers of mathematics have basically assumed one of two positions – Platonist or formalist.\(^{19}\) Davis and Hersh (1981, p. 407; see also Hersh, 1997, p. 16) state two facts about the nature of mathematics: 1) Mathematics is a human invention. 2) Once invented, these mathematical objects have properties which are unknown to us, and which we may only discover by great effort, or sometimes not at all. As Davis and Hersh see it, Platonism is built on fact 2, believing mathematics to have an objective existence outside of human thought, much akin to Plato’s “world of ideas”. Formalism on the other hand is built on fact 1, viewing mathematics as the formal manipulation of symbols according to certain rules and axioms, and rejecting the existence of mathematical objects (Hersh, 1997, pp. 138-139). Davis and Hersh propose an alternative standpoint, what they call a humanist philosophy of mathematics, in an attempt to take into consideration both of the two facts. They take as their starting point the socio-historical character of mathematics as a human endeavour, a human creation. They view mathematical objects as having social reality, meaning that they do not have physical existence, but neither are they purely mental, being rooted in collective ways of acting and thinking, and thus fixed in a way that mental constructs are not (ibid, pp. 13-15). This also serves to explain the perceived contradiction between the two views of mathematics as “invented” and “discovered”. We invent new mathematical objects, whose properties we then proceed to discover. Hersh sees mathematics as one of the humanities, but playing the same role with respect to the social world as the natural sciences do for the physical world: “the study of mental objects with reproducible properties is called mathematics” (ibid, p. 66, emph. in original). This view of the ontological status of mathematics is inspired for instance by Wittgenstein and Lakatos, and similar to the social constructivist philosophy of Ernest (1997). It is the view which underlies the present thesis. More specifically, I view mathematics as a discursive activity, meaning that mathematical objects are seen as discursively constituted, and that doing mathematics is viewed as engaging in mathematical discourse. This position is discussed in more detail in the remainder of the chapter, where the theoretical framework used in the thesis is described.

\(^{19}\) This is of course a gross simplification. The philosophy of mathematics is a research field in its own right, with a large body of literature, and a number of schools of thought, like intuitionism, constructivism, fictionalism, et al. What these have in common, however, is the rejection of Platonism. For the purposes of this thesis, the distinction made above will be adequate.
However, I first present a brief overview of various discursive approaches to research in mathematics education, in order to provide some context for this framework.

### 3.2 Discursive approaches to mathematics education research

A common feature of the theories of concept formation discussed in section 2.1.2 and used in Paper I in this thesis is that they are all what Dörfler calls cognitive theories, “in the sense that they consider the building (of the notion) of a mathematical object as a cognitive process that involves the learner’s construction of adequate cognitive structures. Those cognitive structures in a way are presumed to present the mathematical object to the learners and enable them to participate sensibly in the respective discourse” (Dörfler, 2000a, p. 340). Dörfler contrasts this with a view of doing mathematics as participating in mathematical discourse, and locates himself “within the context of the discursive approach as a general framework for investigating issues of learning mathematics” (ibid, p. 339). This “discursive approach” is the topic of this section.

Put in very general terms, “the study of discourse is the study of human communication; the most unique of this communication is language in use” (Ryve, 2011, p. 169). Ryve, in a review of 108 journal articles using discursive approaches in mathematics education research, follows Wetherell (2001) in highlighting three important principles behind the study of discourse: discourse is constitutive, it is functional and it is co-constructed. Put very briefly, this means that discourse doesn’t just reflect reality – it builds objects, minds, identities and realities; the versions of reality constructed through discourse are designed to accomplish various objectives; and the meanings of language must be understood in their context, both locally in the surrounding interaction, and from a larger socio-historical perspective (Ryve, 2011, p. 170; Wetherell, 2001, pp. 15-18). Differences in the emphasis put on these principles lead to differences in research focus. Wetherell, Taylor and Yates structure their overview of discourse analysis within the social sciences into three topic areas, which are also used by Ryve in his analysis: Social interaction; Minds, selves and sense-making; and Culture and social relations (Wetherell et al, 2001, p. 5; Ryve, 2011, p. 171). Within these topic areas there are various research traditions – Wetherell et al distinguish between at least six. It is clear that a discussion of all these various uses of the notion of discourse in research would lead too far afield20. Instead I will restrict myself to the use of discursive theories within

---

20 For overviews of this field, see e.g. Jaworski & Coupland, 2005; Wetherell et al, 2001.
Still, as pointed out for instance by Ryve (2011) and Jablonka, Wagner and Walshaw (2013), much research within mathematics education makes use of theories explicitly borrowed from other fields. Hence, even restricting oneself to the field of mathematics education, some discussion of these more general theories will be necessary. Moreover I will focus on research relevant to the aims of this thesis. Hence, I will not discuss, for instance, research on the role of discourse in establishing relations of power within the mathematics classroom, or in constructing students’ identities as mathematics learners. Instead, the focus will be on research into what constitutes mathematical discourse, and the relation between mathematical discourses, mathematical classroom discourses, and general educational discourses.

One early attempt at a characterization of mathematical discourse can be found in the work of Halliday (1978). He referred to “the discipline-specific use of language employed in mathematics education” (Jablonka et al., 2013, p. 51) as “the mathematics register”. It should be noted that this does not solely refer to specific vocabulary, but also to meanings, styles and modes of argument (Moschkovich, 2003). The characterization of the mathematics register was later developed and elaborated by Pimm (1987). However it has been pointed out by various researchers that this description is too vague, and that one needs to distinguish between different types of mathematical registers in different contexts (Jablonka et al., 2013, p. 51). For instance, Morgan highlights the obvious differences between a research article and a primary school textbook, say, but also the “considerable cultural differences among those who participate in the exchange of mathematical meanings” (Morgan, 2006, p. 225), suggesting that it might be more useful to speak of a family of mathematical registers. Still, Morgan sees Halliday’s social semiotics as a useful perspective for research in mathematics education, and develops methodological tools based on the three functions of language identified in Halliday’s functional grammar: the ideational, interpersonal and textual functions. These in turn are related to Halliday’s notions of the field, tenor and mode of discourse, where the field refers to the subject matter and institutional setting of the discursive activity; the tenor encompasses the relationships of the participants; and the mode refers to the channel of communication (ibid, p. 221). Jablonka et al sees Morgan’s methodology as useful for “analysing transcripts to identify who or what is doing things in learning contexts, the objects of mathematics in these contexts and the relationships at work” (Jablonka et al., 2013, p. 52).
Another take on Halliday’s work is provided by Moschkovich (2002), who presents three perspectives for studying how bilingual students learn mathematics – one focused on acquisition of vocabulary, one focused on the construction of multiple meanings of words, and one focused on participating in mathematical discourse practices. The second of these is based on Halliday’s notion of the mathematical register, but Moschkovich sees distinct limitations in the usefulness of this perspective in understanding the full complexity of learning in bilingual settings. For instance, although “differences between the everyday and mathematical registers may sometimes present obstacles for communicating in mathematically precise ways and everyday meanings can sometimes be ambiguous, everyday meanings and metaphors can also be resources for understanding mathematical concepts” (Moschkovich, 2002, p. 196). Instead, Moschkovich argues for the third perspective, which she denotes as situated-sociocultural. This perspective is inspired by theories of situated learning (e.g. Lave & Wenger, 1991), and by Gee’s concept of Discourse:

“According to Gee’s definition, Discourses are more than sequential speech or writing and involve more than the use of technical language; they also involve points of view, communities, and values. Mathematical Discourses (in Gee’s sense) include not only ways of talking, acting, interacting, thinking, believing, reading, and writing but also mathematical values, beliefs, and points of view of a situation.” (Moschkovich, 2002, p. 198)

Moschkovich shows how bilingual students participate in various Discourses, which support and reinforce each other. The perspective of bi- or multilinguality in mathematics education has been studied by other researchers (see e.g. Barwell, Barton, & Setati, 2007). Barton (2009), inspired by his involvement in developing mathematical vocabulary in the Maori language of his native New Zealand, discusses various issues concerning the relation between (natural) language and mathematics. Although highly interesting, this perspective on mathematical discourse is of less relevance to the present thesis, except in highlighting the need to execute care when writing in English about analyses of teaching conducted in Swedish (see Paper II for an example).

van Oers (2002), like Moschkovich, is interested in investigating mathematics classroom discourse from a deeper perspective than that of the mathematical register. He applies a Bakhtinian approach to “the discourse in a mathematics classroom, especially focusing on the question of how the participants in this classroom are linked together and what common background is to be constructed in order to constitute a way of speaking and interacting that will be acknowledged as a mathematical discourse” (van Oers, 2002, p. 60, emph. in original). Bakhtin’s notion of ‘speech genre’ is used to bring out the “cultural
historical dimension in the discourse that is supposed to be acted out by the teacher who demonstrates the tools, rules, and norms that are passed on by a mathematical community” (ibid, p. 59). Participants in the discourse value utterances against an implicit cultural-historical background which forms part of the speech genre they share, and it is this shared common background that enables them to communicate effectively. “Bakhtin’s notion of speech genre implies that utterances of the interlocutors in the discourse are not just assessed in terms of their literal meaning, but also valued from a generic background that provides meta-rules and norms which help in defining the utterances involved as mathematical or not” (ibid, p. 70). This view of discourse is similar to the one taken by Sfard (2008), which will be discussed at length in the next section. First, however, I wish to say something about semiotic approaches to mathematical discourse. Researchers such as Duval (2006) and Radford, Bardini and Sabena (2007) “stress that mathematical activities are heavily dependent on the use of different semiotic systems such as mathematical words, algebraic symbols, graphs, and drawings” (Ryve, 2011, p. 173). Radford and colleagues have developed what they call a semiotic-cultural framework in order to investigate how various semiotic resources are employed by learners in order to facilitate knowledge objectification. Again, this approach has similarities with certain aspects of Sfard’s theory. Duval (2006), on the other hand, has focused on the mathematical discourse itself, introducing the notion of ‘registers of semiotic representation’ in order to describe and understand mathematical activity. For Duval, these semiotic representations are the only way for us to get access to the mathematical objects, which are in themselves perceptually inaccessible. However, he very clearly points out that “the mathematical objects must never be confused with the semiotic representations that are used” (Duval, 2006, p. 107). As we shall see in the next section, this stands in contrast to the view taken in this thesis, following Sfard, in which mathematical objects are seen as discursively constituted. This is also the view argued by Dörfler (2000a), who describes the discursive approach as locating “mathematical objects as discursive objects within the mathematical discourse” (Dörfler, 2000a, p. 339). In fact, “the whole discourse of mathematics lends meaning and existence to the mathematical object: They are talked (or written) into reality” (ibid, p. 339). This idea is of course related to Wittgenstein’s (1953) notion of “language games”. As for learning mathematics, this then becomes a process of “learning to use and understand its language, its symbols, its diagrams, its procedures” (Dörfler, 2000a, p. 339)
Having given a brief overview of some theories concerned in various ways with the character of mathematical discourse, we now turn to the framework used in this thesis – Sfard's commognitive framework. As we will see, it has aspects in common with many of the theories discussed in this section. However, I believe that it also has some particular advantages. First, it is a comprehensive theory of thinking and learning developed within the field of mathematics education research itself, and with great care taken in providing operational definitions of all central notions. Second, it comes equipped with a set of methodological tools suitable for my needs.

3.3 Sfard’s commognitive framework

In recent years Sfard has developed a discursive theoretical framework (e.g. Sfard, 2000, 2001, 2006, 2007, 2008) for studying thinking in general, and mathematical thinking, teaching and learning in particular. Underlying the theory is the basic tenet “that patterned, collective forms of distinctly human forms of doing are developmentally prior to the activities of the individual” (Sfard, 2008, p. 78, emph. in original). This assumption is fundamental to what Sfard calls “participationism” (see section 3.1), and it is obviously inspired by the thinking of Vygotsky, and his claim that human intellectual development presupposes a socio-historical context (e.g. Vygotsky, 1978, p. 88).

If we accept this assumption, then it should apply also to the activity we call thinking. Sfard’s solution is to define thinking as “an individualized version of (interpersonal) communicating” (Sfard, 2008, p. 81). She coins the neologism commognition (a composite of “communication” and “cognition”) in order to encapsulate both inter- and intrapersonal communication. But for this definition to tell us anything, we need an operational definition of communication. Sfard (ibid, p. 86f, emph. in original) proposes the following:

“Communication is a collectively performed patterned activity in which action A of an individual is followed by action B of another individual so that

1. A belongs to a certain well-defined repertoire of actions known as communicational
2. Action B belongs to a repertoire of re-actions that fit A, that is, actions recurrently observed in conjunction with A. This latter repertoire is not exclusively a function of A, and it depends, among others, on factors such as the history of A (what happened prior to A), the situation in which A and B are performed, and the identities of the actor and reactor.”

