GARCH (1, 1) with exogenous covariate for EUR/SEK exchange rate volatility

On the Effects of Global Volatility Shock on Volatility

Sarvar Samiev
GARCH (1, 1) with exogenous covariate for EUR/SEK exchange rate volatility

On the Effects of Global Volatility Shock on Volatility

In Partial Fulfillment
of the Requirements for the Degree of
Master of Science in Economics

Supervisor: Professor Kurt Brännäs
Contents

0.1 Introduction .............................................. 1
0.2 Theory and Methodology ................................. 4
  0.2.1 GARCH models ..................................... 4
  0.2.2 GARCH(1,1) with exogenous covariate ............ 4
0.3 Bayesian estimation with Markov Chain Monte Carlo . 5
0.4 Model evaluations ........................................ 7
  0.4.1 Application .......................................... 7
  0.4.2 Estimation Results .................................. 8
0.5 Simulation study .......................................... 8
  0.5.1 Simulation of parameter uncertainty ............... 10
0.6 MCMC estimation results ............................... 11
0.7 Conclusions ............................................... 14
0.8 Appendices ............................................... 16
## List of Tables

1. Estimation results of model parameters ........................................... 9
2. Simulation experiment results of GARCH(1,1) with exogenous covariate .......................................................... 11
3. Simulation experiment results of GARCH(1,1) in-mean with exogenous covariate ..................................................... 12
4. Markov Chain Monte Carlo results for GARCH(1,1) with Student-t distribution ............................................................ 13
5. Descriptive Statistics ........................................................................ 17
List of Figures

1 The exchange rate for Euro and Swedish Krona (EUR/SEK) for the period of January 4, 1993 to February 7, 2012. Source: Datastream. 2
3 The simulated path for conditional variance. 11
4 The simulated path for conditional variance (GARCH (1,1) in-mean). 12
5 Histogram of the posterior density for the persistence. 13
6 Parameter density plots based on simulation 14
7 Parameter density plots based on simulation (GARCH(1,1) in-mean) 15
8 The simulation density for the conditional variance. 16
9 The logistic function 16
10 The simulation density for the conditional variance. 17
11 The simulation path for return series. 18
12 Density plots of estimated MCMC parameters 18
13 Trace plots of estimated MCMC parameters 19
Acknowledgements

The work would not have been successfully accomplished without intellectual and advisory support of a number of people. For that, I have to express my deep gratitude to my supervisor Professor Kurt Brännäs, whose supervision and guidance helped me to gain new insights into econometric modeling and understanding my subject. Department of Economics at Umeå University also deserves a bunch of thanks for providing courses and needed facilities that made this work possible. Moreover, I have to thank my friend, Jonas Westin for helpful discussion, motivation and advices on my topic. Sharing code by Dr. David Ardia (Universität Freiburg, Switzerland) on Bayesian econometrics is greatly appreciated. Last but not least, my thanks go to my parents and my brother for their support and love.

May 27, 2012
Sarvar Samiev
Abstract

This paper develops a GARCH (1,1) with exogenous covariate for EUR/SEK exchange rate volatility. An empirical application of the proposed model is illustrated by including the Chicago Board Options Exchange volatility index (VIX) in the conditional variance equation. This covariate serves as a proxy for global volatility information and has a logistic distribution. Empirical findings in econometric literature suggest the strong influence of US market volatility on exchange rates. So far the global volatility information (shock) affecting the EUR/SEK conditional volatility is not examined in the literature and this paper aims at filling this gap. By accounting for this shock, better volatility forecasts can be obtained, which is of great practical importance in risk management. Estimation results with Gaussian quasi-maximum-likelihood estimator (QMLE) suggest the higher persistence level and statistically significant coefficients for both ARCH and GARCH effects. Effects of global volatility information on conditional volatility of EUR/SEK is significant when the covariate enters the mean equation in multiplicative form, however, it is insignificant when it enters the conditional variance equation. Simulation study is conducted to study the finite-sample properties of model parameters. A comparison is done with estimation results of proposed GARCH(1,1) model with exogenous covariate with Markov Chain Monte Carlo and the findings suggest that persistence level in the proposed model is higher compared with GARCH(1,1) estimated with Markov Chain Monte Carlo. Findings confirm the prior empirical evidence suggesting that due to a structural break the persistence level in GARCH models estimated with Maximum likelihood is upward biased. Implication is that forecasting results based on parameter uncertainty and not accounting for a structural break may not serve valid inputs for risk and portfolio optimization models.

Keywords: GARCH(1,1), logistic and inverse gamma distribution, Monte Carlo simulation, Markov Chain Monte Carlo, Nakatsuma algorithm, bayesian estimation.
0.1 Introduction

A major property of financial time series, is that their volatility varies over time. Describing the volatility of an asset is a principal task in financial economics. Asset pricing models depend on the expected volatility (covariance structure) of the returns and derivative securities depend solely on the covariance structure of their underlying asset. Besides, financial institutions measure the volatility of assets as a proxy for risk and financial theory postulates that investors should be compensated for bearing risk, that is, they require risk premia. Therefore, modeling volatility is a great of practical importance for a wide range of users. Practical risk and portfolio management in asset, insurance and bank industry calculate VaR (Value-at-Risk) as a measure of portfolio loss at a certain probability level over a specified time interval. To provide accurate estimates, which in turn, they are used as inputs in risk management and portfolio optimization models, they should measure the volatility.

