Acknowledgements

The source and inspiration for this report comes to a large extent from my supervisor Stefan Leupold [1], but I take full responsibility for everything that’s in it.
Abstract

In this report we will calculate the amplitude for a scalar-to-scalars \( \phi_3 \to \phi_2 \phi_2 \) decay which involves a triangle loop. We compute the real and imaginary part of the amplitude separately and will argue that this is much more straightforward and practical in this case rather than having to deal with or worry about branch cuts of logarithms.

We will derive simple cutting rules closely related to the imaginary part of the amplitude. In doing this, we derive a formula that deals with expressions of the form \( \delta[ f(x, y)] \delta[ g(x, y)] \), containing two Dirac delta functions.
## Contents

1 Introduction ........................................... 1  
  1.1 The Lagrangian ........................................ 2  

2 The Amplitude ........................................ 3  
  2.1 The Poles of the Integrand ............................. 4  
  2.2 Wick Rotation .......................................... 6  
  2.3 The Real Part of the Amplitude ......................... 9  

3 The Imaginary Part of the Amplitude .................... 11  
  3.1 Cutkosky Cutting Rules ................................. 11  
  3.2 Sokhotsky-Plemelj Formula ............................ 13  
  3.3 Retarded Propagators $\tilde{D}_R$ ....................... 14  
  3.4 Poles of $\tilde{D}_R(k;m)$ .............................. 15  
  3.5 $\tilde{D}_F$ in terms of $\tilde{D}_R$ ....................... 17  
  3.6 $\text{Im} F = \text{Im} F_1 + \text{Im} F_{2+3}$ in terms of $\tilde{D}_R$ .......................... 20  
  3.7 Sweet Dirac-$\delta$ Formula ............................ 23  
  3.8 $\text{Im} F_{2+3} = 0$ .................................. 25  
  3.9 The Cutting Theorem and the Computation of $\text{Im} F$ .................... 28  
    3.9.1 The Zeros and the Jacobi Determinant ................ 29  

4 Results and Summary .................................. 31  

A The MATLAB Integrand ................................ 33  

B The MATLAB Script ..................................... 33  

References ............................................. 35  

List of Figures

1  Feynman diagram for the process $\phi_3 \rightarrow \phi_2\phi_2$ with the momenta and masses indicated. The decaying particle $\phi_3$ (double line) has mass $M$. The two outgoing particles $\phi_2$ and $\phi_2$ are identical with mass $m$. The dashed lines represents virtual particles of mass $\eta$. .................................................. 2

2  The integration path and the poles in the complex $l_0$-plane. ........................................ 5

3  The real part of the integral (2.25) plotted for different values of $-C$ with $A,B$ and $m^2$ fixed. .................................................. 10

4  A cut diagram with the cut indicated by the diagonal dots. ........................................ 12

5  The poles of the retarded propagator in the complex $k_0$-plane. Both lie in the lower half-plane. .................................................. 17

6  A cut diagram with the cut indicated by the diagonal dots. ........................................ 27
1 Introduction

One of the more interesting decays in the Standard Model (SM) of particle physics is the rare decay of the neutral pion into a dielectron

\[ \pi^0 \rightarrow e^+e^- . \]  

(1.1)

This is because apparently, there is a discrepancy of about 3σ between the SM prediction and the experimentally obtained value \[2, 3\]. The decay (1.1) is extremely rare with a branching ratio \[3, 4\]

\[ \text{Br}(\pi^0 \rightarrow e^+e^-) = 7.48 \times 10^{-8} , \]

hence it is sensitive to contributions from possible physics beyond the Standard Model. The decay (1.1) will be studied in more detail in \[5\]. Here we will consider the scalar version of it.

The aim of this report is to obtain an expression for the decay amplitude of the scalar decay

\[ \phi_3 \rightarrow \phi_2\phi_2 \]  

(1.2)

which can be considered as a toy model for the real life decay (1.1). Here, we will derive several results in detail which we will use, not only here to obtain the amplitude, but also throughout \[5\]. Hence this report may be used as an introduction to the techniques used in \[5\].

The Feynman diagram for the pertinent decay is shown in figure 1 below, where the corresponding masses and momenta are also shown. The virtual particles are massless, but for consistency one needs to introduce a small mass \( \eta \) for them also, as we will show later on.

The task is to calculate the amplitude for the scalar decay. We do this by splitting up the job into two parts, namely: calculating the real and imaginary part of the amplitude separately. This is done for several reasons. One reason is that in doing this one needs not worry about the branches of any logarithms that might show up, we simply use the real part of the logarithms. In calculating the imaginary part we essentially have to derive the Cutkosky rules \[6\] for the pertinent diagram. This is the other reason for using this method, because it turns out that one can derive model-independent results from the imaginary part of the amplitude \[5\]. Apart from this as motivation, the Cutkosky rules can also be used in dispersive methods to reconstruct the full amplitude from the imaginary part \[5\]. Hence, understanding where the Cutkosky rules come from and how to use them can be of great advantage in calculations involving loop-diagrams in quantum field theories.
Figure 1: Feynman diagram for the process $\phi_3 \rightarrow \phi_2 \phi_2$ with the momenta and masses indicated. The decaying particle $\phi_3$ (double line) has mass $M$. The two outgoing particles $\phi_2$ and $\phi_2$ are identical with mass $m$. The dashed lines represents virtual particles of mass $\eta$.

1.1 The Lagrangian

The interaction Lagrangian for (1.2) is given by

$$\mathcal{L} = \frac{1}{2} f \phi_3 \phi_2^2 + \frac{1}{2} g \phi_1^2 \phi_3 + \frac{1}{2} h \phi_2^2 \phi_1,$$

(1.3)

where $f, g$ and $h$ are coupling constants and $\phi_1$ is the intermediate particle of mass $\eta$.

The amplitude is diagrammatically given by

$$i\mathcal{M} = \phi_3 \phi_2 + \text{h.o. terms in coupling constants.}$$

The leading order diagram for the decay (1.1) is actually given by the triangle loop and not the diagram on the left. This is because there exists no spinless current coupling between quarks and leptons. Hence, we are here interested in the case where the coupling $f \equiv 0$, otherwise we would have yet another one-loop diagram.
Thus consider the diagram with all the momenta and masses written out in figure 1. We can write down its value in integral form;

\[
\frac{g h^2}{S} \int \frac{d^4 k}{(2\pi)^4} i \int \frac{d^4 k}{(2\pi)^4} i \left( \frac{1}{k^2 - m^2 + i \epsilon (k - p_2)^2 - \eta^2 + i \epsilon (k + p_1)^2 - \eta^2 + i \epsilon} \right) \times \frac{1}{k^2 - m^2 + i \epsilon (k - p_2)^2 - \eta^2 + i \epsilon (k + p_1)^2 - \eta^2 + i \epsilon} \times \frac{1}{k^2 - m^2 + i \epsilon (k - p_2)^2 - \eta^2 + i \epsilon (k + p_1)^2 - \eta^2 + i \epsilon}
\]

\[
= -i gh^2 \frac{1}{S} i \int \frac{d^4 k}{(2\pi)^4} \times F,
\]

where \( S = 1 \) is a symmetry factor associated with the triangle loop.

2 The Amplitude

From now on we’ll concentrate on \( F \) defined above as

\[
F := i \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i \epsilon (k - p_2)^2 - \eta^2 + i \epsilon (k + p_1)^2 - \eta^2 + i \epsilon}
\]

\[
= i \int \frac{d^4 k}{(2\pi)^4} \Delta_0 \Delta_2 \Delta_1.
\]

To make any progress let us introduce Feynman parametrization of the integrand by writing

\[
\Delta_0 \Delta_2 \Delta_1 = \int dF_3 \frac{1}{D^3},
\]

where the Feynman measure is given by

\[
\int dF_n = (n - 1)! \int_0^1 dx_1 \ldots dx_n \delta(x_1 + \ldots + x_n - 1),
\]
normalized such that
\[ \int dF_n 1 = 1. \] (2.4)

The \( D \) in equation (2.2) is given by
\[
D = \begin{aligned}
&x(k^2 - m^2 + i\epsilon) \\
&+ y((k + p_1)^2 - \eta^2 + i\epsilon) \\
&+ z((k - p_2)^2 - \eta^2 + i\epsilon)
\end{aligned}
\]
\[ = \left( k + (yp_1 - zp_2) \right)^2 \] (2.5)
\[
- (yp_1 - zp_2)^2 - m^2 x \\
+ yp_1^2 + zp_2^2 - \eta^2(y + z) + i\epsilon,
\]
where we have shifted \( k \)
\[
k + (yp_1 - zp_2) = l \] (2.6)
such that
\[ d^4k = d^4l. \]

Next we would like to do a Wick rotation of the \( l_0 \) integration contour, thus we need to analyze the pole structure of the integrand with respect to \( l_0 \).

