Operations on ideals in polynomial rings

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Abstract

A ring is an algebraic structure equipped with two binary operations satisfying certain axioms, providing it with specific and highly useful properties. Furthermore, letting the elements of such structure constitute the coefficients in polynomials forms a new set in which the ring properties are preserved – a polynomial ring.

To gain a deeper understanding of the rings, one may consider additively closed and multiplicatively absorbing subsets known as ideals. In the case of polynomial rings, ideals are of particular importance as they often are used to define field extensions to introduce solutions to polynomial equations, one such example being the complex numbers \( \mathbb{C} \). Ideals can be explicitly defined by means of generating sets; a number of key elements that, when applying the rules of operation to them, generates the entire ideal.

Ideals generated solely by single-termed polynomials (monomials) in two or three variables were the focus of this thesis. We outlined a method to graphically visualize these monomial ideals and examined how it could facilitate operations over them. More specifically, we explored basic arithmetics and found that adding two monomial ideals is equivalent to superimposing their graphical representations, and that multiplying them corresponded to point vector addition between the figures. Additionally, we considered an algebraic approach to finding the integral closure of a monomial ideal and derived a graphically analog method generalizable to higher dimensions.

Lastly, we hypothesized how the visualizations could be further utilized to aid in identifying minimal monomial reductions and suggested the incorporation of ideas from linear algebra to further advance such procedure.
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1 Introduction

The importance of polynomial rings is widespread throughout all of mathematics; algebra, linear algebra and even analysis. Over the years, a variety of concepts have been developed in order to let us fully understand the structures of polynomial rings – or rings in general. Initially proposed by Dedekind, one such concept is that of a ring ideal.

In this thesis we dive even deeper, exploring a specific type of ideals in polynomial rings known as monomial ideals. We consider a graphical representation, uniquely applicable to monomial ideals, and examine how it can be helpful to us when performing binary as well as unary operations over the ideals.

Please note that this text aims to present known facts in a summarized, easy-to-follow way. So while nothing new is brought to the table, the author hopes that readers may find the contents engrossing regardless.

2 Preliminaries

In this introductory chapter we will focus on the basic but essential concepts needed to understand the contents of this thesis. These include definitions and properties of polynomial rings and ideals in such. The main source of the material found here is Fraleigh[1].

2.1 Polynomial rings in one variable

We start at the beginning by asking ourselves ”what is a ring?” and answer this very question by stating the so-called ring axioms.

Definition 2.1. A set $R$ is called a ring if it is equipped with and closed under the two binary operations of addition ($+$) and multiplication ($\cdot$) that are satisfying the following properties – the ring axioms – for all $a,b,c \in R$.

Addition is required to be associative and commutative. We also demand an existence of an additive identity (the zero element) and inverse elements.

1. $(a + b) + c = a + (b + c)$
2. $a + b = b + a$
3. There exists $0 \in R$ such that $a + 0 = 0 + a = a$
4. There exists $(-a) \in R$ such that $a + (-a) = (-a) + a = 0$

Multiplication is required to be associative.

5. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Distributivity of multiplication over addition is required to connect the two operations in a meaningful way.

6. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
Moreover, a ring $R$ is called commutative if its multiplication is commutative ($a \cdot b = b \cdot a$) and unital, or ring with unity, if it contains a multiplicative identity element ($1 \in R$ such that $a \cdot 1 = 1 \cdot a = a$).

**Definition 2.2.** A polynomial $p(x)$ is a sum of the form

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n = \sum_{i=0}^{n} a_i x^i, \quad (2.1)$$

where all $a_i$ are elements of some ring $R$ called the polynomial coefficients. If $n$ is the largest exponent such that $a_n \neq 0$ we call $n$ the degree of $p(x)$, denoted by $\deg p(x)$.

The set of all such polynomials is called the polynomial ring in one variable over $R$ and is denoted by $R[x]$. Addition and multiplication of polynomials in $R[x]$ is defined as follows below.

Addition of two polynomials $p(x), q(x) \in R[x]$:

$$p(x) + q(x) = (a_0 + a_1 x + \cdots + a_n x^n) + (b_0 + b_1 x + \cdots + b_n x^n) = (a_0 + b_0) + (a_1 + b_1) x + \cdots + (a_n + b_n) x^n \quad (2.2)$$

Multiplication of two polynomials $p(x), q(x) \in R[x]$:

$$p(x) \cdot q(x) = (a_0 + a_1 x + \cdots + a_n x^n) \cdot (b_0 + b_1 x + \cdots + b_n x^n) = (a_0 b_0) + (a_0 b_1 + a_1 b_0) x + \cdots + (a_0 b_n + \cdots + a_n b_0) x^n \quad (2.3)$$

What are the properties of $R[x]$? It turns out, as the name suggests, that it does satisfy the ring axioms and the following result is due to Theorem 22.2 by Fraleigh[1].

**Theorem 2.3.** If $R$ is a commutative ring with unity 1, then the set $R[x]$ of all polynomials in one variable $x$ with coefficients in $R$ is a commutative ring with the same unity 1.
Proof. The proof is straightforward and we will show that the set $R[x]$ fulfills some of the ring axioms. To prove that $R[x]$ satisfies the axioms omitted here is left as an exercise.

Throughout the proof, let $p(x) = \sum_{i=0}^{n} a_i x^i$, $q(x) = \sum_{i=0}^{n} b_i x^i$ and $r(x) = \sum_{i=0}^{n} c_i x^i$ be three polynomials with coefficients in a commutative, unital ring $R$. Recall that if the degrees of the polynomials are different we simply let $n = \max \{ \deg p(x), \deg q(x), \deg r(x) \}$ and if, for example, $\deg p(x) < n$ then $a_i = 0$ for all $\deg p(x) < i \leq n$.

For polynomial addition we see that the resulting sum is a polynomial with coefficients $a_i + b_i$, but because $a_i, b_i \in R$ and $R$ is a ring it follows that $a_i + b_i \in R$. Hence, $p(x) + q(x) \in R[x]$ and $R[x]$ is closed under addition. With similar reasoning one can show that $R[x]$ satisfies all of the additive ring axioms, thus $\langle R[x], + \rangle$ is an abelian group.

The associativity of the polynomial multiplication is shown in Fraleigh[1].

Next, we will prove the left distributive law:

$$p(x) \cdot (q(x) + r(x)) = \sum_{i=0}^{n} a_i x^i \cdot \left( \sum_{i=0}^{n} b_i x^i + \sum_{i=0}^{n} c_i x^i \right)$$

$$= \sum_{i=0}^{n} a_i x^i \cdot \left( \sum_{i=0}^{n} (b_i + c_i) x^i \right)$$

$$= \sum_{i=0}^{n} \left( \sum_{j+k=i} a_j b_k + a_j c_k \right) x^i$$

$$= \sum_{i=0}^{n} \left( \sum_{j=k=i} a_j b_k \right) x^i + \sum_{i=0}^{n} \left( \sum_{j+k=i} a_j c_k \right) x^i$$

$$= \left( \sum_{i=0}^{n} a_i x^i \cdot \sum_{i=0}^{n} b_i x^i \right) + \left( \sum_{i=0}^{n} a_i x^i \cdot \sum_{i=0}^{n} c_i x^i \right)$$

$$= (p(x) \cdot q(x)) + (p(x) \cdot r(x)).$$

The right distributive law is proved similarly.

