# Hardy and Spectral Inequalities for a Class of Partial Differential Operators

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*Yuri: Mycket manga och på biologiboken förstås, och det är ju helt vetenskapligt orealistiskt att vara så självsäker.*

*Asså enligt kvantfysiken, typ den fetaste fysiken för pyttepyttesmå grejer, så kan man ju inte ens se om partiklar är små punkter, som din baseboll här, eller om de är vågor typ som ljud som bara flowar runt i rummet.*

*Det är ju lite som med kompisar, eller hur?*

*Kompisar är ju bara kompisar men beroende på hur man ser på de så kan man ju börja känna helt andra känslor för de eller hur. De kan liksom vara både bollar och vågor samtidigt enligt den där Heisenbergs Obestämdbarhetsprincip.*

*HIDEO: Eh...? Heisenbergu?*

*Yuri: Asså Heisenberg menar att det kanske inte finns partiklar eller vågor eller så här kompisar eller något annat.*

*Det kanske inte finns någon fix "verklighet" överhuvudtaget utan bara olika sannolikheter.*

*Och då kan ju någonting som man först tog helt för givet plötsligt visa sig vara någonting helt, helt annat än det som man först trodde.*

**-Hamadi Khemiri, *C* (2011)**

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# Abstract


This thesis is devoted to the study of Hardy and spectral inequalities for the Heisenberg and the Grushin operators. It consists of five chapters.

In chapter 1 we present basic notions and summarize the main results of the thesis.

In chapters $2-4$ we deal with different types of Hardy inequalities for Laplace and Grushin operators with magnetic and non-magnetic fields.

It was shown in an article by Laptev and Weidl [LW] that for some magnetic forms in two dimensions the Hardy inequality holds in its classical form. More precisely, by considering the Aharonov-Bohm magnetic potential, we can obtain a non-trivial Hardy inequality.

In chapter 2 we establish an $L^p$-Hardy inequality related to Laplacians with magnetic fields with Aharonov-Bohm vector potentials.

In chapter 3 we introduce a suitable notion of a vector field for the Grushin sub-elliptic operator $G$ and obtain an improvement of the Hardy inequality, which was previously obtained in the paper of N. Garofallo and E. Lanconelli (see [GL]).

In chapter 4 we find an $L^p$-version of the Hardy inequality obtained in chapter 3.

Finally in chapter 5 we aim to find some Lieb-Thirring inequalities for harmonic Grushin-type operators. Since the Grushin operator is non-elliptic, these inequalities do not take their classical form.

# Абстракт

Эта диссертация посвящяется изучению неравенств Харди и спектральных неравенств для операторов Хейзенберга и Грушина Она состоит из пяти глав.

В первой главе мы вводим необходимые обозначения и привидим основные результаты работы.

В главах 2-4 мы получаем различные неравенства Харди для операторов Лапласа и Грушина с магнитными полями. В работе Лаптева и Вейдля [LW] было показано что для некоторово класса магнитных форм в размености два, неравенства Харди имеют классическои вид. Более точно, для операторов с магнитными полями типа Ааронова-Бома получены нетривиальные неравенства Харди.

В главе 2 мы получем $L^p$-неравенство Харди для оператора Лапласа с магнитным полем Ааронова-Бома.

В главе 3 мы определяем некоторое магнитное поле для оператора Грушина и получем некоторое улучшение неравенства Харди по ставнению с неравенством из работы Гарофалло и Ланцонелли [GL].

В главе 4 мы находим $L^p$-версию неравенсва Харди которое было доказано в главе 3.

Наконец в главе 5 мы доказываем неравенство Либа-Тирринга для версии гармонического осциллятора для оператора Грушина. Поскольку оператор Грушина не эллиптичен эти неравенства имеют неклассическую форму.

# Zusammenfassung

Diese Doktorarbeit ist der Untersuchung von Hardy- und Spektralungleichungen für Heisenberg und Grushin Operatoren gewidmet. Die Arbeit besteht aus fünf Teilen.

Im ersten Kapitel stellen wir die Grundbegriffe und eine Zusammenfassung der wichtigsten Resultate der Arbeit vor.

In den Kapiteln 2 bis 4 beschäftigen wir uns mit verschiedenen Arten von Hardy-Ungleichungen für Laplace- und Grushin-Operatoren mit magnetischen und unmagnetischen Feldern.

Im Artikel [LW] zeigen A. Laptev und T. Weidl, daß die klassische Form der Hardy-Ungleichung für einige magnetische Formen in zwei Dimensionen gilt. Genauer gesagt, können wir durch Betrachtung des Aharonov-Bohm magnetischen Potentials die Konstante in der jeweiligen Hardy-Ungleichung verbessern.

