Trudinger–Moser inequality with remainder terms✩,✩✩

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Abstract

The paper gives the following improvement of the Trudinger–Moser inequality:

$$\sup_{\Omega} \int \nabla u^2 \, dx - \psi(u) \leq 1, \quad u \in C_0^\infty(\Omega),$$

related to the Hardy–Sobolev–Mazya inequality in higher dimensions. We show (0.1) with $\psi(u) = \int_{\Omega} V(x) u^2 \, dx$ for a class of $V > 0$ that includes

$$V(r) = \frac{1}{4r^2 (\log \frac{1}{r})^2 \max\{\sqrt{\log \frac{1}{r}}, 1\}},$$

which refines two previously known cases of (0.1) proved by Adimurthi and Druet [2] and by Wang and Ye [23]. In addition, we verify (0.1) for $\psi(u) = \lambda \|u\|_p^2$, as well as give an analogous improvement for the Onofri–Beckner inequality for the unit disk (Beckner [6]).

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1. Introduction

The Trudinger–Moser inequality [24,17,19,22,14]

$$\sup_{\Omega} \int_{\Omega} |\nabla u|^2 \, dx \leq 1, \quad u \in C^\infty_0(\Omega),$$

(1.1)

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, is an analog of the limiting Sobolev inequality in $\mathbb{R}^N$ with $N \geq 3$:

$$\sup_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \leq 1, \quad u \in C^\infty_0(\mathbb{R}^N),$$

(1.2)

We recall that restriction of inequalities involving the gradient norm to bounded domains is of essence when $N = 2$, since the completion of $C^\infty_0(\mathbb{R}^2)$ in the gradient norm is not a function space, and, moreover, since $\int_B |\nabla u|^2 \, dx$ on the unit disk $B \subset \mathbb{R}^2$ coincides with the quadratic form of the Laplace–Beltrami operator on the hyperbolic plane (a complete non-compact Riemannian manifold) when expressed in the coordinates of the Poincaré disk.

Both limiting Trudinger–Moser and Sobolev inequalities are optimal in the sense that they are false for any nonlinearity that grows as $s \to \infty$ faster than $e^{4\pi s^2}$ resp. $|s|^{2^*}$. Inequality (1.2) is also false if the nonlinearity $|u|^{2^*}$ is multiplied by an unbounded radial monotone function, although (1.1) on the unit disk holds also when the integrand is replaced by $e^{4\pi u^2} - 1/(1-r)^2$ [3,10].

This paper studies another refinement of (1.1), whose analogy in the case $N \geq 3$ is the Mazya’s refinement of (1.2), known as Hardy–Sobolev–Mazya inequality [13]:

$$\sup_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u|^{2^*} \, dx < \infty, \quad 2^* = \frac{2N}{N-2},$$

(1.3)

where

$$V_m(x) = \left(\frac{m-2}{2}\right)^2 \frac{1}{|x_1 + \cdots + x_m|^2}, \quad m = 1, \ldots, N-1.$$

It is false when $m = N$, and similarly, inequality (0.1) does not hold with $\psi(u) = \int_{\mathbb{B}} V(|x|)u^2 \, dx$, if $V$ is the two-dimensional counterpart of the Hardy’s radial potential, the Leray’s potential

$$V_{\text{Leray}}(r) = \frac{1}{4r^2(\log \frac{1}{r})^2}.$$

When $\psi(u) = \int_{\Omega} V(x)u^2 \, dx$, inequality (0.1) has been already established for two specific potentials $V$. In one case, proved by Adimurthi and Druet [2], $V(x) = \lambda < \lambda_1$, and $\lambda_1$ is the first eigenvalue of the Dirichlet Laplacian in $\Omega$. Note only that the inequality stated as a main result in [2] is formally weaker, but it immediately implies (0.1) with $V(x) = \lambda < \lambda_1$ via an elementary argument. It was conjectured by Adimurthi [1] that the inequality remains valid whenever one replaces $\int_{\Omega} \lambda u^2 \, dx$ with a general weakly continuous functional $\psi$, as long as $\|\nabla u\|_2^2 - \psi(u) > 0$.
for $u \neq 0$. Another known case of the inequality (0.1), with $\psi(u) = \int_B \frac{u^2}{(1-r^2)^2} \, dx$, is due to Wang and Ye [23]. Note that the result of Wang and Ye involves a non-compact remainder term, and that via conformal maps it extends to general domains.