The multiplicative identity 1 in $R$ is also a unity in $R[x]$ since

$$1 \cdot p(x) = 1 \cdot \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} (1 \cdot a_i) x^i = \sum_{i=0}^{n} a_i x^i = p(x) = p(x) \cdot 1.$$ 

Lastly, we note that polynomial multiplication is commutative if $R$ is commutative by definition and we can conclude that $R[x]$ is a commutative ring with unity 1 – the very fact we sought to prove. \qed
2.2 Polynomial rings in multiple variables

We can expand the notion of single variable polynomials to two variables by combining polynomials \( p(y) \in R_1[y] \) with coefficients \( a_i(x) \in R[x] \); that is \( R_1 = R[x] \). In other words we allow the coefficients to be polynomials in another variable:

\[
p(x, y) = \sum_{i=0}^{n} a_i(x)y^i = \sum_{i=0}^{n} \left( \sum_{j=0}^{m} a_{ij}x^j \right) y^i = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij}x^jy^i \in (R[x])[y].
\]

As one would expect, it turns out that \((R[x])[y]\) is also a ring. It can furthermore be shown that \((R[x])[y] = (R[y])[x]\) and therefore we can use the simpler notation \( R[x, y] \) to denote such two-variable polynomial ring. In a similar fashion we can keep expanding the polynomials to an arbitrary number of variables, eventually resulting in the polynomial ring \( R[x_1, x_2, \ldots, x_n] \) in \( n \) variables. In fact, one could even consider polynomials in infinitely many variables.

2.3 Ring ideals

Now that we understand what a ring is, a natural next step is to consider subsets of a ring. These come in many different "flavors", some examples being regular subsets, subsets satisfying group properties (that is, subgroups) or subsets that are rings in themselves (so-called subrings). An example of the latter can be seen by considering the constant polynomials of \( R[x] \); these constitute the ring \( \mathbb{R} \) and are obviously included in \( R[x] \), hence \( \mathbb{R} \subset R[x] \) is a subring.

A special type of subsets that are of particular interest to us have been named ideals. The meaning behind this term should become apparent in the following definition.

**Definition 2.4.** Let \( I \) be a nonempty subset of a ring \( R \). \( I \) is called an ideal in \( R \) if it is closed under addition and closed under multiplication with ring elements. That is, if

1. For all \( a, b \in I \) : \( a + b \in I \)
2. For all \( a \in I \) and for all \( r \in R \) : \( a \cdot r \in I \)
3. For all \( a \in I \) and for all \( r \in R \) : \( r \cdot a \in I \)

A subset of \( R \), that fulfills condition 1 and 2 only, is called a right ideal in \( R \). Similarly, a subset satisfying the first and third criteria is called a left ideal in \( R \). If \( R \) is commutative the second and third criteria coincide.

A simple example of the ideal concept is the set of all integers divisible by an integer \( n \), commonly denoted \( n\mathbb{Z} \), which constitutes an ideal of \( \mathbb{Z} \).

\[
n\mathbb{Z} = \{ r \in \mathbb{Z} : n|r \}
\]

One way of viewing the ideal is to think of it as a set generated by the integer \( n \). We usually denote this using angular brackets, \( \langle n \rangle \), where \( n \) is called
a *generator*. Observe that the generator in this case must be an element of \( \mathbb{Z} \) itself and that

\[
(0) = \{0\} \text{ since } r \cdot 0 = 0 \cdot r = 0 \text{ for all } r \in \mathbb{Z}, \text{ and}
\]

\[
(1) = \mathbb{Z} \text{ since } r \cdot 1 = 1 \cdot r = r \text{ for all } r \in \mathbb{Z}.
\]

Let us formally define this notion below in order to not limit ourselves to the ring of integers. For now on we will also assume that all rings are commutative, unless we specifically state otherwise.

**Definition 2.5.** Let \( R \) be a commutative ring and define a subset \( I \) of \( R \) as

\[
I := \left\{ \sum_{i=1}^{n} r_i a_i \mid r_i \in R \right\}
\]

for some nonzero positive integer \( n \) and fixed elements \( a_i \in R \). Then \( I \) is an *ideal* generated by \( a_1, \ldots, a_n \) and the set of generators \( \{a_1, \ldots, a_n\} \) is called a *generating set* of \( I \). In shorthand notation, we write

\[
I = \langle a_1, \ldots, a_n \rangle.
\]

In particular, if \( n = 1 \) (that is, if \( I \) is generated by a single element), \( I \) is called a *principal ideal*.

Observe that a generating set need not be unique. Two different set of generators may result in the same ideal, thereby giving rise to a few questions we will deal with in the next chapter.

This concludes the preliminaries necessary to follow the main focus of this thesis. In following chapters we will only be discussing polynomial rings in two or three variables over the real numbers, that is \( \mathbb{R}[x, y] \) and \( \mathbb{R}[x, y, z] \). We will take a closer look at certain types of ideals of these polynomial rings and explore their properties.

## 3 Monomial ideals

We have previously seen that any ring can serve as a basis for polynomials of \( n \) variables and that the resulting set is a ring in itself. We also saw that rings may have subsets of certain properties called ideals. Our next step is then to examine what these ideals, apart for the trivial cases of the ring itself and the zero ideal, can look like in polynomial rings. A good reference for the interested reader is the chapters 1 and 2 by Fröberg[2].

### 3.1 Generating sets of monomial ideals

A simple example shows what a polynomial ring ideal, commonly referred to as a *polynomial ideal*, might look like.
Example 3.1. Let $\mathbb{R}[x, y]$ be the polynomial ring with real coefficients in two variables. A (non-principal) polynomial ideal is then

$$I = \langle x, y \rangle = \{p(x, y)x + q(x, y)y \mid p(x, y), q(x, y) \in \mathbb{R}[x, y]\}$$

since for all $\tilde{p}(x, y), \tilde{q}(x, y) \in I$ and for all $p(x, y) \in \mathbb{R}[x, y]$ it holds that

$$\tilde{p}(x, y) + \tilde{q}(x, y) = (p_1(x, y)x + p_2(x, y)y) + (q_1(x, y)x + q_2(x, y)y)$$

$$= (p_1(x, y) + q_1(x, y))x + (p_2(x, y) + q_2(x, y))y$$

and

$$p(x, y) \cdot \tilde{p}(x, y) = p(x, y)(p_1(x, y)x + p_2(x, y)y)$$

$$= (p(x, y)p_1(x, y))x + (p(x, y)p_2(x, y))y$$

are elements of $I$.

In the previous section we mentioned that a generating set of an ideal does not need to be unique. The following examples illustrate this fact.

Example 3.2. Let $R = \mathbb{R}[x, y]$ be a polynomial ring and consider the ideals

$$I = \langle x, y \rangle = \{p(x, y)x + q(x, y)y \mid p(x, y), q(x, y) \in R\}$$

and

$$J = \langle x + y, y \rangle = \{p(x, y)(x + y) + q(x, y)y \mid p(x, y), q(x, y) \in R\}.$$ 

Our aim is to show that $I = J$ despite the differing generating sets. We do this by first choosing an arbitrary element $p_1(x, y) \in I$:

$$p_1(x, y) = p(x, y)x + q(x, y)y = p(x, y)x + (y - y)y$$

$$= p(x, y)(x + y) - p(x, y)y + q(x, y)y$$

$$= p(x, y)(x + y) + (q(x, y) - p(x, y))y \in J$$

Since $p_1(x, y)$ was chosen arbitrarily, it follows that $I \subseteq J$. Conversely, choosing an element $p_2(x, y) \in J$ we see that

$$p_2(x, y) = p(x, y)(x + y) + q(x, y)y = p(x, y)x + p(x, y)y + q(x, y)y$$

$$= p(x, y)x + (p(x, y) + q(x, y))y \in I$$

so it also holds that $J \subseteq I$. Therefore it must be that $I = J$ and we are done.