Among other things, this definition depends on the notion of communicational action. Sfard does not really try to provide us with a definition of this notion, just stating that a communicational action is an “action that is performed as a
part of the activity of communication” (ibid, p. 296). However, the main point of Sfard’s endeavour is to avoid the typically acquisitionist definitions traditionally given of communication as the exchange of meanings, ideas or information, instead providing a non-objectified definition in terms of observable activities and behaviour. We will thus have to rely on being able to identify communicational actions when we encounter them. In the context of the teaching observed and analysed in the present thesis, this will present no difficulty. I should perhaps emphasize, however, that these communicational actions need not be verbal.

Within the qualifications given above, we thus have an operational definition of communication. Another question is whether this definition is valid for intra-personal communication, or, put somewhat differently, whether Sfard’s definition of thinking really covers the whole of the everyday use of the term thinking. Personally, I am inclined to accept the Vygotskian idea of the social preceding the individual, and also Sfard’s view of thinking as communication. However, I am less convinced that the characterization of communication as a pattern of action and reaction really covers all forms of intra-personal communication. Still, for the type of thinking involved in intellectual, problem-solving activity, like mathematics, the definition probably suffices (see also Kilhamn, 2011, p. 68).

But communication is not the same in all contexts. Indeed, there are many types of communication, “set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors” (Sfard, 2008, p. 93). These different types of communication are called discourses. Hence, for Sfard the notions of discourse and communication are very closely related. A discourse is a specific type of communication, but every act of communication is also part of a certain discourse (or perhaps several overlapping discourses). Indeed, in this thesis I will follow Sfard in using the terms communicational and discursive action as equivalent (ibid, p. 296). Any discourse includes some individuals while excluding others, and we can speak of communities of discourse, made up of those individuals able to participate in certain discourses (ibid, p. 91). Of course these communities of discourse are partially overlapping, with any one individual being member of many such communities. I will use the term discursive practice to denote the collection of various discursive actions employed by any particular individual when participating in a specific discourse.21. Let me

21 For the sake of readability, I will sometimes use, for instance, the term "discourse of the teacher". This should not be interpreted as implying any kind of ownership of the discourse on the part of the
note here that these definitions should not be interpreted as implying a static view of discourse. Boundaries of discourses and discourse communities are seldom clear-cut, and furthermore the discourses themselves should be seen as processes rather than static entities, constantly developing. “Still, human ways of communicating are often distinct enough to justify talking about discourses and their communities as reasonably well-delineated wholes” (ibid, p. 91).

The view of discourse as constantly developing and changing, growing ever more complex, is key to the commognitive view of learning. From a participationist viewpoint, learning is not viewed as change in individuals, but rather in forms of doing, both individual and collective (Sfard, 2007, p. 568). In her book, Sfard argues the claim that “Changes in all forms of human doing are a function of changes in commognition, thus in discourses” (Sfard, 2008, p. 116, emph. in original). Indeed, human development can be seen as the development of discourses. On an individual level, learning may be defined as individualizing discourse, becoming ever more capable at communicating within the discourse, with others as well as with oneself (Sfard, 2006, p. 162). This is achieved through a process of adjusting one’s discursive activities to fit the leading discourse (or, more rarely, the other way around). Central to this process is what Sfard calls commognitive conflict (Sfard, 2008, p. 256). This occurs when different discursants act according to different discursive rules, and is often a necessary feature of discursive change. The teacher takes on the role of facilitator of this process of discursive change, and teaching is seen as the activity of helping the learners turn what is at first a discourse-for-others into a discourse-for-oneself (ibid, p. 284-285). This involves both acting as the “voice” of the leading discourse, and providing the learner with the opportunities of participating in the discourse on her terms, with the goal of turning it into a discourse-for-herself. For this to be possible, teacher and learner need to agree on what the leading discourse is, what their respective roles are, and on the nature of the expected change. This is what Sfard calls the learning-teaching agreement (ibid, p. 282-287).

The unit of commognitive analysis is the discursive activity, the “patterned collective doings” (Sfard 2006, p. 157). Hence, whereas in Paper I the object of study is the concept of function and students’ conceptions of it, in the remainder of the papers the object of study is the discourse of mathematics, specifically functions, and of mathematics teaching, as manifested in the communicative practices of the teachers. One should perhaps note here, that
concepts do have a place within the commognitive framework as well, but they are defined discursively: a concept is a symbol (word or other signifier) together with its discursive uses (Sfard, 2008, p. 111; p. 296). Conceptual development will then be development of the discursive uses of the concept.

In order to be able to study discursive activity, we need to know more of what characterizes different discourses in general, and mathematical discourse specifically. As suggested earlier in this section, what distinguishes one discourse from another are their objects, mediators and rules. Hence, it would be easy to say that mathematics is the discourse about mathematical objects. This is of little value, however, until we have some kind of definition of what we shall mean by mathematical object. Below I will present Sfard’s way of defining the meaning of this concept, but for now I just note that unlike for instance the discourses of biology or chemistry, where the discourse and its objects are distinct, the objects of mathematical discourse are in themselves discursive. Mathematics can be described as an autopoietic system, that is, “a system that produces the things it talks about” (ibid, p. 161). But more than this is needed to be able to determine what distinguishes mathematical discourse, or indeed what sets any one particular discourse apart from others. Sfard presents four characteristics which can be used to describe and distinguish different discourses (ibid, pp. 133-135):

- **word use** - words specific to the discourse or common words used in discourse-specific ways
- **visual mediators** - visual objects operated upon as a part of the discursive process. Examples from mathematical discourse could be diagrams and special symbols.
- **narratives** - Sequences of utterances speaking of objects, relations between and/or processes upon objects, subject to endorsement or rejection within the discourse. Mathematical examples could be theorems, definitions and equations.
- **routines** – Repetitive patterns characteristic of the discourse. Typical mathematical routines are for instance methods of proof, of performing calculations, and so on.

Of these four characteristics, especially the notion of routine needs further elaboration. Routines are central to the discourse, since discourses are by their very definition repetitive and patterned. In fact, the “quest for discursive patterns is the gist of commognitive research” (ibid, p. 200). These patterns are the result of processes governed by rules. Sfard distinguishes between object-level and meta-discursive rules of discourse. The former regard the properties
of the objects of the discourse, while the latter govern the actions of the
discursants. A routine, then, is a set of meta-rules describing a repetitive
discursive action (ibid, p. 208). This set can be divided into the \textit{how} and \textit{when} of
the routine, determining in the first case the course of action and in the second
case the situations in which action would be deemed appropriate. As for
specifically mathematical routines, Sfard distinguishes between \textit{explorations, deeds}
and \textit{rituals} (ibid, ch. 8). Explorations aim at producing new endorsed narratives
within the discourse, while deeds aim at change in objects. Explorations can in
their turn be divided into three types: \textit{construction}, aimed at constructing new
endorsable narratives; \textit{substantiation}, aimed at deciding whether to endorse
previously constructed narratives; and \textit{recall}, aimed at summoning narratives
endorsed in the past (ibid, p. 225). An example of an exploration could thus be
a calculation as part of a piece of mathematic reasoning, whereas a deed would
be a calculation aimed squarely at getting a numerical result. The objects
transformed through the deed can be physical, such as marbles or coins, but
they can also be purely discursive. The crucial difference is that explorations
aim at producing narratives, while deeds aim at producing or transforming
objects (ibid, p. 255). Lastly, rituals are routines performed with the aim of
social approval rather than creating a product, whether narrative or object.
Children’s use of number-chants is an example, as is students’ performing
calculations just to satisfy the teacher, even though they don’t understand what
they are doing. Although students’ use of rituals could be taken as a sign of
failure in teaching, as a matter of fact they are often necessary steps in the
development of routines (ibid, p. 245). That this is so is a consequence of the
fact that mathematics, as any discourse, is fundamentally social. A necessary
step in the process of turning a discourse-for-others into a discourse-for-
one’self is participating in the discourse on others’ terms, without the proper
means of substantiation. The only way of obtaining these means is through
participation in the discourse.
I stated above that mathematics is the discourse of mathematical objects, and
noted that for this to make sense we need an operational definition of
mathematical object. Sfard attempts to give one, or actually of discursive
objects generally. She proposes the following definition: \textit{“The (discursive) object
signified by $S$ (or simply object $S$) in a given discourse on $S$ is the realization tree of
$S$ within this discourse”} (ibid, p. 166, emph. in original). In order to make sense
of this definition, we need to know what is meant by realization tree. At its
most basic, a realization is a procedure that pairs a signifier with a primary (that
is perceptually accessible, discourse-independent) object. But any product of
such a procedure is also a realization, and so on, recursively. Hence we get a tree of realizations, where each node is both a realization of the node directly above it and a signifier realized by the nodes beneath it. This is the realization tree of the signifier at the top node (ibid, pp. 164-166; p. 301). For example, the graph of \( f \), the formula for \( f \) and the table of values of \( f \) are all examples of realizations of the same signifier, the function \( f \).22 Each of these in turn has one or more realizations, and so on. It should perhaps be pointed out that the realization trees of course are purely theoretical constructs, put together by the researcher. They should not be thought of as actually describing the structure of a person’s thinking regarding a given signifier. But nevertheless, realization trees, and hence mathematical objects, are personal constructs, differing from individual to individual, and they can thus be used to assess the quality of a person’s discourse on some mathematical object (ibid, p. 166). This personal, subjective nature of mathematical objects defined through realization trees might seem a little worrying, if one wants to avoid relativism. However, as Sfard points out, “they originate in public discourses that support only certain versions” of trees (ibid, p. 166). Furthermore, it makes sense to speak of “canonical” realization trees, containing all realizations of a given signifier considered valid within mathematical discourse at large. The construction of these canonical trees would then be a socio-historical process (ibid, pp. 174-177). The canonical trees can then be used as yardsticks by which to judge the richness of individual realization trees. However, I believe that there are limitations to the practical use of the theoretical construct of “realization tree”. The construction of such a tree, for a given signifier and a given individual, will probably be less than straightforward, especially if one is dealing with more complex mathematical objects.

22 As seen from this example, the notion of “realization” is related to that of “representation” (see section 2.1.2). However, there is a difference on the ontological level. Representations are the “material incarnations” of some underlying, intangible abstract object (see for instance Duval, 2006), whereas the realizations all belong to the same ontological category. There are no abstract objects behind the realizations. Instead the object is made up of the tree of realization-signifier relations as a whole (Sfard, 2008, p. 155).
4 Methodology

The research reported in this thesis consists of two parts – a preliminary study investigating university students’ conceptions of the function concept (Paper I), and a main study investigating university mathematics teachers’ teaching practices regarding the function concept (Papers II-V). In this chapter I will present the research questions, the design of the studies, as well as the analytical tools used to investigate the empirical data collected, focusing on the main study. There is also some further methodological discussion, including ethical aspects.

4.1 Research questions

Since, as just stated, the aims of the two parts of the thesis differ, as well as the objects of study and the unit of analysis, the research questions the two studies attempt to answer are also different in character. The aims of the preliminary study are described as getting a preliminary understanding of mathematics students’ conceptions of the function concept, and relating these conceptions to the historical development of the function concept. In Paper I, these aims are operationalized in the following two research questions:

What kind of conceptions of the function concept do the participating students have?

What differences and/or similarities can be found between the students’ conceptions and definitions of the function concept, and the formal definitions as they have developed historically?

As stated in section 1.1, the main study has two principal aims: to describe the discourses of function, and of teaching functions, presented by the teachers, and to investigate how the function concept is characterized through the discursive practices of the teachers. The first of these aims have led to the following research question, which is then narrowed down and made more precise in Papers II-IV:

What characterizes the discourses of the teachers regarding functions and the teaching of functions, particularly with respect to narratives and routines?

In the summaries of the papers (chapter 5) the precise formulations of the research questions in each paper can be found.

The second research question concerns the discursive characterization of the function concept, and reads as follows:

How is the function concept characterized, on both object- and meta-level, by the discursive practices of the teachers?

This question is answered in Paper V.
4.2 Research design

Although the two studies in this thesis are united by their focus on the functions concept within a university mathematics education context, their aims differ in important aspects, and these differences lead to differences in research design. In this section the design of the studies is described, and the choices of methods of data collection and handling of data, as well as tools of analysis of data, are discussed in relation to the overall aims of the thesis. Reflecting its role in the thesis as a whole, the emphasis will be on the main study. As discussed in the introduction, two factors which helped shape this thesis was the choice to focus on university mathematics teaching, and the choice of the function concept as the mathematical topic under study. The function concept plays a dual role; in papers I and V it is an aspect of the unit of analysis, whereas in papers II-IV it serves mainly as a demarcation, to help narrow the scope of the study. Teaching practices are central to the main study, but not at all to the preliminary study, which instead focuses on the students. Before turning to the design of the main study, I will now give a brief account of the design of this preliminary study.