In deciding about the natural counterpart of the Hardy–Sobolev–Mazya inequality in the two-dimensional case, we have to make a choice, which is insignificant in the case $N \geq 3$, between using the functional $\int e^{4\pi u^2}$ and the Orlicz norm $\|u\|_{\text{Orl}}$ associated with the integrand (in terms of the standard definition, with the function $e^{4\pi s^2} - 1$). The difference between the case $N \geq 3$ and $N = 2$ is in the fact that (1.3) can be equivalently rewritten as

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} V_m(x) u^2 \, dx \geq C \|u\|^2_{2*},$$

while from

$$\int_{\Omega} |\nabla u|^2 \, dx - \psi(u) \geq C \|u\|^2_{\text{Orl}}$$

(1.4)

for $N = 2$ inequality (0.1) does not follow, and instead one has its weaker version, with the bound on $\int_{\Omega} e^{Cu^2} \, dx$ with some $C$. In particular, in the case of Adimurthi–Druet, $V(x) = \lambda < \lambda_1$, inequality (1.4) is completely trivial while their actual result is very sharp. This example explains why we, following Wang and Ye, treat (0.1), and not (1.4), as a natural counterpart of (1.3).

The objective of this paper is to prove the inequality (0.1) with the more general (and in particular, stronger) remainder term $\psi(u)$ than in the two known cases. In Section 2 we study the case $p = 2$ and the radial potential on a unit disk, in Section 3 we extend the result to general bounded domains and to the values $p > 2$. In Section 4 we give corollaries to the inequalities, prove a related refinement of Onofri–Beckner inequality, and list some open problems.

In what follows, $B$ will denote an open unit disk, $\| \cdot \|_p$ will mean the $L^p(\Omega)$-norm when the domain is specified, and the subspace of radial functions of, say, Sobolev space $H^1_0(B)$ will be denoted $H^1_{0,\text{rad}}(B)$.

2. Remainder with a singular potential

2.1. Ground state alternative

We summarize first some relevant results on positive elliptic operators with singular potentials, drawing upon [18].

Let $\Omega \subset \mathbb{R}^N$ be a domain, and let $V$ be a continuous function in $\Omega$. We consider the functional

$$Q_V(u) = \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} V(x) u^2 \, dx, \quad \psi(u) = \int_{\Omega} V(x) u^2 \, dx, \quad u \in C_0^\infty(\Omega).$$

(2.1)

Assuming that $Q_V \geq 0$, one says that $\varphi \neq 0$ is a ground state of the quadratic form $Q_V$ if there exists a sequence $u_k \in C_0^\infty(\Omega)$, convergent to $\varphi$ in $H^1_{\text{loc}}(\Omega)$, such that $Q_V(u_k) \to 0$. Ground states are sign definite and, up to a constant multiple, unique in the class of positive solutions.
(that is, positive solutions without global integrability requirements or boundary conditions). If, additionally, \( \varphi \in H^1_0(\Omega) \), then \( \varphi \) is a minimizer for the Rayleigh quotient

\[
\inf_{u \in H^1_0(\Omega), u \neq 0} \frac{\| \nabla u \|^2_2}{\int_\Omega V(x) u^2 \, dx}.
\]

There are ground states, however, for which \( \| \nabla \varphi \|_2 = \infty \). This is the case, in particular, for the ground state \( \varphi(x) = \sqrt{\log \frac{1}{|x|}} \) in the case of Leray potential,

\[
Q_V(u) = \int_B |\nabla u|^2 \, dx - \int_B V_{\text{Leray}} u^2 \, dx.
\]

(Leray inequality [9] states that this form is nonnegative.) Similarly, Hardy inequality in \( \mathbb{R}^N \), \( N \geq 3 \), with the radial potential \( V_N \) admits a ground state \( \varphi(x) = |x|^{\frac{2-N}{2}} \), whose gradient norm is infinite as well.