Even generating sets consisting of a different number of generators may result in the same ideal, as example 3.3 below aims to show.

Example 3.3. Let $R = \mathbb{R}[x, y]$ be a polynomial ring and consider the ideals

$$I = \langle x, y \rangle = \{p(x, y)x + q(x, y)y \mid p(x, y), q(x, y) \in R\}$$

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and
\[ K = \langle x, y, x^2 \rangle = \left\{ p(x, y)x + q(x, y)y + r(x, y)x^2 \mid p(x, y), q(x, y), r(x, y) \in R \right\}. \]

Then every element in \( I \) is obviously also an element in \( K \) since

\[ p(x, y)x + q(x, y)y + r(x, y)x^2 \in K \]

but the converse is also true, as

\[
\begin{align*}
p(x, y)x + q(x, y)y + r(x, y)x^2 &= p(x, y)x + q(x, y)y + (xr(x, y))x \\
&= (p(x, y) + xr(x, y))x + q(x, y)y \in I
\end{align*}
\]

and hence \( I = K \) even though \( I \) in this case was two-generated while \( K \) was three-generated.

From the latter example it becomes evident that the generator \( x^2 \) of \( K \) can be removed. The reason behind this lies in how ideals are defined, since by choosing \( p(x, y) = x \in \mathbb{R}[x, y] \) we see that \( x^2 = xp(x, y) + 0 \cdot q(x, y) \) and is therefore contained in \( \langle x, y \rangle \). In other words, \( x^2 \) can be removed from the generating set because \( x \) divides \( x^2 \). This holds for all generators and the principle is further explored in example 3.12. We introduce the following definition for when a generating set cannot be further simplified.

**Definition 3.4.** A generating set \( \{a_1, \ldots, a_n\} \) of an ideal \( I = \langle a_1, \ldots, a_n \rangle \) is said to be **minimal** if no generators can be removed from the set without changing the ideal.

**Remark 3.5.** If all \( a_i \) are monomials (see definition 3.7 below) then it is equivalent to say that \( \{a_1, \ldots, a_n\} \) is minimal if \( a_i \nmid a_j \) for all \( i \neq j \).

**Remark 3.6.** Note that even a minimal generating set need not be unique. For instance, in example 3.2 both \( \langle x, y \rangle \) and \( \langle x, x+y \rangle \) are minimal, yet both generate the same ideal.

As far as notation goes for ideals, we usually resort to regular subset notation. If \( I \) is an ideal of a ring \( R \), we simply write this as \( I \subseteq R \). In the above example \( \langle x, y \rangle \) is a proper ideal, meaning it is a proper subset, of \( \mathbb{R}[x, y] \) and consequently this is denoted as \( \langle x, y \rangle \subset \mathbb{R}[x, y] \). Observe that this notation is not completely reliable as a subset of a ring need not necessarily be an ideal and as such further clarification is usually required. Here, however, the usage of the subset-symbols will always refer to ideals unless something else is specifically stated.

Remember, in a polynomial ring any polynomial can be a generator. The ideal studied in example 3.2 and 3.3 can be generated by different subsets of \( \mathbb{R}[x, y] \), in particular by \( \{x, y\} \) – polynomials consisting of only one term. These types of polynomials have been given their own name and can be used to further classify ideals.

**Definition 3.7.** A polynomial \( p(x_1, \ldots, x_n) \) in \( \mathbb{R}[x_1, \ldots, x_n] \) with coefficients \( a_i \in R \) is called a **monomial** if all the coefficients but one are equal to zero.
That is, if \( p(x_1,\ldots,x_n) \) is a polynomial consisting of one single term. Such polynomial can be written as

\[
p(x_1,\ldots,x_n) = a x_1^{k_1} \cdots x_n^{k_n} = a \prod_{i=1}^{n} x_i^{k_i}
\]

for some non-negative integers \( k_j \) and \( a \in \mathbb{R} \). If \( a = 1 \) we refer to \( p(x_1,\ldots,x_n) \) as a power product.

**Definition 3.8.** An ideal \( I \subseteq \mathbb{R}[x_1,\ldots,x_n] \) is called *monomial* if \( I \) can be generated by a set of monomials. In the case some or all of the generating monomials are power products, \( I \) is still called a monomial ideal.

**Remark 3.9.** An ideal generated by monomials in a polynomial ring over \( \mathbb{R} \) can always be represented by a generating set of power products, since each monomial can be multiplied by the inverse of its coefficient. So for our purposes the notion of generating sets of monomials and of power products will actually coincide.

The goal of this thesis is to explore monomial ideals in three dimensions, and therefore we will mostly deal with power products of the form \( x^a y^b z^c \) with \( a, b, c \) being non-negative integers. However, in many of the examples we will instead resort to analog cases in two dimensions for clarification purposes.

### 3.2 Arithmetics on monomial ideals in three variables

Now let us for a moment consider the "next level" of our binary operations – a step up the abstraction ladder of algebra, if you will – namely the concept of ideal operations. Instead of combining elements of a ring to form new elements, here we wish to define operators using subsets of rings (more specifically monomial ideals) as operands. In other words, how can we, in a logical way, define addition and multiplication of monomial ideals?

One way to tackle this problem is to look at the individual elements, for which the operations are well defined. For instance, let \( I = \langle m_1,\ldots,m_r \rangle \) and \( J = \langle n_1,\ldots,n_s \rangle \) be monomial ideals in \( \mathbb{R}[x,y,z] \) and take an arbitrary element from each of these, \( \sum_{i=1}^{r} p_i m_i \in I \) and \( \sum_{j=1}^{s} q_j n_j \in J \) where \( p_i = p_i(x,y,z) \) and \( q_j = q_j(x,y,z) \). Adding these together and renaming the polynomial coefficients and generators yields:

\[
\sum_{i=1}^{r} p_i m_i + \sum_{j=1}^{s} q_j n_j = p_1 m_1 + \cdots + p_r m_r + q_1 n_1 + \cdots + q_s n_s
\]

\[
= u_1 \tilde{m}_1 + \cdots + u_r \tilde{m}_r + u_{r+1} \tilde{m}_{r+1} + \cdots + u_{r+s} \tilde{m}_{r+s}
\]

\[
= \sum_{i=1}^{r+s} u_i \tilde{m}_i.
\]

It does indeed make sense that this sum should be an element in the resulting set of the ideal operation \( I + J \). Since the two elements were chosen arbitrarily
this further implies that all possible elemental sums should be included in this new set. Let us define this formally (the definition can be made more general, but for our purposes operations on monomial ideals in three variables will suffice).

**Definition 3.10.** Let \( \mathcal{A} \) be the polynomial ring \( \mathbb{R}[x,y] \) or \( \mathbb{R}[x,y,z] \) and then let \( I = \langle m_1, \ldots, m_r \rangle, \ J = \langle n_1, \ldots, n_s \rangle \) be monomial ideals in \( \mathcal{A} \). The sum of the ideals is then defined as

\[
I + J := \left\{ \sum_{i=1}^{r} p_i m_i + \sum_{j=1}^{s} q_j n_j \mid p_i, q_j \in \mathcal{A} \right\} = \langle m_1, \ldots, m_r, n_1, \ldots, n_s \rangle.
\]

From this definition we directly state the following proposition.