The fact is that volatility for financial time-series is not directly observable. One way of overcoming this is constructing proxies which closely approximate the latent (unobservable) volatility. In an empirical modelling context, it is a common practice to use close-to close daily returns. Previous studies show that well-chosen proxies for volatility models enhance their forecasting performance. Good proxies can also add precision to a parameter estimation of volatility models and improve the practical applications of time-varying volatility in the financial industry. For a broad overview of the major developments in this field, see Campbell, Lo, and Mackinlay[29], Gourieroux and Jasiak[23] and Diebold[44].

Research has been active in volatility modelling since Engle[16] proposed the Autoregressive Conditional Heteroskedasticity (ARCH) model. Later, Bollersev[7] generalized the ARCH model into GARCH, which is a more parsimonious specification[9]. A multivariate version of ARCH class models were later been proposed, e.g., VEC model of Bollersev, Engle, and Wooldridge[6] and BEKK coined by Engle and Kroner[5].

In empirical literature volatility clustering indicates that “small changes tend to be followed by small (both negative), and the same is true for large changes (of either sign)” Mandelbrot[35]. As a result, the coefficient is positive for squared returns, with relatively a slow decreasing rate. In other words, it provides a persistence property for volatility and it is possible to predict volatility even though it is noisy. The leptokurtic property of unconditional probability distribution for financial returns is widely observed in practice as having fatter tails and more peaked than the Gaussian distribution. Another feature of financial returns is that they respond asymmetrically to negative and positive shocks of the same magnitude. This is a well-known observed stylized facts, and there are a number models proposed to deal with capturing these asymmetries. Brännäs and DeGooijer[31] combine an asymmetric moving average (asMA) model for the mean equation with an Asymmetric Quadratic GARCH model for the variance equation. Their model (asMA-asQGARCH) is designed to allow the response to shocks to behave asymmetrically. Empirical results have showed that both conditional mean and conditional variance behave asymmetrically to past information. In the econometric literature, there are a number of volatility models accounting for asymmetry, a Quadratic ARCH (GQARCH) of Engle and Ng and Sentana[41], the Asymmetric Power ARCH (APARCH) of Engle, Ding and Granger[12], the Threshold GARCH (TGARCH) of Zakoan[39], the GJR-GARCH (GJR) of Glosten, Jaganathan, and Runkle[22], and the Exponential GARCH of Nelson[9]. For a review of univariate GARCH models, see Teräsvirta[43].

Apart from squared returns and lagged conditional variance as covariates in GARCH(1,1),
researchers include other variables that might add to the explanatory power of the model, for more review of this topic, see Fleming et.al.[18]. The multiplicative error model (MEM) of Engle[15] uses realized variance as a covariate. Following this, Barndorff-Nielsen and Shephard[3] included bipower variation and realized volatility in their proposed specification. For a review, see Shephard and Sheppard[42] and Hansen et.al[25]. In a proposed Realized GARCH model by Hansen[25], he imposed restriction (zero) on the coefficient of past squared returns.

Engle and Patton[17] applied three month US Treasury bill to model the stock return volatility. Glosten[22] et.al has implemented the same approach in volatility modelling. By modifying the GARCH-M, the findings showed that positive unanticipated returns resulted in a downward revision of time-varying volatility and the negative unanticipated returns caused an upward revision in the time-varying volatility. The empirical literature is rich when it comes to linking the explanatory power of covariates to conditional variance. For example, interest rate spreads were used by Hagiwara and Herc[24] and Dominguez[13]. The implied volatility was used by a number of researchers such as Blair, Poon, and Taylor[5], Day and Lewis[8] and Lamoureux and Lastrapes[32].

Figure 1: The exchange rate for Euro and Swedish Krona (EUR/SEK) for the period of January 4, 1993 to February 7, 2012. Source: Datastream.

The purpose of this paper is proposing a GARCH(1,1) with exogenous covariate for EUR/SEK exchange rate volatility. The included exogenous covariate serving as a proxy for global volatility information is expected to affect the conditional variance and deliver better estimates of model parameters. Simulation study is conducted to study the finite-sample properties of model parameters. Besides, the Bayesian approach to estimating the conditional variance of EUR/SEK using Markov Chain Monte Carlo is performed. Figures 2 clearly show the financial time-series’ volatility clustering. Market turbulence and uncertainty driving the trades in the financial markets cause fluctuations in the prices which results in volatility. Exchange rates are not exception here. In the Figure 2, we
see the most turbulent periods for EUR/SEK are 1998 (Asian financial crisis), 2000-2002 (IT bubble) and lately the global credit crunch that happened in 2008. Therefore, proposing a GARCH (1,1) model with exogenous covariate for EUR/SEK exchange rate is of great importance for quantifying the effect of global volatility information (shock) on the dynamics of conditional variance.

This paper contributes to the existing empirical econometric literature in the following ways:

First, I examine conditional volatility of EUR/SEK for longer time horizon compared to prior empirical studies. To the best of our knowledge, this paper examines the covariate Chicago Board Options Exchange volatility index (VIX) for the conditional volatility of SEK/EUR with high frequency data for the first time. Further research focusing on finding dependence structures with covariate can enhance the usefulness of it as an input for volatility modeling.

Second, there is room for extending the model assuming the covariate can be modelled as a function of other variables or may have other distribution functions.

Third, like most previous researchers who tried to link the covariate to the dynamics of conditional variance lacked relevant asymptotic theories, in our case, I have a back-up having a recent proposal of asymptotic theory with persistent covariate for GARCH processes. And finally, the implication of the empirical findings can serve as a crucial input for risk management and design of derivative securities, asset pricing models and foreign exchange industry.