Existence of a ground state is connected to the property of weak coercivity. The form (2.1) is called weakly coercive if there exists an open set \( E \) relatively compact in \( \Omega \) and a constant \( \delta > 0 \), such that

\[
Q_V(u) \geq \delta \left( \int_E u \, dx \right)^2, \quad u \in C_0^\infty(\Omega).
\]

An equivalent criterion of weak coercivity (see [21]) is a seemingly stronger condition that there exists a continuous function \( W > 0 \) such that

\[
Q_V(u) \geq \int_\Omega W(x) (|\nabla u|^2 + u^2) \, dx, \quad u \in C_0^\infty(\Omega).
\]

It is well known that the form (2.1) is nonnegative if and only if it admits a positive solution. However, not any positive solution is a ground state, and in fact, existence of a ground state and weak coercivity for a nonnegative form are mutually exclusive.

**Theorem 2.1** *(Ground state alternative of Murata).* (See [15,18].) A nonnegative functional (2.1) is either weakly coercive or has a ground state.

If the form (2.1) is nonnegative (and thus admits a positive solution \( v \)) it can be represented as an integral of a positive function. This representation is known as *ground state transform* or *Jacobi identity*:

\[
\int_\Omega |\nabla u|^2 \, dx - \int_\Omega V(x) u^2 \, dx = \int_\Omega v^2 \left| \nabla \frac{u}{v} \right|^2 \, dx.
\]
2.2. Remainder in the Trudinger–Moser inequality, radial case

**Definition 2.2.** We say that a radial function on the unit disk \( V(|x|) \in \mathcal{V} \) if \( V(r) \) is a nonnegative continuous function on \((0, 1)\) and the function \( r \mapsto (1 - r^2)^2 V(r) \) is nonincreasing.

**Lemma 2.3.** If \( V \in \mathcal{V} \) then

\[
\sup_{u \in H^1_0(B), Q_V(u) \leq 1} \int_B e^{4\pi u^2} \, dx = \sup_{u \in H^1_0, rad(B), Q_V(u) \leq 1} 2\pi \int_B e^{4\pi u(r)^2} r \, dr. \tag{2.2}
\]

**Proof.** Consider \( B \) as the Poincaré disk representing the hyperbolic plane \( \mathbb{H}^2 \). The quadratic form of Laplace–Beltrami operator on \( \mathbb{H}^2 \) in the Poincaré disk coordinates is \( \int_B |\nabla u|^2 \, dx \).

Let \( u^\# \) denote the spherical decreasing rearrangement of \( u \in H^1_0(B) \) relative to the Riemannian measure of the Poincaré disk, \( d\mu = \frac{4 \, dx}{(1 - r^2)^2} \), and recall that the Hardy–Littlewood and the Polya–Szegö inequalities relative to these rearrangements remain valid [5]. In particular, by the Hardy–Littlewood inequality,

\[
\int_B V(|x|) u(x)^2 \, dx = \int_B \frac{1}{4} (1 - |x|^2)^2 V(|x|) u(x)^2 \, d\mu 
\leq \int_B \frac{1}{4} (1 - r^2)^2 V(r) u^\#(r)^2 \, d\mu = \int_B V(r) u^\#(r)^2 \, dx,
\]

and thus, taking into account the Polya–Szegö inequality, we have \( Q_V(u) \geq Q_V(u^\#) \). From this and the “hyperbolic” Hardy–Littlewood inequality applied to \( \int e^{4\pi u^2} \, dx \) it follows that the right hand side in (2.2) is not less then the left hand side, while the converse is trivial. \( \square \)

**Theorem 2.4.** Let \( N = 2 \), let \( V \in \mathcal{V} \), and assume that, for some \( \alpha > 0 \),

\[
\lim_{r \to 0} r^2 \left( \log \frac{1}{2} \right)^{2+\alpha} V(r) = 0. \tag{2.3}
\]

Then the quantity

\[
S_V = \sup_{u \in H^1_0(B), Q_V(u) \leq 1} J(u), \quad J(u) = \int_B e^{4\pi u^2} \, dx,
\]

is finite if and only if the quadratic form \( Q_V \) is weakly coercive.