Im Kapitel 2 wird eine die mit Laplace-Operatoren mit magnetischem Felt mit Aharonov-Bohm Vektorpotential zusammenhängt $L^p$-Hardy-Ungleichung eingeführt.

Im Kapitel 3 führen wir einen für den Grushin subelliptischen Operator $G$ geeigneten Begriff des Vektorfelds ein, und erhalten eine Verbesserung der Hardy-Ungleichung, die zuvor schon von N. Garofallo und E. Lanconelli hergeleitet wurde ([GL]).

Im Kapitel 4 finden wir eine $L^p$-Version der Hardy Ungleichheit, die im Kapitel 3 erreicht ist.

Schließlich im Kapitel 5 zielen wir darauf ab, die CLR- und Lieb-Thirring Ungleichungen für harmonische Grushin-artige Operatoren herauszufinden. Weil Grushin-Operatoren nicht elliptisch sind, nehmen diese Ungleichungen ihre klassische form nicht an.

# תקציר

התֵזה מוקדשת לחקר אִי־שִׁוויוֹנֵי הארדי ואִי־שִׁווֹיִנים ספֶּקטרַלים לאופרטוֹרים מסוּג הָייזֶנבֶּרְג וּגְרוּשִׁין. היא מורכבת מחמישה פרקים.

בפרק הראשון אנו מציגים מושׂגים בסיסיים וכן את התוצאות העקריות של התזה.

בפרקים 2־4 אנו עוסקים בסוגים שונים של אי־שיוויוני הארדי עם שׂדות מגנטים ושׂדות שאינם מגנטים.

במאמר מאת אָרִי לָאפּטֶב וטִימוֹ וָיְיְדֶל [WL] המחברים הראו, שאי־שיוויון הארדי מיוּשׂם בצורתו הקלאסית עבור שׂדות מגנטיים מסויימים. ליתר דיוק, על־ידי לקיחה בחשבון של הפּוטנציאל המגנטי אהרונוב־בּוֹהם, אנו יכולים לשפר את הקבוע באי־שיוויון הארדי התואם.

בפרק 2 אנו קובעים אי־שיוויון הארדי במרחב $L^p$ שקשור ללפּלאסיאנים עם שׂדות מגנטיים עם פוטנציאל וקטורי מסוג אהרונוב־בוהם.

בפרק 3 אנו מציגים רעיון הולם לשׂדה וקטורי עבור האופרטור הסאב־אליפּטי מסוג גרושין $G$ ומקבלים שיפור של אי־שיויני הארדי, שהושׂג לפנֵי כן במאמר מאת ניקולה גָרוּפָּאלוֹ ואֶרמאנו לָנקוֹנֶלִי ([LG]).

בפרק 4 אנו מוצאים גרסת $L^p$ לאי־שיוויון הארדי שהושׂג בפרק 3.

בפרק 5 אנו שואפים למצוא אי־שיוויונים מסוג צְוִויקֶל־לִיבּ־רוֹזֶנבּלוּם ולִיבּ־תִּירִינג לאופרטורים הרמונים מסוג גְרוּשִׁין. מכיון שאופרטור גרושין אינו אליפּטי, אי־שִׁווֹיִנים אלה לא יתקבלו בצורתם הקלאסית.

# Contents

# 1. Introduction

*In this chapter we describe our main results and also summarize some results known before.*

## 1.1 Classical Hardy inequalities

If $d \geq 3$, then for any function $u$ such that $u \in C_0^\infty(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \geq \left(\frac{d-2}{2}\right)^2 \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} dx. \tag{1.1.1}$$

It is well known that the constant $(d-2)^2/4$ in (1.1.1) is sharp but not achieved.

The inequality (1.1.1) is related to a so-called *Heisenberg uncertainty principle.* In its classical form the uncertainty principle was claimed by Heisenberg in connection with the study of quantum mechanics. According to this principle the position and momentum of a particle could not be defined exactly simultaneously, but only with some uncertainty. On the Euclidien space $\mathbb{R}^d$ the uncertainty principle says that

$$\left(\frac{d-2}{2}\right)^2 \left(\int_{\mathbb{R}^d} |u(x)|^2 dx\right)^2 \leq \left(\int_{\mathbb{R}^d} |x|^2 |u(x)|^2 dx\right) \left(\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx\right). \tag{1.1.2}$$