4.2.1 Preliminary study

The aim of the first study is to investigate students’ conceptions of the function concept, and thus these conceptions are what constitute the unit of analysis. Such studies are commonly conducted through questionnaires complemented by student interviews, aimed at getting insight into students thinking about the concept in question. However, given the character of the study as background for the main study, and considering that the data collection had to be made during a very limited time, it was decided to use only questionnaire data, knowing that this might decrease the validity of the results.

The study was conducted at a major Swedish university. Two groups were selected for participation; on the one hand all pre-service teachers taking their first course in calculus at the time of data gathering (autumn of 2006) and on the other hand one group of first semester civil engineering students. The two groups numbered 35 and 29 students, respectively. The courses the two groups were taking were different, but were intended to cover mostly the same topics. Moreover, the students had recently begun the course at the time the study was conducted. The pre-service teachers had studied more than one semester of mathematics, while the engineering students had only taken a course in algebra. Besides some questions on background data, the questionnaire consisted of 6 items. First, the students were asked to associate freely regarding the concept of...
function, and to construct a “mind map”. The next two items presented the students with 12 mathematical expressions and 4 figures, and asked them to determine which of these represented functions. In the fourth item, they were requested to discuss the possibility of constructing a function with certain given characteristics – having an integer value for every non-integer, and a non-integer value for every integer. The last two items, finally, requested that they state their own definitions of the concepts of “variable” and “function”. The students were allowed about one hour for filling out the questionnaire. It turned out to be impossible to collect answers from all students in the groups, and in the end the total number of questionnaires collected was 34; 14 from pre-service teachers and 20 from engineering students.

The analysis of the questionnaire data was made first by me, and then by the two co-authors independently. The categorizations were then compared, showing a high degree of agreement. In the few cases where there was disagreement, consensus was reached after some discussion. In the analysis of the data use was made of elements from a number of different theories, with the main framework used being concept definition/concept image, process-object duality and Sfard’s three-step model of concept development (see section 2.1.2). The main goal of the analysis was to find categorizations of the students’ definitions and concept images of the function concept. For more detail on these categorizations, see Paper I, or the summary in chapter 5.

4.2.2 Main study

As already mentioned, although the two studies in this thesis have certain aspects in common, they are also very different. In the main study the focus is not on students, but on teachers, and not on individual conceptions, but on teaching activities and the discourse of mathematics and mathematics teaching. Indeed, as described in the previous chapter, the main study is framed within a discursive approach to mathematical teaching and learning. Within a discursive framework such as the commognitive framework used in this study, the unit of analysis is the discursive activity, in this case specifically the teachers’ mathematical and teaching discourse. Furthermore, as already discussed the function concept takes on a dual role in this study. With respect to the first research question it serves primarily as a demarcation, to help limit the scope of the study. With respect to the second research question, dealing with the discursive characterization of the function concept, it is an aspect of the unit of analysis.

23 This example was adapted from Vinner & Dreyfus (1989).
The different theoretical starting point and different object of analysis implies different empirical data. If we want to study discourse, we need access to the discursive activity. As Sfard puts it: “In commognitive research, the data collectors must observe the principle of utmost verbal fidelity: They have to pay uncompromising attention to the verbatim version of the interlocutors’ utterances and document interactions conducted for the sake of data collection in their entirety” (Sfard, 2008, p. 277). But, as discussed in section 3.3, the discursive activity is not only the verbal utterances, but also the use of visual mediators and written language. Thus, if I want to study teaching discourse, video recordings of teaching practice are the natural choice of data. In fact, most published research using the commognitive framework is based on data constructed from video recordings of communicational activities. Given this, I decided to base the data of the main study on video recordings of university lectures. I videotaped lectures given by teachers in freshman year mathematics courses at three Swedish universities. The main grounds for selection was ease of access, but the universities chosen turned out to display a high degree of diversity – one old, large university24, University A; one more recently established, University B; and one smaller, regional university, University C. University A attracts students from all over Sweden, whereas universities B and C have a more regional recruitment base. Hence the educational background of the students at University A is perhaps somewhat stronger than at universities B and C. The teachers giving freshman courses on relevant topics during the time available for data collection were approached, and participants were selected among those willing to participate. It was only at the large university that the number of possible participants was large enough to allow further choice. In making this choice I again aimed for diversity, both in topics covered and in teaching experience. The data were collected over three autumn semesters – 2009-2011. The amount of video recorded of each teacher varied from one to three teaching occasions. Apart from lectures, in a few cases I also made video recordings of other teaching activities – mainly problem solving sessions. In the end, however, I decided against using these recordings in my analyses, instead choosing to focus wholly on lectures. A total of seven teachers participated in the study, four from the large university (labelled A1-A4 in what follows), two from the younger university (B1-B2) and one from the regional university (C1). I will now proceed to give very brief presentations of the seven teachers, and the courses they teach.

24 The same one as in the preliminary study.
Teacher A1 is a woman born in the 1950s, teaching a course preliminary to freshman calculus. She got her doctoral degree in 1990, and has taught at the university since then. Teacher A2 is a male doctoral student in his twenties, teaching a course in introductory algebra. It is his first course as lecturer, having previously only served as a teaching assistant. Teacher A3 is a man born in the 1970s, teaching linear algebra. He got his doctoral degree in 2003, and has just gotten his first position following some years of post-doctoral work. Teacher A4 is male, born in the 1940s, teaching a course in single-variable calculus. He got his doctoral degree in 1979, and is about to go into retirement after having taught at the university for more than 30 years. Teacher B1 is a man in his sixties, also teaching single-variable calculus, and also close to retirement. He was educated as an upper secondary school teacher, and has taught at the university level for about 20 years. Teacher B2 is female, born in the 1970s, and teaches a course in introductory algebra. She got her doctoral degree in 2004, and has taught at the university since. She is the only teacher in the study with a foreign background, having grown up in Eastern Europe. Finally, teacher C1 is a man in his fifties, teaching linear algebra. Educated as an upper secondary school teacher, he has taught at the university for about 20 years. As can be seen from these brief presentations, the participating teachers vary in gender, age, educational background and teaching experience. One thing they all have in common, however, is an active interest in teaching, although none are involved in mathematics education research.

The video-taped teaching comprising the data used in the main study consists of lectures, with audiences ranging from about 30 to about 150 students, mostly engineering students. As in the preliminary study, I would have liked to collect some data from lectures given to prospective teachers, but where there were relevant courses aimed at this group of students, unfortunately the teachers giving the courses declined to participate. For practical reasons I was only able to videotape one lecture (approximately two hours) each for teachers B1, B2 and C1. Of the teachers at university A I made a greater number of recordings, including two lectures each for teachers A1, A3 and A4. However, in the analyses I have only used data from one lecture from each of the seven teachers. Each lecture yielded between one-and-a-half and two-and-a-half hours of video, totally about 12 hours. In all cases, the lectures formed part of the teaching of the course, complemented by problem solving sessions. From the few such sessions I videotaped, it was my impression that the students used these sessions for asking questions, partly explaining the small number of questions asked by students during the lectures. Also, students often
approached the lecturer during breaks, asking for help with problems, or clarification of details of the preceding part of the lecture. The format of the lectures was highly uniform, with all seven teachers spending the whole of the lecture at the blackboard, writing, talking, and asking questions. The only partial exception was teacher B2, who used prepared transparencies complemented by writing. Apparently this mode of teaching originated in the lecturer’s lack of confidence in her grasp of Swedish when she first started teaching, and had remained in use even now when the language was no longer an issue.

The textbooks used in the courses were standard undergraduate textbooks, such as Adams (2006) in calculus and Lay (2006) in linear algebra. As for the topics covered, in the case of the algebra lectures they were similar—an introduction to functions in the basic algebra courses, an introduction to linear transformations in the linear algebra courses. The three calculus lectures were more diverse—teacher A1 gave an introduction to the function concept and trigonometric functions, while teacher A4 covered continuity and teacher B2 the inverse trigonometric functions and their derivatives. The video recordings were made by myself, seated near the front of the auditorium, thus ensuring as good access as possible to the teacher activity, while remaining relatively unobtrusive. In this way I also avoided filming the students, making it easier to get permission for the recordings. The students are present on the video only as “disembodied voices”, when asking or answering questions.

The video recordings were then transcribed verbatim, speech as well as the writing on the board. During the transcription process, I was careful to preserve idiosyncrasies in the teachers’ speech. However, I decided against using a full CA style transcription format, since that level of detail was not deemed necessary for my purposes, which after all are more geared towards the content of the discourse than towards finer details such as inflection and pauses. In cases where for instance emphasis on specific words was deemed relevant for the understanding of the unfolding discourse, I have made note of this in the transcript. The transcribed lectures were then analysed using Sfard’s commognitive theory as a general framework, and more specifically using her model of what characterizes different types of discourse (see section 3.3), trying to distinguish the discursive activities characterizing the teachers’ respective discourses of functions, and of the teaching of functions, paying special attention to repetitive patterns. Seeing as I work within a given theoretical framework, and use a preconceived characterization of discourse, I have definitely not worked within a strict grounded theory approach; however, following for instance Stadler (2009), I have to some extent made use of the
constant comparative method (e.g. Charmaz, 2006) central to grounded theory. I first analysed each lecture separately, trying to identify repetitive patterns, mainly regarding narratives and routines, but also looking at for instance the use of specific terminology (Paper II) and process- and object-oriented utterances (Paper V). Having identified a recurring activity, for instance definition construction, or checking that an example satisfies a given definition, I then identified all instances of this type of activity in the data, and made comparisons, searching for differences and similarities. I then in turn let these comparisons inform the further development of a more fine-grained categorization, in the process testing the validity of my categorizations against the data. Throughout the process I strove to stay close to the data, without rephrasing the actual utterances, again following Sfard: “The commognitive researcher is to begin her report with showing what was done and said, rather than with her own story about it (Sfard, 2008, p. 277). Some words should also be spent on discussing the role of the researcher within commognitive data analysis. When studying the unfolding of discourse, Sfard emphasizes the need for alternating between an insider and an outsider perspective to the discourse under study (ibid, p. 278). In performing my analyses, I have intentionally attempted to adapt an outsider perspective, trying to view the enfolding discourse in as unbiased a way as possible. At the same time, I am of course making use of the fact that my mathematical knowledge, as well as my experience of Swedish university mathematics teaching, both as a student and as a lecturer, makes me an insider to the discourse. This insider perspective is in fact crucial, since a deep understanding of the mathematics involved in the discourse is necessary to be able to understand what is being said and done. Still “what is senseless or inexplicable in the insider’s eyes may become meaningful for an outsider, if only because from the outsider’s perspective, the rules of the discourse in question do have alternatives” (ibid, p. 279). For example, I have specifically tried to avoid making reference to what is not present in the discourse, except in contrasting the teachers’ discursive activities, or in relation to results of previous research. For as Macbeth puts it: “absences (what we don’t do) is a delicate matter; there are too many of them” (Macbeth, 1994, p. 324). There are so many things not being said or done in a specific situation that a discussion of what is absent tends to say more about the presuppositions of the researcher than of the activity under study.
4.3 Further methodological considerations

As seen above, there are major methodological and theoretical differences between Paper I and the remainder of the papers in the thesis. This, together with the fact that the preliminary study was conducted so much earlier, and the somewhat strained circumstances behind its design, made me briefly consider removing it from the thesis. However, I believe that this would present a false picture of the development of my thesis project, and also, more importantly, that the paper contains interesting results and that it can be linked in a natural way to the main study, in particular to Paper V. Still, there are obvious methodological shortcomings of the preliminary study. Already mentioned is the lack of interview data. The rationale behind this was discussed in section 4.2.1, but this, together with the small number of participants in the study, of course limits the conclusions which could be drawn from the data. However, the richness of the questionnaire data made it possible to present worthwhile results. Also, the relatively limited role of Paper I within the thesis as a whole makes its methodological shortcomings less problematic.

