CONVERGENCE OF THE RPEM AS APPLIED TO HARMONIC SIGNAL MODELING *

Emad Abd-Elrady
Department of Systems and Control,
Information Technology, Uppsala University,
P.O. Box 27, SE-751 03 Uppsala, Sweden
ea@SysCon.uu.se

ABSTRACT

Arbitrary periodic signals can be estimated recursively by exploiting the fact that a sine wave passing through a static nonlinear function generates a spectrum of overtones. The estimated signal model is hence parameterized as a real wave with unknown period in cascade with a piecewise linear function. The driving periodic wave can be chosen depending on any prior knowledge. The performance of a recursive Gauss-Newton prediction error identification algorithm for joint estimation of the driving frequency and the parameters of the nonlinear output function is therefore studied. A theoretical analysis of local convergence to the true parameter vector as well as numerical examples are given. Furthermore, the Cramér-Rao bound (CRB) is calculated in this report.

1 INTRODUCTION

The problem of retrieving noisy sinusoidal signals has received a great deal of attention in the literature, see for example [1], [2], and [3]. The algorithm of [4], which is based on the same idea as the algorithm of this report, has the additional property to give information on the underlying nonlinearity, in cases where the overtones are generated by nonlinear imperfections in the system. In some cases it may also be known that the modeled signal is closer to other signals than to sine waves. The existing schemes may then be less efficient than methods utilizing priors like the method of [4] and in this report. The reason is of course the freedom that exists when selecting the driving wave. The method studied here utilizes the fact that a sine wave passing through a static nonlinear function produces a harmonic spectrum of overtones. Hence, a periodic function with unknown fundamental frequency in cascade with a parameterized and unknown nonlinear function can be used as a signal model for an arbitrary periodic signal as shown in Fig. 1. In this report, the nonlinearity is chosen to be piecewise linear exactly as in [4], with the estimated parameters being the function values in a set of user chosen grid points. The difference is that in [4] the differential static gain was fixed in the linear block. Here, however, the differential static gain is fixed in the nonlinear block. This requires a slight modification of the parameterization of the nonlinearity. The

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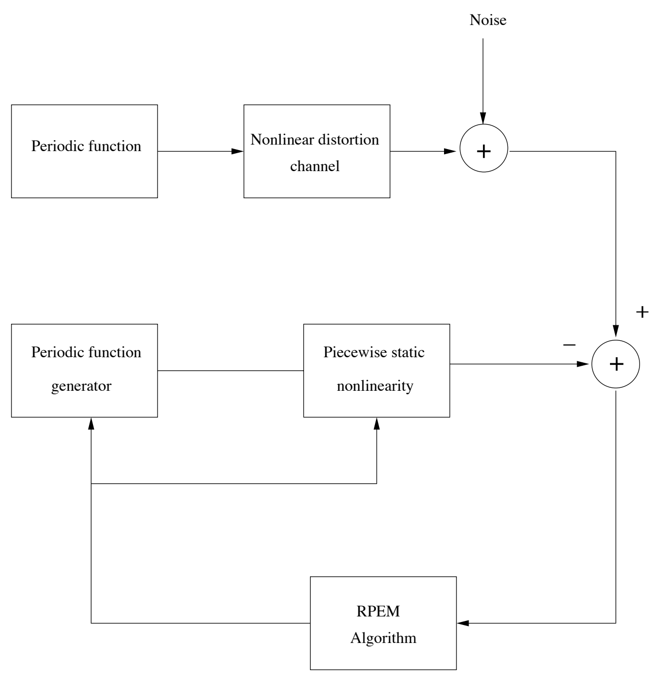

Figure 1: Harmonic signal modeling. The modeled signal *need not* be generated by a cascade structure as shown here, it is sufficient that the signal is periodic.

performance of the proposed RPEM of [4], for joint estimation of the driving frequency and the parameters of the nonlinear output function was studied by numerical examples in [5] and convergence to the true parameter vector was *experimentally observed*. Furthermore, slight modifications in the propsed algorithm was done to improve the ability to track fundamental frequency variations, see the numerical examples of [5].

The modifications in the proposed RPEM are obtained by introducing an interval in the nonlinear block with fixed gain. The modification in the convergence analysis is, however, substantial and allows a complete treatment of the local convergence properties of the algorithm. *This is the main reason for the modification.* The convergence analysis is based on Ljung's method with an associated differential equations [6]. The analysis follows the papers [7] and [8] closely.

The contributions of this report can hence be summarized as follows: Conditions for local convergence to the true parameter vector are derived with averaging theory. Furthermore, the CRB is calculated for the modified algorithm. Finally, the performance of the modified RPEM for joint estimation of the driving frequency and the parameters of the nonlinear output function is studied by numerical examples to investigate convergence to the true parameter vector and the ability of the algorithm to track fundamental frequency variations.

The report is organized as follows. In section 2, a review of the algorithm introduced in [4] is given. Modifications in the algorithm of [4] are discussed in section 3. A local convergence analysis is presented in section 4. Section 5 presents the derivation of the CRB for the modified algorithm. In section 6, numerical examples are given. Conclusions are given in section 7.

2 REVIEW OF THE ALGORITHM OF [4]

In order to define the parametric signal model, a periodic function being the input to the estimated static nonlinearity is needed. This function reflects any prior knowledge that is available. The driving input signal $\widehat{u}(t,\omega)$ is hence modeled as

$$\widehat{u}(t,\omega) = \Lambda(\omega t) \tag{1}$$

where t denotes discrete time, $\omega \in [0,\pi]$ denotes the unknown normalized angular frequency: $\omega = 2\pi f/f_s$ where f is the frequency, and f_s is the sampling frequency. The fact that $\Lambda(.)$ is periodic now means

A1) $\Lambda(\omega(t+\frac{2k\pi}{\omega})) = \Lambda(\omega t), \qquad k \in Z.$

Then let one complete period of $\Lambda(\omega t)$ be divided into L disjoint intervals $I_j, j = 1,\cdots,L$, and assume

A2) $\Lambda(\omega t)$ is a monotone function of ωt on each $I_j,\ j = 1,\cdots,L$.

A2) is introduced to avoid restrictions that would reduce the generality of the approach. This can be explained as follows. Assume that one static nonlinearity is used and $\Lambda(\omega t) = \sin(wt)$ then the model output $f_1(\theta_1, \sin(wt))$ is obtained. If the unknown parameter vector θ_1 of the nonlinear block is fixed, $f_1(\theta_1, \sin(w(\pi/w - t))) = f_1(\theta_1, \sin(wt))$ holds for all t. This means that the model signal in half of the time intervals of length π/w is given by the signal in the remaining time intervals.

In what follows, a note on notation is necessary. Hence by $\widehat{u}(t,\omega) \in I_j$ we mean that the phase ωt is such that I_j is in effect. The reason for this notation is to highlight the switching between nonlinearities in different intervals. However, *it is always the underlying phase of the driving wave that controls this switching.*

Then with $f_j(\theta_j, \Lambda(\omega t))$ denoting the nonlinearity to be used in I_j, the model output becomes

$$\begin{aligned} \widehat{y}(t,\omega,\theta) &= f_j(\theta_j, \Lambda(\omega t)),\ \Lambda(\omega t) \in I_j,\ j = 1,\cdots,L \\ \theta &= (\theta_1^T \cdots \theta_L^T)^T. \end{aligned} \tag{2}$$

A piecewise linear model is used for the parameterization of $f_j(\theta_j, \Lambda(\omega t))$, cf. [4], [7] and [9]. Define a set of grid points

$$\begin{aligned} \mathrm{grid}_j &= (u_1^j\ u_2^j \cdots u_{k_j}^j),\ j = 1,\cdots,L \\ u_1^j &= \inf_{\gamma \in I_j} \gamma,\ j = 1,\cdots,L \\ u_{k_j}^j &= \sup_{\gamma \in I_j} \gamma,\ j = 1,\cdots,L. \end{aligned} \tag{3}$$

The parameters θ_j are then chosen as the values of $f_j(\theta_j, \widehat{u}(t,\omega))$ in the grid points

$$\begin{aligned} \theta_j &= (f_1^j\ f_2^j \cdots f_{k_j}^j)^T,\ j = 1,\cdots,L \\ f_j(\theta_j, u_i^j) &= f_i^j,\ i = 1,\cdots,k_j,\ j = 1,\cdots,L. \end{aligned} \tag{4}$$

A piecewise linear function of $\widehat{u}(t,\omega)$ can now be constructed from the linear segments

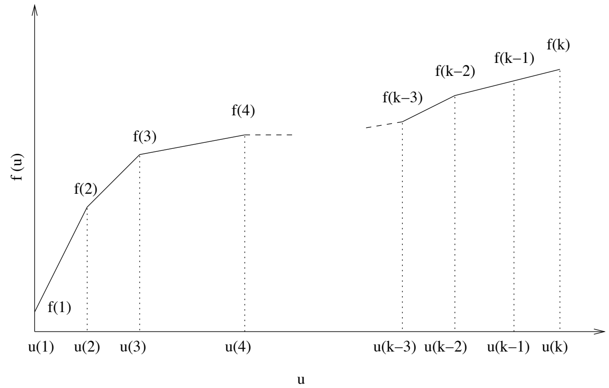

Figure 2: Grids points, parameters and resulting piecewise linear model.

with end points in (u_{i-1}^j, f_{i-1}^j) and (u_i^j, f_i^j) as shown in Fig. 2. A recursive Gauss-Newton prediction error method (RPEM) then follows by a minimization of

$$V(\omega,\theta) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E[\varepsilon^2(t,\omega,\theta)] \tag{5}$$

where $E[.]$ denotes expectation. Here, $\varepsilon(t,\omega,\theta) = y(t) - \widehat{y}(t,\omega,\theta)$ denotes the prediction error and $y(t)$ is the measured signal to be modeled. The negative gradient of $\widehat{y}(t,\omega,\theta)$ is needed in the formulation of the recursive algorithm. It is given by (for $\widehat{u}(t,\omega) \in I_j, j = 1, \cdots, L$)

$$\psi(t,\omega,\theta) = \begin{pmatrix} \frac{\partial f_j(.)}{\partial \widehat{u}} \psi_l(t) & 0\cdots 0 & \frac{\partial f_j(.)}{\partial \widehat{\theta}_j} & 0\cdots 0 \end{pmatrix}^T \tag{6}$$

where

$$\begin{aligned} \frac{\partial f_j(.)}{\partial \widehat{u}} &= \frac{\partial f_j(\theta_j, \widehat{u}(t,\omega))}{\partial \widehat{u}}, \ \widehat{u}(t,\omega) \in I_j, j = 1, \cdots, L \\ \frac{\partial f_j(.)}{\partial \widehat{\theta}_j} &= \frac{\partial f_j(\theta_j, \widehat{u}(t,\omega))}{\partial \widehat{\theta}_j}, \ \widehat{u}(t,\omega) \in I_j, j = 1, \cdots, L \\ \psi_l(t) &= t\frac{d\Lambda(\phi)}{d\phi} \mid_{\phi=\omega t} . \end{aligned} \tag{7}$$

The gradient components of $\frac{\partial f_j(.)}{\partial \widehat{\theta}}$ for a piecewise linear model are shown in Fig. 3. The RPEM that appear in [4] was derived as in [6] and [8]. It is given by

$$\begin{aligned} &\varepsilon(t) = y(t) - \widehat{y}(t) \\ &\lambda(t) = \lambda_o \lambda(t-1) + 1 - \lambda_o \\ &S(t) = \psi^T(t) P(t-1) \psi(t) + \lambda(t) \\ &P(t) = (P(t-1) - P(t-1)\psi(t) S^{-1}(t) \psi^T(t) P(t-1))/\lambda(t) \end{aligned}$$

$$\begin{pmatrix} \widehat{\omega}(t) \\ \widehat{\theta}_1(t) \\ \vdots \\ \widehat{\theta}_L(t) \end{pmatrix} = \left[\begin{pmatrix} \widehat{\omega}(t-1) \\ \widehat{\theta}_1(t-1) \\ \vdots \\ \widehat{\theta}_L(t-1) \end{pmatrix} + P(t)\psi(t)\varepsilon(t) \right]_{D_M} \tag{8}$$