Indeed, the Schwarz inequality applied to (1.1.1) yields

$$\begin{aligned}
\left(\frac{d-2}{2}\right) \int_{\mathbb{R}^d} |u(x)|^2\, dx &= \left(\frac{d-2}{2}\right) \int_{\mathbb{R}^d} |u(x)|^2 \frac{1}{|x|} |x|\, dx \\
&\leq \left(\frac{d-2}{2}\right) \left(\int_{\mathbb{R}^d} |u(x)|^2 |x|^2 dx\right)^{1/2} \left(\frac{|u(x)|^2}{|x|^2} dx\right)^{1/2} \\
&\leq \left(\int_{\mathbb{R}^d} |u(x)|^2 |x|^2 dx\right)^{1/2} \left(\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx\right)^{1/2}.
\end{aligned}$$

This gives (1.1.2).

Applying Parseval's formula for the Fourier transform $\hat{u}$ of the function $u$ in the second integral on the left hand side gives the inequality (1.1.2), particularly by symmetrical form

$$(2\pi)^d \left(\int_{\mathbb{R}^d} |x|^2 |u(x)|^2 dx\right) \left(\int_{\mathbb{R}^d} |\xi|^2 |\hat{u}(\xi)|^2 d\xi\right) \geq \left(\frac{d-2}{2}\right)^2 \left(\int_{\mathbb{R}^d} |u(x)|^2 dx\right)^2,$$

where the Fourier transform of the function $u$ is defined by

$$\hat{u}(\xi) = (2\pi)^{-d/2} \int e^{-ix\xi} u(x) dx.$$

This inequality expresses the Heisenberg uncertainty principle that states that a non-trivial $L^2$-function and its Fourier transform cannot simultaneously be very small near the origin. Of course instead of the origin, we could have taken any other point in $\mathbb{R}^d$.

In two dimensions the uncertainty principle does not hold. However, there is an inequality where the singularity is weakened by adding either a logarithmic term or an extra condition on the function $u$. Namely, we have (see [S])

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, dx \geq C \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2 \left(1 + \ln^2 |x|\right)} \, dx, \quad \text{if} \quad \int_{|x|=1} u(x) \, dx = 0,$$

or

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, dx \geq \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2} \, dx, \quad \text{if} \quad \int_{|x|=r} u(x) \, dx = 0, \quad \forall r > 0.$$

The literature concerning different versions of Hardy inequalities and their applications is extensive. They differ from one another depending on the relation between the parameters, on the weight functions and on the class to which the functions belong. The classical multidimentional $L^p$-Hardy inequality in $\mathbb{R}^d$ reads as follows:

$$\int_{\mathbb{R}^d} |\nabla u|^p dx \geq C_{d,p} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^p} dx, \tag{1.1.3}$$

where $u \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$, $p \geq 1$ and the constant

$$C_{d,p} = \left|\frac{d-p}{p}\right|^p$$

is best possible but not achieved.
The proof of the latter inequality can be found for example in the book “Hardy-type Inequalities” by A. Kufner and B. Opic [KO]. The classical

book “Inequalities” by G. Hardy, J. E. Littlewood and G. Pólya [HLP] has been a source of inspiration for many people. A more modern book, which has been very influential and rewarding for many researchers studying the theory of Hardy-type inequalities, is “Sobolev Spaces” by V. Mazýa [M] (see also [B], [D] and [MMP]).

$L^p$ inequalities are of great importance in the study of the $p$-Laplacian and the $p$-Schrödinger equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + V(x,u) = 0.$$

Note that if $d = p$, then the left hand side of (1.1.3) equals zero.

The literature on different types of Hardy inequalities is vast. Without being able to cover it, we would like to mention the papers [BFT1], [BFT2], [DGN], [T], [MMP], [BM], [DFP], [FMT], [FL], [HHL], [HHLT] and [MMP].

## 1.2 Hardy inequalities related to Heisenberg and Grushin Laplacians

The Hardy inequalities were also studied for some sub-elliptic operators (see for example papers [G], [GL], [A1], [A2], [DGN], [NCH] and [K]) and in particular for the sub-Laplacian on the Heisenberg group $\mathbb{H}$. The latter is the prime example of the non-commutative harmonic analysis and we refer to [Ste] for the background material.