**Proof.** 1. Necessity. Assume that \( Q_V \) is not weakly coercive. If \( Q_V(w) < 0 \) for some \( w \in H^1_0(B) \), then \( J(kw) \to \infty \) and thus \( S_V = +\infty \). Assume now that \( Q_V \geq 0 \). Then by the ground state alternative, \( Q_V \) has a ground state \( \varphi > 0 \) approximated by a \( C_0^\infty \)-sequence \( u_k \to \varphi \) in \( H^1_0(B) \) such that \( Q_V(u_k) \to 0 \). Then, noting that there exist \( \epsilon > 0 \) and \( \delta > 0 \), such that for each \( k \), inequality \( u_k \geq \epsilon \) holds on some set of measure larger than \( \delta \), we have \( J(u_k / \sqrt{Q_V(u_k)}) \to \infty \), which again yields \( S_V = +\infty \). (Of course, \( Q_V(u_k) \neq 0 \) since otherwise \( u_k \) equals \( \varphi \) up to a constant multiple, which is a contradiction since \( \varphi > 0 \) and \( u_k \in C_0^\infty(B) \).)
2. **Sufficiency.** Assume that $Q_V$ is weakly coercive. By Lemma 2.3 it suffices to consider the problem restricted to radial decreasing functions. Since $Q_V$ is nonnegative, equation $Q_V' (u) = 0$ has a positive radial $C^1$-solution $\varphi$. The latter fact can be inferred from the fact that $V$, by (2.3), belongs to the local Kato class $K_2$ (see [4]). Let us normalize $\varphi$ by dividing it by $\varphi(0)$, so that $\varphi(0) = 1$ and $\varphi(r) \leq 1$. Define now

$$s(r) = e^{\int_{[r/e]}^{1/e} \frac{\varphi}{\varphi(r)} t^2}, \quad 0 < r < 1,$$

(2.4)

so that the function $s(r)$ satisfies

$$\frac{s'(r)}{s(r)} = \frac{1}{r \varphi(r)^2}.$$

Since $\varphi(0) = 1$, we have $s(r) = \gamma r + o_{r \to 0}(r)$ with some $\gamma > 0$, which implies that $s(r)$ defines a monotone $C^1$-homeomorphism between $[0, 1)$ and $[0, s(1))$, where $s(1) = \lim_{r \to 1} s(r)$ may be, generally speaking, infinite. Let $w : [0, s(1)) \to [0, 1)$ be the function

$$w(s(r)) = u(r)/\varphi(r).$$

(2.5)

Then, writing $Q_V$ in the ground state transform form and changing the radial integration variable from $r$ to $s(r)$ we get

$$Q_V(u) = \int_{B(s(1))} |w'|^2 dx.$$

Assume first that $s(1) < \infty$. Then, taking into account that $\varphi \leq 1$ and $r \leq s(r)/s(1)$ (which is easy to infer from (2.4)), we have

$$S_V \leq \sup_{|\nabla u|^2 = 1_{B(s(1))}} \int_{B(s(1))} e^{4\pi \varphi(r(s))} w(s)^2 s \, dx \, d\theta \leq \sup_{|\nabla u|^2 = 1_{B(s(1))}} \int_{B(s(1))} e^{4\pi w^2} dx < \infty,$$

which proves the theorem in this case. Assume now that $s(1) = +\infty$. Then $Q_V(u) = \int_{\mathbb{R}^2} |\nabla u|^2 dx$. Let $w_k(s) = 1$ for $r < k$, $w_k(s) = \log \frac{1}{k}$ for $k \leq s < k^2$, $w_k(s) = 0$ for $s \geq k^2$. Then the sequence $\varphi(r) w_k(s(r))$ fulfills the definition of approximating sequence for the ground state $\varphi$ of $Q_V$. This, however, in view of the ground state alternative, contradicts the assumption that $Q_V$ is weakly coercive. Thus $s(1) < \infty$, in which case the theorem is already proved. □

**Example 2.5.** (a) Adimurthi and Druet [2]: the constant potential $V(r) = \lambda < \lambda_1$; where $\lambda_1$ is the first eigenvalue of the Dirichlet Laplacian, satisfies the assumptions of Theorem 2.4.

(b) Potential $V_{\text{Leray}}(r) = \frac{1}{4r^2 \log \frac{1}{r}^2}$ gives $S_V = +\infty$, since $Q_{V_{\text{Leray}}}$ has a ground state $\varphi(r) = \sqrt{\log \frac{1}{r}}$.