Concerning the main study, to specify more clearly what it does and does not set out to achieve, a comparison with two previous similar studies, Weber (2004) and Güçler (2013), might be useful. The results of these papers are discussed at some length in section 2.2.2, but here I wish to draw attention to some methodological aspects of the papers, in relation to the design of my own study. Weber’s paper is one of the first examples of an in-depth analysis of so-called traditional mathematics teaching, with a focus on styles of teaching and ways of handling mathematical content. In this sense, his aims are similar to mine. However, there are also distinct differences, and these lead to differences in design. Weber is much concerned with the reasons the lecturer in his study had for planning and conducting his teaching in the way he did, and he thus interviewed the teacher repeatedly during the course, discussing his aims for the forthcoming lectures, as well as his reasons for teaching the previous lectures the way he had (Weber, 2004, p. 117). My study doesn’t cover this aspect at all. Instead, my focus is solely on what is actually taking place in the classroom. This is partly motivated by the fact that my main concern is the discursive practices, rather than the aims and motives of the individual teachers, and partly by an interest in the teaching as it is perceived by the students. By this I do not mean speculation about what they might and might not learn, but rather that since their only access to the rationales behind the teacher’s way of conducting her teaching is what she might have told them, and what they can infer from the actual teaching, I wish to similarly restrict my data.
Furthermore, where Weber aims at describing and discussing the teaching practices, and the reasons behind these practices, of one particular mathematics teacher, my aim is rather to describe and discuss the range of teaching practices connected with a specific mathematical topic. I make no claim as to being able to characterize any specific teacher’s teaching style. Rather, I am interested in getting data from a wide array of different teachers, teaching about different aspects of the function concept. Hence, where Weber decided to follow one specific teacher over a prolonged period of time, I have instead chosen to collect smaller amounts of data from a larger number of teachers. A further difference which should be mentioned is the fact that the mathematical context is different. In Weber’s study, the teaching documented was part of a course in real analysis, aimed at third year mathematics majors, while in my study the teaching was aimed at first-semester students. It is very likely that this difference in context has an effect on the teaching practices used. I will return to this and other related questions in chapter 6.

As previously mentioned, Güçler’s (2013) study is the only one that I’m aware of besides my own using commognitive theory to investigate university mathematics teaching. It might thus be illuminating to compare how elements of the theory are put to work as analytical tools in the two studies. The discussion of my study in what follows will focus mainly on Papers III and IV. While the overall context of the studies is similar – undergraduate lectures at the freshman level – the aims of the studies are different. Güçler’s aim is to “use the discursive framework to provide further insights about the issues regarding teaching and learning of limits by identifying the instructor’s and students’ discursive patterns and comparing those to highlight communicational links and failures in the classroom” (ibid, p. 440), whereas my aim is to describe and analyse university mathematics teachers’ teaching practices regarding the function concept. Hence an obvious difference between the designs of the studies is that Güçler needs data on the students’ discursive practices. Since the observed lectures were low on explicit student participation, she had to rely on different types of data – a written survey, audio-taped task-based interviews, and students’ written work. As I see it, this might be problematic, since the context of the discursive practices is quite different, making comparisons potentially difficult. In the case of the present study, although the context might be seen as fairly different (different universities, different types of students etc.), in fact the similarities are greater than the differences, with the main difference being in class size. Moreover, my aim is not making comparisons between different teachers, but rather describing a range of discursive activities.
associated with the teaching of the function concept, making possible differences in context less relevant.

Concerning the use of the commognitive framework, there are further differences between the studies. Güçler’s study does include an analysis of the discursive patterns of the teacher, but in contrast to my analyses, which focus mainly on the use of routines, hers focuses mainly on more general meta-rules, as well as on the use of visual mediators and operational and objectified language, aiming at giving a more general characterization of the teachers’ discourse in terms of Sfard’s four characteristics of discourse. However, it appears to me as if we interpret these characteristics somewhat differently in our analysis, giving our results a different character. For Güçler, a statement such as “as \( x \) gets closer and closer to 0, the function values get closer and closer to 1” is indicative of operational word use (ibid, p. 443-444), whereas for me it’s a mathematical narrative, which can then be operational or objectified in character. In contrast, for Güçler the term narrative seems reserved for more large-scale statements, like when she speaks of “two meta-level narratives about the limit concept: ‘limit is a number’, and ‘limit is a process’” (ibid, p. 450). Neither of these statements is actually an explicit utterance within the discourse, but rather they are implicit characterizations inferred from the discursive practices of the teacher and the students. These differences in the interpretation of the framework lead to differences in the analyses. For Güçler, routines are mainly discussed in terms of endorsement of such large-scale narratives, leading to fairly general meta-rules, such as “using objectified utterances”, or “using symbolic representations and graphs” (ibid, p. 445). Detailed descriptions of mathematical and didactical routines of the type presented in papers III and IV are not present in Güçler’s paper. Rather she ends up with a fairly blunt characterization, in the vein of for instance Hora and Ferrare (2013). Still, given the different aims of the papers, Güçler’s analyses are perfectly adequate for her purposes. Further consideration of the results of my study in relation to Güçler’s can be found in chapter 6.

Finally, before moving on to the actual results of the thesis, I wish to discuss some ethical aspects of the study. All participants in the studies have been informed of the purpose of the research, that any participation in the study is entirely voluntary, and that if they so wish, they could withdraw their participation at any time. When doing the video recordings I positioned myself in such a way that no students were caught on film, and although I didn’t actively ask permission of all attending students, they were all informed that the focus of the research was the teaching, and that at most they would be present
in the videos as disembodied voices. The customary precautions were taken to ensure participant confidentiality, with universities, as well as students and teachers, being anonymized.

A less straightforward aspect of research ethics is the possible impact my presence at the lectures being video recorded might have had on the activities being documented. In a few cases, at the beginning of the lectures, I could sense that the teachers were affected by my presence; indeed, a couple of the teachers even made explicit comments to this end. However, I was inconspicuously seated in the auditorium with a small hand-held camera, and my impression is that the teachers soon forgot that I was there. Another aspect of this is whether the students felt less inclined to participate in the lecture, for instance asking or answering questions, from fear of being caught on tape. I find any such effects harder to judge, but having been present at a very large number of mathematics lectures over the years, I can at least say that the student activity in the videotaped lectures were neither obviously greater nor smaller than at an average lecture. Given the character of my research, an ethical aspect needed to be taken into account is how individual participants are portrayed. Is the images given of the participants in the study reasonably accurate, and done in such a way that no participant risks taking offence at how he or she is portrayed? Here, the aims of my study help make this less of a risk. Although I study teaching practices, I make no claims about the quality or character of individual teachers’ teaching, but rather about the range of teaching practices displayed by the teachers. Still, by keeping close to the data, and avoiding making value-laden judgments about the teaching or the teachers, I hope to have presented an image of the observed teaching that the observed teachers would agree with.
5 Overview of the papers

In this chapter, I will present summaries of the five papers comprising part of this thesis (Viirman, Attorps, & Tossavainen, 2010; Viirman, 2011b; 2013a; 2013b; in press). As mentioned above, the papers fall into two groups, with Paper I differing quite a bit from Papers II-V, using different empirical data and a different methodological approach. Papers II-V are more closely related to the principal aims of the thesis as a whole, with Papers II-IV addressing the first aim, and Paper V addressing the second. In fact, Paper V was written earlier than Papers III-IV, but I have chosen to place it last, since it addresses the second aim. Furthermore, Papers III and IV are very much companion pieces, using exactly the same empirical data and tools of analysis, and differing mainly in their objects of study.

5.1 Paper I


This paper should be seen as providing some background to the main study in the thesis, reports of which are given in Papers II-V. It shares with these the focus on the function concept in the context of university mathematics education research, but instead of looking at teaching it aims at investigating university students’ understanding of the function concept. The study was conducted using questionnaires, and 34 students at a large Swedish university participated – 14 pre-service teachers and 20 engineering students. The questionnaire was designed to give insight into the students’ conceptions of the function concept, and questions included giving a definition of the function concept, constructing a “mind map” around the concept of function, determining whether given expressions represented functions, and constructing functions having certain given properties. The analysis of the questionnaires drew on the theory of concept images (Tall & Vinner, 1981), and on Sfard’s (1991) notion of process-object duality and her model of concept formation, with the purpose of addressing two related aims. The first is to derive a categorization of the students’ concept images of the function concept, as expressed in the following two research questions: How do the students define the concept of function? How developed are their concept images for the function concept with respect to the dual nature of mathematical concepts? The second concerns the relation between students’ conceptions and the historical development of the function
concept, and is expressed in the research question: What differences and/or similarities can be found between the students’ conceptions and definitions of the function concept, and the formal definitions as they have developed historically?

The analysis showed that the students’ definitions could be divided into five categories: correspondence/dependence relation, machine, rule/formula, representation, and nonsense. Of these, the first can be said to represent a structural view of the function concept, while the second and third correspond to an operational view. The fourth and fifth categories cannot be said to represent definitions in a strict mathematical sense. Of the students’ definitions the majority fell into one of the two operational categories, with only three giving definitions of the first category. Moreover, more than a third of the students gave definitions falling into the last two categories, thus failing to give mathematically viable definitions. This corresponded well with the more general characterization of the students’ concept images, where the majority were found to express operational conceptions, with a small minority expressing structural conceptions, and as many as a third of the students expressing pre-operational conceptions. Concerning the second aim, the students were seen to give definitions mostly resembling an 18th or 19th century view of functions. Moreover, the ideas behind the modern, structural definition of function were not so commonly displayed by the students in the study. For instance, only eight students mentioned the notions of domain and/or range, aspects central to the 20th century view of functions.

Since this paper was jointly written by me and two other authors, it is appropriate that I specify my contribution. The design of the study and the collection of data were done by me, as well as the first analysis. Some further analysis was then done by the other authors, for the most part corroborating my results, but leading to some refinement of the categorizations. The second author provided some ideas for reworking the paper after the first round of reviews, and the third author supplied parts of the discussion. The rest of the writing was done by me.

5.2 Paper II


This is the first of the four papers relating to the main aims of the thesis, and thus focusing on teaching, and the discursive practices of university
mathematics teachers. In this short paper the focus is mainly on word use, and the empirical data consists of two video-taped 40-minute excerpts from lectures given by two different lecturers, denoted teacher A and teacher B. Both excerpts were taken from lectures in linear algebra, on the topic of linear transformations, allowing me to compare and contrast the teachers’ discursive activities. The analysis was made using the characterisation of discourse central to commognitive theory, paying special attention to the use of central terms, particularly the notion of linear transformation, with the aim of answering the following research question: How is the concept of linear transformation presented in the discursive practices of the two teachers, and what consequences for their students’ learning could this have?

The results show that the way the concept of linear transformation is constituted through the discursive activities of the two teachers is very different. Teacher A starts from an algebraic/geometric definition of the concept, and then proceeds to present examples, both algebraic and geometric, outlining the characteristics of this special type of functions. He also discusses the close relationship between linear transformations and matrices, but still makes a clear distinction between the two. He is generally handling the mathematical content with precision. However, there is very little context, and no discussion of the use of this new notion outside of this specific setting. Teacher B, in contrast, provides a wealth of context, very consciously aiming at connecting the mathematics to the everyday experiences of the students. He introduces linearity by referring back to the notion of linear function in calculus, and then presents a number of examples of linear transformations, paving the way for the definition. However, he is less careful with his use of words than teacher A. For instance, he makes no distinction between the transformation and its matrix, treating them as if they were the same thing. He also mixes the use of the term ‘linear’ in calculus and linear algebra, choosing an example of a linear function which is linear in the calculus sense of having a graph which is a straight line, but which is not linear in the linear algebra sense, since its graph does not pass through the origin. He also uses the term transformation in ways which are correct in everyday Swedish, but not in a mathematical context. In conclusion, the careless handling of words in the discursive practices of teacher B can be expected to be detrimental to student learning, while the lack of contextualization and discussion of usefulness on the

In the terminology of the thesis as a whole, teacher A corresponds to teacher A3, and teacher B corresponds to teacher C1.

However, this definition was not presented until later in the lecture, and was thus not included in the excerpt under scrutiny in this paper.
part of teacher A may cause problems with student motivation. However, the absence of data on the students makes these conclusions fairly speculative.

5.3 Paper III

**Title:** The functions of function discourse – university mathematics teaching from a commognitive standpoint. *International Journal of Mathematical Education in Science and Technology, 2013.* DOI: 10.1080/0020739X.2013.855328

This paper, together with Paper IV, addresses the main aim of the thesis, namely describing and analysing university mathematics teachers’ teaching practices regarding the function concept. As mentioned in the introduction to this chapter, the two papers are closely related, but deal with different aspects of the discursive practices of the teachers. In this paper, the topic is the mathematical discourse, and the aim of the paper is to describe and classify the routines present in the teachers’ mathematical discourses of functions. The empirical data consists of videotaped lectures from seven different teachers, covering a range of topics from freshman mathematics, including basic algebra, linear algebra and calculus. Included in the analysis were approximately 2 hours of video per teacher. The analysis was conducted using commognitive theory, focusing on the use of construction and substantiation routines.

The findings of the study show that, despite great similarities in the outer form of the lectures, with all seven teachers in the study using a traditional lecturing style, standing at the blackboard writing and talking in the mode Artemeva & Fox (2011) denote by “chalk talk”, there are still significant differences in the way the teachers present and do mathematics in their lectures. These differences present themselves both on the level of discursive routines and on a more general level in how the activity of doing mathematics is constituted through the teachers’ discursive practices.