**Proposition 3.11.** Let \( I = \langle m_1, \ldots, m_r \rangle \) and \( J = \langle n_1, \ldots, n_s \rangle \) be monomial ideals in \( \mathcal{A} \), where \( \mathcal{A} \) is \( \mathbb{R}[x,y] \) or \( \mathbb{R}[x,y,z] \). Then \( I + J \) is also a monomial ideal.

**Proof.** To prove this, pick two arbitrary elements of \( I + J \) and add them:

\[
\left( \sum_{i=1}^{r} p_i m_i + \sum_{j=1}^{s} q_j n_j \right) + \left( \sum_{i=1}^{r} u_i m_i + \sum_{j=1}^{s} v_j n_j \right) = \sum_{i=1}^{r} (p_i + u_i) m_i + \sum_{j=1}^{s} (q_j + v_j) n_j
\]

which is an element of \( I + J \) since \( \mathcal{A} \) is a ring, and so \( I + J \) is closed under addition. Now, let \( \tilde{p} \in \mathcal{A} \) and multiply it with one of the elements above:

\[
\tilde{p} \left( \sum_{i=1}^{r} p_i m_i + \sum_{j=1}^{s} q_j n_j \right) = \sum_{i=1}^{r} (\tilde{p}p_i) m_i + \sum_{j=1}^{s} (\tilde{p}q_j) n_j
\]

which, again, is an element of \( I + J \) due to the ring properties of \( \mathcal{A} \). Furthermore, as \( \mathcal{A} \) is commutative we know that the criteria for left and right ideals coincide so that \( \tilde{p}p = p\tilde{p} \) for all \( p \in I \cup J \). Therefore the conditions in definition 2.4 (ring ideals) are satisfied and we can conclude that the sum of two monomial ideals in \( \mathcal{A} \) must also itself be an ideal in \( \mathcal{A} \). Finally, as \( I + J \) clearly can be generated by monomials it is a monomial ideal and our initial claim is proven.

Note that the resulting generating set of \( I + J \) may contain redundant generators. Recall that generators that are divisible by other generators within the same set can be removed without it affecting the ideal. We illustrate this through the following example.

**Example 3.12.** Let \( I = \langle x^3 z^2, x^2 z^3, xyz, y^2 \rangle \) and \( J = \langle x^4 y^3, x^3 y^5, x^2 z^2, xyz, y^4 z^3, y^2 z^4 \rangle \) be monomial ideals in \( \mathbb{R}[x,y,z] \). The sum of \( I \) and \( J \) is then

\[
I + J = \langle x^3 z^2, x^2 z^3, xyz, y^2, x^4 y^3, x^3 y^5, x^2 z^2, xyz, y^4 z^3, y^2 z^4 \rangle
\]
which can be simplified considerably. First we note that $xyz$ appears twice, and since obviously $xyz|xyz$ we can scratch one of them out.

$$I + J = \langle x^3z^2, x^2z^3, xyz, y^2, x^4y^2, x^3y^3, x^2z^2, xyz, y^4z^3, y^2z^4 \rangle$$

Next, we examine which generators can be divided by $y^2$ since it is a simple generator to work with. Any monomial containing a factor $y$ with an exponent of 2 or greater will fit the criterion and can thereby also be canceled.

$$I + J = \langle x^3z^2, x^2z^3, xyz, y^2, x^4y^2, x^3y^3, x^2z^2, xyz, y^4z^3, y^2z^4 \rangle$$

Lastly, by comparing the exponents of the remaining generators it becomes evident that $x^2z^2|x^3z^2$ and $x^2z^2|x^2z^2$ so removing those as well yields

$$I + J = \langle x^3z^2, x^2z^3, xyz, y^2, x^4y^2, x^3y^3, x^2z^2, xyz, y^4z^3, y^2z^4 \rangle$$

meaning that $I + J = \langle xyz, y^2, x^2z^2 \rangle$ and we are done.

We now turn our focus to the operation of multiplication which will serve as a foundation for what eventually will become the definition of the so-called integral closure of an ideal. We use a similar approach to that of addition.

$$\sum_{i=1}^{r} p_i m_i \cdot \sum_{j=1}^{s} q_j n_j = (p_1 m_1 + \cdots + p_r m_r) \cdot \sum_{j=1}^{s} q_j n_j$$

$$= p_1 m_1 \left( \sum_{j=1}^{s} q_j n_j \right) + \cdots + p_r m_r \left( \sum_{j=1}^{s} q_j n_j \right)$$

$$= \sum_{i=1}^{r} \left( \sum_{j=1}^{s} p_i q_j \cdot m_i n_j \right) = \sum_{i=1}^{r} \left( \sum_{j=1}^{s} u_{ij} \tilde{m}_{ij} \right)$$

The last equality follows from the fact that $p_i q_j = u_{ij} (x, y, z) \in \mathcal{A}$ for all $i = 1, \ldots, r$ and $j = 1, \ldots, s$ due to the ring properties of $\mathcal{A}$, and that the product of two monomials always will be a monomial. Note that this is a sum of every possible product of monomials of the initial generating sets with polynomial coefficients. As before, we collect all elements formed in this way and call the resulting set the product of the two monomial ideals.

**Definition 3.13.** Let $\mathcal{A}$ be the polynomial ring $\mathbb{R}[x, y]$ or $\mathbb{R}[x, y, z]$ and let $I = \langle m_1, \ldots, m_r \rangle$ and $J = \langle n_1, \ldots, n_s \rangle$ be monomial ideals in $\mathcal{A}$. The **product of the ideals** is then defined as

$$IJ := \left\{ \sum_{i=1}^{r} a_i m_i \cdot \sum_{j=1}^{s} b_j n_j \mid a_i, b_j \in \mathbb{R} \right\} = \langle m_i n_j \rangle_{1 \leq i \leq r, 1 \leq j \leq s}$$

$$= \langle m_1 n_1, \ldots, m_1 n_s, m_2 n_1, \ldots, m_2 n_s, \ldots, m_r n_s \rangle.$$
Perhaps not too surprisingly, we can once again formulate a proposition based on the definition above.

**Proposition 3.14.** Let \( I = \langle m_1, \ldots, m_r \rangle \) and \( J = \langle n_1, \ldots, n_s \rangle \) be monomial ideals in \( \mathcal{A} \), where \( \mathcal{A} \) is \( \mathbb{R}[x,y] \) or \( \mathbb{R}[x,y,z] \). Then \( IJ \) is also a monomial ideal.