(c) Another potential satisfying the assumptions of Theorem 2.4 is

$$V_\gamma(r) = \frac{1}{4r^2 \log \frac{1}{r}^2 \max\{(\log \frac{1}{r})\gamma, 1\}}, \quad \gamma \in \left(0, \frac{4}{e^2 - 1}\right).$$
Since $V_\gamma < V_{\text{Leray}}$ with the strict inequality on $(0, e^{-1})$, $Q_V$ is weakly coercive. The potential $V(r) = \frac{1}{(1-r^2)^2}$, for which inequality (0.1) was proved in [23], is smaller than $V_\gamma(r)$, which (or comparison with the Hardy inequality) implies that $V_\gamma(r)$ has the optimal multiplicative constant and that the set $\{Q_{V_\gamma}(u) \leq 1\}$ is not bounded in $H^1_0(B)$.

3. The non-radial case and the $L^p$-remainder

We start with an elementary extension of the result of the previous section to the general bounded domain. We recall that $u^\#$ denotes rearrangement with respect to the Riemannian measure on the hyperbolic plane.

**Theorem 3.1.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $R = \sqrt{\frac{\|\Omega\|}{\pi}}$, $V \in L^1_{\text{loc}}(\Omega)$, and let

$$\tilde{V}(r) = \left[\left(1 - |x|^2/R^2\right)^2 V(x/R)\right]^\#(r).$$

**Theorem 3.2.** Assume that $\tilde{V} \in \mathcal{V}$ and satisfies (2.3), with some $\alpha > 0$. If the form $Q_{\tilde{V}} : H^1_{0,\text{rad}}(B) \to \mathbb{R}$, defined as in (2.1), is weakly coercive, then

$$S_V = \sup_{u \in C^\infty_0(\Omega) : Q_V(u) \leq 1} \int_{\Omega} e^{4\pi u^2} \, dx < \infty.$$

**Proof.** Rescale the problem to a domain of the area $\pi$. Reduce the problem to the radial problem on a unit disk by using rearrangements with respect to the Riemannian measure of $\mathbb{H}^2$ and apply Theorem 2.4. \(\square\)

For the rest of the section we consider the maximization problem

$$S_{\lambda,p} = \sup_{u \in C^\infty_0(\Omega) : Q_{\lambda,p}(u) \leq 1} \int_{\Omega} e^{4\pi u^2} \, dx < \infty,$$

where

$$Q_{\lambda,p}(u) = \int_{\Omega} |\nabla u|^2 \, dx - \lambda \|u\|_p^2,$$

and $\Omega \subset \mathbb{R}^2$. We will use the following constant:

$$\lambda_p = \inf_{u \in C_0^1(\Omega^*) : \|u\|_p = 1} \int_{\Omega^*} |\nabla u|^2 \, dx, \quad p > 0,$$

where $\Omega^*$ is the open ball of radius $\sqrt{\frac{\|\Omega\|}{\pi}}$. 

Theorem 3.3. Let $2 < p < \infty$ and $\lambda < \lambda_p$. Then

$$S_{\lambda,p} = \sup_{u \in C_0^\infty(\Omega): \, Q_{\lambda,p}(u) \leq 1} \int_\Omega e^{4\pi u^2} \, dx < \infty.$$ 

Proof. It suffices to verify the assertion in restriction to positive radial decreasing $H_0^1$-functions on $\Omega^*$ when $\Omega^*$ is the unit disk $B$. Let us represent $Q_{\lambda,p}(u)$ as $Q_{Vu}(u)$ with $Vu(u) = \lambda u^{p-2} \|u\|_{p^{-2}}^{p-2}$, $u \in H_0^1$, rad. Observe that by Hölder inequality

$$\int_B u^{p-2} \varphi^2 \, dx \leq \|u\|_{p^{-2}}^{p-2} \|\varphi\|_p^2,$$

and therefore $Q_{Vu}(\varphi) \geq Q_{\lambda,p}(\varphi) \geq 0$. Consequently, there exists a positive radial solution $\varphi_u$ to the linear equation $-\Delta \varphi = Vu \varphi$ in $B$. Since, by the standard radial estimate, $Vu(r) \leq C(\log \frac{1}{r})^{p-2}$, one has $\varphi_u \in C^1(B)$, and the maximum of $\varphi_u$ is at the origin. We assume without loss of generality that $\varphi_u(0) = 1$. By the ground state transform we have for any $v \in C_0^\infty(B)$,

$$Q_{Vu}(v) = \int_B \varphi_u^2 \left| \nabla v \varphi_u \right|^2 \, dx, \quad v \in C_0^\infty(B).$$