### 2.1 The Poles of the Integrand

To see where the poles of the integrand are, we set \( D = l_0 \) and solve for \( l_0^2 \)
\[
l_0^2 = l^2 + y^2 p_1^2 + z^2 p_2^2 - 2p_1 p_2 yz + m^2 x - yp_1^2 - zp_2^2 + \eta^2(y + z) - i\epsilon
\]
\[ = \{-2p_1 p_2 = -(p_1 + p_2)^2 = -p_3^2 + p_1^2 + p_2^2\}
\[ = \begin{aligned}
l^2 - xyp_1^2 - xzp_2^2 - yzp_3^2 + xm^2 + \eta^2(y + z) - i\epsilon
\end{aligned} \] (2.7)
\[ = f(p_1^2, p_2^2, p_3^2, m^2, \eta^2; l^2) - i\epsilon. \]
Notice that the function $f$ is always positive for negative values of the four-momenta squared, thus let’s work in a region where

$$
\begin{align*}
\rho_1^2 &< 0, \\
\rho_2^2 &< 0, \\
\rho_3^2 &< 0,
\end{align*}
$$

this implies that

$$
\begin{align*}
\ell_0^\pm &= \pm \sqrt{f} - i\epsilon \\
&\approx \pm (\sqrt{f} - i\epsilon) \\
&= \pm \sqrt{f} \mp i\epsilon
\end{align*}
$$

lies in the fourth and second quadrant respectively in the complex $l_0$–plane, see figure 2. Also notice that this is irrespective of the $\eta^2$ (since it comes with a positive sign) which means that we can set it to zero at any time we so wish to and it wont invalidate our arguments.
2.2 Wick Rotation

Let us pause for a second and go back to our $F$, equation (2.1), and rewrite
it in terms of the shifted $k$ [equation (2.6)] and Feynman parameters:

$$ F = i \int \frac{d^4k}{(2\pi)^4} \Delta_0 \Delta_2 \Delta_1 $$

$$ = i \int \frac{d^4k}{(2\pi)^4} \int dF_3 \frac{1}{D^3} $$

$$ = \int dF_3 \int \frac{d^3|\mathbf{l}|}{(2\pi)^3} \int_{-\infty}^{\infty} i d\mathbf{l}_0 \frac{1}{2\pi D^3}. $$

(2.10)

Now for the $l_0$-integral consider the closed contour\(^1\) shown in figure 2

$$ \Gamma := C^+_\rho + C^-_\rho + \gamma_1 + \gamma_2, $$

which is pole-free as we argued above [equations (2.8) and (2.9)]. The con-
tribution from the quarter circles $C^+_\rho$ can be shown to be zero as follows: for
sake of clarity let’s work out the $C^+_\rho$-integral; on $C^+_\rho$ we have that

$$ l_0 = \rho e^{i\theta} \Rightarrow dl_0 = i\rho e^{i\theta} d\theta $$

where $\theta$ runs from 0 to $\pi/2$ so that

$$ J := \left| \int_{C^+_\rho} \frac{dl_0}{(l_0^2 - f)^3} \right| $$

$$ = \left| \int_0^{\pi/2} \frac{i\rho e^{i\theta} d\theta}{(\rho^2 e^{i2\theta} - f)^3} \right| $$

$$ \leq \frac{\rho}{(\rho^2 - f)^3} \times \ell(C^+_\rho), $$

(2.11)

\(^1\) This contour actually consist of two simply closed contours, namely the quarter circles
in the first and third quadrant, thus the Residue Theorem applies.
where $\ell(C^+_{\rho}) = \frac{1}{4}2\pi \rho$ is the length of the quarter circle $C^+_{\rho}$. Thus

$$J \leq \frac{\rho \cdot 2\pi \rho}{(\rho^2 - f)^3} = \frac{2\pi}{\rho^4 (1 - f/\rho^2)^3} \quad \text{(2.12)}$$

$\to 0$ as $\rho \to \infty$.

The $C^-_{\rho}$-integral is treated similarly with the only difference that $\theta$ runs from $3\pi/2$ to $\pi$. We should also mention that we’ve used a well know theorem on integral estimates\footnote{Keyword Estimation Lemma.} in the third step of equation (2.11), see for instance p. 170 in [11]. Using these results and the Residue Theorem we find that (schematically)

$$\sum_{\text{Residues inside } \Gamma} = 0$$

$$= \oint_{\Gamma} \, dl_0$$

$$= \int_{C^+_{\rho}} \int_{-i\infty}^{+i\infty} + \int_{-\infty}^{+\infty}$$

so that

$$\int_{-\infty}^{+\infty} = - \int_{-i\infty}^{+i\infty} = + \int_{-i\infty}^{+i\infty} \quad \text{(2.14)}$$

hence (2.10) can be written as

$$F = \int dF_3 \int \frac{d^3|u|}{(2\pi)^3} \int_{-i\infty}^{+i\infty} i \, dl_0 \, \frac{1}{2\pi} \frac{1}{D^3}. \quad \text{(2.15)}$$

Now let us introduce the Euclidean vector

$$\vec{l} := \begin{pmatrix} l_E \\ l \end{pmatrix}$$

with

$$\vec{l}^2 = l_E^2 + l^2$$
and make a change of variables
\[ l_E = il_0 \Rightarrow idl_0 = dl_E \]
so that (2.15) becomes
\[
F = \int dF_3 \int \frac{d^3|l|}{(2\pi)^3} \int_{-\infty}^{\infty} dl_E \frac{1}{2\pi D_E^3} \\
= \int dF_3 \int \frac{d^4l}{(2\pi)^4} \frac{1}{D_E^3} \tag{2.16}
\]
where \( D_E \) is just \( D \) with \( l^2 \) replaced by \(-l^2\)
\[
l_0^2 - l^2 = l_E^2 - l^2 = -l^2.
\]
Notice that we can now set \( \epsilon \) to zero since we’re no longer integrating near the poles. Plugging this into (2.16) and using the following master formula [7]
\[
\int \frac{d^d l}{(2\pi)^d} \left( \frac{t^2}{\ell^2 + B} \right)^a = \frac{\Gamma(b - a - \frac{d}{2}) \Gamma(a + \frac{1}{2}d)}{(4\pi)^{d/2} \Gamma(b) \Gamma(1 + d/2)} \frac{1}{B^{b-a-d/2}} \tag{2.17}
\]
we can write \( F \) as
\[
F = -\int dF_3 \int \frac{d^4l}{(2\pi)^4} \\
\times \frac{1}{\left[ \frac{l^2 - xyp_1^2 - xzp_2^2 - yzp_3^2 + xm^2 + \eta^2(y + z)}{3} \right]^3} \\
= -\frac{1}{32\pi^2} \int_0^1 dx \int_0^{1-x} dy \\
\times \frac{2 \cdot \delta(x + y + z - 1)}{-xy^2p_1^2 - xz^2p_2^2 - yz^2p_3^2 + xm^2 + \eta^2(y + z)} \\
= -\frac{1}{16\pi^2} \mathcal{I} \tag{2.18}
\]
where we’ve defined\(^3\) \( \mathcal{I} \) as
\[
\mathcal{I} := \int_0^1 dx \int_0^{1-x} dy \\
\times \frac{1}{y^2p_3^2 + y \left[ x(p_2^2 - p_1^2) - p_3^2(1 - x) \right] + x \left[ m^2 - p_2^2(1 - x) \right]} \\
\]
\(^3\) We’ve suppressed the \( \eta^2 \) term. The denominator is \( y^2p_3^2 + \cdots + \eta^2(1 - x) \).
2.3 The Real Part of the Amplitude