Let us consider $\mathbb{H}$ as $\mathbb{R}^3$ with coordinates $(x,y,t)$ and the (non-commutative) multiplication $(x,y,t)\circ(x',y',t') = (x+x', y+y', t+t'-2(xy'-yx'))$. The vector fields

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}\,, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$$

are left-invariant and the sub-Laplacian on $\mathbb{H}$ is given by

$$H = -X^2 - Y^2 = -\left(\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}\right)^2. \tag{1.2.1}$$

The quadratic form $h$ of the operator $H$ is defined by the equality

$$h[u] = \int_{\mathbb{R}^3} (|Xu|^2 + |Yu|^2)\, dx\, dy\, dt. \tag{1.2.2}$$

Let $z = (x, y)$, $|z| = \sqrt{x^2 + y^2}$ and let us consider the so-called Kaplan distance function from $(z, t)$ to the origin

$$d(z,t) = (|z|^4 + t^2)^{1/4}. \tag{1.2.3}$$

The function $d$ is positively homogeneous with the property

$$d(\lambda z, \lambda^2 t) = \lambda\, d(z,t), \quad \lambda > 0$$

and it has a singularity at zero.

The Grushin operator (see [Gr]),

$$G = -\Delta_z - 4|z|^2 \partial_t^2, \tag{1.2.4}$$

gives another example of a sub-elliptic operator. Its quadratic form $g$ respectively equals

$$g[u] = \int_{\mathbb{R}^3} (|\nabla_z u|^2 + 4|z|^2 |\partial_t u|^2)\, dzdt. \tag{1.2.5}$$

For the forms (1.2.2) and (1.2.5) the following sharp Hardy inequalities were discussed in details in [G] and [GL]:

$$h[u] = \int_{\mathbb{R}^3} (|Xu|^2 + |Yu|^2)\, dzdt \geq \int_{\mathbb{R}^3} \frac{|z|^2}{d^4}\, |u|^2\, dzdt \tag{1.2.6}$$

and

$$g[u] = \int_{\mathbb{R}^3} (|\nabla_z u|^2 + 4|z|^2 |\partial_t u|^2)\, dzdt \geq \int_{\mathbb{R}^3} \frac{|z|^2}{d^4}\, |u|^2\, dzdt. \tag{1.2.7}$$

The inequalities (1.2.6) and (1.2.7) are related. Indeed, the operator $H$ defined in (1.2.1) could be rewritten in the form

$$Hu = -\Delta_z u - 4|z|^2 \partial_t^2 - 4\partial_t\, Tu = Gu - 4\partial_t\, Tu, \tag{1.2.8}$$

where $T = y\,\partial_x - x\,\partial_y$. In particular, if $u(z,t) = u(|z|, t)$, then $Tu = 0$ and on this subclass of functions the inequalities (1.2.6) and (1.2.7) coincide.

In [A1] and [A2] D'Ambrosio obtained a number of Hardy inequalities generalizing (1.2.6) and (1.2.7). In particular, he proved an $L^p$-version of these inequalities.
Let $p \geq 1$. Then for any $u \in C_0^\infty(\mathbb{R}^3 \setminus 0)$ we have

$$h_p[u] = \int_{\mathbb{R}^3} |(Xu, Yu)|^p\, dzdt \geq \frac{|4-p|^p}{p^p} \int_{\mathbb{R}^3} \frac{|z|^p}{d^{2p}}\, |u|^p\, dzdt \tag{1.2.9}$$

and

$$g_p[u] = \int_{\mathbb{R}^3} |\nabla_G u|^p \, dzdt \geq \frac{|4-p|^p}{p^p} \int_{\mathbb{R}^3} \frac{|z|^2}{d^{2p}} \, |u(z,t)|^p \, dzdt, \qquad (1.2.10)$$

where

$$\nabla_G = (\partial_x, \partial_y, 2x\partial_t, 2y\partial_t). \qquad (1.2.11)$$

Note that the Grushin operator defined in (1.2.4) satisfies

$$G = -|\nabla_G|^2.$$

The constant $|4-p|^p \, p^{-p}$ in (1.2.9) and (1.2.10) is sharp but not achieved.

## 1.3 Hardy inequalities for operators with magnetic fields

In 1959 Yakir Aharonov and David Bohm [AhB] observed the phenomenon where a charged particle is affected by electromagnetic fields, despite being confined to regions where both the magnetic field and the electric field are zero (such effects may arise in both electric and magnetic fields, but the latter is easier to study). An important consequence of this effect is that understanding of the classical electromagnetic field acting locally on a particle is not enough in order to predict the quantum mechanical behaviour of a particle.
A simple example of the Aharonov-Bohm vector potential is given by the vector-function

$$\vec{A} = \beta \, \frac{(-x_2, x_1)}{|x|^2}, \qquad |x|^2 = x_1^2 + x_2^2. \qquad (1.3.1)$$

Note that the value of $\operatorname{curl} \vec{A}$ coincides with the value of $\beta \, \Delta \ln |x| = \beta \, \delta(x)$, where $\delta$ is the Dirac $\delta$-function

$$\operatorname{curl} \vec{A}(x) = \beta \, \delta(x). \qquad (1.3.2)$$

This means that the respective vector field

$$B = \operatorname{curl} \vec{A}$$

is concentrated only at the origin of $\mathbb{R}^2$.