The structure of the paper is as follows: The Chapter 2 and 3 presents the theory and model specification. Chapter 4, 5 and 6 covers the estimation results together with simulation study. The final Chapter concludes the paper.

Figure 2: Log returns of Swedish Krona and Euro (SEK/EUR) exchange rate for the period of January 4, 1993 to February 7, 2012. Source: Datastream.
0.2 Theory and Methodology

0.2.1 GARCH models

The theoretical framework of the present study is basically referring to a class of ARCH models. Let \( \Omega_{t-1} \) be a set of past generated \( \varepsilon_t \) values and \( \sigma_t \) is measured with respect to \( \Omega_{t-1} \). Then, the univariate ARCH model is written as follows:

\[
\varepsilon_t | \Omega_{t-1} \sim N(0, \sigma_t^2), \quad \sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \ldots + \alpha_p \varepsilon_{t-p}^2 \tag{1}
\]

This model is called ARCH of order \( p \). The contribution of Bollerslev[7] is generalizing the second part of (1) by allowing for past conditional variance in the variance equation:

\[
\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-q}^2 + \ldots + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-p}^2 \tag{2}
\]

Where \( \alpha_0, \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) are non-negative. Stationarity requires that \( p \sum_{i=1}^{i=p} \alpha_i + \sum_{j=1}^{j=q} \beta_j < 1 \).

This is called a generalized ARCH (\( q,p \)) or GARCH(\( q,p \)). The log-likelihood function for GARCH(1,1) with Gaussian distribution becomes:

\[
L = -\frac{T}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^{T} [\ln(\sigma_t^2)] + \frac{\varepsilon_t^2}{\sigma_t^2} \tag{3}
\]

0.2.2 GARCH(1,1) with exogenous covariate

Consider the volatility model which is given by

\[
y_t = \sigma_t \varepsilon_t \tag{4}
\]

and let \( (\Omega_t) \) be the filtration representing the information set at time \( t \).

\[
Var(y_{t+1}|\Omega_t) = \sigma_{t+1}^2 \tag{5}
\]

We assume \( (\varepsilon_t) \) is iid (0,1) and is adapted to information set available at time \( t(\Omega_t) \). And \( \sigma_{t+1} \) is given by

\[
\sigma_{t+1}^2 = w + \alpha y_t^2 + \beta \sigma_t^2 + f(z_t, \pi) \tag{6}
\]

for parameters \( w, \alpha, \beta \in J \subset R_+ \) and it is assumed that \( \sum_{i=1}^{\infty} \alpha_i + \sum_{j=1}^{\infty} \beta_j < 1, \pi \in R^d \). The \( f(z_t, \pi) \) is assumed to be strictly positive. In the context of the model, the covariate \( z_t \) is a nearly integrated process. That is given a univariate time-series with deterministic trend and if disturbance terms is regressed on first lag of its series, in case of the coefficient is nearly to one, this process is called a nearly integrated process. To ensure a positive effect of \( z_t \) a logistic distribution function is employed

\[
F_x = \frac{1}{1 + e^x} \tag{7}
\]

and the density by \( \frac{d}{dx}F_x = \frac{e^{-x}}{(1 + e^{-x})} \). Prior studies have used other distributions for covariates such as beta and gamma distributions.

In order to present the model in ARMA(1,1) form, we define \( k_{t+1} = y_{t+1} - \sigma_{t+1}^2 \) as a martingale difference sequence. Moreover, \( \sigma_{t+1}^2 = y_{t+1}^2 - k_{t+1} \).
\[ y_{t+1}^2 - k_{t+1} = w + \alpha y_t^2 + \beta (y_t^2 - k_t) + f(z, \pi) \]
\[ y_{t+1}^2 - k_{t+1} = w + \alpha y_t^2 + \beta y_t^2 - \beta k_t + f(z, \pi) \]
\[ y_{t+1}^2 - k_{t+1} = (w + \alpha + \beta) y_t^2 - \beta k_t + f(z, \pi) \]
\[ y_{t+1}^2 = w + (\alpha + \beta) y_t^2 + f(z, \phi) + (k_{t+1} - \beta k_{t-1}) \]

With a covariate it is given in a non-linear ARMAX(1,1). If the covariate is not included it reduces to GARCH(1,1) process. For most GARCH models, the quasi-maximum likelihood estimation procedure was empirically used which is discussed by Bollerslev and Wooldridge[6]. Lumsdaine[34] and Lee and Hansen[33] have formally established the asymptotic distribution theory of the quasi-maximum likelihood estimator (QMLE) in the GARCH(1,1) model, stationary GARCH(1,1) as well as IGARCH(1,1). Their assumptions consist of covariance-stationarity or strict stationarity (in case of IGARCH(1,1)) of the GARCH(1,1) process. In case of an integrated or nearly integrated covariate included in the GARCH(1,1), the process loses its stationarity property which undermines the suggested theories.