Let now

$$s_u(r) = e^\int_0^r \frac{dr}{\varphi_u(t)}^2, \quad 0 < r < 1,$$

and note that this function satisfies

$$\frac{s_u'(r)}{s_u(r)} = \frac{1}{r \varphi_u(r)^2}.$$ 

Observe that since $\varphi_u(0) = 1$ and $\varphi_u$ is a classical solution, we have $s_u(r) = \gamma r + o_{r \to 0}(r)$ with some $\gamma > 0$, and thus the mapping $r \mapsto s_u(r)$ is a monotone $C^1$-homeomorphism between $[0,1]$ and $[0, s_u(1))$. We will show now that $\varphi_u$ is bounded away from zero near $r = 1$, uniformly in an $H^1_{0,\text{rad}}(B)$-ball of $u$. First note that if for some $u \in H^1_{0,\text{rad}}(B)$ one has $\varphi_u(1) = 0$, then $\varphi_u$ is the first eigenfunction for the Dirichlet eigenvalue problem $-\Delta \varphi = Vu \varphi$ in $B$. From the Hölder inequality and the definition of $\lambda_p$ we get:

$$\int_B |\nabla \varphi_u|^2 \, dx = \int_B Vu \varphi^2 \, dx \leq \lambda \left( \int_B \left( \frac{u}{\|u\|_p} \right)^p \right)^{1-2/p} \left( \int_B \varphi_u^p \right)^{2/p} \leq \lambda \lambda_p^{-1} \int_B |\nabla \varphi_u|^2 \, dx < \int_B |\nabla \varphi_u|^2 \, dx,$$

a contradiction. Thus $\varphi_u(1) > 0$ for any $u$, and it remains to show that $\varphi_u(r)$ has a common positive lower bound for all $u$ and all $r$ near 1. Indeed, assume that there is a sequence $u_k$ with
Q_{\lambda,p}(u_k) \leq 1$, and a sequence $r_k \to 1$ such that $\varphi_{u_k}(r_k) \to 0$ and $-\Delta \varphi_{u_k} = \lambda u_k^{p-2} \varphi_{u_k}$. Note that since $\lambda < \lambda_p$, the sequence $u_k$ is bounded in $H^1_0(B)$, and without loss of generality we may assume that $u_k \to u$ in $H^1_0(B)$ with $Q_{\lambda,p}(u) \leq 1$. From here one can easily derive that $\varphi_{u_k}$ converges uniformly to some nonnegative $\varphi$ with $\varphi(1) = 0$, and that $\varphi$ satisfies the equation $-\Delta \varphi = V u \varphi$. In other words, $\varphi = \varphi_u$ and we have $\varphi_u(1) = 0$, which is a contradiction. We conclude that there exists $\epsilon > 0$ and $\delta > 0$, such that $\inf_{r \in [1-\epsilon,1]} \varphi_u(r) \geq \delta$. This implies that there is a number $S$ such that $s_u(1) \leq S$ for all $u$ satisfying $Q_{\lambda,p}(u) \leq 1$.

For each $v \in H^1_{0,\text{rad}}(B)$ define the following function on $[0, s_u(1))$:

\[
w_{v;u}(s_u(r)) = v(r).
\]

Then, applying the ground state transform and changing the radial integration variable from $r$ to $s_u$, we have

\[
Q_{V_u}(v) = \int_B \varphi_u^2 \left| \nabla \frac{v}{\varphi_u} \right|^2 \, dx = \int_{B_{s_u(1)}} \left| w_{v;u}'(|x|) \right|^2 \, dx, \quad v \in H^1_{0,\text{rad}}(B).
\]

By setting $v = u$, we get from here

\[
Q_{\lambda,p}(u) = \int_{B_{s_u(1)}} \left| w'_{u;u}(|x|) \right|^2 \, dx, \quad u \in H^1_{0,\text{rad}}(B).
\]

Then, taking into account that $\varphi_u \leq 1$ for every $u$, we arrive at

\[
S_{\lambda,p} \leq S^2 \sup_{\int_B |\nabla w|^2 = 1} \int_B e^{4\pi w(|x|)^2} \, dx < \infty,
\]

which proves the theorem.