**Proof.** As with addition, it is straightforward to prove that the product \( IJ \) is a monomial ideal, \( IJ \subseteq \mathcal{A} \). However, with multiplication it is considerably more tedious, hence we leave the proof as an exercise and jump straight into the following example by Fröberg[2]. \( \square \)

**Example 3.15.** Let \( I = \langle x^3, xy^4 \rangle \) and \( J = \langle x^2, xy^2 \rangle \) be two monomial ideals in \( \mathbb{R}[x,y] \). We calculate their product:

\[
IJ = \langle x^3, xy^4 \rangle \langle x^2, xy^2 \rangle = \langle x^5, x^4 y^2, x^3 y, x^2 y^3, x^2 y^4, x y^6 \rangle
\]

Comparing exponents among the generators quickly reveals that \( x^2 y^3|x^2 y^4 \) and \( x^3 y|x^4 y^2 \) effectively letting us simplify the generating set.

\[
IJ = \langle x^5, x^3 y^3, x^2 y^3, x^2 y^4, x y^6 \rangle = \langle x^5, x^3 y^3, x y^6 \rangle
\]

A special case of the ideal multiplication which will be of special interest to us is when \( I = J \), that is, when \( IJ = I^2 \). We see that

\[
I^2 = \langle m_1, \ldots, m_p \rangle \langle m_1, \ldots, m_p \rangle = \langle m_1^2, \ldots, m_1 m_p, m_2, \ldots, m_2 m_p, \ldots, m_p^2 \rangle = \langle m_i m_j \rangle_{1 \leq i \leq r, 1 \leq j \leq r}
\]

or more generally, for some integer \( l \) and integers \( l_i \in \{1, 2, \ldots, l \} \),

\[
I^l = \langle m_1, \ldots, m_r \rangle^l = \left( \prod_{i=1}^r m_i^{l_i} \right) \sum_{i=1}^r l_i = l
\]

When adding or (especially when) multiplying ideals we have learned that the number of generators grow. We have also seen examples when a few of the generators can be removed from the "new" generating set. This becomes particularly useful when \( l \) becomes large, as seen in this next example.

**Example 3.16.** Let \( I = \langle xy^2, y^3, xz \rangle \subseteq \mathbb{R}[x,y,z] \) be a monomial ideal. We calculate the second and third power of \( I \):

\[
I^2 = \langle xy^2, y^3, xz \rangle^2 = \langle x^2 y^4, x y^5, x^2 y^2 z, y^6, x y^3 z, x^2 z^2 \rangle
\]

Clearly, there are no generators dividing other generators in this set so no simplification is possible.

\[
I^3 = \langle xy^2, y^3, xz \rangle \langle x^2 y^4, x y^5, x^2 y^2 z, y^6, x y^3 z, x^2 z^2 \rangle
\]

\[
= \langle x^3 y^6, x^2 y^7, x^3 y^4 z, x y^8, x^2 y^5 z, x^3 y^2 z, x^2 y^7 z, x y^9, x y^6 z, x^2 y^3 z, x^3 z^2 \rangle
\]

\[
= \langle x^3 y^6, x^2 y^7, x^3 y^4 z, x y^8, x^2 y^5 z, x^3 y^2 z, x^2 y^7 z, x y^9, x y^6 z, x^2 y^3 z, x^3 z^2 \rangle
\]

\[
= \langle x^3 y^6, x^2 y^7, x^3 y^4 z, x y^8, x^2 y^5 z, x^3 y^2 z, x^2 y^7 z, x y^9, x y^6 z, x^2 y^3 z, x^3 z^2 \rangle
\]

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A total of 18 generators! Luckily we can cancel a few this time as many of them are duplicates (and therefore divisible by each other):

\[ I^1 = \langle x^3y^6, x^2y^7, x^3y^1z, xy^8, x^2y^5z, x^3y^2z^2, x^2y^9, x^2y^7z, x^3y^4z, x^2y^3z^2, x^3z^3 \rangle = \langle x^3y^6, x^2y^7, x^3y^1z, xy^8, x^2y^5z, x^3y^2z^2, x^3z^3 \rangle \]

This example aims to show how even with cancellation, the size of the generating set might quickly get out of hand as the exponent grows. This fact will be of central importance in the next chapter when we introduce the notion of an integral closure. But before then we want one more tool at our disposal, namely a way of visualizing a monomial ideal in \( \mathbb{R}[x, y] \) or \( \mathbb{R}[x, y, z] \) graphically.

### 3.3 Graphical representation

The following idea is simply a reiteration of the concept of a graphical representation of monomial ideals as presented by Cox[3] and Villarreal[4]. A monomial ideal \( I \) in three variables will have generators of the form \( x^ay^bz^c \) for some non-negative integers \( a, b, c \) since, as the reader might recall, generators of monomial ideals in \( \mathbb{R}[x, y] \) and \( \mathbb{R}[x, y, z] \) can be multiplied by the inverse of their coefficient to form power products. With this in mind, each generator can therefore be represented by lattice points \((a, b, c) \in \mathbb{N}^3\) in the 3-dimensional space \( \mathbb{R}^3 \). This will provide us with a framework for a graphical representation of all the monomials in the entire ideal \( I \). Of course, a similar approach will work equally well with an ideal in \( \mathbb{R}[x, y] \), resulting in a two-dimensional representation. For the reason of simplicity (three-dimensional images tend to get cluttered and difficult to understand) we will only present examples of this kind.

In this context the coordinate axes will each correspond to a variable and each integer point along the axes will refer to its exponent. Connecting these integer points with vertical and horizontal lines will result in a stair-like structure that clearly shows which monomials are included in the set and which are not. We attempt to display this by means of an example.

**Example 3.17.** Let \( I = \langle x^3, xy, y^2 \rangle \subset \mathbb{R}[x, y] \) be a monomial ideal. The lattice points in \( \mathbb{R}^2 \) corresponding to the generators – which we can rewrite as \( x^3y^0, x^1y^1, x^0y^2 \) – will then be \((3, 0), (1, 1)\) and \((0, 2)\) respectively.
We highlight these points in a regular coordinate system and connect the dots using the method described previously, giving us the image above. In this figure, any monomial \( m \in I \) is represented by a dot – for instance, the point \((2,3)\) corresponds to \(x^2y^3\) which can be constructed as \((xy^2 \cdot xy) \in I\) or \((x^2y \cdot y^2) \in I\).

So what about, for instance, the monomial \(2x^2y^3\)? It is clearly included in \(I\) and as a monomial it should be represented in the above figure. Recall our previously touched-upon idea that power products and monomials in polynomial ideals over \(\mathbb{R}\) are closely related due to the fact that one can always be made into the other, given the multiplicative properties of an ideal. Therefore, in the graphical representation, each dot will actually correspond not only to a power product but also to all monomials created by multiplying that power product with a real nonzero constant. So in fact all monomials \(ax^2y^3 \in I\) where \(a \in \mathbb{R}\), including \(2x^2y^3\), is visualized by the point \((2,3)\).

While the representation is restricted to monomials only (recall that \(I\) also contains linearly combined polynomials) it provides us with useful insight into the structure of \(I\), in particular which monomials are not included. In our case, these are the power products corresponding to \((1,0)\), \((2,0)\) and \((0,1)\) meaning that \(x, x^2, y \notin I\).

Next we will investigate how addition and multiplication of two ideals is represented graphically.

**Example 3.18.** Let \(I = (x^3, y^2)\) and \(J = (xy)\) be ideals of \(\mathbb{R}[x,y]\). Representing these individually gives the following images.
We know that \( I + J = \langle x^3, y^2 \rangle + \langle xy \rangle = \langle x^3, xy, y^2 \rangle \) from the definition of monomial ideal addition, which happens to be the same ideal as presented in example 3.17. This means we can simply superimpose the ideal "staircases" to create a new one representing the sum \( I + J \):

Regarding ideal multiplication we can make a similar observation. By the definition we can find the points of interest by adding all the point vectors corresponding to the generators in one generating set to each and every point vector of the other generating set. This makes sense as the point vectors are directly related to the exponents of the generators that, when multiplied, has their exponents added together. We know that \( IJ = \langle x^3, y^2 \rangle \langle xy \rangle = \langle x^4y, xy^3 \rangle \) so in the context of point vectors we see this as \((3,0) + (1,1) = (4,1)\) and \((0,2) + (1,1) = (1,3)\) respectively, giving the staircase shown below.
Note that the staircase is translated further away from the origin when two ideals $I$ and $J$ are multiplied, meaning we get a larger set of monomials not included in the product ideal. This is of course also true for the special case when $I = J$ considered previously, and is particularly obvious when considering the graphical representation of $I^l$ for increasing values of $l$.