Recently, Han and Park[27] propose asymptotic theory of maximum likelihood estimator for ARCH/GARCH model with persistent covariate, establish consistency and obtain limit distribution. They show that limit distribution is non-Gaussian and takes the functional form of Brownian motions. However, the limit distribution becomes Gaussian if the covariate has innovation not correlated with squared innovation of the model. In other words, volatility function is linear in parameter. The estimation of the proposed volatility model is carried out with maximum likelihood estimator. Under a set of regularity conditions, estimation through Maximum Likelihood method, provides optimal estimators. They are asymptotically efficient and consistent when Cramer-Rao bound is reached[21]. Let \( \psi = (w, \alpha, \beta, \pi) \) and the parameter set is \( \Theta = A \times \Pi \). Besides, it can be written as \( \sigma_{t+1}^2(\psi) = w + \alpha y_t^2 + \beta \sigma_{t}^2 + f(z_t, \pi) \). The conditional log-likelihood for \( y_{t+1} \) given \( \Omega_t \) for \( t=1,...,n \) is \( \ell_t(\psi) \).

For the entire sample \( (y_1, ..., y_n) \), we define the log-likelihood function as:

\[
\sum_{t=1}^{n} \ell_t(\psi) = -\frac{1}{2} \sum_{t=1}^{n-1} \left( \log \sigma_{t+1}(\psi) + \frac{y_{t+1}^2}{\sigma_{t+1}^2(\psi)} \right)
\]

The MLE \( \hat{\psi}^n \) is defined as \( \hat{\psi}^n = \arg\max \sum_{t=1}^{n} \ell_t(\psi) \).

By denoting \( \sigma_{t+1}^2(\psi) = \sigma_{t+1}(\psi) \) the score vector \( \hat{S}^n(\psi) \) and Hessian \( \hat{H}(\psi) \) are given by

\[
\hat{S}^n(\psi) = \sum_{t=1}^{n} \frac{\partial \ell_t(\psi)}{\partial (\psi)} = \frac{1}{2} \sum_{t=1}^{n} \left[ \frac{y_{t+1}^2}{\sigma_{t+1}^2(\psi)} - 1 \right] \frac{1}{\sigma_{t+1}^2(\psi)} \frac{\partial \sigma_{t+1}^2(\psi)}{\partial (\psi)},
\]
\[
\hat{H}^n = \sum_{t=1}^{n} \frac{\partial^2 \ell_t(\psi)}{\partial (\psi) \partial (\psi)} = \frac{1}{2} \sum_{t=1}^{n} \left[ (1 - 2 \frac{y_{t+1}^2}{\sigma_{t+1}^2(\psi)}) \frac{1}{\sigma_{t+1}^2(\psi)} \frac{\partial \sigma_{t+1}^2(\psi)}{\partial (\psi)} \frac{\partial \sigma_{t+1}^2(\psi)}{\partial (\psi)} + \left( \frac{y_{t+1}^2}{\sigma_{t+1}^2(\psi)} - 1 \right) \frac{1}{\sigma_{t+1}^2(\psi)} \frac{\partial^2 \sigma_{t+1}^2(\psi)}{\partial (\psi) \partial (\psi)} \right]
\]

### 0.3 Bayesian estimation with Markov Chain Monte Carlo

Now we present the procedure applied for conducting Markov Chain Monte Carlo estimation of GARCH(1,1) with Student-t innovations. Let’s write the GARCH model with Student-t innovations for log returns \( y_t \) via data augmenting:

\[ y_t = \varepsilon_t \left( \frac{S-2}{S} \right) w_th_t \]
\[ \varepsilon_t \sim N(0,1) \]
\[ w_t \sim IG\left( \frac{S}{2}, \frac{S}{2} \right) \]
where \( w > 0, \alpha, \beta \geq 0, \text{and} s > 2; N(0, 1) \) IG is the inverse Gamma distribution. Here estimation is carried with Student-t distribution and the restriction on \( s \) (degrees of freedom of parameter) ensures the conditional variance to be finite. For the rest, applies the same notion explained above. Let’s define the vectors \( y = (y_1, ..., y_T) \), \( w = (w_1, ..., w_T) \) and \( \alpha(\alpha_0, \alpha_1) \). It can be regrouped into the vector \( \theta = (\alpha, \beta, s) \). Now we need to define the diagonal matrix which is given by: \( \Sigma = \Sigma(\theta, w) = \text{diag}(w_i - S h_i(\alpha, \beta)) \) where \( n=1, ..., N \) where, \( h_i(\alpha, \beta) = \alpha_0 + \alpha_1 + y_i^2 \) and \( \alpha(\alpha_0, \alpha_1) \). The loglikelihood function for \((\theta, w)\) can be written as

\[
L(\theta, w \mid y) \propto (\text{det} \Sigma)^{-1/2} \exp\left[-\frac{1}{2} \Sigma^{-1} y\right]
\] (9)

In Bayesian analysis framework \((\theta, w)\) is considered to be a random variable. It can be characterized by a prior density given by \(p(\theta, w)\). As a standard procedure in econometrics, to come a posterior density the probability density is transformed applying Bayes rule. The posterior density is then given by:

\[
p(\theta, w \mid y) = \frac{L(\theta, w \mid y)p(\theta, w)}{\int L(\theta, w \mid y)p(\theta, w)d\theta dw}
\] (10)

We use truncated normal priors on the parameters \((\alpha, \beta)\) of GARCH model

\[
p(\alpha) \propto \theta_{N_2}(\alpha \mid \mu_\alpha, \Sigma_\alpha) 1\{\alpha \in R^2_+\} \\
p(\beta) \propto \theta_{N_1}(\alpha \mid \mu_\beta, \Sigma_\beta) 1\{\alpha \in R_+\}
\]

where \(\mu_\alpha\) and \(\Sigma_\alpha\) are hyperparameters. There is an indicator function, which is given by \(1\{\circ\}\) and \(\theta_{N_1}\) is the \(d\)-dimensional normal density. To obtain the prior distribution of vector \(w\) conditional on \(\nu\), the assumption is that components of \(w_t\) are independent and identically distributed from inverted gamma density

\[
p(w \mid \nu) = \left(\frac{\nu}{2}\right)^\frac{T}{2} [\Gamma(\frac{\nu}{2})]^{-T} \exp\left[-\frac{1}{2} \sum_{t=1}^{T} \frac{\nu}{w_t}\right]
\]