Recall that our $p^2$'s are all negative (2.8) so let us write them as
\begin{align}
    p_1^2 &= -A \\
    p_2^2 &= -B \\
    p_3^2 &= -C
\end{align}
(2.20)

with $A, B, C$ all positive (for now), we can then write $I$ as
\begin{align}
    I &= \frac{1}{-C} \int_0^1 dx \int_0^{1-x} dy \frac{1}{y^2 - y \left( 1 + \frac{x(A-B-C)}{C} \right) + \frac{m^2 + B(1-x)}{-C}}.
\end{align}
(2.21)

The zeros of the denominator are found to be
\begin{align}
    y^\pm &= \frac{1 + \frac{x(A-B-C)}{C} \pm \sqrt{\Delta}}{2},
\end{align}
(2.22)
where the discriminant is given by
\begin{align}
    \Delta &= \left( 1 + \frac{x(A-B-C)}{C} \right)^2 + \frac{4x(m^2 + B(1-x))}{C} > 0.
\end{align}
(2.23)

Thus the integrand in (2.21) can be written as
\begin{align}
    \frac{1}{(y-y^+)(y-y^-)} = \frac{1}{y^+ - y^-} \left[ \frac{1}{y-y^+} - \frac{1}{y-y^-} \right]
\end{align}
(2.24)
and recalling that we have a factor of
\begin{align}
    \frac{1}{-C}
\end{align}
in front we can finally write
\begin{align}
    I &= \frac{1}{C} \int_0^1 \frac{dx}{y^+ - y^-} \int_0^{1-x} \left[ \frac{1}{y-y^+} - \frac{1}{y-y^-} \right] \\
    &= \frac{1}{C} \int_0^1 \frac{dx}{y^- - y^+} \log \left( \frac{(1-x-y^+)y^-}{(1-x-y^-)y^+} \right),
\end{align}
(2.25)
notice that we’re only interested in the real part here, the imaginary part will be dealt with in the following section.
Figure 3: The real part of the integral (2.25) plotted for different values of \(-C\) with \(A, B\) and \(m^2\) fixed.

Remark 1. Notice that \(I\) is derived in a specific region of the complex \(A, B, C\) plane, namely for \(A, B, C\) real and positive. But since \(I\) is an analytic expression, it is valid for all values of \(A, B, C \in \mathbb{C}\) by analytic continuation. The analytic continuation in this case is straightforward since \(\text{Re} \log(\ldots) = \log|\ldots|\).

The integral \(I\) can be seen as a function of the external momenta-squared

\[
I = I(A, B; C; m^2, \eta^2),
\]

which we cannot solve in closed form but what we can do is to fix some of the parameters and solve it numerically as a function of \(C\) say, and check whether it is at least well behaved. The MATLAB code for this can be found
in Appendix A. In order to do this we fix the mass
\[ m = 1 \]
and analytically continue the integrand to a region where
\[ A = B = -1 \]
this corresponds to \( p_1^2 \) and \( p_2^2 \) taking on the physical value of \( m^2 \)
\[ p_1^2 \mapsto m^2 \]
\[ p_2 \mapsto m^2. \]
In doing this the discriminant (2.23) might become negative hence \( y^\pm \) complex etc. but we don’t care about that since we will take the real part of whatever number we get.

The plot for different values of \(-C\) ranging from 1 to 200 is shown in figure 3. We must add that MATLAB couldn’t handle the integration limit \( x = 0 \) exactly and we had to cut the integral off at \( x = 0.000001 \). The reason for this can be understood if we expand the integrand \( J \) of (2.25) near the origin (for these specific values of \( A, B \) and \( m^2 \)):
\[
J = \frac{1}{C} \frac{1}{y^- - y^+} \log \left[ \frac{(1 - x - y^+)y^-}{(1 - x - y^-)y^+} \right]
\]
\[
= \frac{-2}{C} \left( \log(2) + \log(x^2) - \log(-2C) \right) + \mathcal{O}(x)
\]
\[
\sim \log(x),
\]
hence it blows up, but nonetheless it is integrable. Finally, notice that the real part goes to zero for very large \(-C\) i.e. for large \( M^2 \), the squared mass of the decaying particle.

Next we will consider the imaginary part of the amplitude which we will find in a completely different way than the present one, namely by utilizing some cutting rules which we also will derive.

3 The Imaginary Part of the Amplitude

3.1 Cutkosky Cutting Rules

In the following sections we will derive and prove a rule or algorithm that can vastly facilitate the calculation of the imaginary part of the amplitude
in a Feynman diagram. This so called cutting rule goes as follows: starting

\[ p_3, M \]
\[ k + p_1, \eta \]
\[ k, m \]
\[ k - p_2, \eta \]
\[ p_1, m \]
\[ p_2, m \]

Figure 4: A cut diagram with the cut indicated by the diagonal dots.

with the expression for the amplitude

\[
F = i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(k-p_2)^2 - \eta^2 + i\epsilon} \frac{1}{(k+p_1)^2 - \eta^2 + i\epsilon}
\]

the imaginary part of a cut-diagram is given by the imaginary part of \( F \) with the cut propagators replaced by delta functions in conjunction with step functions with the remaining propagator replaced by a principal value one i.e. \( \epsilon \mapsto 0 \).

The step functions reflect energy flow through the cut, for instance for the cut in figure 4 we see that momentum is flowing into the cut from the left (which we will define as positive) and out through the cut to the left. Hence for the particular cut in figure 4 one of the thetas should be

\[ \Theta(k_0 + p_{1,0}) \]

(since \( k + p_1 \) is flowing in the positive direction we must use \( +(k_0 + p_{1,0}) \) for the argument of this \( \Theta \) ) and the other should be

\[ \Theta(-(k_0 - p_{2,0})) \]
(since \( k - p_2 \) is flowing in the negative direction we must use \(-(k_0 - p_{2,0})\) for the argument of this \( \Theta \)). Thus for the particular cut shown in figure 4 with the \( k + p_1 \) and \( k - p_2 \)-propagators cut we can write

\[
\text{Im } F_1 \propto \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2} \delta((k - p_2)^2 - \eta^2) \delta((k + p_1)^2 - \eta^2) \times \Theta(-(k_0 - p_{2,0})) \Theta(k_0 + p_{1,0}).
\]

(3.1)

It turns out\(^4\) that this is the only non-zero contribution to the imaginary part of the amplitude, the other zero contributions corresponds to the diagram cut diagonally such that the \( k \) and \( k - p_2 \)-propagators are cut

\[
\text{Im } F_2 \propto \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k + p_1)^2 - \eta^2} \delta(k^2 - m^2) \delta((k - p_2)^2 - \eta^2) \times \Theta(k_0) \Theta(-(k_0 - p_{2,0}))
\]

(3.2)

or such that the \( k \) and \( k + p_1 \)-propagators are cut, which yields

\[
\text{Im } F_3 \propto \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k - p_2)^2 - \eta^2} \delta(k^2 - m^2) \delta((k + p_1)^2 - \eta^2) \times \Theta(-k_0) \Theta(k_0 + p_{1,0}).
\]

(3.3)

again notice the signs of the arguments of the \( \Theta \)'s. Also notice that the integrals are now principal value integrals. We will now in the following sections prove all the assertions made above and in particular find the imaginary part of the amplitude for the decay \( \phi_3 \rightarrow \phi_2 \phi_2 \).

3.2 Sokhotsky-Plemelj Formula

We now turn to the task of computing the imaginary part of the amplitude and to this end we go back to our original expression for \( F \), equation (2.1)

\[
F = i \int \frac{d^4k}{(2\pi)^4} \tilde{D}_F(k; m) \tilde{D}_F(k + p_1; \eta) \tilde{D}_F(k - p_2; \eta),
\]

(3.4)

and analyze the propagators more directly using the Sokhotsky-Plemelj Theorem [11].