The examples displayed in this section have all been focusing on monomial ideals in two variables, but the principles can be generalized to three (or more) variables. Upon doing so one would simply add a third coordinate axis to represent the exponents of the third variable, and the resulting staircases would then become three-dimensional—somewhat reminiscent of a Rubik’s Cube missing some of its cubelets, making one corner of the cube incomplete. As we are mainly interested in three-variable monomial ideals in this thesis, it is this image we should keep in the back of our mind as we continue into the next chapter.

### 4 Operations on monomial ideals

In this finishing chapter we will do mainly three things. Firstly, we will define the monomial ideal extension touched upon earlier known as the integral closure. Secondly, we will consider two different methods of finding the integral closure of a monomial ideal; one utilizing algebraic manipulations and one utilizing the graphical representation introduced in section 3.3. Thirdly, we will attempt to simplify the resulting generating set once the integral closure has been found. The main source of reference here is Crispin Quiñonez[5] and most of the ideas presented are directly based upon theirs.

#### 4.1 Integral closure of ideals

Let us begin by stating the formal definition of the integral closure of an ideal. Note that this definition can be extended to be valid for any ring, not just an ideal, but that for our purposes this more limited version will suffice.
**Definition 4.1.** Let \( x \) be an element of a commutative ring \( R \) and let \( I \subseteq R \) be an ideal. If there exists \( a_j \in I \) such that the equation

\[
x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0
\]

is satisfied for some \( n \geq 1 \), then \( x \) is said to be **integral** over \( I \). The set of all such \( x \in R \) is called the **integral closure** of \( I \) and is denoted \( \bar{I} \).

**Remark 4.2.** Clearly \( I \) is always contained in \( \bar{I} \), since for any \( x \in I \) we can choose \( a_0 = -x \) and see that \( x^1 + a_0 = x - x = 0 \), implying that \( x \in \bar{I} \).

As the reader might have guessed from the finishing words of the preamble of this chapter, it turns out that the integral closure of a monomial ideal is, too, a monomial ideal. We omit the proof here and instead simply state a slightly modified version of proposition 2.7 found in Crispin Quiñonez\[5\].

**Proposition 4.3.** Let \( A \) be the polynomial ideal \( \mathbb{R}[x, y] \) or \( \mathbb{R}[x, y, z] \) and let \( I \subseteq A \) be a monomial ideal. Then the integral closure \( \bar{I} \) is also a monomial ideal and

\[
\bar{I} = \langle m | m^l \in I^l \text{ for some } l \geq 1 \rangle
\]

for power products \( m \).

**Proof.** The reader may consult Crispin Quiñonez\[5\] for further information.

**Remark 4.4.** Proposition 4.3 is only valid for monomial ideals – not for ideals in general.

From above facts it becomes evident that in order to find \( \bar{I} \) we need to collect all power products that, to some power \( l \), is included in \( I^l \) and let them form a generating set. Obviously this means that all generators of \( I \) will also be in a generating set of \( \bar{I} \). In other words, the real challenge lies in finding the generators of \( \bar{I} \) which are not in the set generating \( I \). In the next section we will look into two possible approaches to this problem.

It is also worth mentioning that the integral closure of an ideal is unique, while its generating set (as discussed earlier) is not. Because of this we wish to not only find the integral closure of any monomial ideal in three variables, but also to represent it with a generating set that is as manageable and “nice” as possible.

### 4.2 Finding the integral closure

Determining \( \bar{I} \) of an arbitrary monomial ideal will be a new kind of operation, very different from the basic arithmetics covered earlier. As opposed to addition and multiplication, calculating the integral closure is obviously not a binary operation as it does not require two separate operands (monomial ideals). Here we suggest two different methods of how it can be achieved.
4.2.1 The algebraic method

One, and possibly the most straightforward, way of determining the generators of the integral closure is to directly use proposition 4.3 above. That is, by considering all possible power products not in \( I \) raised to the power \( l \) and see which of them are included in \( I^l \). This gives a set of inequalities for each power product that can be solved algebraically.

Because we focus on monomial ideals in three variables, any power product will be of the form \( m = x^a y^b z^c \) where \( a, b, c \) are non-negative integers. Recall that if we have an ideal \( I = \langle m_1, \ldots, m_r \rangle \subseteq \mathbb{R}[x,y,z] \) with power products \( m_i \), we can write \( I^l \) as \( \langle m_1^l, \ldots, m_r^l | l_1 + \cdots + l_r = l \rangle \), and that any element of \( I^l \) will be a linear combination of these generators with polynomial coefficients from \( \mathbb{R}[x,y,z] \). This means that if \( m^l \in I^l \), then \( m^l \) can be written as such a combination consisting of only one term with a power product coefficient \( \tilde{m} \):

\[
m^l = (x^a y^b z^c)^l = x^{al} y^{bl} z^{cl} = \tilde{m} m_1^{l_1} \cdots m_r^{l_r}
\]

Now let \( x_i, y_i, z_i \) be the exponents of \( x, y \) and \( z \) respectively in \( m_i \). We see that if the three inequalities

\[
\begin{cases}
al & \geq x_1 l_1 + \cdots + x_r l_r \\
bl & \geq y_1 l_1 + \cdots + y_r l_r \\
c_l & \geq z_1 l_1 + \cdots + z_r l_r
\end{cases}
\]

are satisfied for some \( l = l_1 + \cdots + l_r \geq 1 \), then only then is \( m^l \) an element of \( I^l \) and therefore \( m \) is an element of the integral closure \( \bar{I} \). We get inequalities, not equalities, because \( \tilde{m} \in \mathbb{R}[x,y,z] \) can be any arbitrary power product. Of course, when considering the case of two variables the above also holds with the minor difference that the last inequality is ignored. This next example will illustrate just that.

**Example 4.5.** Let \( I = \langle x^3, y^2 \rangle \subset \mathbb{R}[x,y] \) be a monomial ideal in two variables. From the generators we conclude that \( x, x^2, y, xy \) and \( x^2 y \) are the only power products not belonging to \( I \). Our aim is then to determine which, if any, of these belongs to the integral closure \( \bar{I} \). Referring to proposition 4.3 we note that \( \bar{I} \) of \( I \) is given as the monomial ideal generated by all the power products satisfying

\[
(x^a y^b)^l = x^{al} y^{bl} = \tilde{m} (x^{3l_1} y^{2l_2} = \tilde{m} x^{3l_1} y^{2l_2}
\]

for some \( l \geq 1 \) where \( l_1 \) and \( l_2 \) are non-negative integers and \( l_1 + l_2 = l \).