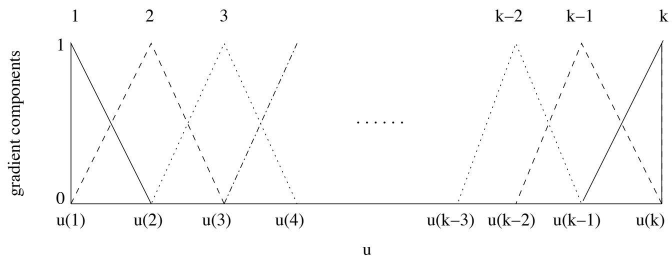

Figure 3: The gradient components for (8)

$\widehat{u}(t+1) = \Lambda(\widehat{\omega}(t)(t+1))$
$\psi_l(t+1) = (t+1)\frac{d\Lambda(\phi)}{d\phi}\mid_{\phi=\widehat{\omega}(t)(t+1)}$
when $\widehat{u}(t+1) \in I_j,\ j = 1, \cdots, L$
  when $\widehat{u}(t+1) \in [u_i^j, u_{i+1}^j],\ i = 1, \cdots, k_j - 1$
    $\widehat{y}(t+1) = \frac{\widehat{f}_i^j(t)u_{i+1}^j - \widehat{f}_{i+1}^j(t)u_i^j}{u_{i+1}^j - u_i^j} + \frac{\widehat{f}_{i+1}^j(t) - \widehat{f}_i^j(t)}{u_{i+1}^j - u_i^j}\widehat{u}(t+1)$
    $\frac{\partial f_j(.)}{\partial \widehat{u}} = \frac{\widehat{f}_{i+1}^j(t) - \widehat{f}_i^j(t)}{u_{i+1}^j - u_i^j}$
    $\frac{\partial f_j(.)}{\partial \widehat{f}_i^j} = \frac{u_{i+1}^j - \widehat{u}(t+1)}{u_{i+1}^j - u_i^j}$
    $\frac{\partial f_j(.)}{\partial \widehat{f}_{i+1}^j} = \frac{\widehat{u}(t+1) - u_i^j}{u_{i+1}^j - u_i^j}$
    $\frac{\partial f_j(.)}{\partial \widehat{f}_l^j} = 0, \qquad l \neq i, i+1$
  end
$\frac{\partial f_j(.)}{\partial \widehat{\theta}_j} = \begin{pmatrix} \frac{\partial f_j(.)}{\partial \widehat{f}_1^j} \cdots \frac{\partial f_j(.)}{\partial \widehat{f}_{k_j}^j} \end{pmatrix}$
$\psi(t+1) = \begin{pmatrix} \frac{\partial f_j(.)}{\partial \widehat{u}}\psi_l(t+1) & 0 \cdots 0 & \frac{\partial f_j(.)}{\partial \widehat{\theta}_j} & 0 \cdots 0 \end{pmatrix}^T$
end.

where D_M indicates that the algorithms described in [6] are used to keep the predictor in the model set.

It was shown in [4] that the minimum of $V(\omega, \theta)$ is unaffected by *colored* measurement disturbances. Also, the CRB for $(\omega\ \theta^T)^T$ was given in that paper. As shown also in [5], the algorithm works well and can be easily modified for tracking. However, it seems to be difficult to analyse. This, and a desire to investigate and compare with alternatives, is the main motivation for the development in the subsequent section.

3 MODIFIED ALGORITHM

In order to fix the static gain in an amplitude subinterval I_o *contained in exactly one* of the nonlinear blocks, the driving input signal in this case is modeled as

$$\widehat{u}(t, X, \omega) = X\Lambda(\omega t) \tag{9}$$

where X is a (possibly time varying) parameter recursively estimated to allow the linear block of the model to adapt its static gain so that the data in I_o can be explained. Choosing I_o to

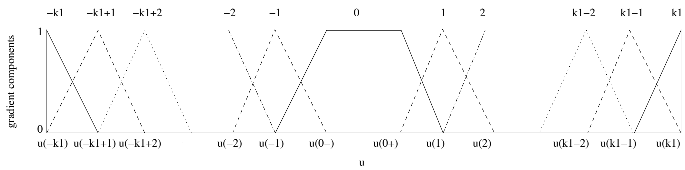

Figure 4: The gradient components for I_1 (Assuming $k_1^+ = k_1^- = k_1$).

be contained in the first interval I_1, $f_1(\theta_1, \widehat{u}(t, X, \omega))$ is defined as is done in [7] to become

$$f_1(f_o, \theta_1, \widehat{u}(t, X, \omega)) = K_o \widehat{u}(t, X, \omega) + f_o, \quad \widehat{u}(t, X, \omega) \in I_o \subset I_1 \tag{10}$$

so

$$\frac{\partial f_1(f_o, \theta_1, \widehat{u}(t, X, \omega))}{\partial \widehat{u}} = K_o, \qquad \widehat{u}(t, X, \omega) \in I_o. \tag{11}$$

Here K_o is a user chosen constant. Also, the grid points become

$$\text{grid}_j = \begin{cases} (u^1_{-k_1^-}\, u^1_{-k_1^-+1} \cdots u_{o^-}\, u_{o^+} \cdots u^1_{k_1^+-1}\, u^1_{k_1^+}), & j = 1 \\ (u^j_{-k_j^-}\, u^j_{-k_j^-+1} \cdots u^j_{-1} u^j_1 \cdots u^j_{k_j^+-1}\, u^j_{k_j^+}), & j = 2, \cdots, L. \end{cases} \tag{12}$$

Thus equation (4) is transformed into

$$\begin{aligned} \theta_j &= (f^j_{-k_j^-} \cdots f^j_{-1} f^j_1 \cdots f^j_{k_j^+})^T, \qquad j = 1, \cdots, L. \\ f_j(\theta_j, u^j_i) &= \begin{cases} K_o \widehat{u}(t, X, \omega) + f_o, & \widehat{u}(t, X, \omega) \in I_o,\ j = 1 \\ f^j_i, & i = -k_j^-, \cdots, -1, 1, \cdots, k_j^+,\ j = 1, \cdots, L. \end{cases} \end{aligned} \tag{13}$$

In this case there are no parameters corresponding to u_{o^-} and u_{o^+}, since

$$\begin{aligned} f_1(f_o, \theta_1, u_{o^-}) &= K_o u_{o^-} + f_o \\ f_1(f_o, \theta_1, u_{o^+}) &= K_o u_{o^+} + f_o. \end{aligned} \tag{14}$$

The parameters vector takes the form

$$\begin{aligned} \theta &= (\theta_l^T\ \theta_n^T)^T \\ \theta_l &= (X\ \omega)^T \\ \theta_n &= (f_o\ \tilde{\theta}_n^T)^T \\ \tilde{\theta}_n &= (\theta_1^T \cdots \theta_L^T)^T \\ \theta_j &= (f^j_{-k_j^-} \cdots f^j_{-1}\ f^j_1 \cdots f^j_{k_j^+})^T, \qquad j = 1, \cdots, L. \end{aligned} \tag{15}$$

In this case, the gradient components in the interval I_1 are shown in Fig. 4 and the gradient components for the other intervals are similar to Fig. 3. Taking into account that $\psi_l(t)$ in this case is defined as

$$\psi_l(t) = \begin{pmatrix} \Lambda(\phi)\mid_{\phi=\omega t} & Xt\frac{d\Lambda(\phi)}{d\phi}\mid_{\phi=\omega t} \end{pmatrix}^T \tag{16}$$

the RPEM algorithm becomes after comparing with [4] and [7]

$$\varepsilon(t) = y(t) - \widehat{y}(t)$$
$$\lambda(t) = \lambda_o \lambda(t-1) + 1 - \lambda_o$$
$$S(t) = \psi^T(t)P(t-1)\psi(t) + \lambda(t)$$
$$P(t) = (P(t-1) - P(t-1)\psi(t)S^{-1}(t)\psi^T(t)P(t-1))/\lambda(t)$$

$$\begin{pmatrix} \widehat{\theta}_l(t) \\ \widehat{\theta}_n(t) \end{pmatrix} = \left[\begin{pmatrix} \widehat{\theta}_l(t-1) \\ \widehat{\theta}_n(t-1) \end{pmatrix} + P(t)\psi(t)\varepsilon(t) \right]_{D_M} \tag{17}$$

$$\widehat{u}(t+1) = \widehat{X}(t)\Lambda(\widehat{\omega}(t)(t+1))$$
$$\psi_l(t+1) = \begin{pmatrix} \Lambda(\widehat{\omega}(t)(t+1)) & \widehat{X}(t)(t+1)\frac{d\Lambda(\phi)}{d\phi}\,|_{\phi=\widehat{\omega}(t)(t+1)} \end{pmatrix}^T$$

when $\widehat{u}(t+1) \in I_1$

when $\widehat{u}(t+1) \in [u^1_{-i}, u^1_{-i+1}], \qquad i = k_1^-, \cdots, 2$

$$\widehat{y}(t+1) = \frac{\widehat{f}^1_{-i}(t)u^1_{-i+1} - \widehat{f}^1_{-i+1}(t)u^1_{-i}}{u^1_{-i+1} - u^1_{-i}} + \frac{\widehat{f}^1_{-i+1}(t) - \widehat{f}^1_{-i}(t)}{u^1_{-i+1} - u^1_{-i}}\widehat{u}(t+1)$$
$$\frac{\partial f_1(.)}{\partial \widehat{u}} = \frac{\widehat{f}^1_{-i+1}(t) - \widehat{f}^1_{-i}(t)}{u^1_{-i+1} - u^1_{-i}}$$
$$\frac{\partial f_1(.)}{\partial \widehat{f}^1_{-i}} = \frac{u^1_{-i+1} - \widehat{u}(t+1)}{u^1_{-i+1} - u^1_{-i}}$$
$$\frac{\partial f_1(.)}{\partial \widehat{f}^1_{-i+1}} = \frac{\widehat{u}(t+1) - u^1_{-i}}{u^1_{-i+1} - u^1_{-i}}$$
$$\frac{\partial f_1(.)}{\partial \widehat{f}^1_l} = 0, \qquad l \neq -i, -i+1$$

end

when $\widehat{u}(t+1) \in [u^1_{-1}, u_{o^-}]$

$$\widehat{y}(t+1) = \frac{\widehat{f}^1_{-1}(t)u_{o^-} - (K_o u_{o^-} + \widehat{f}_o(t))u^1_{-1}}{u_{o^-} - u^1_{-1}} + \frac{K_o u_{o^-} + \widehat{f}_o(t) - \widehat{f}^1_{-1}(t)}{u_{o^-} - u^1_{-1}}\widehat{u}(t+1)$$
$$\frac{\partial f_1(.)}{\partial \widehat{u}} = \frac{K_o u_{o^-} + \widehat{f}_o(t) - \widehat{f}^1_{-1}(t)}{u_{o^-} - u^1_{-1}}$$
$$\frac{\partial f_1(.)}{\partial \widehat{f}^1_{-1}} = \frac{u_{o^-} - \widehat{u}(t+1)}{u_{o^-} - u^1_{-1}}$$
$$\frac{\partial f_1(.)}{\partial \widehat{f}_o} = \frac{\widehat{u}(t+1) - u^1_{-1}}{u_{o^-} - u^1_{-1}}$$
$$\frac{\partial f_1(.)}{\partial \widehat{f}^1_l} = 0, \qquad l \neq -1, 0$$

end

when $\widehat{u}(t+1) \in [u_{o^-}, u_{o^+}]$

$$\widehat{y}(t+1) = K_o\widehat{u}(t+1) + \widehat{f}_o(t)$$
$$\frac{\partial f_1(.)}{\partial \widehat{u}} = K_o$$
$$\frac{\partial f_1(.)}{\partial \widehat{f}_o} = 1$$
$$\frac{\partial f_1(.)}{\partial \widehat{f}^1_l} = 0, \qquad l \neq 0$$