Deschamps[10] procedure is followed to choose the prior distribution on the degrees of freedom parameters. The distribution can be represented as a translated exponential with parameters \(\lambda > 0\) and \(\tau \geq 2\)

\[
p(\nu) = \lambda \exp[-\lambda(\nu - \tau)] 1\{\nu > \tau\}
\]

As pointed out by Ardia[2], the mass of the prior is concentrated in the neighbourhood of \(\tau\) and the restriction is imposed on the degrees of freedom. Deschamps[10] emphasized that this prior is useful for two reasons. First, for bounding the degrees of freedom parameter for numerical reasons and preventing conditional variance to explode. Second, it improves the convergence of the sampler. To form the joint distribution as shown by Ardia[2], prior independence between parameters was assumed

\[
p(\theta, w) = p(\alpha)p(\beta)p(w \mid \nu)p(\nu)
\]

Application of Bayesian methods using Markov Chain Monte Carlo simulation has substantially increased in econometric research because they are able to estimate complicated models and produce finite sample inference. This procedure takes its root since Metropolis-Hastings algorithm was suggested by Metropolis et.al[26]. The choice of proposal (target) distribution plays a crucial role whether and at what speed the chain converges to the posterior distribution. Adaptive schemes were later proposed in the literature, for more
review of statistical properties, see Andrieu[1]. Nakatsuma[38] has developed Markov
Chain Monte Carlo for GARCH($q$, $p$). This laid foundation for applying Bayesian meth-
ods in GARCH models. Bayesian inference methods based on GARCH class models have
been performed with importance sampling by Kleibergen and van Dijk[30] and Griddy-
Gibbs sampler by Bauwens and Lubrano[4]. Several researchers including Bauwens and
Lubrano[4], Nakatsuma[38], Mitsui and Watanabe[37] have implemented Markov Chain
Monte Carlo in drawing inference while GARCH class volatility models were used. A
Taylored approach based on the acceptance-rejection Metropolis-Hastings (ARMH) was
developed by Mitsui and Watanabe[37]. This method is well-known and can work with
any parametric ARCH class models. The following steps are followed when this method is
applied in practice. The maximization is carried out for the sum of log-prior coupled with
log-likelihood with respect to all parameters. In the next step, sampling with all parame-
ters is performed using ARMH algorithm from a multivariate Student-t distribution with
mean that maximizes the inverse of negative hessian matrix and objective function. In the
third step, the proportional constant is defined that can serve as a candidate generating
density to the full conditional distribution. It is performed until the former evaluated at
some parameter is equal to the latter. Iteratively, the second step is repeated and outputs
are retained after the Markov Chain has converged. To implement the bayesian approach
we employ the package in R proposed by Ardia[2]. For approximating the posterior den-
sity, Markov Chain Monte Carlo is used. The algorithm is made up of a MH algorithm
where the GARCH parameters are updated by blocks while the degrees of freedom pa-
rameter is sampled using an optimized rejection technique from a translated exponential
source density Ardia[2].

0.4 Model evaluations

0.4.1 Application

The data used in the empirical application of the proposed model is daily closing price
of EUR/SEK exchange rate and Chicago Board Options Exchange volatility index (VIX)
for the period 1993 and 2010. The time-series for the variables were obtained from Datas-
tream. We take log difference of both series to convert them into stationary series. The
total number of observations for modeling is 4655. The sample ranges from January 04,
1993 to November 3 2010.

Table 5 in the Appendix reports summary statistics for the SEK/EUR and VIX return
series. The presented statistics cover first and second coupled with third and fourth
order moments. The Jarque-Berra test rejects, at the 1 percent significance level the null
hypothesis of normality for both return series. Welch Two Sample t-test did reject of true
difference in means equal to zero. The Ljung-Box statistic rejects the null hypothesis at
10 significance for both return series. Looking at the table, we observe that both series
are positively skewed. This explains the fact that extreme returns are probable to happen
in positive values.

Figure 1 shows the exchange rate series for SEK/EUR for the period of 1992 to 2012. It
reveals the abrupt rise during the global credit crunch causing SEK to depreciate against
EUR. During this period of time fluctuations in many European financial markets occurred
and prior empirical evidence links this to fluctuations at the exchange rate. The EUR falls
to its minimum rate in the course of IT bubble. A close look at Figure 1 indicates that
after the IT bubble bursted there is a slight upward trend for EUR until the global credit crunch. We observe the downward trend for SEK after the crisis that implies that EUR was depreciating.

0.4.2 Estimation Results

As a starting point, we estimated the GARCH(1,1) model with Gaussian distribution for SEK/EUR for the period 1992-2010. Table 1 (Panel A) presents the estimation results for the parameters of the model with Maximum Likelihood estimator. The value for constant appears to be zero there because of very small value. We document positive and statistically significant coefficients both for ARCH and GARCH effects. 1 percent increase in shocks effects the conditional variance to increase by 0.06 percent. And 1 percent increase in one period lagged conditional variance effects the conditional variance to increase by 0.9 percent. Summing up two coefficients, we obtain a persistence level, which is equal to 0.996. In other words, past shocks and variances have longer effect on the future conditional variance. One can also describe it as the degree of persistence in the autocorrelation of squared returns and in turn it controls the intensity of volatility clustering. Apparently, this value is similar to previous findings in empirical literature.