It is fairly obvious that the "pure" power products \( x, x^2 \) and \( y \) cannot be expressed in this way: for instance, consider the power product \( y \). For \( y^l = x^0 y^l = \tilde{m} x^{3l_1} y^{2l_2} \) we have that \( 0 \geq 3l_1 \) and \( l \geq 2l_2 \). From the first inequality it follows that \( l_1 = 0 \), which means that \( l_2 = l \). But the second inequality then states that \( l \geq 2l \); an obvious contradiction. The same argument holds true for \( x \) and \( x^2 \), and so \( x, x^2, y \notin \bar{I} \).

Regarding the mixed power products \( xy \) and \( x^2 y \) things are not quite as obvious. We solve the inequalities separately for each of them. For \( xy \) we have
\[ a = b = 1, \text{ giving us the following inequalities:} \]

\[
\begin{aligned}
\{ & l \geq 3l_1 \\
& l \geq 2l_2 
\end{aligned}
\]

We rewrite the second inequality using \( l = l_1 + l_2 \) and get

\[ l \geq 2l_2 = 2(l - l_1) = 2l - 2l_1 \iff l \leq 2l_1. \]

Combining this with the first inequality yields

\[ 3l_1 \leq l \leq 2l_1 \]

which is impossible since \( l \geq 1 \) and \( l_1 \) is nonnegative. It follows that \((xy)^l \notin I^l\) for all \( l \geq 1 \) and therefore \( xy \notin \bar{I} \). For the last power product \( x^2y \) we have \( a = 2 \) and \( b = 1 \):

\[
\begin{aligned}
\{ & 2l \geq 3l_1 \\
& l \geq 2l_2 
\end{aligned}
\]

Using the same approach as above:

\[ l \geq 2l_2 = 2(l - l_1) = 2l - 2l_1 \iff l \leq 2l_1 \iff 2l \leq 4l_1 \]

meaning that

\[ 3l_1 \leq l \leq 4l_1. \]

Since \( l \geq 1 \) we first try \( l = 1 \). But this means that \( l_1 = 0 \) or \( l_1 = 1 \) resulting in either \( 0 \leq 2 \leq 0 \) or \( 3 \leq 2 \leq 4 \), none of which holds. However, by letting \( l = 2 \) and \( l_1 = 1 \) we get \( 3 \leq 4 \leq 4 \) and the two-sided inequality is satisfied! Apparently, \((x^2y)^2 \in I^2\); let us verify this. From the definition of monomial ideal multiplication we know that \( I^2 = \langle x^6, x^3y^2, y^4 \rangle \). Furthermore \((x^2y)^2 = x^4y^2\), which actually can be written as \( x^4y^2 = x \cdot x^3y^2 \) and so it must be an element of \( I^2 \). The conclusion is therefore that \( x^2y \in \bar{I} \).

Having dismissed or confirmed all of the power products not in \( I \) to be elements for the integral closure \( \bar{I} \) we can now describe \( \bar{I} \) explicitly. We found that \( x, x^2, y, xy \notin \bar{I} \), while \( x^2y \in \bar{I} \), which combined with the fact that \( I \subseteq \bar{I} \) gives us

\[ \bar{I} = \langle x^3, x^2y, y^2 \rangle. \]

For clarity we also provide an example in three variables, but due to the vast number of power products necessary to check we refrain from considering all of them here and leave the rest as an exercise.

**Example 4.6.** Let \( I = \langle x^2, xy^3, y^4, z^2 \rangle \subseteq \mathbb{R}[x, y, z] \). Below we list the power products not present in \( I \):

pure power products: \( x, y, y^2, y^3, z \)

mixed power products: \( xy, xy^2, xz, yz, y^2z, y^3z, xyz, xy^2z \).
By the arguments presented in the previous example the pure power products will trivially not be in \(\bar{I}\). Out of the rest we focus on the last two: \(xyz\) and \(xy^2z\). For the former we get:

\[
\begin{cases}
  l \geq 2l_1 + l_2 \\
  l \geq 3l_2 + 4l_3 \\
  l \geq 2l_4
\end{cases}
\]

With \(2l_4 = 2(l - l_1 - l_2 - l_3) = 2l - 2l_1 - 2l_2 - 2l_3\), we rearrange the third inequality

\[
\begin{cases}
  l \geq 2l_1 + l_2 \\
  l \geq 3l_2 + 4l_3 \\
  l \leq 2l_1 + 2l_2 + 2l_3
\end{cases}
\]

that combined with the first and second inequality yields

\[
\begin{cases}
  2l_1 + l_2 \leq l \leq 2l_1 + 2l_2 + 2l_3 \\
  3l_2 + 4l_3 \leq l \leq 2l_1 + 2l_2 + 2l_3
\end{cases}
\]

Letting \(l = 2\) and \(l_1 = l_4 = 1\) means that \(l_2 = l_3 = 0\) and

\[
\begin{cases}
  2 \leq 2 \leq 2 \\
  0 \leq 2 \leq 2
\end{cases}
\]

which certainly holds, and so \((xyz)^2 \in I^2\) and \(xyz \in \bar{I}\). Following the exact same formula for \(xy^2z\) eventually gives the system

\[
\begin{cases}
  2l_1 + l_2 \leq l \leq 2l_1 + 2l_2 + 2l_3 \\
  3l_2 + 4l_3 \leq 2l \leq 4l_1 + 4l_2 + 4l_3
\end{cases}
\]

Here we can for example choose \(l = 2\) and \(l_1 = l_3 = 1\) (implying that \(l_2 = l_4 = 0\)) to satisfy the system since

\[
\begin{cases}
  2 \leq 2 \leq 4 \\
  4 \leq 4 \leq 8
\end{cases}
\]

is valid. We conclude that also \(xy^2z \in \bar{I}\).

It’s easy to confirm these results by plugging in the values of \(l\) and \(l_i\) in both cases, as

\[
(xy^2z)^2 = x^2 y^4 z^2 = m_2 (x^2)^1 (y^4)^1 (z^2)^1 = m_2 x^2 y z^2
\]

and

\[
(xy^2z)^2 = x^2 y^4 z^2 = m_2 (x^2)^1 (y^4)^0 (z^2)^0 = m_2 x^2 y z^2
\]

are satisfied for \(m_1 = y^2\) and \(m_2 = z^2\), which obviously are monomials and \(m_1, m_2 \in \mathbb{R}[x, y, z]\).
4.2.2 The graphical method

Another way of identifying $\bar{I}$ is to do so graphically. To do this we attempt to translate each step of the algebraic method presented above into a graphical one. Let us once again start with a monomial ideal in $\mathbb{R}[x, y, z]$ – denote it by $I = \langle m_1, \ldots, m_r \rangle$. We know how to visualize this ideal (or at least the monomials of it) and we want to see what its integral closure will look like in the same graphical representation.

In this setting, what happens to a power product $m \in I$ when raised to the power $l$? If we write $m$ as $x^a y^b z^c$, we know that it is represented by the lattice point $(a, b, c)$. Moreover, when taking $m$ to the $l$th power we get $m^l = (x^a y^b z^c)^l = x^{al} y^{bl} z^{cl}$ which naturally corresponds to the point $(al, bl, cl) = l(a, b, c)$ in $\mathbb{R}^3$. In other words, the point vector of $m$ is multiplied by the scalar $l$ to give a new vector pointing at $m^l$. This also explains why ideals are translated away from the origin when multiplied, as seen earlier.