\(^4\) We will prove this in the coming sections.
Lemma 1 (Sokhotsky-Plemelj Theorem (weak version)). Let \( f \) be a complex-valued function, continuous on \( \mathbb{R} \) and let the constants \( a, b \in \mathbb{R} \) with \( a < 0 < b \). Then

\[
\lim_{\epsilon \to 0^+} \int_a^b dx \frac{f(x)}{x \pm i \epsilon} = \mathcal{P} \int_a^b dx \frac{f(x)}{x} \mp i \pi f(0). \tag{3.5}
\]

Here, \( \mathcal{P} \) denotes the Cauchy Principal Value integral. Symbolically we can write (3.5) as

\[
\frac{1}{x \pm i \epsilon} = \mathcal{P} \frac{1}{x} \mp i \pi \delta(x). \tag{3.6}
\]

Using this, we can rewrite propagators in a different form, for instance the Feynman propagators in (3.4) can be written as

\[
\tilde{D}_F(k; m) = \frac{1}{k^2 - m^2 + i \epsilon} = \mathcal{P} \frac{1}{k^2 - m^2} - i \pi \delta(k^2 - m^2). \tag{3.7}
\]

Thus if we write out all the propagators of (3.4) in this form (which we soon will) we will get \( 2 \times 2 \times 2 \) terms and in particular one term containing three principal value integrals:

\[
\mathcal{P} \frac{1}{k^2 - m^2} \mathcal{P} \frac{1}{(k - p)^2 - \eta^2} \mathcal{P} \frac{1}{(k + p)^2 - \eta^2}. \tag{3.8}
\]

We will now show how we can avoid such a term by introducing retarded propagators.

### 3.3 Retarded Propagators \( \tilde{D}_R \)

These are defined as

\[
\tilde{D}_R(k; m) := \frac{1}{k^2 - m^2 + i \sigma k_0}, \quad \text{with } \sigma \downarrow 0 \tag{3.9}
\]

thus

\[
\text{Re } \tilde{D}_R = \text{Re } \tilde{D}_F
\]

while

\[
\text{Im } \tilde{D}_R(k; m) = -\pi \text{sgn}(k_0) \delta(k^2 - m^2).
\]
Using these we can write
\[ \tilde{D}_F(k; m) - \tilde{D}_R(k; m) = -i\pi \delta(k^2 - m^2) \left( 1 - \text{sgn}(k^0) \right) \]
thus
\[ \tilde{D}_F(k; m) = \tilde{D}_R(k; m) - i2\pi \Theta(-k^0) \delta(k^2 - m^2). \]
Notice that using this form for the Feynman propagator in
\[ \tilde{D}_F(k; m) \tilde{D}_F(k + p_1; \eta) \tilde{D}_F(k - p_2; \eta) \]
we will no longer get the term (3.8), but rather
\[ \tilde{D}_R(k; m) \tilde{D}_R(k + p_1; \eta) \tilde{D}_R(k - p_2; \eta) \]
which looks at least as complicated but fortunately we can show that it vanishes, as we now will prove.

### 3.4 Poles of \( \tilde{D}_R(k; m) \)

Let us analyze the pole structure of the retarded propagator in the \( k^0 \)-plane:
the simplest one
\[ \tilde{D}_R(k; m) = \frac{1}{k^2 - m^2 + i\sigma k^0} \]
has poles when
\[ 0 = k_0^2 - k^2 - m^2 + i\sigma k_0 \]
\[ \approx \left( k_0 + \frac{1}{2}i\sigma \right)^2 - (k^2 + m^2) \]
this implies that
\[ k_0^\pm = \pm \sqrt{k^2 + m^2} - \frac{1}{2}i\sigma \Rightarrow \text{both in the lower half-plane!} \]
For a more general one of mass \( \eta \)
\[ \tilde{D}_R(k \pm p; \eta) = \frac{1}{(k \pm p)^2 - \eta^2 + i\sigma (k_0 \pm p_0)} \]
we find
\[ 0 = (k_0 \pm p_0)^2 - (k \pm p)^2 - \eta^2 + i\sigma(k_0 \pm p_0) \]
\[ \approx \left[(k_0 \pm p_0) + \frac{1}{2}i\sigma\right]^2 - (k \pm p)^2 - \eta^2 \]
this implies that
\[ k_0^\pm = \mp p_0 \pm \sqrt{(k \pm p)^2 + \eta^2} - \frac{1}{2}i\sigma \Rightarrow \text{both in the lower half-plane!} \quad (3.13) \]

Apparently, the extra momentum can only shift the poles parallel to the real \(k_0\)-axis. This means that the integral of (3.11) - call it\(^5\) ACE - vanishes because we can close the \(k_0\) integration path with the semi-circle \(C_\rho^+\) in the upper half-plane as shown in figure 5. Thus let \(\Gamma_\rho := C_\rho + [-\rho, \rho]\) and notice that just as in (2.11) the \(C_\rho^+\)-integral doesn’t contribute as \(\rho \to \infty\) so that we can replace
\[
\int_{[-\rho,\rho]} \text{with } \oint_{\Gamma_\rho} : \]

\[
ACE = \int \frac{d^4k}{(2\pi)^4} \tilde{D}_R(k; m) \tilde{D}_R(k + p_1; \eta) \tilde{D}_R(k - p_2; \eta) = \int \frac{d^3|k|}{(2\pi)^3} \int_{-\infty}^\infty \frac{dk_0}{2\pi} \tilde{D}_R(k; m) \tilde{D}_R(k + p_1; \eta) \tilde{D}_R(k - p_2; \eta) \]
\[
= \lim_{\rho \to \infty} \int \frac{d^3|k|}{(2\pi)^3} \int_{-\rho}^\rho \frac{dk_0}{2\pi} \tilde{D}_R(k; m) \tilde{D}_R(k + p_1; \eta) \tilde{D}_R(k - p_2; \eta) \]
\[
= \lim_{\rho \to \infty} \int \frac{d^3|k|}{(2\pi)^3} \left( \int_{-\rho}^\rho + \int_{C_\rho^+} \right) \frac{dk_0}{2\pi} \]
\[
\times \tilde{D}_R(k; m) \tilde{D}_R(k + p_1; \eta) \tilde{D}_R(k - p_2; \eta) \]
\[
= \lim_{\rho \to \infty} \int \frac{d^3|k|}{(2\pi)^3} \int_{\Gamma_\rho} \frac{dk_0}{2\pi} \tilde{D}_R(k; m) \tilde{D}_R(k + p_1; \eta) \tilde{D}_R(k - p_2; \eta) \]
\[
= 2\pi i \sum \text{residues inside } \Gamma_\rho \]
\[
= 0 \]

\(^5\) For reasons that will be clear in a moment.
\[ \rho - \rho \]

\[ \text{poles} \]

\[ J J \]

\[ \Gamma_\rho := C_\rho + [-\rho, \rho] \]

Figure 5: The poles of the retarded propagator in the complex \( k_0 \)-plane. Both lie in the lower half-plane.

as claimed.

### 3.5 \( \tilde{D}_F \) in terms of \( \tilde{D}_R \)

Finally let us now replace all the Feynman propagators in (3.4) with (3.10)

\[
\tilde{D}_F(k; m) \tilde{D}_F(k + p_1; \eta) \tilde{D}_F(k - p_2; \eta) = \\
\left[ \tilde{D}_R^A(k; m) - i2\pi \Theta(-k_0) \delta(k^2 - m^2) \right] \\
\times \left[ \tilde{D}_R^B(k + p_1; \eta) - i2\pi \Theta(-k_0 - p_{1,0}) \delta \left( (k + p_1)^2 - \eta^2 \right) \right] \\
\times \left[ \tilde{D}_R^C(k - p_2; \eta) - i2\pi \Theta(+p_{2,0} - k_0) \delta \left( (k - p_2)^2 - \eta^2 \right) \right],
\]
the $ACE$-term vanishes as we showed above so what is left is an expression of the form

$$F = i \int \frac{d^4k}{(2\pi)^4} [ACF + ADE + ADF + BCE + BCF + BDE + BDF].$$

Doing the algebra, the above expression becomes

$$F = \int \frac{d^4k}{(2\pi)^4} \left\{ (2\pi) \tilde{D}_R(k; m) \tilde{D}_R(k - p_2; \eta) \right.$$ 

$$\times \Theta \left(-k_0 - p_{1,0}\right) \delta \left((k + p_1)^2 - \eta^2\right)$$ 