end

when $\widehat{u}(t+1) \in [u_{o^+}, u^1_1]$

$$\widehat{y}(t+1) = \frac{(K_o u_{o^+} + \widehat{f}_o(t))u^1_1 - \widehat{f}^1_1(t)u_{o^+}}{u^1_1 - u_{o^+}} + \frac{\widehat{f}^1_1(t) - K_o u_{o^+} - \widehat{f}_o(t)}{u^1_1 - u_{o^+}}\widehat{u}(t+1)$$
$$\frac{\partial f_1(.)}{\partial \widehat{u}} = \frac{\widehat{f}^1_1(t) - K_o u_{o^+} - \widehat{f}_o(t)}{u^1_1 - u_{o^+}}$$
$$\frac{\partial f_1(.)}{\partial \widehat{f}_o} = \frac{u^1_1 - \widehat{u}(t+1)}{u^1_1 - u_{o^+}}$$
$$\frac{\partial f_1(.)}{\partial \widehat{f}^1_1} = \frac{\widehat{u}(t+1) - u_{o^+}}{u^1_1 - u_{o^+}}$$
$$\frac{\partial f_1(.)}{\partial \widehat{f}^1_l} = 0, \qquad l \neq 0, 1$$

end

when $\widehat{u}(t+1) \in [u^1_{i-1}, u^1_i], \qquad i = 2, \cdots, k^+_1$

$$\widehat{y}(t+1) = \frac{\widehat{f}^1_{i-1}(t)u^1_i - \widehat{f}^1_i(t)u^1_{i-1}}{u^1_i - u^1_{i-1}} + \frac{\widehat{f}^1_i(t) - \widehat{f}^1_{i-1}(t)}{u^1_i - u^1_{i-1}}\widehat{u}(t+1)$$

$$\frac{\partial f_1(.)}{\partial \widehat{u}} = \frac{\widehat{f}^1_i(t) - \widehat{f}^1_{i-1}(t)}{u^1_i - u^1_{i-1}}$$

$$\frac{\partial f_1(.)}{\partial \widehat{f}^1_{i-1}} = \frac{u^1_i - \widehat{u}(t+1)}{u^1_i - u^1_{i-1}}$$

$$\frac{\partial f_1(.)}{\partial \widehat{f}^1_i} = \frac{\widehat{u}(t+1) - u^1_{i-1}}{u^1_i - u^1_{i-1}}$$

$$\frac{\partial f_1(.)}{\partial \widehat{f}^1_l} = 0, \qquad l \neq i-1, i$$

end

when $\widehat{u}(t+1) \in I_j, \qquad j = 2, \cdots, L$

when $\widehat{u}(t+1) \in [u^j_i, u^j_{i+1}], \qquad i = -k^-_j, \cdots, k^+_j - 1$

$$\widehat{y}(t+1) = \frac{\widehat{f}^j_i(t)u^j_{i+1} - \widehat{f}^j_{i+1}(t)u^j_i}{u^j_{i+1} - u^j_i} + \frac{\widehat{f}^j_{i+1}(t) - \widehat{f}^j_i(t)}{u^j_{i+1} - u^j_i}\widehat{u}(t+1)$$

$$\frac{\partial f_j(.)}{\partial \widehat{u}} = \frac{\widehat{f}^j_{i+1}(t) - \widehat{f}^j_i(t)}{u^j_{i+1} - u^j_i}$$

$$\frac{\partial f_j(.)}{\partial \widehat{f}^j_i} = \frac{u^j_{i+1} - \widehat{u}(t+1)}{u^j_{i+1} - u^j_i}$$

$$\frac{\partial f_j(.)}{\partial \widehat{f}^j_{i+1}} = \frac{\widehat{u}(t+1) - u^j_i}{u^j_{i+1} - u^j_i}$$

$$\frac{\partial f_j(.)}{\partial \widehat{f}_o} = 0$$

$$\frac{\partial f_j(.)}{\partial \widehat{f}^j_l} = 0, \qquad l \neq i, i+1$$

end

$$\frac{\partial f_j(.)}{\partial \widehat{\theta}_j} = \left(\frac{\partial f_j(.)}{\partial \widehat{f}^j_{-k^-_j}} \cdots \frac{\partial f_j(.)}{\partial \widehat{f}^j_{-1}} \quad \frac{\partial f_j(.)}{\partial \widehat{f}^j_1} \cdots \frac{\partial f_j(.)}{\partial \widehat{f}^j_{k^+_j}} \right)$$

$$\psi(t+1) = \left(\frac{\partial f_j(.)}{\partial \widehat{u}}\psi^T_l(t+1) \quad \frac{\partial f_j(.)}{\partial \widehat{f}_o} \quad 0 \cdots 0 \quad \frac{\partial f_j(.)}{\partial \widehat{\theta}_j} \quad 0 \cdots 0 \right)^T, \ \widehat{u} \in I_j, \ j = 1, \cdots, L.$$

end.

4 LOCAL CONVERGENCE

Local convergence of the RPEM to the true parameter vector is here analysed with the linearized, associated differential equation, see e.g. [6] and [7]. In fact, because of the structural similarities, much of the analysis is exactly similar to [7]. *However, here there are L intervals to handle and another linear block.* The analysis follows the following general lines. First, it is investigated when the true parameter vector is a stationary point to the differential equations. The local stability of this point is then studied.

The analysis relies on the fact that the Ljung's original method of analysis is also applicable to a Wiener model structure. This was shown formally in [8]. However, here the driving linear block is slightly different from [8]. This fact is not discussed further here, but will be treated by future work.

The average updating directions that define the associated ODE are calculated from the model and gradient relations, using a fixed parameter vector $\theta \in D_M$ and a fixed R (where $P(t) = (tR(t)^{-1})$. When (17) is compared to the algorithm of [7], it is found that the averaging updating directions are

$$
\begin{aligned}
f(\theta) &= \lim_{t\to\infty} E\psi(t,\theta)\varepsilon(t,\theta) \\
&= \lim_{t\to\infty} E \begin{pmatrix} \frac{\partial f_j(.)}{\partial \widehat{u}}\psi_l(t,\theta)\varepsilon(t,\theta) \\ \frac{\partial f_j(.)}{\partial \theta_n}^T \varepsilon(t,\theta) \end{pmatrix}, \ \widehat{u} \in I_j, \ j = 1, \cdots, L.
\end{aligned} \tag{18}
$$

$$
F(R,\theta) = G(\theta) - R \tag{19}
$$

$$
\begin{aligned}
G(\theta) &= \lim_{t\to\infty} E\psi(t,\theta)\psi^T(t,\theta) \\
&= \lim_{t\to\infty} E \begin{pmatrix} [\frac{\partial f_j(.)}{\partial \widehat{u}}]^2 \psi_l \psi_l^T & \frac{\partial f_j(.)}{\partial \widehat{u}} \psi_l \frac{\partial f_j(.)}{\partial f_o} & \frac{\partial f_j(.)}{\partial \widehat{u}} \psi_l \frac{\partial f_j(.)}{\partial \tilde{\theta}_n} \\ \frac{\partial f_j(.)}{\partial f_o} \frac{\partial f_j(.)}{\partial \widehat{u}} \psi_l^T & \frac{\partial f_j(.)}{\partial f_o} \frac{\partial f_j(.)}{\partial f_o} & \frac{\partial f_j(.)}{\partial f_o} \frac{\partial f_j(.)}{\partial \tilde{\theta}_n} \\ \frac{\partial f_j(.)}{\partial \tilde{\theta}_n}^T \frac{\partial f_j(.)}{\partial \widehat{u}} \psi_l^T & \frac{\partial f_j(.)}{\partial \tilde{\theta}_n}^T \frac{\partial f_j(.)}{\partial f_o} & \frac{\partial f_j(.)}{\partial \tilde{\theta}_n}^T \frac{\partial f_j(.)}{\partial \tilde{\theta}_n} \end{pmatrix}, \ \widehat{u} \in I_j.
\end{aligned} \tag{20}
$$

Proceeding with the analysis, the following assumption is introduced,

A3) The linear block and the static nonlinearity of the system are contained in the model set.

Then there are vectors θ^o such that the output of the static nonlinearity of the system is described by

$$
y(t) = f_j(\theta_n^o, \widehat{u}(t, X^o, \omega^o)) + w(t), \ \widehat{u} \in I_j, \ j = 1, \ldots, L. \tag{21}
$$

where *w(t)* is the disturbance which satisfies the following assumption; cf. [6] and [7],

A4) *w(t)* is a bounded, strictly stationary, zero mean stochastic process that fulfills $E \mid w(t) - w_s^o(t) \mid^4 \leq c\lambda^{t-s}, c < \infty, \mid \lambda \mid < 1$.

Since there is no use of $w(t)$ in the input signal generation, the following assumption is satisfied

A5) $\psi_l(t, \theta^o)$ and $w(t)$ are independent.

By choosing $\theta = \theta^o$, it is concluded from (17) that $\varepsilon(t, \theta^o) = w(t)$. When this is inserted into (18) the result $f(\theta^o) = 0$ is obtained. A4), together with (19) implies that the ODE associated with (17) has a stationary point described by

$$
\begin{pmatrix} \theta \\ colR \end{pmatrix} = \begin{pmatrix} \theta^o \\ colG(\theta^o) \end{pmatrix} \tag{22}
$$

Following [6] and [7], we need to prove that $G^{-1}(\theta^o)\frac{df(\theta)}{d\theta} \mid_{\theta=\theta^o}$ has all eigenvalues in the left half plane to prove local convergence. The derivative can be calculated from (18) by straight forward differentiation. When the true parameter vector is inserted in the resulting expression, the equation (20), together with A4), gives

$$
G^{-1}(\theta^o)\frac{df(\theta)}{d\theta} \mid_{\theta=\theta^o} = -I \tag{23}
$$

provided that the inverse exists. Since $G(\theta^o)$ is positive semidefinite by construction, conditions that imply the positive definiteness of $G(\theta^o)$ are now needed in order to prove that $G^{-1}(\theta^o)$ exists.

In order to prove positive definiteness of $G(\theta^o)$ it is convenient to extract the contribution to $G(\theta^o)$ from the subinterval I_o and to study this contribution separately. Introducing the gate function for this purpose, where

$$\text{gate}(\widehat{u}(t,X^o,\omega^o)) = \begin{cases} 1 & \widehat{u}(t,X^o,\omega^o) \in I_o \subset I_1 \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

Again, by $\widehat{u}(t,X^o,\omega^o) \in I_o$ we mean that the phase ωt is such that I_0 is in effect. Thus $G(\theta^o)$ can then be expressed as

$$G(\theta^o) = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} + \begin{pmatrix} \tilde{A} & 0 \\ 0 & 0 \end{pmatrix} \tag{25}$$

where

$$\tilde{A} = \lim_{t\to\infty} E\text{gate}(\widehat{u}) \begin{pmatrix} K_o^2 \psi_l \psi_l^T & K_o \psi_l \\ K_o \psi_l^T & 1 \end{pmatrix} \tag{26}$$

$$\begin{aligned} C &= \lim_{t\to\infty} E(1-\text{gate}(\widehat{u})) \frac{\partial f_j(.)}{\partial \tilde{\theta}_n}^T \frac{\partial f_j(.)}{\partial \tilde{\theta}_n} \\ &= \lim_{t\to\infty} E(1-\text{gate}(\widehat{u})) \begin{pmatrix} \frac{\partial f_1(.)}{\partial \theta_1}^T \frac{\partial f_1(.)}{\partial \theta_1} & \cdots & \frac{\partial f_1(.)}{\partial \theta_1}^T \frac{\partial f_L(.)}{\partial \theta_L} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_L(.)}{\partial \theta_L}^T \frac{\partial f_1(.)}{\partial \theta_1} & \cdots & \frac{\partial f_L(.)}{\partial \theta_L}^T \frac{\partial f_L(.)}{\partial \theta_L} \end{pmatrix}_{|\theta=\theta^o} . \end{aligned} \tag{27}$$

Now use lemma 2 in [7], which can be formulated as follows.

Lemma. Consider the block-matrix decomposition

$$G = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} + \begin{pmatrix} \tilde{A} & 0 \\ 0 & 0 \end{pmatrix}$$

where both terms on the right-hand side are symmetric. Assume that the first term of G is positive semidefinite and that $\tilde{A}$ and C are positive definite. Then G is also positive definite.
Proof. See [7].