Considering the mathematical routines, a classification of construction and substantiation routines is presented, with the construction routines including

- definition construction, including definition by stipulation, exemplar, contrast, saming and naming
- example construction
- counter-example construction

The above routines are not specifically connected to the topic of functions. There are also a number of topic-specific constructions, including routines for
graph and domain construction, for construction of a linear transformation from a geometric situation, as well as construction algorithms, for instance for determining the derivative of an inverse function, and for constructing the standard matrix of a linear transformation. Many of the construction routines act as their own substantiations, but the specific substantiation routines include

- definition verification
- proof
- claim contradiction

The substantiation routines tend to be less frequent than the construction routines, and with more pronounced differences between the teachers, for instance in the prominence given to substantiation. Some of the teachers strongly emphasize the need for substantiating claims, whereas others hardly use any substantiation routines at all. This in turn is related to how the activity of doing mathematics is constituted in the discursive practices of the teachers, with, for instance, teachers A3 and A4 emphasizing the cumulative nature of mathematics, and the need to build on previous knowledge, whereas for teacher C1 doing mathematics is constituted as a natural aspect of solving problems and getting to know more about the world around us. The observation is also made that despite the fact that the function concept was used as a focus in the collection and analysis of data, many of the routines described have little to do with the teaching of functions per se, and it is conjectured that the classification of routines presented might be useful in studying the teaching of other mathematical topics.

5.4 Paper IV

**Title:** A commognitive analysis of the didactical aspects of teachers’ discursive practices in university mathematics lectures. *Manuscript submitted for publication.*

As previously mentioned, this paper is a companion piece to Paper III, using the same methodology and data. But whereas Paper III dealt with the mathematical discourse, in this paper the topic is the discourse of mathematics teaching. When conducting the analyses presented in Paper III, a number of repetitive patterns in the discursive activities of the teachers were found which did not fit into the general classification of mathematical routines into construction, substantiation and recall routines presented by Sfard (2008, p. 225). Instead they seemed to belong to a different, but related, discourse of
mathematics teaching, a didactical discourse. Analogously with Paper III, the aim of this paper is to describe and classify the routines present in the teachers’ didactical discourse, particularly regarding functions.

Given that the mathematical content which is the object of the discourse is the same in both the mathematical and the didactical discourse it is not surprising that they are highly similar as regards word use and visual mediators. Instead the major differences occur on the level of routines. The main result of Paper IV is a classification of the didactical routines into five types, present in the discourses of all seven teachers:

- explanation routines
- motivation routines
- engagement routines
- question routines
- recall routines

These five types play the same role in the description of the discourse of mathematics teaching as the construction, substantiation and recall routines do for the mathematical discourse. Like them they are ordered according to the functions they are meant to serve within the discourse, with the exception of the question routines, which can serve many different functions, but which are instead grouped together because of their easily distinguishable form. But just as we saw in Paper III that there were different types of construction and substantiation routines within the mathematical discourse, so there are different types of routines within the five categories of didactical routines. In Table 1 an overview of the different types of didactical routines found in the discursive practices of the teachers can be found.

Another result of the study is that while these five general types of didactical routines are used by all seven teachers, there are major differences between the teachers regarding the more specific types of such routines. These differences will be discussed in more detail in section 6.1.2, but types showcasing great differences are, for instance, motivational and question routines. Regarding rhetorical questions, the results in this paper suggest that such questions are used by many of the teachers as a tool to help them model general patterns of mathematical activity. This use of rhetorical questions complements those already highlighted in the literature (e.g. Artemeva & Fox, 2011; Hora & Ferrare, 2013).
Table 1: classification of didactical routines

<table>
<thead>
<tr>
<th>Explanation routines</th>
<th>Motivation routines</th>
<th>Engagement routines</th>
<th>Question routines</th>
<th>Recall routines</th>
</tr>
</thead>
<tbody>
<tr>
<td>Everyday language</td>
<td>usefulness</td>
<td>“mathematical we”</td>
<td>Control questions</td>
<td>summary</td>
</tr>
<tr>
<td>metaphor</td>
<td>interest</td>
<td>surprise</td>
<td>Fact questions</td>
<td>repetition</td>
</tr>
<tr>
<td>concretization</td>
<td>Nature of mathematics</td>
<td>humour</td>
<td>Conceptual questions</td>
<td>Reference to known facts</td>
</tr>
<tr>
<td>Known mathematical facts</td>
<td>lack</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Different representations</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The description of the five general types of didactical routines also contributes to methodological development, since it provides a partial categorization in commognitive terms of the discourse of university mathematics teaching. I make no claim to having developed a complete typology of didactical routines, but I am fairly confident that the five types presented in the paper can be used to describe university mathematics teaching practice more generally.

5.5 Paper V

**Title:** What we talk about when we talk about functions: Characteristics of the function concept in the discursive practices of three university teachers. To appear in *Proceedings of the eighth congress of the European society for research in mathematics education*. Ankara: ERME.

In this, the final paper of the thesis, I address the second part of the general aim, namely that concerning how the function concept is constituted through the discursive practices of the teachers. Thus, after having served mainly to limit the scope of Papers III and IV, the function concept is now once again an aspect of the unit of analysis. The aim is operationalized in the paper by the following research question: *How is the function concept characterized, on both object- and meta-level, by the discursive practices of the teachers?* In order to be able to answer
this question, data from lectures in calculus,\textsuperscript{27} given by three different teachers (A1, A4 and B1 in the notation of this thesis) at two different Swedish universities were analysed using commognitive theory. Additionally I made use of the notion of patterns of variation, borrowed from variation theory (Marton et al, 2004), as well as results on the various characteristics of the function concept from a didactical point of view (see section 2.1.2 for details).

Starting at the object level, the results show that all three teachers use patterns of variation in order to highlight different aspects of the function concept, characterizing functions through rule, domain and range. Still, certain important aspects of the more general function concept are not made explicit. For instance, almost all functions considered in the lectures are real-valued functions of one real variable, making the notion of arbitrariness of domain and range impossible to discern. Since all data were from lectures in single-variable calculus, this is easily understood, however. Perhaps more interesting are the results on the use the three teachers make of different realizations in the characterization of the function concept. Here, teacher A1 is found to give precedence to the algebraic realization, whereas teachers B1 and, particularly, A4 are much less reliant on algebraic realizations, for instance introducing new functions through graphs.

Regarding the meta-level, the process-object duality of the function concept can be seen in various ways in the discursive practices of the teachers, and sometimes the distinction is not made clear. All three teachers speak of functions as objects, but differences can be seen in how they use functions in doing mathematics. Generally, the process aspect takes precedence in the discourse of teacher A1, who speaks of functions as objects performing processes on numbers. Teacher A4 uses slightly more objectified language, for instance making no distinction between the function and is values, whereas teacher B1 treats functions very much like objects. This seems to be connected to the fact that the lectures given by the three teachers are situated at different points in the calculus course, with the conjecture being that a move towards a more objectified discourse of function takes place as the course progresses. However, this conjecture cannot be answered without recourse to different data. Indeed, this would be an interesting topic for further research.

\textsuperscript{27} These lectures form part of the data analysed in Papers III and IV as well.
6 Conclusions

In this chapter I discuss the results of the papers in relation to the aims of the thesis, and also present some conclusions which didn’t fit within the separate papers, relating the results to previous research. I will start with discussing the first aim of the thesis.

6.1 Teaching practices regarding the function concept

The first aim of the thesis is to describe and analyse the participating university teachers’ teaching practices regarding the function concept. Various aspects of this aim are addressed in Papers II-IV in the thesis. To this end I have applied the commognitive framework, describing the teaching in terms of the teachers’ discursive practices. Within the commognitive framework, what is being studied is the discursive activity, and one major finding of this thesis is that the discursive activities performed when teaching about functions actually belong to two distinct but intertwined discourses: a mathematical discourse, and a didactical discourse or discourse of mathematics teaching. Looking at the characteristics used within the commognitive framework to distinguish different discourses, these two discourses use mainly the same words and visual mediators, but differ to some extent regarding narratives, and to a larger extent regarding routines. In what follows, I will present these two discourses as articulated through the discursive practices of the teachers and in the process try to highlight the similarities and differences in the discursive practices of the different teachers.

6.1.1 Mathematical discourse

The mathematical discourse regarding functions uses the expected mathematical terminology, words such as ‘function’, ‘domain’, ‘range’, ‘linear map’, ‘set’, ‘coordinate system’ and the like, but also words describing the role of the mathematical objects within the discourse, such as ‘example’, ‘definition’, ‘property’, ‘formula’, etc. Still, there are instances of idiosyncratic and potentially problematic word use. As described in Paper II, in a lecture on linear transformations, teacher C1 uses the term ‘linear function’ in a manner which causes it to change meaning midway through the lecture. He also uses the term ‘transformation’ (or rather its Swedish counterpart, ‘avbildning’ or ‘avbildla’) in a manner mixing the mathematical and everyday use of the term in a possibly confusing way.
The visual mediators used are also the ones commonly associated with the topic: function graphs, set diagrams (two blobs side by side, representing domain and codomain, with an arrow between them representing the function), vector diagrams, geometric diagrams, etc. Symbol use is also conventional, with for instance $f(x)$-notation and arrows representing functions used by all seven teachers in the study. Also, there is some more context-specific symbol use, like for instance standard vector- and matrix notation used in the linear algebra lectures. There is very little, if any, idiosyncratic or non-standard symbol use. In fact, several of the teachers, for instance teacher A1, A3 and B1, make explicit reference to adhering to the symbol use in the textbook and in exams. The types of visual mediators used are similar to those found by Güçler (2013) in her analysis of university mathematics teacher discourse on limits. The way the visual mediators are used by the teacher in Güçler’s study also agrees with how some of the teachers in this study use them: A main focus on symbolic representation, where graphs are used mainly for explanatory purposes, not for solving problems. For instance, teacher A1 relies to a large extent on symbolic representation. For other teachers in the study, however, graphical realizations play a much larger role. Teachers A3, A4 and B1, for instance, use graphs to introduce new functions, derive properties of functions, and solve problems.

Since it is, to my knowledge, the only published study besides mine using commognitive theory to investigate university mathematics teaching, further comparison between Güçler’s (2013) characterization of teacher discourse and the one presented in this thesis would be of interest. However, differences both in the aims of the studies and in the interpretation of the commognitive framework make such comparisons difficult. As discussed in section 4.3, Güçler mainly describes fairly general routines, such as “using symbolic representation”. A more detailed characterization of what constitutes this use is lacking. I will return to Güçler’s paper in section 6.2, when discussing the use of objectification in the teachers’ discursive constitution of the function concept.

The most common narratives in the mathematical discourse of the teachers are definitions and examples aimed at motivating or exemplifying definitions. Theorems, the other type of narrative commonly associated with traditional mathematics teaching of the type Weber (2004) labels the DTP (definition-theorem-proof) feature to a much lesser extent. Teachers A1, B2 and C1 state no theorems at all, while teachers A2, A3 and B1 state one each, and only teacher A4 states more than one.

Finally, routines were the main focus of my analysis in Paper III. Sfard divides explorative routines into three categories (see section 3.3): Construction,
substantiation and recall routines. Since the context of prepared lectures leave little need for recall routines, these are not present among the mathematical routines of the teachers in the study (a different type of recall routines does figure among the didactical routines however, as we will see in the next section). As for construction and substantiation routines, in Paper III a categorization of the ones found in the discourses of the teachers is presented. This categorization is briefly described in the summary of Paper III in section 5.3, and I will not reiterate it here, beyond stating that the construction routines include

- definition construction, including definition by stipulation, exemplar, contrast, sameing and naming
- example construction
- counter-example construction

Since they are potentially more generally applicable, I believe that these more general types of routines are the most interesting. This holds in particular for the classification of types of definition construction. In the literature on definitions within mathematics education, the focus is mostly on the characteristic features of definitions and what constitutes a good definition (e.g. Van Dormolen & Zaslavsky, 2003), on students’ constructions of definitions (e.g. Ouvrier-Buffet, 2006), or on discrepancies between students’ definitions and the established ones (e.g. Vinner & Dreyfus, 1989). The classification given in Paper III achieves something different. It doesn’t deal with the qualities and characteristics of the definitions themselves and whether they live up to the standards set by the mathematical community (I take it as a given that university mathematics teachers, presenting definitions in the context of prepared lectures, strive to present mathematically sound definitions). Neither does it deal with “proper” construction of definitions, since the definitions presented are well-known to the teachers, in contrast to the situation in the literature on students’ definition construction. Instead, the present categorization focuses on how the essential aspects of the definitions are constituted through the discursive activities of the teachers, and on to which extent these discursive activities serve to highlight the mathematical reasons for constructing the definitions the way they are constructed, including these specific aspects. This is a perspective I have not found in the literature.