After including the covariate (Chicago Board Options Exchange volatility index (VIX)), we have estimated the eq (6) with Gaussian distribution. In Table 1 (Panel B) we display the estimated parameters for GARCH(1,1) with exogenous covariate. If we compare the obtained results with GARCH(1,1) without covariate, it reveals that the coefficient for one-period lagged conditional variance ($\beta$) increase by 0.34 percent and in absolute term is 0.00323. The case is slightly different for $\alpha$. The effect is -0.67 percent decrease in the coefficient value. The main findings which are of particular interest is the coefficient for the covariate in the eq (6). Estimation results show that it is very small and does not significantly effect the conditional variance of SEK/EUR. The Likelihood increased by 0.27 percent. A close look at Table 1 (Panel B) indicates that the persistence level ($\alpha + \beta$) is equal to 0.999, which is extremely high. The increase in percentage is 0.28. However, when one consider to select the one of two models, one that has higher Akaike criterion is preferred. In this case, the GARCH(1,1) with covariate is selected.

Our expectation was that the covariate would have statistically significant effect on the conditional variance, however, it turned out not to be the case. One way to overcome this problem is including the covariate in mean equation in multiplicative form. After performing this task, the estimation results is presented in Table 1 (Panel C). The coefficient ($\eta$) for in-mean effect, is negative and statistically significant. In contrast, the effect of covariate is positive $\vartheta$ with statistically significant coefficient. The loglikelihood is 22883.90 and higher than that of estimated with GARCH(1,1) and GARCH(1,1)in-mean with covariate and Akaike criterion is -9.8277. There is no any slight change observed in persistence level.

0.5 Simulation study

I perform simulation on the proposed GARCH(1,1) model with exogenous covariate for SEK/EUR and GARCH(1,1) in-mean with exogenous covariate in order to examine the finite-sample properties of model parameters and extreme scenarios. Simulation is based on the volatility equation, which is given by
### Table 1: Estimation results of model parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>t value</th>
<th>p value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.00000</td>
<td>0.00002</td>
<td>0.14342</td>
<td>0.88596</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.02652</td>
<td>0.97884</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>0.06259</td>
<td>0.02673</td>
<td>2.34146</td>
<td>0.01920</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.93406</td>
<td>0.02467</td>
<td>37.8555</td>
<td>0.00000</td>
</tr>
<tr>
<td>Panel B:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.00000</td>
<td>0.00002</td>
<td>0.05187</td>
<td>0.95863</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.02942</td>
<td>0.97653</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>0.06217</td>
<td>0.00665</td>
<td>9.34417</td>
<td>0.00000</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.93727</td>
<td>0.00566</td>
<td>165.658</td>
<td>0.00000</td>
</tr>
<tr>
<td>$\vartheta$</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.16853</td>
<td>0.86616</td>
</tr>
<tr>
<td>Panel C:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.00001</td>
<td>0.000035</td>
<td>2.82614</td>
<td>0.00471</td>
</tr>
<tr>
<td>$\eta$</td>
<td>-0.06158</td>
<td>0.019484</td>
<td>-3.16044</td>
<td>0.00157</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>-0.00005</td>
<td>0.000105</td>
<td>0.47828</td>
<td>0.63244</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.00000</td>
<td>0.000000</td>
<td>0.14123</td>
<td>0.88768</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>0.06178</td>
<td>0.000121</td>
<td>509.255</td>
<td>0.00000</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.93451</td>
<td>0.001835</td>
<td>509.252</td>
<td>0.00000</td>
</tr>
<tr>
<td>$\vartheta$</td>
<td>0.00000</td>
<td>0.000000</td>
<td>61663.9</td>
<td>0.00000</td>
</tr>
</tbody>
</table>

The estimation of all parameters is carried out with Maximum Likelihood Estimator with Gaussian distribution. Panel A represents estimation results for $\sigma_{t+1}^2 = w + \alpha_1 y_t^2 + \beta \sigma_t^2$ equation. Panel B represents estimated parameters for $\sigma_{t+1}^2 = w + \alpha_1 y_t^2 + \beta \sigma_t^2 + f(z_t, \pi)$ equation. While covariate enters the mean equation in multiplicative form is given in Panel C. $p$ values are given in the last column. Significant levels are at 1%, 5%, 10% when $p < 0.01, p < 0.05$ and $p < 0.1$. 

The estimation of all parameters is carried out with Maximum Likelihood Estimator with Gaussian distribution. Panel A represents estimation results for $\sigma_{t+1}^2 = w + \alpha_1 y_t^2 + \beta \sigma_t^2$ equation. Panel B represents estimated parameters for $\sigma_{t+1}^2 = w + \alpha_1 y_t^2 + \beta \sigma_t^2 + f(z_t, \pi)$ equation. While covariate enters the mean equation in multiplicative form is given in Panel C. $p$ values are given in the last column. Significant levels are at 1%, 5%, 10% when $p < 0.01, p < 0.05$ and $p < 0.1$. 