Now let us consider the inequalities from the algebraic method.

\[
\begin{cases}
al \geq x_1 l_1 + \cdots + x_r l_r \\
bl \geq y_1 l_1 + \cdots + y_r l_r \\
cl \geq z_1 l_1 + \cdots + z_r l_r
\end{cases}
\]

On the left hand side we have the $x$, $y$ and $z$ coordinates in $\mathbb{R}^3$, all multiplied by $l$ as discussed above. Dividing both sides by $l$ yields

\[
\begin{cases}
a \geq \frac{l_1}{l} x_1 + \cdots + \frac{l_r}{l} x_r = \lambda_1 x_1 + \cdots + \lambda_r x_r \\
b \geq \frac{l_1}{l} y_1 + \cdots + \frac{l_r}{l} y_r = \lambda_1 y_1 + \cdots + \lambda_r y_r \\
c \geq \frac{l_1}{l} z_1 + \cdots + \frac{l_r}{l} z_r = \lambda_1 z_1 + \cdots + \lambda_r z_r
\end{cases}
\]

where $\lambda_i \in \mathbb{Q}$ and $0 \leq \lambda_i \leq 1 = \sum_{i=1}^{r} \lambda_i$. We write the system on vector form:

\[
(a, b, c) \geq (\lambda_1 x_1 + \cdots + \lambda_r x_r, \lambda_1 y_1 + \cdots + \lambda_r y_r, \lambda_1 z_1 + \cdots + \lambda_r z_r) = (\lambda_x(x_1, y_1, z_1) + \cdots + \lambda_r(x_r, y_r, z_r))
\]

Imagine for a moment that the inequality was an equality instead. If this was the case, the above expression would describe an $r$-sided, solid, three-dimensional polygon. At every corner $(x_i, y_i, z_i)$ of the polygon, $a$, $b$ and $c$ would be integers and therefore correspond to monomials $m^l$ in $I^l$. There could possibly also be integer lattice points inside the polygon which then also would represent such monomials.

So what changes in this image when there is an inequality instead? Since we have a lower, closed bound for $(a, b, c)$ we know that the sides of the polygon facing the origin would remain. The other sides, however, would expand infinitely
along the positive $x$, $y$- and $z$-axes, resulting in a semi-infinite, three-dimensional space. All points of $\mathbb{N}^3$ in this space would symbolize all the monomials $m$ that, when raised to the power of $l$, is in $I^l$, and therefore also all $m \in \bar{I}$. Of course, many of these points already belong to $I$ and are as such trivially included in $\bar{I}$, hence we are interested in the lattice points inside the expanded polygon that are not in $I$. And by the definition of the integral closure of a monomial ideal, it is exactly these points we need to add to the generating set of $I$ to get the generating set of $\bar{I}$.

To get a more intuitive way to think about this, recall the Rubik’s cube analogy we introduced at the end of the previous chapter. The cube, which has an infinite number of cubelets with a side length of 1, is first placed with one of its corners at the origin so that it fills the first octant of $\mathbb{R}^3$. Then some of the cubelets near the origin are removed, each cube (since the side length is 1) representing a monomial not included in the ideal we are interested in. We get the semi-infinite cube with a severed corner described earlier; the graphical representation of the monomial ideal $I$ in $\mathbb{R}[x, y, z]$. Now to find $\bar{I}$, imagine wrapping the entire cube tightly in plastic. By adding all the lattice points that are inside the plastic but outside the cube to the set generating $I$ (the cube), we end up with the wanted generating set of $\bar{I}$.

**Example 4.7.** We attempt to derive the conclusion of example 4.5 graphically. Let $I = (x^3, y^2) \subset \mathbb{R}[x, y]$ be a monomial ideal in two variables and draw its graphical representation.

Next we ”wrap the ideal in plastic”, which in two dimensions is equivalent to stretching a rubber band around the entire set. In addition, we also highlight the monomials not included in $I$ and get the following figure:
It is immediate that the only monomial inside of the rubber band that is not already included in $I$ is $x^2y$, so adding this to the generating set of $I$ gives us the integral closure: $\bar{I} = \langle x^3, x^2y, y^2 \rangle$.

We refrain from giving a similar example for a monomial ideal in three variables, partly because the procedure is almost identical to the above example and partly because it is a difficult task to visualize a three-dimensional ideal clearly. Yet it is still apparent that the graphical representation is of great aid when seeking the integral closure $\bar{I}$. It provides us with a convenient way of eliminating some – in some cases all – of the generator candidates $m$ that otherwise needs to be confirmed ($m^l \in \bar{I}^l$) or dismissed ($m^l \notin \bar{I}^l$) algebraically. Furthermore it can theoretically be generalized to higher-dimensional systems, although we would then need to utilize methods from linear algebra since conventional visualization no longer would be possible.

We have seen that two different generating sets $\{m_1, \ldots, m_r\}$ and $\{n_1, \ldots, n_s\}$ might produce the same monomial ideal $I = \langle m_1, \ldots, m_r \rangle = \langle n_1, \ldots, n_s \rangle$, and that we handle this by finding the minimal set generating $I$. The generators of the minimal set are always included in the generating set of $\bar{I}$ along with some other monomial(s) – assuming $I \neq \bar{I}$. Therefore, since there is no distinction whether the generators of $\bar{I}$ originally were generators of $I$ or not, we could find another monomial ideal $J$ such that $J \neq I$ and $\bar{J} = \bar{I}$. In other words there might be different monomial ideals having the same integral closure, and the most simple among those ideals is called the minimal monomial reduction.

### 4.3 Minimal monomial reductions

We shall not go too deep into the theory, but rather keep a short and general discussion regarding some of the aspects of reductions. We generally want our ideals to be as simple as possible, but what does “simple” mean? For one, we obviously want the generating sets to be minimal. But given a situation where multiple monomial ideals, all represented by minimal generating sets, share the same integral closure we are forced to evaluate simplicity based on other properties of the set. We can for example do this by comparing the total
degrees of the generators, or by comparing the total number of generators. In our case we resort to the latter, which is directly connected to the number of steps in our graphical staircase. So when the number of steps is minimized (without it affecting the integral closure) we consider the generating set to be as simple as possible, and so the ideal generated by it is the minimal reduction.

Usually we only have one monomial ideal to start out with, however, and in such case we cannot simply compare the cardinality of generating sets. Instead we need to find the integral closure and figure out which monomials are necessary and sufficient as generators for the ideal to have that exact integral closure. Here is where the graphical representation becomes even more valuable, as it provides us with the means to find those generators without the need to calculate the integral closure explicitly. This is important because, as shown in this thesis, finding the integral closure algebraically can be significantly troublesome. From how the graphical representation and the integral closure are interconnected, it follows that the desired monomials are exactly those represented by points changing the direction of the plastic wrap. The set of those specific monomials will be the generating set of the minimal monomial reduction.

While this might sound easy enough, it is still a somewhat limited solution. It works exceptionally well in two and even three dimensions; we can easily draw the ideals and literally "plastic wrap" them. But what happens when we move into four or more dimensions? The analogy still holds true, but since we cannot visualize it any longer we lose the main advantage it provided us with in the first place. As mentioned earlier, one solution could then be the introduction of linear algebra into the mix – but that is for another thesis to investigate.

4.4 Finishing words

Monomial ideals have the unique property as subsets of polynomial rings to be visualized graphically. Such representation facilitates new perspectives on operations on these ideals and can help us understand the inner workings behind them. Moreover, it can aid in defining new operations and thus widening the arsenal of tools at our disposal, ultimately bringing us deeper insight. When seen in this light, one realizes that the ideas and methods covered in this thesis are just fragments of a foundation waiting to be further built upon.
References