$G(\theta^o)$ is positive definite provided that $\tilde{A} > 0$ and $C > 0$. Thus the analysis will be divided as follows:

4.1 Positive definiteness of $\tilde{A}$

The first step of the analysis is to find conditions that guarantee that $\tilde{A} > 0$. Note that it can not be assumed that

$$\lim_{t\to\infty} E\ \psi_l = 0$$

because of possible bias in the driving signal. The idea now is to compose $\tilde{A}$ into two parts, where the first part is determined by the bias in the input signal, and where the second part comes from the variation. The same technique is used in [7]. In order to overcome the technical problem with the gate-function that appears in $\tilde{A}$, ψ_l^o is introduced according to

$$\lim_{t\to\infty} E\ gate(\widehat{u})\ \psi_l = \psi_l^o \tag{28}$$

where ψ_l^o is a constant vector. Furthermore, assume

A6) $\alpha_o = \lim_{t\to\infty} E\ gate(\widehat{u}) > 0$.

which means that there must be signal energy in the mid-subinterval I_o. Then ψ_l can be written as

$$\psi_l = \frac{1}{\alpha_o}\psi_l^o + \triangle\psi_l \tag{29}$$

where $\triangle\psi_l$ is the variation of ψ_l around $\frac{1}{\alpha_o}\psi_l^o$. This implies the following equalities, cf. [7]

$$\begin{aligned}
\psi_l^o &= \lim_{t\to\infty} E\ gate(\widehat{u})\ \psi_l \\
&= \lim_{t\to\infty} E\ gate(\widehat{u})\ (\frac{1}{\alpha_o}\psi_l^o + \triangle\psi_l) \\
&= (\lim_{t\to\infty} E\ gate(\widehat{u}))\ \frac{1}{\alpha_o}\psi_l^o + \lim_{t\to\infty} E\ gate(\widehat{u})\triangle\psi_l \\
&= \psi_l^o + \lim_{t\to\infty} E\ gate(\widehat{u})\triangle\psi_l.
\end{aligned}$$

which implies that

$$\lim_{t\to\infty} E\ gate(\widehat{u})\triangle\psi_l = 0 \tag{30}$$

The blocks of $\tilde{A}$ can now be calculated using A6), (29) and (30). The (1,1)-block becomes

$$\begin{aligned}
\lim_{t\to\infty} Egate(\widehat{u})K_o^2\psi_l\psi_l^T &= \lim_{t\to\infty} Egate(\widehat{u})K_o^2\{\frac{1}{\alpha_o^2}\psi_l^o\psi_l^{oT} + \frac{1}{\alpha_o}\psi_l^o\triangle\psi_l^T \\
&\quad + \frac{1}{\alpha_o}\triangle\psi_l\ \psi_l^{oT} + \triangle\psi_l\triangle\psi_l^T\} \\
&= K_o^2\frac{1}{\alpha_o}\psi_l^o\psi_l^{oT} + K_o^2\lim_{t\to\infty} Egate(\widehat{u})\triangle\psi_l\triangle\psi_l^T.
\end{aligned} \tag{31}$$

Thus $\tilde{A}$ can be written as

$$\begin{aligned}
\tilde{A} &= \alpha_o \begin{pmatrix} \frac{K_o}{\alpha_o}\psi_l^o \\ 1 \end{pmatrix} \begin{pmatrix} \frac{K_o}{\alpha_o}\psi_l^o & 1 \end{pmatrix} + \begin{pmatrix} \lim_{t\to\infty} Egate(\widehat{u})K_o^2\triangle\psi_l\triangle\psi_l^T & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} F & M \\ M^T & H \end{pmatrix} + \begin{pmatrix} \tilde{F} & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{32}$$

Applying lemma 2 in [7] again and taking into account that $H = \alpha_o > 0$ by assumption, positive definiteness of $\tilde{A}$ follows, if conditions implying that

$$\tilde{F} = \lim_{t\to\infty} Egate(\widehat{u})K_o^2\triangle\psi_l\triangle\psi_l^T > 0$$

can be found. In order to proceed, assume that

A7) $\lim_{t\to\infty} E\text{gate}(\widehat{u})\triangle\psi_l\triangle\psi_l^T \geq \delta \lim_{t\to\infty} E\triangle\psi_l\triangle\psi_l^T, \delta > 0$.

This assumption means that the contribution to the expectation from I_o, should not be negligible as compared to the whole contribution to the expectation. It is a condition on the amplitude distribution of the driving signal, i.e, the driving signal should be such that a sufficient amount of signal energy is located in the subinterval I_o (interpreted via the phase condition). Also, assume that

A8) $|K_o| > 0.$

It remains to investigate when

$$\lim_{t\to\infty} E\triangle\psi_l\triangle\psi_l^T > 0$$

holds. Since (30) does *not* imply that $\triangle\psi_l$ has zero mean, it is necessary to write

$$\triangle\psi_l = \triangle\psi_l^o + \triangle\tilde{\psi}_l$$

where

$$\triangle\psi_l^o = \lim_{t\to\infty} E\triangle\psi_l$$

Then $\triangle\tilde{\psi}_l$ has zero mean, and

$$\lim_{t\to\infty} E\triangle\psi_l\triangle\psi_l^T = \triangle\psi_l^o\triangle\psi_l^{oT} + \lim_{t\to\infty} E\triangle\tilde{\psi}_l\triangle\tilde{\psi}_l^T \geq \lim_{t\to\infty} E\triangle\tilde{\psi}_l\triangle\tilde{\psi}_l^T$$

Since $\triangle\tilde{\psi}_l$ has zero mean, and

$$\psi_l = \frac{1}{\alpha_o}\psi_l^o + \triangle\psi_l^o + \triangle\tilde{\psi}_l$$

the effect of the transformation is to remove possible bias in ψ_l. Therefore, the following condition is introduced.

A9) $\lim_{t\to\infty} E\triangle\tilde{\psi}_l\triangle\tilde{\psi}_l^T > 0$

It is motivated by an example below. Formal and general conditions implying A9) are presently under investigation.

***Example* 1:** To investigate the conditions needed to secure A9), consider the following

$$\begin{aligned} \widehat{u}(t, X^o, \omega^o) &= X^o\Lambda(\omega^o t) \\ \Lambda(\omega^o t) &= sin(\omega^o t) \end{aligned} \tag{33}$$

and assume that the phase $\omega^o t$ is uniformly distributed in the interval $[0, 2\pi]$. In this case

$$\triangle\tilde{\psi}_l = \psi_l$$

this is because there is no bias in the driving signal. Thus

$$E\triangle\tilde{\psi}_l\triangle\tilde{\psi}_l^T = E\begin{pmatrix} \Lambda^2(\phi) & \frac{X^o}{\omega^o}\ \phi\ \Lambda(\phi)\frac{d\Lambda(\phi)}{d\phi} \\ \frac{X^o}{\omega^o}\ \phi\ \Lambda(\phi)\frac{d\Lambda(\phi)}{d\phi} & [\frac{X^o}{\omega^o}]^2\ \phi^2\ [\frac{d\Lambda(\phi)}{d\phi}]^2 \end{pmatrix}_{\phi=\omega^o t} \tag{34}$$

Straightforward caculations gives

$$
\begin{aligned}
E[\Lambda^2(\phi)] &= 1/2 \\
E[\phi\ \Lambda(\phi)\frac{d\Lambda(\phi)}{d\phi}] &= -1/4 \\
E[\phi^2\ [\frac{d\Lambda(\phi)}{d\phi}]^2] &= \frac{2\pi^2}{3} + \frac{1}{4}
\end{aligned}
\tag{35}
$$

thus (34) becomes

$$
E\triangle\tilde{\psi}_l\triangle\tilde{\psi}_l^T = \begin{pmatrix} \frac{1}{2} & \frac{-X^o}{4\omega^o} \\ \frac{-X^o}{4\omega^o} & [\frac{X^o}{\omega^o}]^2\ (\frac{2\pi^2}{3} + \frac{1}{4}) \end{pmatrix}
\tag{36}
$$

which is a positive definite matrix. Thus A9) is a valid assumption.

4.2 Positive definiteness of C

It remains to analyse equation (27) to prove the positive definiteness of $G(\theta^o)$. Since only two of the components of $\frac{\partial f_j(.)}{\partial \theta_j}$ are nonzero in each of the subintervals and $\frac{\partial f_m(.)}{\partial \theta_m}\frac{\partial f_n(.)}{\partial \theta_n}^T = 0,\ n \neq m$, C can be written as follows

$$
C = \begin{pmatrix} C_{11} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & C_{LL} \end{pmatrix}
\tag{37}
$$

where

$$
C_{jj} = \lim_{t\to\infty} E(1 - \mathrm{gate}(\widehat{u}))\frac{\partial f_j(.)}{\partial \theta_j}^T \frac{\partial f_j(.)}{\partial \theta_j} = \lim_{t\to\infty} E\frac{\partial f_j(.)}{\partial \theta_j}^T \frac{\partial f_j(.)}{\partial \theta_j}
\tag{38}
$$

Equation (38) results from the fact that (cf. the phase condition)

$$
\frac{\partial f_j(.)}{\partial \theta_j} = 0, \qquad \widehat{u}(t, X^o, \omega^o) \in I_o
\tag{39}
$$

Also, C_{jj} can be written as

$$
C_{jj} = \begin{pmatrix} C_{jj^-} & 0 \\ 0 & C_{jj^+} \end{pmatrix}
\tag{40}
$$

where C_{jj^-} and C_{jj^+} are band-matrices give by

$$
C_{jj^-} = \lim_{t\to\infty} E \begin{pmatrix}
\frac{\partial f_j}{\partial f^j_{-k_j^-}}\frac{\partial f_j}{\partial f^j_{-k_j^-}} & \frac{\partial f_j}{\partial f^j_{-k_j^-}}\frac{\partial f_j}{\partial f^j_{-k_j^-+1}} & 0 & \cdots & 0 \\
\frac{\partial f_j}{\partial f^j_{-k_j^-+1}}\frac{\partial f_j}{\partial f^j_{-k_j^-}} & & & \ddots & \vdots \\
0 & & \ddots & & \\
\vdots & \ddots & & & 0 \\
 & & & & \frac{\partial f_j}{\partial f^j_{-2}}\frac{\partial f_j}{\partial f^j_{-1}} \\
0 & \cdots & 0 & \frac{\partial f_j}{\partial f^j_{-1}}\frac{\partial f_j}{\partial f^j_{-2}} & \frac{\partial f_j}{\partial f^j_{-1}}\frac{\partial f_j}{\partial f^j_{-1}}
\end{pmatrix}
\tag{41}
$$

$$C_{jj^+} = \lim_{t \to \infty} E \begin{pmatrix} \frac{\partial f_j}{\partial f_1^j} \frac{\partial f_j}{\partial f_1^j} & \frac{\partial f_j}{\partial f_1^j} \frac{\partial f_j}{\partial f_2^j} & 0 & \cdots & 0 \\ \frac{\partial f_j}{\partial f_2^j} \frac{\partial f_j}{\partial f_1^j} & & & \ddots & \vdots \\ 0 & & \ddots & & \\ \vdots & \ddots & & & 0 \\ & & & & \frac{\partial f_j}{\partial f_{k_j^+-1}^j} \frac{\partial f_j}{\partial f_{k_j^+}^j} \\ 0 & \cdots & 0 & \frac{\partial f_j}{\partial f_{k_j^+}^j} \frac{\partial f_j}{\partial f_{k_j^+-1}^j} & \frac{\partial f_j}{\partial f_{k_j^+}^j} \frac{\partial f_j}{\partial f_{k_j^+}^j} \end{pmatrix} \tag{42}$$