Turning to substantiation routines, it should be mentioned that many of the construction routines act as their own substantiations. But looking at the
discursive practices of the teachers, a number of specific substantiation routines can be found, including

- definition verification
- proof
- claim contradiction

Again, the routines are described in greater detail in Paper III. Compared to the construction routines, the substantiations are both less frequent and more varied, the differences between teachers being more pronounced. This can be seen both in types of substantiations used, and in the prominence given to substantiation, and validating claims. For instance, teachers A3 and A4 emphasize the substantiation of claims as central to mathematical practice, whereas teacher C1 hardly uses substantiation routines at all. Given the prominent role of definitions and examples as mathematical narratives in the teachers’ discourses, it is not surprising that the most common substantiations concern the verification of whether a given example satisfies a definition. On the other hand, proofs, crucial to the deductive reasoning central to mathematical practice, are much less common. At first glance this might seem surprising, but given the relative dearth of theorems in the teachers’ discourses, proofs naturally play a lesser role. Still, a few teachers (A2, A3 and A4) do prove theorems, although in no case more than one. There are also a few instances (teachers A3, A4 and B1) of using proof-like techniques to verify claims. But despite this the lack of proofs also serves to contrast the teaching practices documented in this study from the DTP format of teaching. Indeed, it would be difficult to claim that the teaching documented in this study adheres to the DTP format at all. A probable reason for this fact is the difference in the level of the mathematics taught, as well in the type of students participating in the courses. It is hardly surprising that courses aimed at first year engineering students would have less focus on proof than a course aimed at third year mathematics majors, as in Weber’s (2004) study. Still, given the central role of the process of proving claims within scholarly mathematics discourse, one might view the lack of theorems and proofs in the discourses of the teachers in this study as suggestive of a mathematical practice with less emphasis on rigour. In section 2.2.1 I quoted Greiffenhagen on the topic of university mathematics teaching practice: “the lecturer is predominantly not talking "about" mathematics, but actually "doing" mathematics at the board” (Greiffenhagen, 2008, par. 35). From a discursive perspective this is almost a truism, since doing
mathematics is precisely engaging in mathematical discourse, and mathematics lecturing of course is a form of mathematical discourse. Still, different types of discursive activities articulate different ways of viewing the process of doing mathematics. In this study, this can for instance be seen in how substantiation routines are used differently in the discursive practices of the teachers. I have already mentioned how the need for substantiation of claims is emphasized by teachers A3 and A4, and this contributes to establishing a view of mathematics as a cumulative process, where statements build on previous statements, and where validity is judged within the mathematical discourse itself. In the light of this, it is not surprising that teachers A3 and A4 are among the few in the study who actually prove theorems. On a related note, the way teacher A2 avoids the use of mathematical terms not already defined emphasizes a characterization of mathematics as a deductive activity, consisting of the investigation of mathematical objects, the origins of which are not a topic of discussion. As we shall see in the next section, this has consequences for the way teacher A2 gives motives for the mathematical activities he engages in. In contrast, as discussed mainly in papers 2 and 4, teacher C1 does not emphasize the cumulative or deductive aspects of mathematical activity. Instead his discursive activities suggest a view of doing mathematics as a natural part of solving problems and understanding the world around us. He uses examples from outside the mathematical discourse to introduce new mathematical notions, and the problems he presents to argue the need for new mathematical activities also mostly originate outside of mathematics.

Having given an account of some characteristic aspects of the mathematical discourse of the teachers, I now turn to the didactical discourse, which is the topic of the next section.

6.1.2 Discourse of mathematics teaching

As mentioned above, the two discourses described in this thesis are very similar regarding vocabulary and visual mediators. Given that the teaching discourse concerns the teaching of the mathematical content which is the subject of the mathematical discourse, this fact is hardly surprising. There are however some words specific to the teaching discourse, such as ‘important’, ‘useful’, ‘understand’, ‘questions’, etc. There is also some discourse-specific use of everyday language, such as the use of the word ‘machine’ as a metaphoric description of the idea of a mathematical function. As for the didactical narratives, they are most easily described as the results of the didactical routines – explanations, motives, questions etc.
As was the case for the mathematical discourse, the didactical routines have been studied in the greatest detail. In Paper IV of this thesis, I present a classification of didactical routines in the manner of the classification of mathematical routines into construction, substantiation and recall routines given by Sfard. I distinguish between five types of didactical routines:

- explanation routines
- motivation routines
- engagement routines
- question routines
- recall routines

These are described in greater detail in Paper IV. All these types are present in the discourses of all seven teachers in the study, and I suspect that they are fairly general aspects of mathematics lecturing. Looking closer at these general types of routines, however, one finds marked differences between the teachers in the study. Just like there were different examples of construction and substantiation routines found in the teachers’ discourses, so there are different examples of the various didactical routines, indicating differences in the teaching practices of the teachers. An overview of the different types of didactical routines found in the discursive practices of the teachers can be found in table 1 in section 5.4. Again, these various examples are described in greater detail in Paper IV, but in what follows I will highlight some of the more interesting aspects of the use of these didactical routines in the discursive practices of the teachers.

Starting with the explanation routines, an interesting feature of these is the relationship between formal and informal language in explaining mathematical facts and phenomena. This is something that has been discussed in previous research on university mathematics teaching (e.g. Greiffenhagen, 2008; Güçler, 2013; Wood & Smith, 2004), where it is found that the oral part of the lecture tends to be more informal and personal in tone, whereas the written part is formal and depersonalized. This agrees with the findings of this study, where the various explanation routines generally use informal oral language to explain written mathematical narratives, with explanations through reference to other aspects of the mathematical discourse being less common. However, as was shown in Paper II, the use of informal language also carries risks, where mixing mathematical and everyday meanings of words can lead to potentially confusing results.
As for the motivation routines, these are not so frequent, with the most common one being reference to usefulness. However, despite the fact that the lectures in the study were mostly aimed at engineering students, the usefulness referred to was mainly intra-mathematical; in other parts of mathematics, in future courses, or to pass the exam. Only two of the teachers gave examples of the use of mathematics in areas outside of mathematics itself. Of the other motivational routines, I find one to be of specific interest. This is the one which I have denoted by “lack”, that is, providing motivation for some mathematical activity through reference to something lacking, something which is needed in order to move forward with the mathematical topic under discussion. The technique is exemplified in the following excerpt, taken from the beginning of teacher A2’s lecture, where after having recalled various operations on sets he says the following:

“What we haven't learned is how to connect two sets in the sense that to every element in one set we associate an element of the other set. And this is what functions do.”

This type of routine is used frequently by teacher A2, but of the other teachers only A4 uses it on one occasion. I believe that the need for this type of routine can be explained by examining the discursive practices of teacher A2. In the previous section I mentioned how he avoids making reference to mathematical terms not previously defined. But if you introduce every new term through a definition, then you need some way of arguing why this newly defined concept is interesting; referring to something lacking, something which the new mathematical object can help remedy, seems to be how teacher A2 handles this problem. On the other hand, if you introduce new concepts through examples before defining them, like most of the other teachers in the study, then there is less need for this type of motivational routine, since the examples themselves serve as motivation.

The use of questions in lecturing has been widely discussed in the literature (e.g. Artemeva & Fox, 2011; Hora & Ferrare, 2013; Knott, Sriraman, & Jacob, 2008; Martin et al 2005), but then mostly in the context of teacher-student interaction, thus focusing on the role of questions as vehicles for affecting student discourse. This is not the main purpose of the question routines found in the teaching practices documented in this study. Of the four types of question routines described in Paper IV, only the conceptual questions, which are by far the rarest, actually aim at initiating student discourse beyond the brief statement of facts. Indeed, two of the question types are actually not intended to be answered at all – the control and rhetorical questions. The use of rhetorical
questions deserves further discussion. These are discussed by both Artemeva and Fox (2011) and Hora and Ferrare (2013). Artemeva and Fox find that university mathematics teachers use rhetorical questions to signal transitions or to pause the action to make room for reflection. Based on the results of this study, to these functions I would like to add one. In the discursive practices of many of the teachers in the study, rhetorical questions serve to make the process of doing mathematics more explicit. The rhetorical questions asked by the teachers are the questions you would have to ask yourself if it was you doing the reasoning. Thus, by asking these questions, the teachers model at least one aspect of the process of doing mathematics. I will return to this aspect of lecturing in more detail in the next section, where I will discuss some more general features of the discursive practices of the teachers in the study.

6.1.3 General features of the teaching practices

Having described the two types of discourse present in the teaching practices of the teachers in the study, I now wish to discuss some general features of these practices, in relation to previous research. Saroyan and Snell (1997) distinguish between three types of lecturing styles: content-, context- and pedagogy-driven lectures. With the possible exception of teacher C1, who makes some attempt at situating his teaching within a context other than mathematics, all teaching practices analysed in this thesis can be said to be content-driven. However, this characterization is not at all concerned with the type of content. A characterization of this type is presented by Weber (2004) who distinguishes between three lecturing styles – logico-structural, procedural, and semantic – specific to mathematics and focusing on how the mathematics is presented and handled by the teacher. However, differences in mathematical context as well as in the character of the empirical data make this categorization less applicable to the teaching practices documented in this thesis.

Approaching the topic of characterising university mathematics teaching from a different perspective, Artemeva and Fox (2011) identify chalk talk – “writing out mathematical narrative on the board while talking aloud” – as a central pedagogical genre of university mathematics teaching. In fact, it was observed in all 50 lecture classrooms observed in their study. Given this pervasiveness of the genre, it is not surprising that all seven teachers in the present study also adhere closely to it in their teaching practices. In section 2.2.2 a number of

28 With the possible exception of teacher B2, who uses prepared transparencies complemented by whiteboard. However, since Swedish is not her first language, and she says that she doesn’t feel entirely confident writing Swedish “on the fly”, this choice is probably more a solution to a language
characteristics of the genre are mentioned. Perhaps the most prominent of these are the use of running commentary and meta-commentary; that is, verbalizing what is written on the board, as well as talking about it. The presence of these discursive practices can be clearly seen in the excerpts provided in papers II-V. As mentioned in the previous section, rhetorical questions, the use of which is highlighted by Artemeva and Fox as characteristic of chalk talk, are a distinctive feature of the discursive practices of all teachers in the present study. In conclusion, it is reasonable to claim that the overall form of the teaching practices used by all seven teachers is similar: content-driven lectures conducted using chalk talk.

Artemeva and Fox describe chalk talk as a means of providing students with an experience of the processes of doing mathematics (Artemeva & Fox, 2011, p. 356), and in this context I want to continue the discussion of the role of mathematics teaching in modelling mathematical discourse, and thus mathematical activity, mentioned in the previous section. This important function of the mathematical lecture has been somewhat overlooked in university education research, with its criticism of lectures as inefficient vehicles for the transmission of factual knowledge, turning students into passive recipients (e.g. Bligh, 2000; Biggs & Tang, 2011). But, as shown by Hora and Ferrare (2013), the conveying of facts is just one part of what lecturers do in their teaching, with many other teaching practices being just as prominent. In fact, it has been argued by a number of researchers (e.g. Artemeva & Fox, 2011; Fukawa-Connelly, 2012; Greiffenhagen, 2008; Knott et al, 2008; Pritchard, 2010) that the primary purpose of mathematics lectures is not the transmission of factual knowledge, but rather the modelling of mathematical reasoning. The results of the present study regarding the discursive practices of the teachers lend further support to this claim. We have seen in the previous sections how the various types of construction and substantiation routines, as well as many of the didactical routines, such as explanation routines, the use of rhetorical questions, and the specific motivational routine of teacher A2 discussed in section 6.1.2, all serve to make the process of mathematical reasoning explicit. One aspect of mathematics which is not particularly emphasized by the teachers in the study, however, is its connectedness; that different areas of mathematics are interrelated in various ways. As we have seen, explanations are rarely made through reference to other aspects of the mathematical discourse, and although reference to intra-mathematical usefulness is the main motivational routine it is problem than a primarily pedagogical decision. Also, she uses the transparencies very much in the same way as the other teachers use their writing on the board.
still not very frequently used, and moreover the connections are rarely made explicit beyond simply stating that “you will be using functions a lot in the calculus course”. This fragmented nature of university mathematics teaching has been discussed in the literature (e.g. Burton, 2004; Nardi, 2008) as an obstacle to student learning.