9
\begin{align*}
\sigma^2_{t+1} &= 0.062169\sigma^2_t + 0.937275\sigma^2_t + f(.) \tag{11} \\
\sigma^2_{t+1} &= 0.061785\sigma^2_t + 0.934515\sigma^2_t + f(.) \tag{12}
\end{align*}

The innovation process \( \varepsilon_t \) is generated from iid standard Gaussian distribution. I replicate this task for five thousand times. Then, I look at the mean of the simulated path series coupled with densities to draw conclusions. The parameters of the laptop used for simulation are: model=ASUS, Processor: Celeron (R)Dual Core CPU, T300@ 1.80GHz (2 CPUs), 1.8GHz,2048MB RAM, OS: Windows 7 Home Premium. The software used in simulation experiment is R version 2.12.2 distributed free by (The R Foundation for Statistical Computing). The package is provided by [20].

Figure 3 depicts the simulated mean for conditional variance. This corresponds to the eq (11). We observe the highest value for conditional variance in the horizon of five thousand and one hundred and then the path takes the downward shift. Table 2 presents the results of the empirical value for conditional and their means of simulated paths. To further extend the analysis, simulation results for the eq (12) is presented in Table 3. There is also evidence showing the simulated means closer to their empirical means. Figure 4 plots the simulated path for conditional variance. Observe that the pattern of simulation path is different compared with eq(11). Covariate that enters mean equation in multiplicative form resulted in different pattern in the simulated path of conditional variance. This can be explained by the negative effect of the covariate in mean equation. To have a better picture of the conditional variance, there are Figure 8 and 10 that plot the density of conditional variance coupled with simulated density. A careful examination of the figures indicate that they are skewed on the right side. Both of them are not symmetric as well. The case of simulation experiments with GARCH(1,1) in-mean with covariate show that the simulated density nearly approximate the actual density of conditional variance. Those densities can be applied in probability models accounting for uncertainty of conditional variance.

0.5.1 Simulation of parameter uncertainty

Understanding the sources of GARCH parameters uncertainty is crucial for accurate forecasting purposes. It is often the case when the forecasted latent conditional variance is biased due to a parameter uncertainty. Simulation-based of establishing the consistency of estimators is practiced in econometrics. This is carried out in the context of the difference of simulated and hypothesized true values. Figure 6 depicts the simulated density plots for estimate parameters of GARCH(1,1) with exogenous covariate. Note that the constant term and coefficient for exogenous covariate in the conditional variance equation given in Table 1 (Panel B) appeared to be zero because of very small value. However, the true value is 4.26E-09 and we conducted simulation experiment by fitting the model multiple times under different window size. The densities for all parameters are presented in Figure 6 under GARCH(1,1) with exogenous covariate. The simulated density plot for exogenous covariate shows the uncertainty which can cause to fluctuate in given density and the true is 1.78E-08, which is very small. In addition, the figure shows the simulated densities both for \( \alpha_1 \) and \( \beta_1 \). And Figure 7 displays the simulated densities under GARCH(1,1) in-mean with exogenous covariate. One can observe that the density patterns for \( \alpha_1 \) and \( \beta_1 \) are different under two various simulation experiments states. When the covariate enters in the conditional mean equation in multiplicative form, I found that they have fatter tails and probability of extreme values to occur is greater. Based on these results,
it is obvious that we can model the uncertainty of GARCH model parameters and cor-
respondingly adjust our future conditional variance. The risk of uncertainty arising from
parameters in GARCH model can also be quantified once these simulated densities used
to give the parameter value with their probabilities. The forecasted future volatility is
widely used in risk management as we noted before. Better forecasts deliver more precise
VoR (Value-at-Risk) estimation and correct risk variable inputs in risk models prevent
financial institutions to incur bigger losses.

0.6 MCMC estimation results

We applied bayesian estimation method for GARCH(1,1) estimated with Student-t distri-
bution. The data set for the analysis is 4655 daily closing exchange for SEK/EUR. The
number of chains used for the analysis is two and each chain has length equal to thirty
thousand. Figure 13 in the Appendix plots the trace for estimated parameters coupled
with density plots of each parameter for thirty thousand iterations. Table 4 presents the
results of posterior statistics for GARCH(1,1) estimated with Student-t distribution using

<table>
<thead>
<tr>
<th>T</th>
<th>Empirical</th>
<th>Simulated mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>0.000004460</td>
<td>0.000014400</td>
</tr>
<tr>
<td>2000</td>
<td>0.000004460</td>
<td>0.000005440</td>
</tr>
<tr>
<td>3000</td>
<td>0.000004460</td>
<td>0.000006520</td>
</tr>
<tr>
<td>4000</td>
<td>0.000004460</td>
<td>0.000006020</td>
</tr>
<tr>
<td>5000</td>
<td>0.000004460</td>
<td>0.000006090</td>
</tr>
</tbody>
</table>