If C_{jj^-} and C_{jj^+} are positive definite then also C_{jj} is positive definite and positive definiteness of C is consequently proved. C_{jj^-} and C_{jj^+} have exactly the same structure which means that it is sufficient to investigate one of them [7]. In order to analyse, for example, C_{jj^+} consider

$$D^T \; C_{jj^+} \; D$$

where D is an arbitrary vector. Since C_{jj^+} is positive semidefinite by construction, positive definiteness of C_{jj^+} can be proved by showing that

$$D^T \; C_{jj^+} \; D = 0 \implies D = 0.$$

Equation (42) gives

$$D^T \; C_{jj^+} \; D = \lim_{t \to \infty} E \left(\sum_{m=1}^{k_j^+} d_m \; \frac{\partial f_j}{\partial f_m^j} \right)^2 \tag{43}$$

where d_m are the components of the vector D. The expression

$$f_{C_{jj^+}}(D, \widehat{u}) = \sum_{m=1}^{k_j^+} d_m \; \frac{\partial f_j}{\partial f_m^j} \tag{44}$$

can be interpreted as a piecewise linear curve (see [7]), that can be nonzero only when

$$\widehat{u} \; \in \; [u_1^j, u_{k_j^+}^j]$$

The function values of $f_{C_{jj^+}}(D, \widehat{u})$ in the grid points $u_i^j, \; i = 1, \ldots, k_j^+$ are the corresponding components of the vector D, i.e.

$$f_{C_{jj^+}}(D, u_i^j) = d_i \;, \qquad i = 1, \ldots, k_j^+. \tag{45}$$

This is becuase when $\widehat{u} \in [u_i^j, u_{i+1}^j], \; i = 1, \ldots, k_j^+ - 1$ the function $f_{C_{jj^+}}(D, \widehat{u})$ can be written as

$$f_{C_{jj^+}}(D, \widehat{u}) = d_i \; \frac{u_{i+1}^j - \widehat{u}}{u_{i+1}^j - u_i^j} + d_{i+1} \; \frac{\widehat{u} - u_i^j}{u_{i+1}^j - u_i^j}$$

which is a linear function of $\widehat{u}$ that satisfies

$$\begin{aligned} f_{C_{jj^+}}(D, u_i^j) &= d_i \\ f_{C_{jj^+}}(D, u_{i+1}^j) &= d_{i+1}. \end{aligned}$$

Let $I_i^j, i = -k_j^-, \ldots, k_j^+, \ j = 1, \ldots, L$ denote the (open) subintervals of the piecewise linear model of the static nonlinearity including I_o. To proceed with the analysis, the following assumption is needed

A10) The probability density function $h_{\widehat{u}}(\widehat{u})$ of $\widehat{u}(t, X, \omega)$ fulfills

$$h_{\widehat{u}}(\widehat{u}) \geq \delta_1 > 0.$$

in at least one nonzero interval $[a_i^j, b_i^j] \in I_i^j$ for all $i = -k_j^-, \ldots, k_j^+, \ j = 1, \ldots, L.$

This assumption means that there must be signal somewhere in every subinterval of the model. Thus equation(43) can be written as

$$\begin{aligned} D^T \ C_{jj^+} \ D &= \lim_{t\to\infty} E f_{C_{jj^+}}(D, \widehat{u})^2 \\ &= \int_{u_1^j}^{u_{k_j^+}^j} f_{C_{jj^+}}(D, \widehat{u})^2 h_{\widehat{u}}(\widehat{u}) d\widehat{u} \end{aligned} \tag{46}$$

The condition

$$D^T \ C_{jj^+} \ D = 0$$

can now be investigated using assumption A10) and equation (46). Since $f_{C_{jj^+}}(D, \widehat{u})^2$ is nonnegative and continuous on $[u_1^j, u_{k_j^+}^j]$, it follows from equation (46) and A10) that

$$f_{C_{jj^+}}(D, \widehat{u}) = 0 \ , \ \widehat{u} \in [a_i, b_i] \subset I_i^j, \ i = 1, \ldots, k_j^+, \ j = 1, \ldots, L$$

Thus $f_{C_{jj^+}}(D, \widehat{u})$ is

- analytic in $I_i^j, \ i = 1, \ldots, k_j^+, \ j = 1, \ldots, L$
- zero in nonzero intervals $[a_i, b_i] \subset I_i^j, \ i = 1, \ldots, k_j^+, \ j = 1, \ldots, L.$

It then follows from a well known theorem for analytic functions (see, for example, [10]) that

$$f_{C_{jj^+}}(D, \widehat{u}) = 0 \ , \ \widehat{u} \in I_i^j, \ i = 1, \ldots, k_j^+, \ j = 1, \ldots, L.$$

The continuity of $f_{C_{jj^+}}(D, \widehat{u})$ together with equation(45) finally gives

$$f_{C_{jj^+}}(D, u_i^j) = d_i = 0 \ , \ i = 1, \ldots, k_j^+.$$

Thus it has been shown that

$$D^T \ C_{jj^+} \ D = 0 \implies D = 0$$

This proves the positive definiteness of C_{jj^+}. Since C_{jj^-} has the same structure as C_{jj^+}, it follows that C_{jj} is positive definite. This leads to the positive definiteness of C. The fact that $\tilde{A}$ and C are positive definite now implies the positive definiteness of $G(\theta^o)$. This completes the proof of the following theorem:

***Theorem* 1.** Assume that the assumptions A1)-A10) holds. Then the RPEM algorithm given in (17) converges locally to

$$\begin{pmatrix} \theta_l^o \\ \theta_n^o \end{pmatrix} \in D_M.$$

$\square$

5 THE CRAMÉR-RAO BOUND

Assume that A1)-A10) holds. Further assume that

A11) $E[w(t)w(s)] = \sigma^2 \delta_{t,s}$ and $w(t)$ is Gaussian.

A12) $N > N_o$ such that there exist a time instant $t < N_o$ where $\widehat{u} \in I_j$ and $\widehat{u} \in [u_i^j, u_{i+1}^j]\ \forall i, j \in \{i = -k_j^-, \dots, k_j^+ - 1,\ j = 1, \dots, L\}$.

A12) means that there has been signal energy in each subinterval of the model, cf. [7] and A10). Then the following theorem holds for the signal model of A3):

***Theorem* 2.** Under the assumptions A1)-A12), the CRB for $(\theta_l^T\ \theta_n^T)^T$ is given by

$$CRB(\theta) = \sigma^2 \left(\sum_{t=1}^{N} I(t) \right)^{-1} \tag{47}$$

where

$$I(t) = \begin{pmatrix} I_{X,X} & I_{X,w} & I_{X,f_o} & 0 \cdots 0 & I_{X,f_i^j} & I_{X,f_{i+1}^j} & 0 \cdots 0 \\ I_{w,X} & I_{w,w} & I_{w,f_o} & 0 \cdots 0 & I_{w,f_i^j} & I_{w,f_{i+1}^j} & 0 \cdots 0 \\ I_{f_o,X} & I_{f_o,w} & I_{f_o,f_o} & 0 \cdots 0 & I_{f_o,f_i^j} & I_{f_o,f_{i+1}^j} & 0 \cdots 0 \\ 0 & 0 & 0 & & 0 & 0 & \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & 0 & & 0 & 0 & \\ I_{f_i^j,X} & I_{f_i^j,w} & I_{f_i^j,f_o} & 0 \cdots 0 & I_{f_i^j,f_i^j} & I_{f_i^j,f_{i+1}^j} & 0 \cdots 0 \\ I_{f_{i+1}^j,X} & I_{f_{i+1}^j,w} & I_{f_{i+1}^j,f_o} & 0 \cdots 0 & I_{f_{i+1}^j,f_i^j} & I_{f_{i+1}^j,f_{i+1}^j} & 0 \cdots 0 \\ 0 & 0 & 0 & & 0 & 0 & \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & 0 & & 0 & 0 & \end{pmatrix} \tag{48}$$

$\widehat{u} \in [u_i^j, u_{i+1}^j] \in I_j,\ i = -k_j^-, \dots, k_j^+ - 1,\ j = 1, \dots, L.$

$$\begin{aligned}
I_{X,X} &= \left[\frac{\partial f_j(.)}{\partial u}\right]^2 \ \Lambda^2(\phi) \mid_{\phi=\omega t} \\
I_{w,w} &= \left[\frac{\partial f_j(.)}{\partial u}\right]^2 \ X^2 t^2 \left[\frac{d\Lambda(\phi)}{d\phi} \mid_{\phi=\omega t}\right]^2 \\
I_{f_o,f_o} &= \left[\frac{\partial f_j(.)}{\partial f_o}\right]^2 \\
I_{X,w} &= \left[\frac{\partial f_j(.)}{\partial u}\right]^2 \ Xt\ \Lambda(\phi)\frac{d\Lambda(\phi)}{d\phi} \mid_{\phi=\omega t} \\
I_{X,f_o} &= \frac{\partial f_j(.)}{\partial u}\frac{\partial f_j(.)}{\partial f_o}\Lambda(\phi) \mid_{\phi=\omega t} \\
I_{w,f_o} &= \frac{\partial f_j(.)}{\partial u}\frac{\partial f_j(.)}{\partial f_o}Xt\frac{d\Lambda(\phi)}{d\phi} \mid_{\phi=\omega t} \\
I_{X,f_i^j} &= \frac{\partial f_j(.)}{\partial u}\frac{\partial f_j(.)}{\partial f_i^j}\Lambda(\phi) \mid_{\phi=\omega t}
\end{aligned}$$

$$
\begin{aligned}
I_{X,f_{i+1}^j} &= \frac{\partial f_j(.)}{\partial u}\frac{\partial f_j(.)}{\partial f_{i+1}^j}\Lambda(\phi)\mid_{\phi=\omega t} \\
I_{w,f_i^j} &= \frac{\partial f_j(.)}{\partial u}\frac{\partial f_j(.)}{\partial f_i^j}Xt\frac{d\Lambda(\phi)}{d\phi}\mid_{\phi=\omega t} \\
I_{w,f_{i+1}^j} &= \frac{\partial f_j(.)}{\partial u}\frac{\partial f_j(.)}{\partial f_{i+1}^j}Xt\frac{d\Lambda(\phi)}{d\phi}\mid_{\phi=\omega t} \\
I_{f_o,f_i^j} &= \frac{\partial f_j(.)}{\partial f_o}\frac{\partial f_j(.)}{\partial f_i^j} \\
I_{f_o,f_{i+1}^j} &= \frac{\partial f_j(.)}{\partial f_o}\frac{\partial f_j(.)}{\partial f_{i+1}^j} \\
I_{f_i^j,f_i^j} &= \frac{\partial f_j(.)}{\partial f_i^j}\frac{\partial f_j(.)}{\partial f_i^j} \\
I_{f_{i+1}^j,f_{i+1}^j} &= \frac{\partial f_j(.)}{\partial f_{i+1}^j}\frac{\partial f_j(.)}{\partial f_{i+1}^j} \\
I_{f_i^j,f_{i+1}^j} &= \frac{\partial f_j(.)}{\partial f_i^j}\frac{\partial f_j(.)}{\partial f_{i+1}^j} \qquad (49)
\end{aligned}
$$

***Proof*:** The log-likelihood function is given by

$$
l(\theta) = \kappa - \frac{1}{2\sigma^2}\ \sum_{t=1}^{N}(y(t) - \widehat{y}(t,\theta))^2 \qquad (50)
$$

where κ is a constant does not effect the further calculation. Let,

$$
\frac{\partial l(\theta)}{\partial \theta} = \left(\frac{\partial l(\theta)}{\partial \theta_l}\ \frac{\partial l(\theta)}{\partial \theta_n}\right) \qquad (51)
$$

where

$$
\begin{aligned}
\frac{\partial l(\theta)}{\partial \theta_l} &= \left(\frac{\partial l(\theta)}{\partial X} \quad \frac{\partial l(\theta)}{\partial \omega}\right) \\
\frac{\partial l(\theta)}{\partial \theta_n} &= \left(\frac{\partial l(\theta)}{\partial f_o} \quad \frac{\partial l(\theta)}{\partial \theta_1}\ \cdots\ \frac{\partial l(\theta)}{\partial \theta_L}\right) \\
\frac{\partial l(\theta)}{\partial \theta_j} &= \left(\frac{\partial l(\theta)}{\partial f_{-k_j^-}^j}\ \cdots\ \frac{\partial l(\theta)}{\partial f_{-1}^j}\ \frac{\partial l(\theta)}{\partial f_1^j}\ \cdots\ \frac{\partial l(\theta)}{\partial f_{k_j^+}^j}\right) \qquad (52)
\end{aligned}
$$