\section{Related inequalities}

The arguments in Sections 2 and 3 allow to give the following refinement of the Onofri–Beckner inequality [16,6] for the unit disk (there is an earlier analogous refinement of Onofri inequality for $S^2$ by Chongwei Hong [8, Theorem 1.6]), see [10,11] for the hyperbolic space. The original inequality for the unit disk is

\[
\log \left( \frac{1}{\pi} \int_B e^u \, dx \right) + \left( \frac{1}{\pi} \int_B e^u \, dx \right)^{-1} \leq 1 + \frac{1}{16\pi} \int_B |\nabla u|^2 \, dx, \quad u \in C^\infty_0(B). \tag{4.1}
\]

\textbf{Theorem 4.1.} Let $\Omega = B$ and assume that $\psi(u) = \int_B V u^2 \, dx$ with $V$ as in Theorems 2.4 and 3.1, or that $\psi(u) = \lambda \| u \|^2_{L^p}, \lambda < \lambda_p, p > 2$, as in Theorem 3.3. Then for every $u \in C^\infty_0(B)$,

\[
\log \left( \frac{1}{\pi} \int_B e^u \, dx \right) + \left( \frac{1}{\pi} \int_B e^u \, dx \right)^{-1} \leq 1 + \frac{1}{16\pi} \left( \int_B |\nabla u|^2 \, dx - \psi(u) \right). \tag{4.2}
\]
Proof. We give the proof for the case of the remainder term $\psi$ as in Theorem 2.4. The proofs in other cases are analogous. By the standard rearrangement argument it suffices to consider the radially symmetric functions.

Assume first that $u \geq 0$. Without loss of generality we may assume that $u$ is radial. Let us use the coordinate transformation (2.4) and the substitution (2.5). Taking into account that the function $F(t) := \log t + t^{-1}$ is increasing on $(1, \infty)$, that the function $\varphi$, involved in the transformation, does not exceed 1, and that, as it is immediate from (2.4), $s(r)/s(1) \geq r$ we have from (4.1)

$$F\left(\frac{1}{\pi} s(1)^2 \int_{B_s(1)} e^{\varphi(r(s))} w(s) \frac{r(s)^2 \varphi(r(s))^2}{s^2} \, dx(s)\right) \leq 1 + \frac{1}{16\pi} \int_{B_{s(1)}} |\nabla w|^2 \, dx, \quad w \in H^1_{0,\text{rad}}(B_{s(1)}).$$

Using (2.5) in order to return to the original variable $u$, we immediately have (4.2) for $u \geq 0$.

Consider now the case $u \leq 0$. Without loss of generality we again assume that $u$ is radial. Then, taking into account (2.4), (2.5), $\varphi \leq 1$, $r \leq s(r)$, and the fact that the function $F$ is decreasing on $(0, 1)$, we have

$$F\left(\frac{1}{\pi} \int_{B} e^{u} \, dx\right) \leq F\left(\frac{1}{\pi} \int_{B} e^{w(s(r))} \, dx\right)$$

$$= F\left(\frac{1}{\pi} s(1)^2 \int_{B_s(1)} e^{w(s)} \frac{s^2}{r(s)^2 \varphi(r(s))^2} \, dx(s)\right)$$

$$\leq F\left(\frac{1}{\pi} s(1)^2 \int_{B_s(1)} e^{w(s)} \, dx(s)\right)$$

$$\leq 1 + \frac{1}{16\pi} \int_{B_{s(1)}} |\nabla w|^2 \, dx = 1 + \frac{1}{16\pi} Q_V(u).$$