$$+ (2\pi) \tilde{D}_R(k; m) \tilde{D}_R(k + p_1; \eta) \right.$$ 

$$\times \Theta \left(p_{2,0} - k_0\right) \delta \left((k - p_2)^2 - \eta^2\right)$$ 

$$+ (2\pi) \tilde{D}_R(k - p_2; \eta) \tilde{D}_R(k + p_1; \eta) \right.$$ 

$$\times \Theta \left(-k_0\right) \delta \left(k^2 - m^2\right)$$ 

$$- i(2\pi)^2 \tilde{D}_R(k; m) \Theta \left(p_{2,0} - k_0\right) \Theta \left(-k_0 - p_{1,0}\right)$$ 

$$\times \delta \left((k - p_2)^2 - \eta^2\right) \delta \left((k + p_1)^2 - \eta^2\right)$$ 

$$- i(2\pi)^2 \tilde{D}_R(k - p_2; \eta) \Theta \left(-k_0\right) \Theta \left(-k_0 - p_{1,0}\right)$$ 

$$\times \delta \left(k^2 - m^2\right) \delta \left((k - p_2)^2 - \eta^2\right)$$ 

$$- i(2\pi)^2 \tilde{D}_R(k + p_1; \eta) \Theta \left(-k_0\right) \Theta \left(p_{2,0} - k_0\right)$$ 

$$\times \delta \left(k^2 - m^2\right) \delta \left((k - p_2)^2 - \eta^2\right)$$

$$+ R \right\},$$

where $R$ is a purely real term which we don’t need since we’re only interested in the imaginary part anyway, but for completeness here it is

$$R = -(2\pi)^3$$ 

$$\times \Theta \left(-k_0\right) \Theta \left(p_{2,0} - k_0\right) \Theta \left(-k_0 - p_{1,0}\right)$$ 

$$\times \delta \left(k^2 - m^2\right) \delta \left((k - p_2)^2 - \eta^2\right) \delta \left((k + p_1)^2 - \eta^2\right).$$
Thus we have in (3.15) terms of the form $\tilde{D}_R(k; m)\tilde{D}_R(q; \eta)$ and $-i\tilde{D}_R(q; \eta)$ and in the end we want to take the imaginary part of it so let us consider $\text{Im}\{\tilde{D}_R(k; m)\tilde{D}_R(q; \eta)\}$ and $\text{Im}\{-i\tilde{D}_R(q; \eta)\}$ for generic masses $\eta$, $m$ and momenta $k, q$. Now,

\[
\text{Im}\{\tilde{D}_R(k; m)\tilde{D}_R(q; \eta)\} = \text{Im}\left\{\frac{1}{k^2 - m^2 + i\sigma_k k_0 q^2 - \eta^2 + i\sigma_q q_0}\right\}
\]

\[
= \text{Im}\left\{\left[\mathcal{P}_k - i\pi \text{sgn} k_0 \delta(k^2 - m^2)\right]\left[\mathcal{P}_q - i\pi \text{sgn} q_0 \delta(q^2 - \eta^2)\right]\right\}
\]

\[
= \text{Im}\left\{\text{real terms such as } \mathcal{P}_k \mathcal{P}_q - \pi^2 (\cdots) + \text{ etc.}\right\}
\]

\[
+ \text{Im}\left\{i\left[\mathcal{P}_k \pi \text{sgn} q_0 \delta(q^2 - \eta^2) - \mathcal{P}_q \pi \text{sgn} k_0 \delta(k^2 - m^2)\right]\right\}
\]

\[
= -\left(\mathcal{P}_k \pi \text{sgn} q_0 \delta(q^2 - \eta^2) + \mathcal{P}_q \pi \text{sgn} k_0 \delta(k^2 - m^2)\right)
\]

and

\[
\text{Im}\{-i\tilde{D}_R(k; m)\} = \text{Im}\left\{-i\mathcal{P}_k + \text{ real stuff}\right\}
\]

\[
= -\mathcal{P}_k.
\]

In the above we’ve shorten the notation by defining

\[
\mathcal{P}_q := \mathcal{P} \frac{1}{q^2 - \eta^2}.
\]
3.6 \( \text{Im } F = \text{Im } F_1 + \text{Im } F_{2+3} \) in terms of \( \tilde{D}_R \)

For the upcoming monstrous expression to be apprehensible, we must abuse notation slightly\(^6\) and let

\[
\begin{align*}
\mathcal{P}_1 &:= \mathcal{P} \frac{1}{(k + p_1)^2 - \eta^2}, \\
\mathcal{P}_2 &:= \mathcal{P} \frac{1}{(k - p_2)^2 - \eta^2}, \\
\mathcal{P}_k &:= \mathcal{P} \frac{1}{k^2 - m^2}, \\
\delta_1 &:= \delta((k + p_1)^2 - \eta^2), \\
\delta_2 &:= \delta((k - p_2)^2 - \eta^2), \\
\delta_k &:= \delta(k^2 - m^2).
\end{align*}
\]

Notice the overall negative sign all of the terms in (3.15) has, factoring it, taking the imaginary part and using equations (3.17) and (3.18) we get

\[
\text{Im } F = -\int \frac{d^4 k}{(2\pi)^4} \left\{ 2\pi^2 \left[ \mathcal{P}_k \text{sgn}(k_0 - p_{2,0}) \delta_2 + \mathcal{P}_2 \text{sgn}(k_0) \delta_k \right] \cdot \Theta(-k_0 - p_{1,0}) \delta_1 \\
+ 2\pi^2 \left[ \mathcal{P}_k \text{sgn}(k_0 + p_{1,0}) \delta_1 + \mathcal{P}_1 \text{sgn}(k_0) \delta_k \right] \cdot \Theta(+p_{2,0} - k_0) \delta_2 \\
+ 2\pi^2 \left[ \mathcal{P}_2 \text{sgn}(k_0 + p_{1,0}) \delta_1 + \mathcal{P}_1 \text{sgn}(k_0 - p_{2,0}) \delta_2 \right] \cdot \Theta(-k_0) \delta_k \\
+ (2\pi)^2 \mathcal{P}_k \Theta(p_{2,0} - k_0) \Theta(-k_0 - p_{1,0}) \delta_2 \delta_1 \\
+ (2\pi)^2 \mathcal{P}_2 \Theta(-k_0) \Theta(-k_0 - p_{1,0}) \delta_k \delta_1 \\
+ (2\pi)^2 \mathcal{P}_1 \Theta(-k_0) \Theta(p_{2,0} - k_0) \delta_k \delta_2 \right\}. \tag{3.20}
\]

\(^6\) We will strictly work with the \text{sgn}'s and \Theta's while all the other fellows (\delta's and \mathcal{P}'s) are merely spectators for now.

20
We can best organize this by factoring each $P$ into separate terms

$$\text{Im } F = -\int \frac{d^4k}{(2\pi)^4} 2\pi^2 P_1 \delta_1 \delta_2 \left\{ \text{sgn}(k_0 - p_{2,0})\Theta(-k_0 - p_{1,0}) \\
+ \text{sgn}(k_0 + p_{1,0})\Theta(p_{2,0} - k_0) + 2\Theta(p_{2,0} - k_0)\Theta(-k_0 - p_{1,0}) \right\}_1$$

$$-\int \frac{d^4k}{(2\pi)^4} 2\pi^2 P_2 \delta_2 \delta_1 \left\{ \text{sgn}(k_0)\Theta(-k_0 - p_{1,0}) \\
+ \text{sgn}(k_0 + p_{1,0})\Theta(-k_0) + 2\Theta(-k_0)\Theta(-k_0 - p_{1,0}) \right\}_2$$

$$-\int \frac{d^4k}{(2\pi)^4} 2\pi^2 P_1 \delta_2 \delta_k \left\{ \text{sgn}(k_0)\Theta(p_{2,0} - k_0) \\
+ \text{sgn}(k_0 - p_{2,0})\Theta(-k_0) + 2\Theta(-k_0)\Theta(p_{2,0} - k_0) \right\}_3,$$