Then, the Fisher informatiom matrix can be written as

$$
\begin{aligned}
J &= -E\ \frac{\partial l(\theta)}{\partial \theta}^T\ \frac{\partial l(\theta)}{\partial \theta} \\
&= -E\ \begin{pmatrix} \frac{\partial l(\theta)}{\partial \theta_l}^T \frac{\partial l(\theta)}{\partial \theta_l} & \frac{\partial l(\theta)}{\partial \theta_l}^T \frac{\partial l(\theta)}{\partial \theta_n} \\ \frac{\partial l(\theta)}{\partial \theta_n}^T \frac{\partial l(\theta)}{\partial \theta_l} & \frac{\partial l(\theta)}{\partial \theta_n}^T \frac{\partial l(\theta)}{\partial \theta_n} \end{pmatrix} \qquad (53)
\end{aligned}
$$

In order to calculate J, note that

$$
\begin{aligned}
\frac{\partial l(\theta)}{\partial X} &= \frac{1}{\sigma^2} \sum_{t=1}^{N} (y(t) - \widehat{y}(t,\theta)) \frac{\partial \widehat{y}(t,\theta)}{\partial X} \\
\frac{\partial l(\theta)}{\partial w} &= \frac{1}{\sigma^2} \sum_{t=1}^{N} (y(t) - \widehat{y}(t,\theta)) \frac{\partial \widehat{y}(t,\theta)}{\partial w} \\
\frac{\partial l(\theta)}{\partial f_o} &= \frac{1}{\sigma^2} \sum_{t=1}^{N} (y(t) - \widehat{y}(t,\theta)) \frac{\partial f_j(.)}{\partial f_o} \\
\frac{\partial l(\theta)}{\partial f_i^j} &= \frac{1}{\sigma^2} \sum_{t=1}^{N} (y(t) - \widehat{y}(t,\theta)) \frac{\partial f_j(.)}{\partial f_i^j} \\
\frac{\partial l(\theta)}{\partial f_{i+1}^j} &= \frac{1}{\sigma^2} \sum_{t=1}^{N} (y(t) - \widehat{y}(t,\theta)) \frac{\partial f_j(.)}{\partial f_{i+1}^j}
\end{aligned} \tag{54}
$$

For $\widehat{u} \in [u_i^j, u_{i+1}^j] \in I_j,\ i = -k_j^-, \ldots, k_j^+ - 1,\ j = 1, \ldots, L$ it holds that

$$
\begin{aligned}
\frac{\partial \widehat{y}(t,\theta)}{\partial X} &= \frac{\partial f_j(.)}{\partial u}\ \Lambda(\phi) \ |_{\phi=\omega t} \\
\frac{\partial \widehat{y}(t,\theta)}{\partial w} &= \frac{\partial f_j(.)}{\partial u}\ Xt \frac{d\Lambda(\phi)}{d\phi} \ |_{\phi=\omega t}
\end{aligned} \tag{55}
$$

Thus using the signal model A3) and A11)

$$
\begin{aligned}
E\left[\frac{\partial^2 l(\theta)}{\partial X^2}\right] &= -\frac{1}{\sigma^2} \sum_{t=1}^{N} \left[\frac{\partial f_j(.)}{\partial u}\right]^2 \ \Lambda^2(\phi) \ |_{\phi=\omega t} \\
E\left[\frac{\partial^2 l(\theta)}{\partial w^2}\right] &= -\frac{1}{\sigma^2} \sum_{t=1}^{N} \left[\frac{\partial f_j(.)}{\partial u}\right]^2 \ X^2 t^2 \left[\frac{d\Lambda(\phi)}{d\phi} \ |_{\phi=\omega t}\right]^2 \\
E\left[\frac{\partial^2 l(\theta)}{\partial f_o^2}\right] &= -\frac{1}{\sigma^2} \sum_{t=1}^{N} \left[\frac{\partial f_j(.)}{\partial f_o}\right]^2 \\
E\left[\frac{\partial^2 l(\theta)}{\partial X \partial w}\right] &= -\frac{1}{\sigma^2} \sum_{t=1}^{N} \left[\frac{\partial f_j(.)}{\partial u}\right]^2 \ Xt\ \Lambda(\phi) \frac{d\Lambda(\phi)}{d\phi} \ |_{\phi=\omega t} \\
E\left[\frac{\partial^2 l(\theta)}{\partial X \partial f_o}\right] &= -\frac{1}{\sigma^2} \sum_{t=1}^{N} \frac{\partial f_j(.)}{\partial u} \frac{\partial f_j(.)}{\partial f_o} \Lambda(\phi) \ |_{\phi=\omega t} \\
E\left[\frac{\partial^2 l(\theta)}{\partial w \partial f_o}\right] &= -\frac{1}{\sigma^2} \sum_{t=1}^{N} \frac{\partial f_j(.)}{\partial u} \frac{\partial f_j(.)}{\partial f_o} Xt \frac{d\Lambda(\phi)}{d\phi} \ |_{\phi=\omega t} \\
E\left[\frac{\partial^2 l(\theta)}{\partial X \partial f_i^j}\right] &= -\frac{1}{\sigma^2} \sum_{t=1}^{N} \frac{\partial f_j(.)}{\partial u} \frac{\partial f_j(.)}{\partial f_i^j} \Lambda(\phi) \ |_{\phi=\omega t} \\
E\left[\frac{\partial^2 l(\theta)}{\partial X \partial f_{i+1}^j}\right] &= -\frac{1}{\sigma^2} \sum_{t=1}^{N} \frac{\partial f_j(.)}{\partial u} \frac{\partial f_j(.)}{\partial f_{i+1}^j} \Lambda(\phi) \ |_{\phi=\omega t}
\end{aligned}
$$

$$
\begin{aligned}
E\left[\frac{\partial^2 l(\theta)}{\partial w \partial f_i^j}\right] &= -\frac{1}{\sigma^2}\sum_{t=1}^{N}\frac{\partial f_j(.)}{\partial u}\frac{\partial f_j(.)}{\partial f_i^j} X t \frac{d\Lambda(\phi)}{d\phi} \mid_{\phi=\omega t} \\
E\left[\frac{\partial^2 l(\theta)}{\partial w \partial f_{i+1}^j}\right] &= -\frac{1}{\sigma^2}\sum_{t=1}^{N}\frac{\partial f_j(.)}{\partial u}\frac{\partial f_j(.)}{\partial f_{i+1}^j} X t \frac{d\Lambda(\phi)}{d\phi} \mid_{\phi=\omega t} \\
E\left[\frac{\partial^2 l(\theta)}{\partial f_o \partial f_i^j}\right] &= -\frac{1}{\sigma^2}\sum_{t=1}^{N}\frac{\partial f_j(.)}{\partial f_o}\frac{\partial f_j(.)}{\partial f_i^j} \\
E\left[\frac{\partial^2 l(\theta)}{\partial f_o \partial f_{i+1}^j}\right] &= -\frac{1}{\sigma^2}\sum_{t=1}^{N}\frac{\partial f_j(.)}{\partial f_o}\frac{\partial f_j(.)}{\partial f_{i+1}^j} \\
E\left[\frac{\partial^2 l(\theta)}{\partial f_i^j \partial f_i^j}\right] &= -\frac{1}{\sigma^2}\sum_{t=1}^{N}\frac{\partial f_j(.)}{\partial f_i^j}\frac{\partial f_j(.)}{\partial f_i^j} \\
E\left[\frac{\partial^2 l(\theta)}{\partial f_{i+1}^j \partial f_{i+1}^j}\right] &= -\frac{1}{\sigma^2}\sum_{t=1}^{N}\frac{\partial f_j(.)}{\partial f_{i+1}^j}\frac{\partial f_j(.)}{\partial f_{i+1}^j} \\
E\left[\frac{\partial^2 l(\theta)}{\partial f_i^j \partial f_{i+1}^j}\right] &= -\frac{1}{\sigma^2}\sum_{t=1}^{N}\frac{\partial f_j(.)}{\partial f_i^j}\frac{\partial f_j(.)}{\partial f_{i+1}^j} \qquad (56)
\end{aligned}
$$

Introduce the notation in (49) and use the facts that $\frac{\partial^2 l(\theta)}{\partial\theta_m \partial\theta_n} = 0$ for $m \neq n$ and $J = 1/\sigma^2 \sum_{t=1}^{N} I(t)$. Then (47) directly follows from A12).

Remark. $\frac{\partial f_j(.)}{\partial u}$, $\frac{\partial f_j(.)}{\partial f_o}$, $\frac{\partial f_j(.)}{\partial f_i^j}$ and $\frac{\partial f_j(.)}{\partial f_{i+1}^j}$ can be calculated for different subintervals using (17). Thus the matrix $I(t)$ takes the following forms:

(i) When $\widehat{u} \in [u_i^j, u_{i+1}^j]$ $\forall i = \begin{cases} -k_1^-, \dots, -2, 1, \dots, k_1^+ - 1, \; j = 1 \\ -k_j^-, \dots, k_j^+ - 1, \; j = 2, \dots, L. \end{cases}$

$$
I(t) = \begin{pmatrix}
I_{X,X} & I_{X,w} & 0 \cdots 0 & I_{X,f_i^j} & I_{X,f_{i+1}^j} & 0 \cdots 0 \\
I_{w,X} & I_{w,w} & 0 \cdots 0 & I_{w,f_i^j} & I_{w,f_{i+1}^j} & 0 \cdots 0 \\
0 & 0 & & 0 & 0 & \\
\vdots & \vdots & & \vdots & \vdots & \\
0 & 0 & & 0 & 0 & \\
I_{f_i^j,X} & I_{f_i^j,w} & 0 \cdots 0 & I_{f_i^j,f_i^j} & I_{f_i^j,f_{i+1}^j} & 0 \cdots 0 \\
I_{f_{i+1}^j,X} & I_{f_{i+1}^j,w} & 0 \cdots 0 & I_{f_{i+1}^j,f_i^j} & I_{f_{i+1}^j,f_{i+1}^j} & 0 \cdots 0 \\
0 & 0 & & 0 & 0 & \\
\vdots & \vdots & & \vdots & \vdots & \\
0 & 0 & & 0 & 0 &
\end{pmatrix} \qquad (57)
$$

where

$$
\begin{aligned}
I_{X,X} &= \left[\frac{f_{i+1}^j - f_i^j}{u_{i+1}^j - u_i^j}\right]^2 \Lambda^2(\phi) \mid_{\phi=\omega t} \\
I_{w,w} &= \left[\frac{f_{i+1}^j - f_i^j}{u_{i+1}^j - u_i^j}\right]^2 X^2 t^2 \left[\frac{d\Lambda(\phi)}{d\phi} \mid_{\phi=\omega t}\right]^2
\end{aligned}
$$

$$
\begin{aligned}
I_{X,w} &= \left[\frac{f_{i+1}^j - f_i^j}{u_{i+1}^j - u_i^j}\right]^2 \;\; Xt\; \Lambda(\phi)\frac{d\Lambda(\phi)}{d\phi}\; |_{\phi=\omega t} \\
I_{X,f_i^j} &= \frac{f_{i+1}^j - f_i^j}{u_{i+1}^j - u_i^j}\;\; \frac{u_{i+1}^j - u(t,X,\omega)}{u_{i+1}^j - u_i^j}\Lambda(\phi)\; |_{\phi=\omega t} \\
I_{X,f_{i+1}^j} &= \frac{f_{i+1}^j - f_i^j}{u_{i+1}^j - u_i^j}\;\; \frac{u(t,X,\omega) - u_i^j}{u_{i+1}^j - u_i^j}\Lambda(\phi)\; |_{\phi=\omega t} \\
I_{w,f_i^j} &= \frac{f_{i+1}^j - f_i^j}{u_{i+1}^j - u_i^j}\;\; \frac{u_{i+1}^j - u(t,X,\omega)}{u_{i+1}^j - u_i^j}Xt\frac{d\Lambda(\phi)}{d\phi}\; |_{\phi=\omega t} \\
I_{w,f_{i+1}^j} &= \frac{f_{i+1}^j - f_i^j}{u_{i+1}^j - u_i^j}\;\; \frac{u(t,X,\omega) - u_i^j}{u_{i+1}^j - u_i^j}Xt\frac{d\Lambda(\phi)}{d\phi}\; |_{\phi=\omega t} \\
I_{f_i^j,f_i^j} &= \left[\frac{u_{i+1}^j - u(t,X,\omega)}{u_{i+1}^j - u_i^j}\right]^2 \\
I_{f_{i+1}^j,f_{i+1}^j} &= \left[\frac{u(t,X,\omega) - u_i^j}{u_{i+1}^j - u_i^j}\right]^2 \\
I_{f_i^j,f_{i+1}^j} &= \frac{u_{i+1}^j - u(t,X,\omega)}{u_{i+1}^j - u_i^j}\;\; \frac{u(t,X,\omega) - u_i^j}{u_{i+1}^j - u_i^j}
\end{aligned}
\tag{58}
$$