Finally, we write a general $u$ as $u = u^+ + (-u^-)$ and note that the function $\log t + 1/t$ is subadditive on $(0, \infty)$. We leave it to the reader to prove the subadditivity with help of the following sketch: collect the logarithmic terms in the subadditivity inequality into a single logarithm, invert the logarithm, and replace the resulting exponential function by its Taylor polynomial up to the order 2. Inequality (4.2) is then immediate from the cases where $u \geq 0$ and $u \leq 0$. \qed

Corollary 4.2 (Inequality of Adimurthi–Druet type). Let $Q(u) = \|\nabla u\|^2_2 - \psi(u)$ be any of the functionals $Q_V$ as in Theorems 2.4 and 3.1, or the functional $Q_p$, as in Theorem 3.3. Then

$$\sup_{\|\nabla u\|_2 \leq 1} \int_{\Omega} e^{4\pi(1+\psi(u))u^2} \, dx \leq \sup_{\|\nabla u\|_2 \leq 1} \int_{\Omega} e^{\frac{4\pi u^2}{1-\psi(u)^2}} \, dx < \infty.$$
Proof. Note first that the integral in the left hand side is smaller than the integral in the right hand side by the inequality $(1 + \psi)(1 - \psi) < 1$. Let $u = \sqrt{\gamma} v$ with $\|\nabla v\|_2^2 = 1$. Then $Q(u) \leq 1$ is equivalent to $\gamma - \gamma \psi(v) \leq 1$, i.e. $\gamma \leq \frac{1}{1 - \psi(v)}$. Write (0.1), substitute $u^2 = \gamma v^2$ into the integral and rename $v$ as $u$.

Corollary 4.3. Let $\| \cdot \|_{\text{Orl}}$ denote the Orlicz norm associated with the Trudinger–Moser functional on a bounded domain $\Omega \subset \mathbb{R}^2$, and let $Q(u) = \|\nabla u\|_2^2 - \psi(u)$ be any of the functionals $Q_V$ as in Theorems 2.4 and 3.1, or the functional $Q_p$, as in Theorem 3.3. Then there exists a $C > 0$ such that

$$\int_{\Omega} |\nabla u|^2 \, dx - \psi(u) \geq C\|u\|_{\text{Orl}}^2.$$ 

Proof. Assume first that $Q(u) = 1$. From the uniform bound on $\int_{\Omega} (e^{4\pi u^2} - 1) \, dx$ in (0.1) follows a uniform bound for the Orlicz norm, which yields the inequality under the constraint $Q(u) = 1$. It remains to use the standard homogeneity argument.

4.1. Open problems

(1) Does the inequality (0.1) hold for general bounded $\Omega$, all potentials $V$ of the local Kato class $K_2$ and all $p \in (0, \infty)$, as long as the constraint functional $Q$ remains weakly coercive?

(2) When $\Omega = \mathbb{R}^2$, inequality (0.1) with $Q(u) = \|\nabla u\|_2^2$ is false, since the form $\|\nabla u\|_2^2$ on the whole $\mathbb{R}^2$ admits a ground state 1. On the other hand, the inequality holds when $Q(u) = \|\nabla u\|_2^2 + \|u\|_2^2$ (Ruf [20]). Furthermore, as it is shown in [10], inequality (0.1) with $Q(u) = \|\nabla u\|_2^2$ holds for a simply connected (generally unbounded) domain $\Omega \subset \mathbb{R}^2$ if and only if $\|\nabla u\|_2^2 \geq \lambda \|u\|_2^2$ with some $\lambda > 0$. In both results the condition is $L^2$-coercivity, $Q(u) \geq C\|u\|_2^2$. It is natural then to ask, for unbounded domains, if there are weaker coercivity conditions on $Q$ that yield (0.1)?

(3) Since Hardy–Sobolev–Mazya inequalities can be derived from Caffarelli–Kohn–Nirenberg inequalities [7] via the ground state transform, it is natural to ask what could be an analog of Caffarelli–Kohn–Nirenberg inequalities related to the remainder estimates of the Hardy–Moser–Trudinger type.

(4) Our reduction to the radial case is of tentative character, as it is based on rearrangements specific to the hyperbolic plane which resulted in a restrictive condition of weighted monotonicity on the potential. Perhaps more general rearrangements satisfying Polia–Szegö and Hardy–Littlewood inequalities (see [12]) can be used to relax the monotonicity condition on the potential.

References