Figure 3: The simulated path for conditional variance.
MCMC method. It clearly shows that \( \alpha_1 \) is equal to 0.24 and the parameter value for the \( \beta \) is 0.67. We are interested in drawing inference about parameter distributions based on a random sample drawn from joint posterior density of \((\alpha_1+\beta)\). In other words, our inference gives us an estimate of persistence level. To carry out this task, we randomly sample two thousand from joint posterior density of \( \alpha \) and \( \beta \). A closer look at Figure 5 reveals that the persistence level is lower compared to previous estimated values. It is based on two thousand random draws from joint posterior distribution of \((\alpha_1+\beta)\). According to empirical findings, this fact can be explained by the presence of structural break and estimation through Maximum Likelihood cannot capture and delivers higher persistence level. The persistence level is equal to 0.79. The notion behind taking into account for structural breaks while modeling time-varying volatility is that sudden large shocks can result in abrupt breaks in the unconditional variance of exchange rate returns as pointed out by Rapach et.al[40]. He argues that this is equivalent to structural breaks in the parameters of GARCH processes. Theoretical studies indicate the existence of structural breaks in an unconditional variance have major implications for estimated parameters of GARCH models. For more review of this topic refer to Diebold[19] and Henry[11]. Empirical findings of studies conducted by Mikosch and Stáríča[36] and Hillebrand[14] show that there is
upward biases in estimate of the persistence of GARCH processes when structural breaks are neglected in the parameters of GARCH processes. Consequently, poor estimates of the unconditional volatility of exchange rate returns are obtained in case of not taking into account for a structural break in evaluating the forecasts. West and Chou[45] argue that forecasting performance of GARCH(1,1) can be improved by allowing for structural breaks in the unconditional variance of exchange rate returns. A bayesian comparison of multivariate ARCH-type models was performed by Osiewalski and Pipien[28]. Their findings suggest that model parsimony is equally important as flexibility of covariance structure. They emphasize that parsimonious specification comes at the cost of strong assumptions about the covariance structure and in turn model have little explanatory power. Moreover, they rejected the hypotheses of constant correlations. Finally, our findings are in line with previous empirical literature.

Figure 5: Histogram of the posterior density for the persistence.
0.7 Conclusions

In this paper, we have developed a GARCH(1,1) with exogenous covariate for EUR/SEK volatility. The included covariate in the conditional variance equation was Chicago Board Options Exchange volatility index (VIX). This covariate assumed to be a proxy for global volatility information. In this way, we have attempted to capture the effect of the covariate to the dynamics of EUR/SEK conditional volatility. Empirical literature shows that various covariates add to the explanatory power of model and help to better capture the dynamics of conditional volatility. Moreover, the covariate was included in mean equation in multiplicative form to further extend the analysis.

As a starting point, the estimation of GARCH(1,1) with Gaussian distribution was carried out for SEK/EUR and empirical findings suggest the existence of volatility clustering and this promotes an argument that accounting for such features of financial time-series volatility based models, such as GARCH serves our purpose well. Findings indicate positive and statistically significant coefficients for both ARCH and GARCH effects. Besides, our model GARCH(1,1) with exogenous covariate was estimated and the effect of Chicago Board Options Exchange volatility index (VIX) on conditional variance is not significant. However, when the covariate entered the conditional mean equation in multiplicative form, the effect is statistically significant. Effects for persistence level were quantified as a result of estimating model under two different states. In fact, in-mean effect is negative and at the same time there is a positive effect of exogenous covariate in conditional variance equation with statistically significant coefficient.

To explore the bayesian method in estimating GARCH class models, we have empirically applied the Markov Chain Monte Carlo to approximate the posterior density and
empirical facts substantiate that persistence level under bayesian estimation is lower compared with prior estimation results. This confirms the previous studies showing higher persistence level due to a structural break when Maximum Likelihood estimation was applied. In addition, simulation study was conducted to investigate extreme scenarios.

Summing up, accounting for global volatility information (shock) for EUR/SEK exchange rate volatility can help to design better risk models within financial institutions.
0.8 Appendices

Figure 8: The simulation density for the conditional variance.

Figure 9: The logistic function
Table 5: Descriptive Statistics

<table>
<thead>
<tr>
<th></th>
<th>SEK/EUR</th>
<th>VIX</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observations</td>
<td>4655.000000</td>
<td>4655.000000</td>
</tr>
<tr>
<td>Minimum</td>
<td>-0.012600</td>
<td>-0.152259</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.015943</td>
<td>0.215413</td>
</tr>
<tr>
<td>1 Quartile</td>
<td>-0.001032</td>
<td>-0.015305</td>
</tr>
<tr>
<td>3 Quartile</td>
<td>0.001037</td>
<td>0.013532</td>
</tr>
<tr>
<td>First moment</td>
<td>0.000008</td>
<td>0.000024</td>
</tr>
<tr>
<td>Median</td>
<td>0.000000</td>
<td>-0.001381</td>
</tr>
<tr>
<td>Sum</td>
<td>0.036628</td>
<td>0.113180</td>
</tr>
<tr>
<td>SE Mean</td>
<td>0.000031</td>
<td>0.000379</td>
</tr>
<tr>
<td>LCL Mean</td>
<td>-0.000052</td>
<td>-0.000719</td>
</tr>
<tr>
<td>UCL Mean</td>
<td>0.000068</td>
<td>0.000767</td>
</tr>
<tr>
<td>Second moment</td>
<td>0.000004</td>
<td>0.000668</td>
</tr>
<tr>
<td>Stdev</td>
<td>0.002084</td>
<td>0.025852</td>
</tr>
<tr>
<td>Third moment</td>
<td>0.114405</td>
<td>0.518890</td>
</tr>
<tr>
<td>Fourth moment</td>
<td>4.294212</td>
<td>3.521415</td>
</tr>
</tbody>
</table>
Figure 11: The simulation path for return series.

Figure 12: Density plots of estimated MCMC parameters
Figure 13: Trace plots of estimated MCMC parameters
Bibliography


[34] R.L. Lumsdaine. Consistency and asymptotic normality of the quasi-maximum likelihood estimator in igarch(1,1) and covariance stationary garch(1,1) models. Econometrica, 64:575–596, 1996.