(ii) When $\widehat{u} \in [u_{-1}^1, u_{o^-}] \subset I_1$.

$$
I(t) = \begin{pmatrix}
I_{X,X} & I_{X,w} & I_{X,f_o} & 0 \cdots 0 & I_{X,f_{-1}^1} & 0 \cdots 0 \\
I_{w,X} & I_{w,w} & I_{w,f_o} & 0 \cdots 0 & I_{w,f_{-1}^1} & 0 \cdots 0 \\
I_{f_o,X} & I_{f_o,w} & I_{f_o,f_o} & 0 \cdots 0 & I_{f_o,f_{-1}^1} & 0 \cdots 0 \\
0 & 0 & 0 & & 0 & \\
\vdots & \vdots & \vdots & & \vdots & \\
0 & 0 & 0 & & 0 & \\
I_{f_{-1}^1,X} & I_{f_{-1}^1,w} & I_{f_{-1}^1,f_o} & 0 \cdots 0 & I_{f_{-1}^1,f_{-1}^1} & 0 \cdots 0 \\
0 & 0 & 0 & & 0 & \\
\vdots & \vdots & \vdots & & \vdots & \\
0 & 0 & 0 & & 0 &
\end{pmatrix}
\tag{59}
$$

where

$$
\begin{aligned}
I_{X,X} &= \left[\frac{K_o u_{o^-} + f_o - f_{-1}^1}{u_{o^-} - u_{-1}^1}\right]^2 \;\; \Lambda^2(\phi)\; |_{\phi=\omega t} \\
I_{w,w} &= \left[\frac{K_o u_{o^-} + f_o - f_{-1}^1}{u_{o^-} - u_{-1}^1}\right]^2 \;\; X^2 t^2 \left[\frac{d\Lambda(\phi)}{d\phi}\; |_{\phi=\omega t}\right]^2 \\
I_{f_o,f_o} &= \left[\frac{u(t,X,\omega) - u_{-1}^1}{u_{o^-} - u_{-1}^1}\right]^2 \\
I_{X,w} &= \left[\frac{K_o u_{o^-} + f_o - f_{-1}^1}{u_{o^-} - u_{-1}^1}\right]^2 \;\; Xt\; \Lambda(\phi)\frac{d\Lambda(\phi)}{d\phi}\; |_{\phi=\omega t}
\end{aligned}
$$

$$
\begin{aligned}
I_{X,f_o} &= \frac{K_o u_{o^-} + f_o - f_{-1}^1}{u_{o^-} - u_{-1}^1}\ \frac{u(t,X,\omega) - u_{-1}^1}{u_{o^-} - u_{-1}^1}\Lambda(\phi)\ |_{\phi=\omega t} \\
I_{w,f_o} &= \frac{K_o u_{o^-} + f_o - f_{-1}^1}{u_{o^-} - u_{-1}^1}\ \frac{u(t,X,\omega) - u_{-1}^1}{u_{o^-} - u_{-1}^1} Xt\frac{d\Lambda(\phi)}{d\phi}\ |_{\phi=\omega t} \\
I_{X,f_{-1}^1} &= \frac{K_o u_{o^-} + f_o - f_{-1}^1}{u_{o^-} - u_{-1}^1}\ \frac{u_{o^-} - u(t,X,\omega)}{u_{o^-} - u_{-1}^1}\Lambda(\phi)\ |_{\phi=\omega t} \\
I_{w,f_{-1}^1} &= \frac{K_o u_{o^-} + f_o - f_{-1}^1}{u_{o^-} - u_{-1}^1}\ \frac{u_{o^-} - u(t,X,\omega)}{u_{o^-} - u_{-1}^1} Xt\frac{d\Lambda(\phi)}{d\phi}\ |_{\phi=\omega t} \\
I_{f_o,f_{-1}^1} &= \frac{u(t,X,\omega) - u_{-1}^1}{u_{o^-} - u_{-1}^1}\ \frac{u_{o^-} - u(t,X,\omega)}{u_{o^-} - u_{-1}^1} \\
I_{f_{-1}^1,f_{-1}^1} &= \left[\frac{u_{o^-} - u(t,X,\omega)}{u_{o^-} - u_{-1}^1}\right]^2
\end{aligned}
\tag{60}
$$

(iii) When $\widehat{u} \in [u_{o^-}, u_{o^+}] \subset I_1$.

$$
I(t) = \begin{pmatrix}
I_{X,X} & I_{X,w} & I_{X,f_o} & 0 & \cdots & 0 \\
I_{w,X} & I_{w,w} & I_{w,f_o} & 0 & \cdots & 0 \\
I_{f_o,X} & I_{f_o,w} & I_{f_o,f_o} & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & & \\
0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\tag{61}
$$

where

$$
\begin{aligned}
I_{X,X} &= K_o^2\ \Lambda^2(\phi)\ |_{\phi=\omega t} \\
I_{w,w} &= K_o^2\ X^2 t^2 \left[\frac{d\Lambda(\phi)}{d\phi}\ |_{\phi=\omega t}\right]^2 \\
I_{f_o,f_o} &= 1 \\
I_{X,w} &= K_o^2\ Xt\ \Lambda(\phi)\frac{d\Lambda(\phi)}{d\phi}\ |_{\phi=\omega t} \\
I_{X,f_o} &= K_o\ \Lambda(\phi)\ |_{\phi=\omega t} \\
I_{w,f_o} &= K_o\ Xt\frac{d\Lambda(\phi)}{d\phi}\ |_{\phi=\omega t}
\end{aligned}
\tag{62}
$$

(iv) When $\widehat{u} \in [u_{o^+}, u_1^1] \subset I_1$.

$$
I(t) = \begin{pmatrix}
I_{X,X} & I_{X,w} & I_{X,f_o} & 0 & \cdots & 0 & I_{X,f_1^1} & 0 & \cdots & 0 \\
I_{w,X} & I_{w,w} & I_{w,f_o} & 0 & \cdots & 0 & I_{w,f_1^1} & 0 & \cdots & 0 \\
I_{f_o,X} & I_{f_o,w} & I_{f_o,f_o} & 0 & \cdots & 0 & I_{f_o,f_1^1} & 0 & \cdots & 0 \\
0 & 0 & 0 & & & & 0 & & & \\
\vdots & \vdots & \vdots & & & & \vdots & & & \\
0 & 0 & 0 & & & & 0 & & & \\
I_{f_1^1,X} & I_{f_1^1,w} & I_{f_1^1,f_o} & 0 & \cdots & 0 & I_{f_1^1,f_1^1} & 0 & \cdots & 0 \\
0 & 0 & 0 & & & & 0 & & & \\
\vdots & \vdots & \vdots & & & & \vdots & & & \\
0 & 0 & 0 & & & & 0 & & &
\end{pmatrix}
\tag{63}
$$

where

$$
\begin{aligned}
I_{X,X} &= \left[\frac{f_1^1 - K_o u_{o^+} - f_o}{u_1^1 - u_{o^+}}\right]^2 \ \Lambda^2(\phi)\mid_{\phi=\omega t} \\
I_{w,w} &= \left[\frac{f_1^1 - K_o u_{o^+} - f_o}{u_1^1 - u_{o^+}}\right]^2 \ X^2 t^2 \left[\frac{d\Lambda(\phi)}{d\phi}\mid_{\phi=\omega t}\right]^2 \\
I_{f_o,f_o} &= \left[\frac{u_1^1 - u(t,X,\omega)}{u_1^1 - u_{o^+}}\right]^2 \\
I_{X,w} &= \left[\frac{f_1^1 - K_o u_{o^+} - f_o}{u_1^1 - u_{o^+}}\right]^2 \ Xt\ \Lambda(\phi)\frac{d\Lambda(\phi)}{d\phi}\mid_{\phi=\omega t} \\
I_{X,f_o} &= \frac{f_1^1 - K_o u_{o^+} - f_o}{u_1^1 - u_{o^+}} \ \frac{u_1^1 - u(t,X,\omega)}{u_1^1 - u_{o^+}}\Lambda(\phi)\mid_{\phi=\omega t} \\
I_{w,f_o} &= \frac{f_1^1 - K_o u_{o^+} - f_o}{u_1^1 - u_{o^+}} \ \frac{u_1^1 - u(t,X,\omega)}{u_1^1 - u_{o^+}} Xt\frac{d\Lambda(\phi)}{d\phi}\mid_{\phi=\omega t} \\
I_{X,f_1^1} &= \frac{f_1^1 - K_o u_{o^+} - f_o}{u_1^1 - u_{o^+}} \ \frac{u(t,X,\omega) - u_{o^+}}{u_1^1 - u_{o^+}}\Lambda(\phi)\mid_{\phi=\omega t} \\
I_{w,f_1^1} &= \frac{f_1^1 - K_o u_{o^+} - f_o}{u_1^1 - u_{o^+}} \ \frac{u(t,X,\omega) - u_{o^+}}{u_1^1 - u_{o^+}} Xt\frac{d\Lambda(\phi)}{d\phi}\mid_{\phi=\omega t} \\
I_{f_o,f_1^1} &= \frac{u_1^1 - u(t,X,\omega)}{u_1^1 - u_{o^+}} \ \frac{u(t,X,\omega) - u_{o^+}}{u_1^1 - u_{o^+}} \\
I_{f_1^1,f_1^1} &= \left[\frac{u(t,X,\omega) - u_{o^+}}{u_1^1 - u_{o^+}}\right]^2
\end{aligned}
\tag{64}
$$

6 NUMERICAL EXAMPLES

In order to study the performance of the modified RPEM algorithm suggested for joint estimation of the driving frequency and the parameters of the nonlinear output function, the following simulations were performed.

***Example* 2: Convergence to the true parameter vector.**

The data were generated according to the following description: the driving wave was given by $u(t,X,\omega) = X\ \sin\omega t$ where $\omega = 2\pi 0.05$. Two static nonlinearities ($L = 2$) were used as shown in Fig. 5,

$$
\begin{aligned}
\text{grid}_1 &= (-1, -0.3, -0.15, 0.15, 0.3, 1) \\
\text{grid}_2 &= (-1, -0.3, 0.3, 1) \\
\theta_1 &= (-0.8, -0.3, 0.3, 0.8),\ \widehat{u}(t,X,\omega) \in I_1 \\
\theta_2 &= (-0.8, -0.5, 0.5, 0.8),\ \widehat{u}(t,X,\omega) \in I_2
\end{aligned}
\tag{65}
$$

where $\widehat{u}(t,X,\omega) \in I_1$ for positive slopes and $\widehat{u}(t,X,\omega) \in I_2$ for negative slopes, respectively. The additive noise was white zero mean Gaussian with variance $\sigma^2 = 0.01$.

The algorithm was initialized with $\lambda(0) = 0.95,\ \lambda_o = 0.99,\ P(0) = 0.01I,\ X = 1,\ K_0 = 1,\ f_0 = 0$ and $\omega(0) = 2\pi 0.02$. Further, the grid points in (65) were used, and the initial values for the nonlinearities were given by straight lines with unity slope.