and then using the identity

$$\text{sgn } k_0 = \Theta(k_0) - \Theta(-k_0),$$

we can write all the signum functions in terms of $\Theta$’s starting with the first integral:

$$\left\{ \cdots \right\}_1 = \left[ \Theta(k_0 - p_{2,0}) - \Theta(p_{2,0} - k_0) \right] \Theta(-k_0 - p_{1,0})$$

$$+ \left[ \Theta(k_0 + p_{1,0}) - \Theta(-k_0 - p_{1,0}) \right] \Theta(p_{2,0} - k_0)$$

$$+ 2\Theta(p_{2,0} - k_0)\Theta(-k_0 - p_{1,0})$$

$$\Rightarrow k_0 \geq p_{2,0} > 0 \quad k_0 \leq -p_{1,0} < 0$$

$$+ \Theta(k_0 + p_{1,0})\Theta(p_{2,0} - k_0)$$

$$= \Theta(k_0 + p_{1,0})\Theta(p_{2,0} - k_0)$$

$$= \Theta(k_0 + p_{1,0})\Theta(-(k_0 - p_{2,0})).$$

\footnote{Fortunately we can do this.}
Notice how we’ve written
\[ \Theta(p_{2,0} - k_0) \text{ as } \Theta(-(k_0 - p_{2,0})) \]
to emphasize the cutting rules we introduced in section 3.1. This is starting to look like what we asserted in section 3.1, all that is left to do is to show that \( \{ \}_1 \) is the only contribution to \( \text{Im } F \).

Using the identity (3.22), we can write the rest of \( \text{Im } F \) as
\[
\text{Im } F_{2+3} = \frac{-2\pi^2}{(2\pi)^4} \int d^3||k|| \, dk_0 \, \delta(k^2 - m^2)\delta((k + p_1)^2 - \eta^2) \\
\times \frac{1}{(k - p_2)^2 - \eta^2} \times \Theta(-k_0)\Theta(k_0 + p_{1,0})
\]
\[+ \frac{-2\pi^2}{(2\pi)^4} \int d^3||k|| \, dk_0 \, \delta(k^2 - m^2)\delta((k - p_2)^2 - \eta^2) \\
\times \frac{1}{(k + p_1)^2 - \eta^2} \times \Theta(k_0)\Theta(-k_0 + p_{2,0}),
\]
now let’s work specifically in the \( p_3 \)-COM frame.

**Remark 2** (\( p_3 \)-COM Frame). By momentum conservation we have that
\[
(p_{3,0}, p_3) = (p_{1,0} + p_{2,0}, p_1 + p_2)
\]
with \( p_{1,0} = \sqrt{p_1^2 + m^2} \) and \( p_{2,0} = \sqrt{p_2^2 + m^2} \). Then in the \( p_3 \)-COM frame where \( p_3 = (M, 0) \) we have
\[
p_2 = -p_1 \\
\Rightarrow \\
p_{2,0} = p_{1,0} := E, \text{ in } p_3 \text{-COM frame.}
\]
With this, equation (3.24) becomes
\[
\text{Im } F_{2+3} = \frac{-1}{8\pi^2} \int d^3||k|| \, dk_0 \, \delta(k^2 - m^2) \\
\times \frac{\Theta(-k_0)\Theta(k_0 + E) \delta((k_0 + E)^2 - (k + p)^2 - \eta^2)}{(k_0 - E)^2 - (k + p)^2 - \eta^2} \\
+ \frac{-1}{8\pi^2} \int d^3||k|| \, dk_0 \, \delta(k^2 - m^2) \\
\times \frac{\Theta(+k_0)\Theta(-k_0 + E) \delta((k_0 - E)^2 - (k + p)^2 - \eta^2)}{(k_0 + E)^2 - (k + p)^2 - \eta^2} \\
= \frac{-2}{8\pi^2} \int d^3||k|| \, dk_0 \, \delta_a(k^2 - m^2)\delta_b((k_0 - E)^2 - (k + p)^2 - \eta^2) \\
\times \frac{\Theta(+k_0)\Theta(-k_0 + E)}{(k_0 + E)^2 - (k + p)^2 - \eta^2},
\]
where we changed \( k_0 \mapsto -k_0 \) in the first integral and added subscripts to the deltas for reference only.

### 3.7 Sweet Dirac-\( \delta \) Formula

Now \( \delta_a \) and \( \delta_b \) have functions inside their argument and not only that but they are also coupled in a sense so we cannot just go on and use
\[
\delta(f(x)) = \sum_{\text{zeros } x_i \text{ of } f(x)} \frac{\delta(x - x_i)}{|f'(x_i)|},
\]
but rather we have to derive a slightly more general formula. Thus consider the expression
\[
\delta_a(f(x, y))\delta_b(g(x, y))
\]
with
\[
f(x_0(y), y) = 0
\]
and where we assume that there is only one zero\(^8\) in the integration range of interest. Then we have that

\[
\delta_a(f(x, y))\delta_b(g(x, y)) = \frac{\delta(x - x_0(y))}{\left| \frac{\partial f}{\partial x}(x_0(y), y) \right|} \delta_b(g(x_0(y), y)),
\]

where \(\delta_b\) involves

\[
\frac{d}{dy} g(x_0(y), y) = \frac{\partial g}{\partial x}(x_0(y), y) \frac{dx_0(y)}{dy} + \frac{\partial g}{\partial y}(x_0(y), y).
\]

We can obtain \(dx_0(y)/dy\) from \(f(x_0(y), y) = 0\) (which holds for any \(y\)) thus

\[
0 = \frac{d}{dy} f(x_0(y), y) = \frac{\partial f}{\partial x}(x_0(y), y) \frac{dx_0(y)}{dy} + \frac{\partial f}{\partial y}(x_0(y), y)
\]

which implies that

\[
\frac{dx_0(y)}{dy} = -\frac{\partial f}{\partial y}(x_0(y), y) \left( \frac{\partial f}{\partial x}(x_0(y), y) \right)^{-1},
\]

hence

\[
\frac{d}{dy} g(x_0(y), y) = \left( \frac{\partial f}{\partial x}(x_0(y), y) \right)^{-1} \left( -\frac{\partial g}{\partial y} \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x} \frac{\partial f}{\partial y} \right). \tag{3.27}
\]

Collecting all the above we can finally write [1]:

**Lemma 2.** Let \(f\) and \(g\) be sufficiently smooth functions of \((x, y)\) at least once differentiable in each variable with non vanishing Jacobian. Assume further that \(f\) and \(g\) has at most a single zero \(x_0\) and \(y_0\) respectively in the integration region of interest. Then

\[
\delta(f(x, y))\delta(g(x, y)) = \frac{\delta(x - x_0)\delta(y - y_0)}{\left| \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y} \right|},
\]

with \(f(x_0, y_0) = 0 = g(x_0, y_0).\)

\(^8\) This suffices for the task at hand.
3.8 \( \text{Im} F_{2+3} = 0 \)

Let us now go back to equation (3.26) and our Dirac deltas:

\[
\delta_a(f(k_0, \|k\|)) \delta_b(g(k_0, \|k\|)) \text{ with}
\]

\[
f(k_0, \|k\|) = k^2 - m^2 = k_0^2 - k^2 - m^2, \quad (3.28)
\]

\[
g(k_0, \|k\|) = (k_0 - E)^2 - (k + p)^2 - \eta^2.
\]

Notice that because of the two step functions in (3.26), we only need to keep zeros of \( f \) and \( g \) that are in the interval \([0, E]\), thus

\[
f(k_0, \|k\|) = 0 \iff k_0^\pm = \pm \sqrt{k^2 + m^2} := \pm E_k = \mp E_k
\]

\[
g(k_0, \|k\|) = 0 \iff (k_0 - E)^2 - (k - p)^2 - \eta^2 = 0. \quad (3.29)
\]

Substituting the first into the second we find that

\[
0 = (E_k - E)^2 - (k - p)^2 - \eta^2
\]

\[
\iff 0 = 2m^2 - 2EE_k - 2\|k\|\|p\| \cos \theta - \eta^2
\]

\[
\iff E_kE = m^2 - \eta^2/2 - \|k\|\|p\| \cos \theta
\]

\[
\Rightarrow E_k^2E^2 = m^4 + \eta^4/4 - m^2\eta^2 + k^2p^2 \cos^2 \theta - 2\|k\|\|p\| \cos \theta(m^2 - \eta^2/2)
\]

\[
\iff 0 = k^2p^2 \sin^2 \theta + m^2k^2 + m^2p^2 + \eta^2(m^2 - \eta^2/4)
\]

\[
+ 2\|k\|\|p\| \cos \theta(m^2 - \eta^2/2).
\]