As for the view of lectures as encouraging students to be passive, this has been criticized by Rodd (2003), who argues that it is false to assume that you are necessarily passive just because you are not visibly or audibly participating, comparing the audience of a lecture with the audience of a play. This can also be seen in the discursive activities of the teachers, where for instance the motivation and engagement routines clearly serve to counteract passivity in the students. This also relates to Rodd’s notion of “mathematical awe and wonder”, where for instance surprise can be used to convey such a feeling, as in the following example from teacher A4, who when discussing limits, and after having drawn the graph of the function \( f(x) = \frac{1}{x} \) on the blackboard, says the following:

“And finally, far off at infinity, if we say that infinity lies there, [he draws a point about a metre to the right of the coordinate system and marks it with an infinity symbol] then it becomes zero.”

This could also be related to Bergsten’s (2007) criteria for quality lectures, where he emphasizes “teacher immediacy”, including personalization and humour, as one important aspect. Several of the teachers in the study have a high degree of teacher immediacy, using humour and personalized delivery to a great extent. This notion of teacher immediacy also relates to Rodd’s (2003) claims regarding the role of lectures in establishing a community of practice, in helping students develop an identity as participants in a mathematical community. The use of “mathematical we” discussed in paper IV also helps serve this end, with the “we” used in an inclusive sense, to help create a feeling of shared purpose.

So far, I have mostly discussed the similarities in form of the teachers’ teaching practices. A major finding of the thesis, however, is that despite these outer similarities the discursive practices of the teachers are very different. Although the various general types of discursive routines are present in the teaching practices of all the teachers, the way they use them differs greatly. This can be seen for instance in the different ways in which they handle definition construction, in the different emphasis they place on substantiations, in their different uses of questions, and so on. As already mentioned elsewhere in this thesis, in recent years a number of studies (e.g. Artemeva & Fox, 2011; Hora &
Ferrare, 2013; Weber, 2004) have been published arguing the inadequacy of traditional categorizations of teaching according to outer form – lecture, small group work, group discussion, tutorial, etc. – in understanding teaching practice. I would argue that the results of this study show that even the more fine-grained characterizations provided by, for instance, Saroyan and Snell (1997), Artemeva and Fox (2011) or Hora and Ferrare (2013), might fail in conveying the scope of teaching practices possible within, say, content-driven lectures, chalk talk, or the lecture-problem solving chalkboard triad. In fact, the lectures of for instance teachers A3 and C1 documented in this study (and subject of comparison in Paper II), despite sharing overall form, and covering similar mathematical content, convey vastly different views of mathematical practice to the students partaking of them. It is a strength of the commognitive framework that it enables me as a researcher to analyse these differences, providing me with the means for discerning the ways in which the teachers’ discursive practices are similar and different.

Having dealt at some length with the first aim of the thesis, I will now proceed to discuss the second aim, concerning the discursive constitution of the function concept.

6.2 The discursive constitution of the function concept

As previously mentioned, the second aim is dealt with in Paper V. In the paper I make use of data from the three calculus lectures to analyse how the function concept is discursively constituted, on the object- and meta-levels. Given the smaller amount of data used, and the more limited scope of the analyses, the results presented in this section are more tentative than the results regarding the first aim. Starting with the object-level, in section 2.1.1 a number of characteristics of the function concept as it is described within modern scholarly mathematical discourse are presented. These include the Dirichlet- and Bourbaki-type definitions, emphasizing rule, domain and range; one-valuedness; and arbitrariness, both in terms of values at different points being independent of one another, and in terms of domain and range being arbitrary sets (e.g. Even, 1993). Looking at the function discourses of the calculus teachers in the present study, all three speak of functions as determined by the triad of rule, domain and range, and all three use the notion of one-valuedness. This is done more or less explicitly, however. Teacher A1, when formulating a definition of the function concept, explicitly states the need for the one-valuedness requirement, whereas it is present in a more implicit way when teacher B1 discusses the need to restrict the domain of the function $f(x) = \cos(x)$
in order to make it invertible. Regarding the arbitrariness requirement, it is not so clearly expressed, especially as regards the arbitrariness of domain and range. With one exception (teacher A1 uses an example of a binary function defined on the natural numbers, with the purpose of discussing arbitrariness of values), all examples of functions given during the lectures are real-valued functions of one real variable. This is not surprising, however, given that all data in paper V were from lectures in single-variable calculus. Still this aspect of the function concept is not part of the discourse of function as it is expressed by the three teachers in their lectures.

Another interesting aspect of the object-level discourse of functions is the use of different realizations. As seen in section 2.1.2, this has been the topic of much research regarding the teaching and learning of functions (e.g. Gagatsis & Shaikalli, 2004; Hitt, 1998; Keller & Hirsch, 1998; Schwarz & Dreyfus, 1995), and knowledge of these different realizations (or representations, which is the more common term within this line of research), as well as the ability to move between them, is deemed highly important for understanding the function concept. Looking at the role of different realizations of the function concept in the discursive practices of the teachers in this study, certain differences can be seen. As briefly mentioned in section 6.1.1 teacher A1 gives precedence to algebraic realizations, to the extent that, for instance, graphs are spoken of in a manner suggesting they are merely illustrations of the function given by the formula. In contrast to this, teacher A4 often introduces new functions through graphs, placing equal emphasis on graphical and algebraic realizations, and also making the relationship between them more explicit.

Turning to the meta-level, in section 2.1.2 the notion of process-object duality was discussed. This way of viewing functions as processes or objects is an important meta-level aspect of the function concept, and it has been the topic of much research (e.g. Breidenbach et al, 1992; Güçler, 2013; Sfard, 1991, 1992). Sfard (1991) claims that a well-developed process conception of the function concept is necessary to be able to reach an objectified conception, and this view is generally acknowledged in the literature. In her later writings (e.g. Sfard, 2008), she instead speaks of more or less objectified discourse, but the difficulties for learners in moving from less to more objectified discourse still remain. Looking at the discursive practices of the teachers in this study, the process-object duality of the function concept is expressed in various ways, with all three teachers using objectified language, but to different extent and with varying degrees of clarity. All three teachers speak of functions as objects which can, for instance, be moved around or split into smaller parts. Still, when
working with functions, Teacher A1 uses mainly operational language, speaking of functions as objects performing processes on numbers. In the discourse of teacher A4, objectification is taken a step further, often making no distinction between the function and its values, but still using metaphors of moving, suggesting a view of the function as changing, rather than as an integrated whole. For teacher B1, finally, objectification is taken even further, with functions spoken of in a highly object-like fashion. He is very much concerned with the global characteristics of functions, for instance in determining invertibility.

The topics covered by the teachers – an introduction to the function concept by teacher A1; continuity by teacher A4; and the inverse trigonometric functions by teacher B1 – are typically situated at different points in the calculus course\(^{29}\), with functions introduced early and continuity covered somewhat later, while the inverse trigonometric functions are treated fairly late. This suggests a move towards a more objectified discourse as the course progresses. Given the data available I have no way of determining whether this conjecture holds, but based on what is known regarding the difficulties that the process of objectification presents to students it seems reasonable. In this context one could also mention the results of Paper I\(^{30}\), which show that students generally view functions as processes rather than objects. If the move from a process-oriented to a more objectified discourse during the course which can be seen in this study is indeed typical, then it is not at all surprising that the students participating in the study reported in Paper I, who were in the middle of their first calculus course, had not yet completed it.

The notion of objectification of discourse is central to the study by Güçler (2013) already discussed at various points in this thesis. The teaching documented in Güçler's study concerned limits of functions, and she found that the discourse of the teacher was mainly objectified, with a notable exception being the computation of limits, where he used a more operational discourse. Still, when discussing the results of these computations he again used an objectified discourse. Indeed, there were two distinct meta-level narratives within the teacher's discourse of limits – limit as a process, and limit as a number – but Güçler found that the shifts between these two narratives remained implicit for the students. She also found that the written language of the teacher was highly objectified, with the operational aspects of the discourse

\(^{29}\) Of course the three lectures are taken from different calculus courses, in fact even from two different universities. Still, at least at Swedish universities, calculus courses tend to be relatively similar in this respect, covering roughly the same topics in about the same order; thus it makes sense to compare the three lectures in this way.

\(^{30}\) And many other studies, including Breidenbach et al, 1992; Even, 1993; Vinner & Dreyfus, 1989.
limited to the spoken utterances. The same type of strict distinction between spoken and written language regarding objectification cannot be found among the teachers in the present study. However, like in Güçler’s study, when the teachers used operational language, it was mostly in the context of working with the functions, whereas the characteristics of the functions themselves were discussed in more objectified terms. Also, the shift between operational and objectified discourse was mostly implicit, lending support to the results in Güçler’s study, as well as to Sfard’s more general claims about the mainly tacit character of meta-level learning (Sfard, 2008, p. 202).

Having discussed the results of the thesis in relation to its aims, in the next chapter I will consider the contributions of the thesis to mathematics education research and practice.
7 Discussion

In this, the last chapter of the thesis, I will discuss the contributions of the thesis to university mathematics teaching research and practice. Furthermore I will present some reflections on the quality of the research presented in the thesis, and discuss some directions for future research suggested by the present study.

7.1 Contributions of the thesis

7.1.1 Contributions to research on university mathematics teaching practice

As stated in the introduction, while there has been heated debate about the qualities, or lack thereof, of the lecture as a format for teaching university mathematics, this debate is seldom, if ever, informed by results of empirical research. The research presented in Papers II-V in this thesis contributes to addressing this lack of empirical studies of university teaching practice, expressed for instance by Speer et al (2010). The various classifications of mathematical and didactical practice presented in the thesis can be helpful in describing and analysing university mathematics teaching practice. The teaching practices described in the thesis, while concerned with the teaching of functions, nevertheless in many ways appear to be fairly general, and might be used to investigate and classify university mathematics teaching practice more generally.

In particular, I believe that the classification of didactical routines given in Paper IV might be applicable to university mathematics lecturing in general. This is a type of classification which I have not found in the literature. I find this somewhat surprising, but given the lack of empirical research into traditional university mathematics teaching, perhaps not so much so. Regarding teaching at the K-12 level, there are a number of similar classifications of teaching practices in the classroom (e.g. Davis, 1997; Knott et al, 2008; Krussel, Edwards, & Springer, 2004; Martin et al, 2005; Voigt, 1985). However, although these studies do discuss some aspects of teaching relevant to the present thesis, they are mainly concerned with teacher-student interaction, and the teacher’s ways of affecting student discourse. This is to be expected, given the differences in character between school and university mathematics teaching, but it still makes these earlier classifications less useful in a university context.

31 In fact, this paper contains a number of examples from collegiate mathematics teaching. Still, they mainly concern the teacher as a mediator of student discourse.
setting, while making my classification more so. Furthermore, the results of this thesis lend further support to the claim, put forward by Artemeva and Fox (2011), Grieffenhagen (2008), and others, that mathematics lectures are not primarily concerned with the transmission of factual knowledge, as has been claimed by critics of the lecture format, but rather with the modelling of mathematical reasoning. Many of the discursive practices of the teachers in this study can be claimed to serve this purpose.

As has been discussed repeatedly in this thesis, much research on university mathematics teaching tends to categorize teaching according to outer form only, treating, for instance, lectures as if they were all essentially similar. As mentioned in section 6.1.3, this way of categorizing teaching has been criticized as insufficient (e.g. Hora & Ferrare, 2013; Weber, 2004), and more fine-grained characterizations have been presented. This is a much needed development. However, the results of this thesis show that even teaching which would be categorized as similar according to these more precise characterizations can still display great differences in teaching practices, both regarding the treatment of the mathematical content and in the various didactical routines employed. Hopefully, having read this thesis you will realize that describing a mathematics course by stating that “the teaching took the form of lectures and tutorials” is not much more informative than stating that the teaching was conducted in Swedish.

7.1.2 Methodological contributions

As previously noted, over the last few years studies have started to appear using the commognitive framework to investigate the learning of more advanced mathematical topics, at the upper secondary and tertiary levels. However, there is still very little research on university mathematics teaching practice from a commognitive perspective, and this thesis contributes to the methodological development of the framework by showing how aspects of it can be used for studying more advanced mathematical discourse, particularly in a context where the discursive practices under investigation are not explicitly dialogical in character. The analysis of discursive practice in this thesis is conducted mainly using Sfard’s (2008) characterization of different discourses through word use, visual mediators, narratives, and routines. In her book, Sfard uses this characterization to provide a detailed analysis of mathematical discourse, and what distinguishes it from other discourses. This thesis, and in

particular paper IV, contributes to the development of this aspect of the framework by extending this description of the characteristics of discourse to cover more than just mathematical discourse. A discourse of mathematics teaching, closely related to and intertwined with but still distinct from the mathematical discourse, is found in the discursive practices of the teachers, and the beginnings of a description of this discourse is presented, with a set of didactical routines (explanation, motivation, engagement, question and recall routines) corresponding to Sfard’s categorization of mathematical explorative routines into construction, substantiation and recall routines.