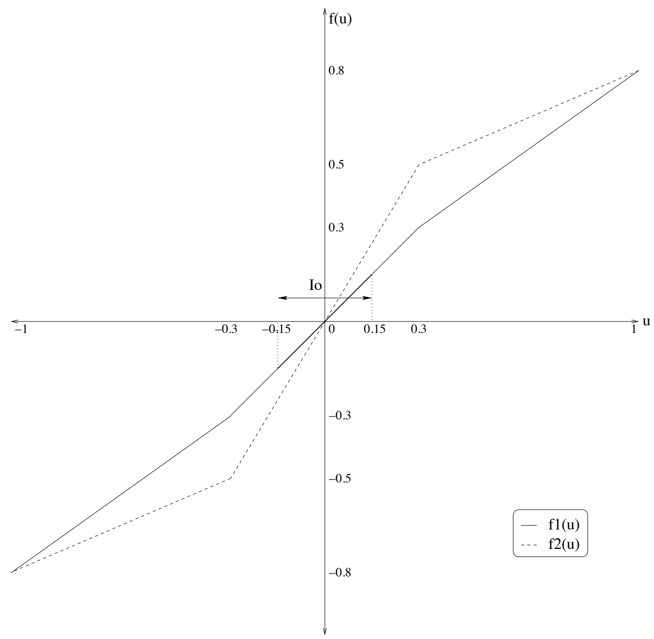

Figure 5: Static nonlinearities of Example 2.

The modeled signal and the estimated signal are given in Fig. 6. Also, the estimate of the driving frequency, the parameter estimates, and the prediction error are given in Figures 7-10. After 1000 samples the following estimates were obtained:

$$\begin{aligned}
\widehat{\omega} &= 0.3141 \\
\widehat{f}_o &= -0.0094 \\
\widehat{\theta}_1 &= (-0.8705, -0.2869, 0.3024, 0.8625) \\
\widehat{\theta}_2 &= (-0.8214, -0.4910, 0.5087, 0.8280)
\end{aligned}$$

and it can be concluded that the convergence to the true parameter vector is taking place.

***Example* 3: Tracking fundamental frequency variations.**

As in [5], to improve the ability of the modified algorithm to track fundamental frequency variations, the algorithm given in (17) is modified to

$$\begin{aligned}
\varepsilon(t) &= y(t) - \widehat{y}(t) \\
S(t) &= \psi^T(t)P(t-1)\psi(t) + R_2(t) \\
P(t) &= P(t-1) - P(t-1)\psi(t)S^{-1}(t)\psi^T(t)P(t-1) + R_1(t)
\end{aligned}$$

$$\begin{pmatrix} \widehat{\theta}_l(t) \\ \widehat{\theta}_n(t) \end{pmatrix} = \left[\begin{pmatrix} \widehat{\theta}_l(t-1) \\ \widehat{\theta}_n(t-1) \end{pmatrix} + P(t)\psi(t)\varepsilon(t) \right]_{D_M} \tag{66}$$

where $R_1(t)$ and $R_2(t)$ are the gain design variables (see [6] page 273). This modification transform the problem into an extended Kalman filter formulation.

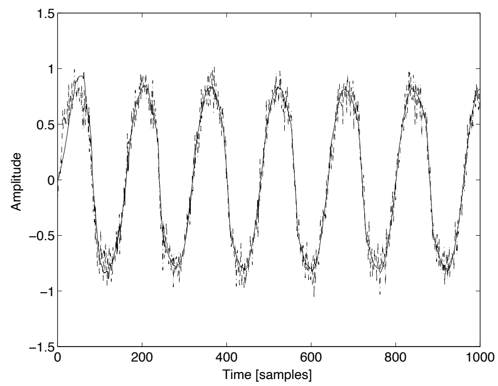

Figure 6: The modeled signal (dashed) and the model output (solid).

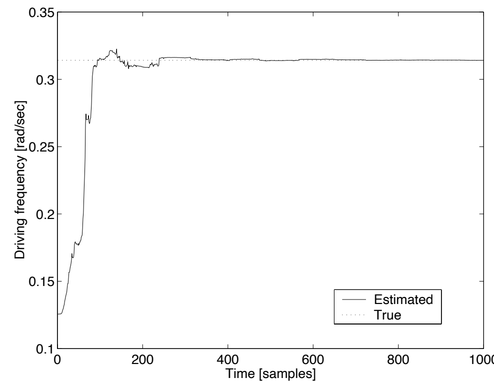

Figure 7: Convergence of the fundamental frequency.

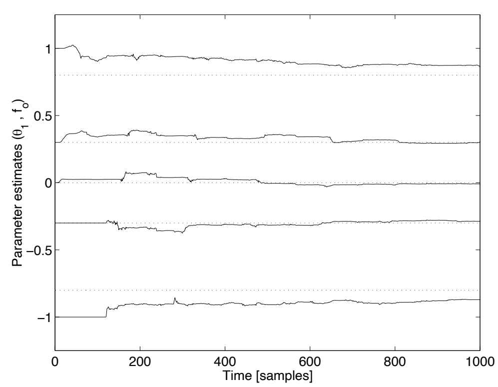

Figure 8: Parameter convergence.

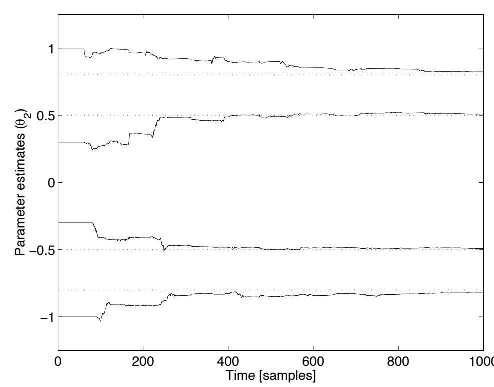

Figure 9: Parameter convergence.

The data were generated as in the last example and the algorithm was initialized with $P(0) = 0.001I$, $X = 1$, $K_0 = 1$, $f_0 = 0$ and $\omega(0) = 2\pi 0.02$ and the design variables were $R_1(t) = 0.0001I$ and $R_2(t) = 0.25$. Also, the grid points in (65) were used, and the initial values for the nonlinearities were given by straight lines with unity slope.

The modeled signal and the estimated signal are given in Fig. 11. Also, the true and estimated fundamental frequency are shown in Fig. 12. The parameter estimates, and the prediction error are given in Figures 13-15. After 2400 samples the following estimates were obtained:

$$\begin{aligned}
\widehat{\omega} &= 0.3474 \\
\widehat{f}_o &= -0.0499 \\
\widehat{\theta}_1 &= (-0.9169, -0.2709, 0.3477, 0.9313) \\
\widehat{\theta}_2 &= (-0.8897, -0.5422, 0.4102, 0.9036)
\end{aligned}$$

and it can be concluded that the algorithm of (66) has the ability to track the fundamental frequency variations.

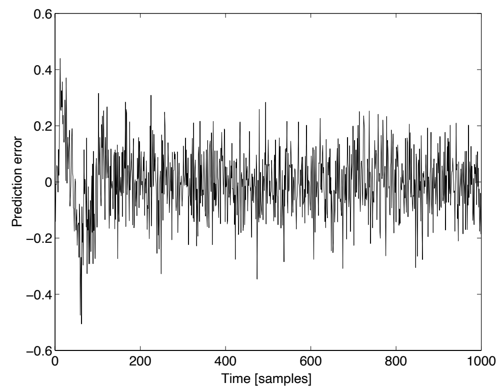

Figure 10: Prediction error.

***Example* 4: Performance of the RPE algorithm as compared to the CRB.**

In order to compare the performance of the algorithm with the derived CRB for the fundamental frequency estimation, 100 Monte Carlo simulations were performed with different noise realizations. The data were generated and the algorithm was initialized as in example 2. The statistics is based on excluding simulations that did not satisfy a margin of 5 standard deviations (as predicted by the CRB) from the true fundamental frequency. Both the CRB for the fundamental frequency estimate and the mean square error (MSE) value were evaluated for N=2000 and for different signal to noise ratios (SNR). The statistical results are plotted in Fig. 16.

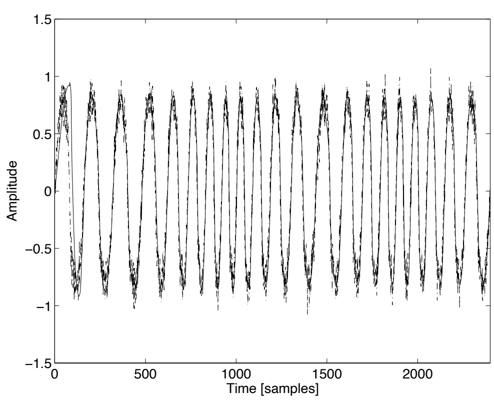

Figure 11: The modeled signal (dashed) and the model output (solid).

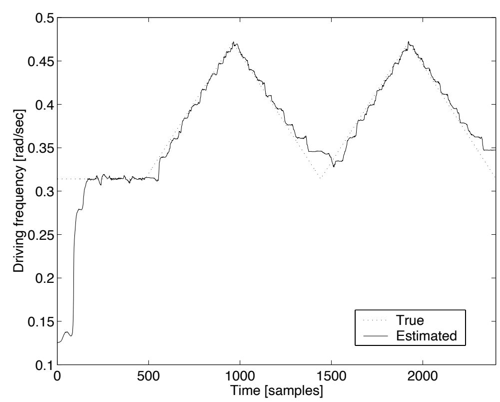

Figure 12: Tracking of the fundamental frequency.

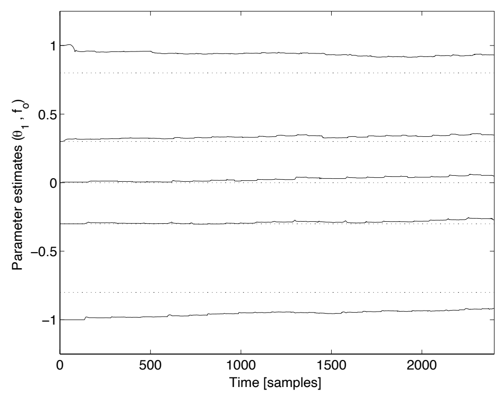

Figure 13: Parameter estimates.

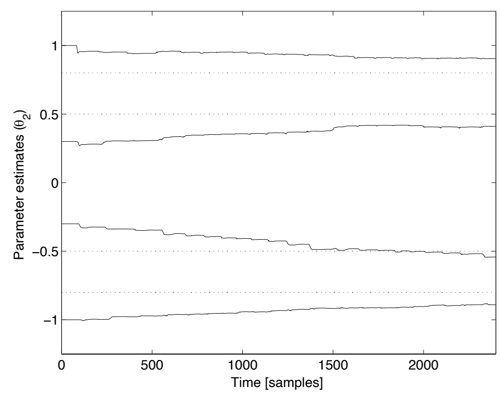

Figure 14: Parameter estimates.

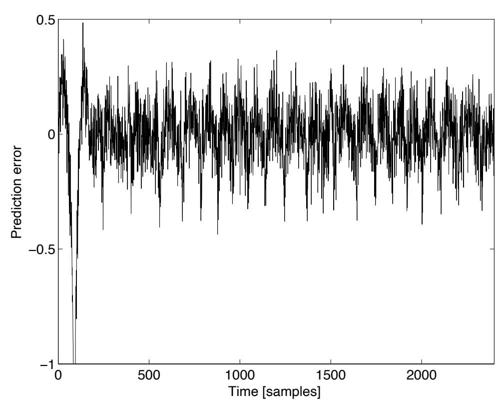

Figure 15: Prediction error.

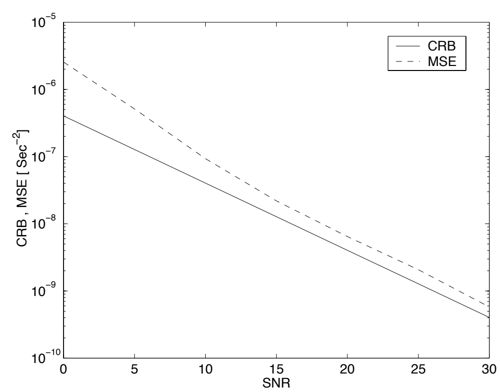

Figure 16: Statistical results.

7 CONCLUSIONS

A recursive harmonic signal estimation scheme has been modified by introducing an interval in the nonlinear block with fixed static gain. Then the modified algorithm was studied by numerical examples and it was proven that the algorithm is locally convergent to the true parameter vector and can easily modified to track fundamental frequency variations. Also, the CRB was calculated for the modified scheme. Monte Carlo experiments shows that the modified algorithm gives good results, in particular for moderate values of the SNR.

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