We get two cases from this depending on how the factor \((m^2 - \eta^2/2)\) next to \(\cos \theta\) behaves:

**Case 1** \((m^2 - \eta^2/2 < 0 \Rightarrow \eta > \sqrt{2}m)\). In this case we write equation (3.30) as

\[
k^2p^2 \sin^2 \theta + m^2k^2 + m^2p^2 + \eta^2(m^2 - \eta^2/4) - 2\|k\|\|p\| \cos \theta \left(\frac{\eta^2}{2} - m^2\right) = 0
\]
with \( \left( \frac{\eta^2}{2} - m^2 \right) > 0 \). Now, the \( \cos \theta \) term is most negative when \( \cos \theta = 1 (\theta = 0, \pm 2\pi, \ldots \Rightarrow \sin \theta = 0 ) \) thus

\[
LHS \geq m^2(p^2 + k^2 + 2\|k\|\|p\|) + \eta^2 \left( m^2 - \frac{\eta^2}{4} \right) - \eta^2\|k\|\|p\| \geq -4m^2\|k\|\|p\|
\]

\[
\eta < 2m
\]

\[
\geq m^2(p^2 + k^2 + 2\|k\|\|p\|) + \eta^2 \left( m^2 - \frac{\eta^2}{4} \right) - 4m^2\|k\|\|p\|
\]

\[
= m^2(p - k)^2 + \eta^2 \left( m^2 - \frac{\eta^2}{4} \right)
\]

\[
> 0.
\]

Notice that we have assumed that \( \eta < 2m \) and with this assumption one of the deltas is zero because its argument has no roots, thus we have proved the assertion made in section 3.1.

The above assumption is not so restrictive since \( \eta \) is infinitesimal anyway and can be understood in the following way: suppose we cut the diagram such that the \( k \)- and \( (k - p_2) \)-propagators are cut as shown in figure 6; the cut diagram now consists of \( \phi_3 \) branching off to two trees, one of which is just a line namely the line with momentum \( (k - p_2) \) and mass \( \eta \). The other is a line with momentum \( (k + p_1) \) of mass \( \eta \) which then splinters off to two lines each of mass \( m \) and momentum \( k \) and \( p_1 \) see figure 6. Now if \( \eta < 2m \) as we assumed above then \( \eta \to 2m \) is not even kinematically allowed hence we should expect to get zero contribution (as we got). If \( \eta > 2m \) on the other hand we see that \( \eta \to 2m \) is at least kinematically allowed hence a nonzero contribution should not come as a surprise.

Case 2 \( (m^2 - \eta^2/2 > 0 \Rightarrow \eta < \sqrt{2m}). \) In this case we write the last line of equation (3.30) as it stands and notice that the cosine term is most negative when \( \cos \theta = -1 \) i.e. when \( \theta = \pi, \pm 3\pi, \cdots \Rightarrow \sin \theta = 0 \) so that the l.h.s. satisfy

\[
LHS \geq m^2(k^2 + p^2 - 2\|k\|\|p\|) + \eta^2 \left( m^2 - \frac{\eta^2}{4} \right) + \eta^2\|k\|\|p\| \\
= m^2(\|k\| - \|p\|)^2 + \eta^2 \left( m^2 - \frac{\eta^2}{4} \right) + \eta^2\|k\|\|p\| \\
> 0.
\]
Thus, also in this case we get that one of the arguments of the deltas doesn’t have any roots/zeros hence the delta is zero and therefore the contribution to the imaginary part is zero, which is what we stated without proof in section 3.1.

Thus we have shown that

$$\text{Im } F_{2+3} = 0$$

and therefore

$$\text{Im } F = \text{Im } F_1,$$

recall that from equation (3.23) we have an explicit expression for $\text{Im } F_1$ with an equality sign as opposed to the beginning of section 3.1, with the constant of proportionality $(2\pi)^2/2$. 

Figure 6: A cut diagram with the cut indicated by the diagonal dots.
3.9 The Cutting Theorem and the Computation of \( \text{Im} F \)

Recalling from section 1.1 that

\[
iM = i\frac{gh^2}{S} F = i^3\frac{gh^2}{S} F
\]

i.e.

\[
M = \frac{i^2 gh^2}{S} F,
\]

we can summarize all the results above regarding the cutting rule for this particular diagram in a theorem.

**Theorem 1** (Cutkosky Cutting Rules [6]). *Starting with the Feynman amplitude \( M \) for our diagram in figure 1, then (twice) the imaginary part is given by the Principal Value Integral*

\[
2 \text{Im} M = \frac{gh^2}{S} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 - m^2} \times (2\pi i) \Theta(k_0 + p_{1,0}) \delta ((k + p_1)^2 - \eta^2) \times (2\pi i) \Theta(p_{2,0} - k_0) \delta ((k - p_2)^2 - \eta^2)
\]

*which corresponds to the cut shown in figure 4 with left-to-right flow through the cut defined as positive.*

All that is left to do now is to actually compute (3.32), we do this in the \( p_3 \)-COM frame [see equation (3.25) for details] and suppress the \( \eta \) for notational convenience. We find

\[
\text{Im} M = \frac{-gh^2}{S} \text{Im} F
\]

where

\[
\text{Im} F = -\int \frac{d^4 k}{(2\pi)^4} \frac{2\pi^2}{k^2 - m^2} \delta_1((k + p_1)^2 - \eta^2) \delta_2((k - p_2)^2 - \eta^2)
\]

\[
\times \Theta(k_0 + p_{1,0}) \Theta(-k_0 - p_{2,0})
\]

\[
= -\frac{1}{8\pi^2} \int d^4 k \frac{1}{k_0^2 - k^2 - m^2} \Theta(k_0 + E) \Theta(E - k_0)
\]

\[
\times \delta_1 ((k_0 + E)^2 - (k + p)^2)
\]

\[
\times \delta_2 ((k_0 - E)^2 - (k + p)^2),
\]
shifting $k \mapsto k + p$, the integral becomes
\[
\text{Im } F = -\frac{1}{8\pi^2} \int d^4k \frac{1}{k_0^2 - (k - p)^2 - m^2} \Theta(k_0 + E)\Theta(E - k_0)
\times \delta_1 ((k_0 + E)^2 - k^2)
\times \delta_2 ((k_0 - E)^2 - k^2),
\]
where the subscripts on the deltas are for reference only and we have suppressed $\eta$.

3.9.1 The Zeros and the Jacobi Determinant

Since $\delta_1 = \delta_1(f(k_0, \|k\|))$ and $\delta_2 = \delta_2(g(k_0, \|k\|))$ we must use formula (3.27) of Lemma 2, thus let us find the zeros of $f$ and $g$ and their Jacobian.