7.1.3 Contributions to university mathematics teaching practice

In the introduction I referred to Speer et al (2010), who stress that any attempt to improve teaching practice ought to be grounded in a thorough understanding of the teaching actually conducted, and that so far there has been little empirical research in this vein. The results of this thesis are a small contribution to such an understanding. Furthermore, as has been previously suggested in the literature (e.g. Dörfler, 2003; Nardi, 2008), bridging the gap between mathematics and mathematics education, and encouraging closer collaboration between mathematicians and mathematics education researchers ought to be beneficial for both parties. This is especially true concerning university mathematics teaching and university mathematics education research. As Dörfler (2003) points out, many mathematicians feel that research in mathematics education has little if anything to do with their everyday lives as mathematicians and teachers of university mathematics. Judging from the reactions of the teachers involved in this study, the very act of conducting empirical research on university teachers’ teaching practices encourages them to reflect on their own practice, and might help to create a feeling of a shared endeavour.

More specifically, Nardi (2008) has found that the notion of concept image has proven very useful in facilitating discussion among university mathematicians regarding student thinking. Although the commognitive framework as a whole is far too complex to be able to function in this manner, I believe that the commognitive characterization of discourse through word use, visual mediators, narratives, and routines, and the notions of object-level and meta-level rules of discourse, can be used to facilitate discussion regarding teacher practice and teacher-student communication. It is often said that teachers lack a vocabulary for discussing the didactical aspects of teaching. This is probably even more true regarding university mathematics teachers, who often lack
pedagogical training. The results of this thesis regarding the types of mathematical and didactical routines present in university mathematics teachers’ teaching discourse could be a small step towards developing such a vocabulary, helping university teachers to reflect on what it is they actually do in their teaching.

7.2 Reflections on the quality of the study

Traditionally, the criteria of validity, generalizability, reliability and objectivity have been employed to assess quality in research. However, these criteria were developed within a positivist research paradigm, and have proved less well suited for judging the quality of qualitative research of the type presented in this thesis. Thus, various alternative sets of criteria have been suggested, for instance, by Lincoln and Guba (1985) and Sierpinska (1993). The criteria proposed by Lincoln and Guba – credibility, transferability, dependability, and confirmability – are intended to correspond to the four traditional criteria, whereas Sierpinska operates with a larger set of criteria – relevance, validity, objectivity, originality, rigor and precision, predictability, reproducibility, and relatedness – meant to cover a wider spectrum, both of forms of research and aspects of quality. In recent years, however, the notion of fixed sets of criteria of quality has been criticized, for instance by Seale (1999) and Simon (2004). As Simon puts it: “The strengthening of research in mathematics education rests not on acceptance of a set of criteria, but rather on a dynamic and ongoing discussion of research quality” (Simon, 2004, p. 157). He makes a comparison with the notion of proof in mathematical discourse. While there are no set criteria for what characterizes a proof, let alone a good proof, there are nevertheless well-established norms for judging proofs (ibid, p. 157). In this section I will discuss various aspects of the quality of the study, taking as my starting point the issues raised by Simon, but also drawing on the criteria proposed by Sierpinska (1993).

Simon formulates two basic assumptions underlying his discussion of quality in research, namely that research should be authentic inquiry, and serve to advance knowledge in the field, where the field could be taken to mean practitioners as well as mathematics education researchers. These requirements are obviously related to Sierpinska’s relevance criterion. The relevance of the research reported in this thesis has already been discussed, and at this point it suffices to point out the need for empirical research concerning university mathematics teaching articulated, for instance, by Speer et al (2010), as well as the lively debate about the value of lectures, argued more on the basis of strong
opinions than on strong empirical results. As for the contribution to knowledge in the field, this has been discussed earlier in this chapter. The main issue raised by Simon is the character of the empirical research study as an argument: “A research study, from question to conclusion, can be thought of as the construction and presentation of a warranted argument” (Simon, 2004, p. 159). It is the researcher’s responsibility to ensure that the important aspects of the study are justified, and to provide a coherent chain of reasoning. This issue is connected to several of Sierpinska’s criteria: mainly validity, but also objectivity, reproducibility and precision. Focusing on the main study, the justification for the research questions can be argued from several standpoints: There is an interest within the field for studies of university mathematics teaching practice, and given the basic theoretical assumption of mathematics as a discursive activity, the unit of analysis then becomes the discursive practices of the teachers. Characterizations of such discursive practices are needed to further our knowledge of university mathematics teaching. The function concept is a mathematical topic which has generated much research activity, but more knowledge concerning university lecturers’ teaching of function is needed. Concerning the justification of the methodology, given the focus on teachers’ discursive practices, video data provides the possibilities for detailed analysis of both spoken and visual aspect of the discursive activity, and the commognitive framework supplies useful theoretical tools for the analysis of such discursive activities. It could be argued that this framework is better suited for studying communicational settings involving dialogue between discursants, and this is how it has most often been used. However, key aspects of the framework are designed for the characterization and analysis of discourse, and do not depend on the presence of explicit interpersonal communication. I have provided a detailed account of the theoretical framework to help make the basis for my analyses and conclusions explicit. This, together with a detailed account of the context in which the study was conducted, contributes to the reproducibility of the study, making it possible for the results of the study to be compared, extended or challenged by others conducting studies in similar contexts using similar methodology.

Regarding the analysis of data, I have tried to justify my results through a transparent description of the reflexive process of categorization and comparison leading up to the final categorizations. In the papers I have also provided a large amount of examples illustrating the various types of discursive activities, in order to provide a convincing argument for the validity of the results. I have also aimed at a precise and consistent use of technical
terminology, with central terms being operationally defined, thus helping to clarify how this study builds on previous studies using the same theoretical apparatus or studying similar phenomena. The conclusions of the study are derived from the analysis of the data, but also informed by previous research, and I have attempted to state clearly how the study contributes to knowledge of university mathematics teaching practice. The validity of the analyses is also increased by the fact that preliminary results of the study have been presented at various conferences (e.g. Viirman 2011a; 2012), and thus been subject to criticism and suggestions. In this way, the analyses and the results, but also the research questions and the aims of the study have been continually revised.

A criticism which could be levelled at this study is that it is descriptive to a large extent. Both Sierpinska (1993) and Simon (2004) question the relevance of studies that are solely descriptive, stating the need for conclusions which go beyond the mere characterization of the situation under study. However, although my results are largely descriptive, I do go beyond the descriptions in my conclusions, discussing how various aspects of the teachers discursive practices can be understood, for instance, in how specific mathematical objects, as well as the very practice of doing mathematics, are constituted through the teaching practices. Also, the classifications of mathematical and didactical routines presented in the thesis are potentially useful as analytical tools in future research on university mathematics teaching.

7.3 Further research

During a research project of this type, undertaken over several years, a number of interesting ideas appear which have to be left unexplored. I wish to conclude this thesis by discussing some possible routes of further research which have opened up during the work on the present study.

In the process of analysis of the video recordings, it soon occurred to me that the role of gestures as visual mediators in the discursive practices of the teachers was something which needed further investigation. There were obvious patterns to be found in the way the different teachers used for instance pointing gestures. Indeed, this is highlighted by Artemeva and Fox (2011) as one characteristic feature of chalk talk. Still, a more thorough treatment of this was deemed to be beyond the scope of the thesis. The use of gestures in mathematics education has been widely studied, for instance, by Radford and others using a semiotic perspective (e.g. Radford, Bardini, & Sabena, 2007; Radford, Schubring, & Seeger, 2008), and it would be interesting to investigate university mathematics teachers’ use of gestures in their teaching from a
commognitive perspective, but informed by previous research such as that of Radford and colleagues.
In Paper V, and in section 6.2, I made a tentative conjecture about the discursive development of the function concept towards a greater degree of objectification during the course of a typical course in single variable calculus. However, the data collected for this thesis did not allow me to investigate this further, and I believe that it would be an interesting topic for further research. This could be done, for instance, through following a smaller number of teachers throughout a calculus course, tracking the discursive development of the function concept as it is constituted through the discursive practices of the teachers.

As also previously mentioned, certain types of discursive activities central to scholarly mathematical discourse, for instance the stating and proving of theorems, were largely absent from the teaching documented in this thesis. Indeed, little was seen of the style of teaching denoted by Weber (2004) as the definition-theorem-proof (DTP) format, and often (e.g. Davis & Hersh, 1981; Dreyfus, 1991) described, and maligned, as typical of so-called traditional university mathematics teaching. I suggested in Paper III that this might be due to the fact that all data was taken from first-semester courses, occupying some middle ground between elementary and more advanced mathematical practice. Also, the choice of the function concept as the mathematical focus of the thesis may have had consequences for the discursive activities undertaken by the teachers, since functions are mostly not extensively treated as a topic in their own right, with their own theorems and proofs. Still, the activity of proving is central to mathematical discourse, and has been the subject of much research.
An analysis of the discursive activities connected to the teaching of topics involving proof to a greater extent might not only extend the results of this thesis, but also be a worthwhile addition to the body of research of proof more generally. On a related note, performing analyses of the type undertaken in this thesis on lectures in more advanced courses such as, for instance, abstract algebra or general topology, could help determine to what extent the classifications presented in this thesis are generally applicable, and to what extent they are context-dependent. Specifically, a worthwhile topic of future research would be to examine to what extent the general classification of didactical routines presented in this thesis applies to university mathematics lecturing in general.

The analyses of teaching activity in this thesis have centred on teaching practices regarding the mathematical topic of functions. However, there are of
course many aspects of the mathematical and didactical discourses of the teachers that have not been the subject of investigation in this thesis, but that are still worthy of further research. One such aspect, briefly touched upon in Chapter 6, is the ontological and epistemological assumptions suggested by the discursive practices of the teachers. For instance, the different approaches to definition construction and to the substantiation of claims contribute to different ways of constituting mathematical practice. Similarly, the way the teachers talk about, for instance, mathematical objects as being discovered or invented, suggest different views of the ontology of mathematics.

Finally, the choice to focus on the teachers’ discursive practices meant that the other important group of discursants within a university mathematics teaching setting – the students – were largely absent from the study. The data collected for this study does include some teacher-student interaction, but little was made of these data in the analyses. Still, there were features of this interaction which it would be interesting to study further. This might be done either by documenting a larger amount of teaching practice, and focusing specifically on those situations where there are explicit teacher-student interaction, or by collecting data from those settings within traditional university mathematics teaching practice where teacher-student interaction is most prominent, for instance, problem-solving sessions and tutorials. Data of this type would allow for comparison between teacher and student discourse, for instance regarding the level of objectification, and it might also let us capture commognitive conflict, thus getting closer to the crucial question of the relationship between teaching and learning in university teaching practice.
8 References


discovery. In M.P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education. MAA Notes #73.* (pp 167-177). Washington DC: Mathematical Association of America.


Oehrtman, M., Carlson, M., & Thompson, P.W. (2008). Foundational reasoning abilities that promote coherence in students’ function understanding. In M.P. Carlson & C. Rasmussen (Eds.), Making the connection: Research and teaching in undergraduate mathematics education. MAA Notes #73. (pp. 27-41). Washington DC: Mathematical Association of America.


---


Sinclair, N. & Moss, J. (2012). The more it changes, the more it becomes the same: The development of the routine of shape identification in dynamic geometry environment. *International Journal of Educational Research, 51–52*, 28–44.


Viirman, O. (2011a). Discourses of functions: University mathematics teaching through a commognitive lens. In M. Pytlak, T. Rowland & E. Swoboda (Eds.), *Proceedings of CERME7 (7th Conference of European Research in Mathematics Education)* (pp. 2103-2112). University of Rzeszów, Poland.


of Mathematical Education in Science and Technology, 2013. DOI: 10.1080/0020739X.2013.855328.

Viirman, O. (2013b). What we talk about when we talk about functions: Characteristics of the function concept in the discursive practices of three university teachers. In B. Ubuz, Ç. Haser, & M.A. Mariotti (Eds.), Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education. (pp. 2466-2475). Ankara: ERME.


The function concept and university mathematics teaching

This thesis concerns the teaching of mathematics at university level, with a particular focus on the teaching of the function concept. The main aim of the thesis is describing and analysing the teaching practices of university mathematics teachers regarding the function concept, and how this concept is constituted through these practices. To this end, video recordings of lectures by seven mathematics teachers at three Swedish universities were analysed using a discursive perspective, Sfard’s commognitive framework. The observed teaching was traditional in form, with teachers using “chalk talk” – simultaneously talking and writing on the board. The results show that the teaching practices of the teachers belong to two distinct but intertwined discourses – a mathematical discourse, and a discourse of mathematics teaching. Classifications of important aspects of these discourses are presented, and it is found that the teachers’ discursive practices, while sharing overall form, still display considerable differences. Other results include an analysis of the levels of objectification displayed by the teachers in their discursive constitution of the function concept. The study contributes to a small but growing body of empirical research on university mathematics teaching practice.