\[
f = 0 \Leftrightarrow k_0^\pm = -E \pm \|k\| = -E + \|k\|
\]
where we only kept $k_0^+$ because of the theta functions, plugging this $k_0^+$ into $0 = g(k_0^+, \|k\|)$ we get
\[
0 = (-2E + \|k\|)^2 - \|k\|^2 \Leftrightarrow 0 = E(E - \|k\|)
\]
which implies that
\[
E = \|k\| \Rightarrow k_0^+ = 0
\]
therefore the derivatives at $k_0^+$ become quite simple expressions:

\[
\frac{\partial f}{\partial k_0} = 2k_0 + 2E = 2E \quad \text{(at } k_0 = k_0^+) \ ,
\]
\[
\frac{\partial f}{\partial \|k\|} = -2\|k\| = -2E
\]
\[
\frac{\partial g}{\partial k_0} = 2k_0 - 2E = -2E,
\]
\[
\frac{\partial g}{\partial \|k\|} = -2\|k\| = -2E
\]

hence
\[
\left| \frac{\partial f}{\partial k_0} \frac{\partial g}{\partial \|k\|} - \frac{\partial g}{\partial k_0} \frac{\partial f}{\partial \|k\|} \right| = 8E^2.
\]
Thus

\[ \text{Im } F = -\frac{1}{8\pi^2} \frac{1}{8E^2} \int d^4k \frac{\delta(k_0)\delta(||k|| - E)\Theta(k_0 + E)\Theta(E - k_0)}{k_0^2 - (k - p)^2 - m^2} \]

\[ = \frac{1}{8\pi^2} \frac{1}{8E^2} \int d\Omega \frac{\delta(||k|| - E)}{||k||^2 + ||p||^2 + m^2 - 2||k||||p|| \cos \theta} \]

\[ = \frac{2\pi}{8\pi^2} \frac{1}{8E^2} \int_{-1}^{1} d\cos \theta \frac{E^2}{2E^2 - 2E||p|| \cos \theta} \]

\[ = \frac{1}{64\pi E} \int_{-1}^{1} dz \frac{1}{E - ||p|| z} \]

\[ = \left\{ E - ||p|| z = \sqrt{||p||^2 + m^2} - ||p|| z > 0 \right\} \]

\[ = \frac{1}{64\pi E ||p||} \log \frac{E + ||p||}{E - ||p||} \]

\[ = \left\{ \frac{E + ||p||}{E - ||p||} > 0 \Rightarrow \log = \ln \right\} \]

\[ = \frac{1}{64\pi E ||p||} \ln \frac{E + ||p||}{E - ||p||} \]

and so

\[ \text{Im } \mathcal{M} = -\frac{g\hbar^2}{S} \frac{1}{64\pi E ||p||} \ln \frac{E + ||p||}{E - ||p||} \] (3.33)

Notice that since \( M = 2E \) and

\[ ||p|| = \sqrt{E^2 - m^2} \]
we can rewrite the fraction in a more convenient form

\[
\frac{E + \|p\|}{E - \|p\|} = \frac{E + \sqrt{E^2 - m^2}}{E - \sqrt{E^2 - m^2}}
\]

\[
= \frac{1 + \sqrt{1 - \frac{m^2}{E^2}}}{1 - \sqrt{1 - \frac{m^2}{E^2}}}
\]

\[
= \frac{1 + \sqrt{1 - \frac{4m^2}{M^2}}}{1 - \sqrt{1 - \frac{4m^2}{M^2}}}
\]

\[
= \frac{1 + \beta_m(M^2)}{1 + \beta_m(M^2)}
\]

(3.34)

where

\[
\beta_m(M^2) := \sqrt{1 - \frac{4m^2}{M^2}}
\]

is the speed of the final state particles in the COM frame. Similarly replacing the \(E\|p\|\) in the prefactor, we can finally write

\[
\text{Im} \mathcal{M} = \frac{-g \hbar^2}{S} \frac{1}{16\pi M^2} \frac{1}{\beta} \ln \frac{1 + \beta}{1 - \beta}
\]

(3.35)

Finally we notice that

\[
\lim_{M \to \infty} \text{Im} \mathcal{M} = 0
\]

just as the real part did as we saw at the end of section 2.3.

4 Results and Summary

We have studied a toy model for the decay \(\pi^0 \to e^+ e^-\) and found the real and imaginary part separately. The real part was treated in section 2.3 where we numerically integrated our obtained expression for a relatively large range of incoming momentum squared \(p_3^2\), which is shown in figure 3. As can be seen the numerical integration procedure seems stable, although the integrand
goes like \( \log x \) near \( x = 0 \) and we had to cut the lower integration limit off at 0.000001 (see Appendix A).

The Feynman diagram for the decay is shown in figure 1 and we have obtained the following formulas for the corresponding Feynman amplitude

\[
\text{Re } \mathcal{M} = \frac{-gh^2}{S} \frac{1}{16\pi^2} \frac{1}{M^2} \text{ Re } \int_0^1 \frac{dx}{y^- - y^+} \log \frac{(1 - x - y^+)y^-}{(1 - x - y^-)y^+},
\]

(4.1)

with notation\(^9\) explained in section 2.3 and

\[
\text{Im } \mathcal{M} = \frac{-gh^2}{S} \frac{1}{16\pi M^2} \frac{1}{\beta_m(M^2)} \ln \frac{1 + \beta_m(M^2)}{1 - \beta_m(M^2)}.
\]

(4.2)

Having obtained these two quantities, it is straightforward to obtain the decay rate. This will be addressed in [5].

We also derived a formula dealing with two coupled Dirac deltas which can be found in Lemma 2. We suspect that this lemma can be made stronger to deal with functions that have several zeros inside the integration region (ours had only single zeros). Furthermore, after stating the Cutkosky cutting rules in section 3.1, we spent the next sections deriving and proving them and they can be found summarized in Theorem 1.

\[^9\text{ Essentially } C = -p_s^2 = -M^2 = -(2E)^2 \text{ so that} \]

\[
\text{Re } \mathcal{M} = \frac{gh^2}{S} \frac{-1}{64\pi^2} \frac{1}{E^2} \text{ Re } \int_0^1 \frac{dx}{y^- - y^+} \log \frac{(1 - x - y^+)y^-}{(1 - x - y^-)y^+}
\]
A The MATLAB Integrand

The integrand of equation (2.21) defined here in a separate MATLAB function file.

```matlab
function [ y ] = Integrand( x,A,B,C,m )
%INTEGRAND(x,A,B,C,m) m=mass of e+, A=−p₁^2, B=−p₂^2 and C=−p₃^2.
% This is the integrand of the real part of the amplitude for ?→e+e−
% although this is only a Toy-Model with no spins involved.
% To be more specific the integrand is:
% 1/(C*(ym−yp))*log|1−x−yp)*ym/((1−x−y)*yp| we only need
% the real part, hence the absolute bars in LOG.
Delta = (1+x.*(A−B−C)./C).^2+4*x.*(m^2+B.*(1−x))./C;%The discriminant of
...y+, y− which are the roots of the denominator of the integrand.
yp = (1+x.*(A−B−C)/C+sqrt(Delta))./2; %This is y+
ym = (1+x.*(A−B−C)/C−sqrt(Delta))./2; %This is y−
y = 1./(C.*(ym−yp)).*log(((1−x−yp).*ym/((1−x−y)*yp)));%This is the
...integrand.
end
```

B The MATLAB Script

This script takes care of everything [but it needs the function file of Appendix A] by saving the value of the integrand for each value of C in a vector dubbed I as can be seen on row 22 in the code below. Notice that the integration limit are from

\[ \int_{0.000001}^{1} \]

rather than \( \int_{0}^{1} \). The reason being the logarithmic behaviour of the integrand near \( x = 0 \) as explained in section 2.3.

```matlab
% Real Part Numerical Integration ?0→e+e−
%A short script to test the stability of the numerical integration of the
% formula for the real part of the amplitude
m = 1;%mass of electron/positron
A = −m^2;% A = −p₁^2 = −m^2
B = −m^2;% B= −p₂^2
Q = integral(@(y)Integrand(y,A,B,−5,m),0,1);% Notice that using "integral"
```
...instead of "quad" here is better. quad(0,1) returns NaN
...EDIT: Actually in this case the integrand is divergent near x=0
...(for C<0)
...anyway so it doesn't matter!

I = zeros(1,200); % Define a vector full of zeros in order to later
...store the information about the different C's in place of the zeros.
...One could introduce an empty vector instead but then this would grow
...inside the for-loop and would need larger amount of computing time/
...resources
L = length(I);

tic;%Starts Clock!
for n = 1:L
   I(n) = integral(@(y)Integrand(y,A,B,-n,m),0.000001,1);%Notice that
   ...for large -n the lower boundary gets problematic. For instance try
   ...0.0000001 and notice that for small -n the integral is OK. It
   ...only breaks for larger values....see the plot!
end
t = toc;% Amount of time the loops takes to calculate
R = real(I);
T = real(exp(exp((I))));
semilogy(R,'LineWidth',2)
title('The Cut-Off-Integral for Different Values of -C')
xlabel('\text{\texttt{C = p.3^2 [m^2] }}'),ylabel('Re (I(-C))')
References