It has been noticed in the paper of A. Laptev and T. Weidl [LW] that an introduction of an Aharonov-Bohm vector-potential makes the classical

Hardy ineequality (1.1.1) valid even in the two dimensional case. Indeed, let

$$\vec{A} = \beta \, \frac{(-x_2, x_1)}{|x|^2}, \qquad |x|^2 = x_1^2 + x_2^2. \tag{1.3.3}$$

Then

$$\int_{\mathbb{R}^2} |(\nabla + i\vec{A})\, u(x)|^2 \, dx \geq \min_{k\in\mathbb{Z}} (k-\beta)^2 \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2} \, dx, \tag{1.3.4}$$

where $u \in C_0^\infty(\mathbb{R}^2 \setminus 0)$.

Symbolically this inequality could be rewritten as

$$-(\nabla + i\vec{A})^2 - \frac{\min_{k\in\mathbb{Z}} (k-\beta)^2}{|x|^2} \geq 0. \tag{1.3.5}$$

It is well known that the standard Laplacian $-\Delta$ in $L^2(\mathbb{R}^2)$ has a resonance state at the spectral point zero; namely, any perturbation by a non-positive electric potential $V$ generates at least one negative eigenvalue. The inequality (1.3.5) shows that this is not the case for the Aharonov-Bohm magnetic Laplacian $-(\nabla + i\vec{A})^2$.
This fact allowed A. A. Blinsky, W. D. Evans and R. T. Lewis [BEL] to obtain a Cwikel-Lieb-Rozenblum ineaquality for the Schrödinger operators in $L^2(\mathbb{R}^2)$ with a class of potentials depending only on the radial variable. The sharp constant in such an inequality is due to Laptev [L1]. Some inequalities for Laplacians with magentic field were also obtained in [Bal] and [BLS].

The main result of chapter 1 is an inequality that generalizes (1.3.4) to $L^p$ spaces. We obtain:

**Theorem 1.3.1.** *Let the vector field $\vec{A}$ be defined by* (1.3.3) *and let $-1/2 \leq \beta \leq 1/2$. Then for any $u \in C_0^\infty(\mathbb{R}^2 \setminus 0)$ we have*

$$||(\nabla + i\vec{A})u||_{L^p} + ||(\nabla + i\vec{A})\bar{u}||_{L^p} \geq 2\, \frac{\left((2-p)^2 + p^2\,\beta^2\right)^{1/2}}{p} \left(\int_{\mathbb{R}^2} \frac{|u|^p}{|x|^p}\, dx\right)^{\frac{1}{p}}. \tag{1.3.6}$$

If $\beta = 0$ in (1.3.6), then this inequality coincides with the classical $L^p$-Hardy inequality which in this case takes the form

$$2\left(\int_{\mathbb{R}^2} |\nabla u|^p dx\right)^{1/p} \geq 2\, \frac{|2-p|}{p} \left(\int_{\mathbb{R}^2} \frac{|u|^p}{|x|^p} dx\right)^{1/p}.$$

If $p = 2$ and $\beta \neq 0$, then we obtain

$$||(\nabla + i\vec{A})u||_{L^2} + ||(\nabla + i\vec{A})\bar{u}||_{L^2} \geq 2\,|\beta| \left( \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2} dx \right)^{1/2},$$

which is "almost" the same as (1.3.4).

In the paper [BE] the authors considered an $L^p$-Hardy inequality for the Aharonov-Bohm magnetic gradient with

$$\vec{A}(x) = \Psi(x/|x|)\, \frac{(-x_2, x_1)}{|x|^2},$$

where $\Psi$ is not necessarily a constant function. They studied mainly the circular part of the operator. Gauging away the magnetic field, the authors reduce the problem to a problem for the $p$-Laplacian on the interval $(0, 2\pi)$ with some boundary conditions. The sharp constant in the respective inequality is still unknown.

In Theorem 1.3.1 we consider a special case in which $\Psi \equiv$ constant, but obtain a result which gives sharp constants and moreover, shows the interplay between the classical $L^p$-type Hardy inequality and its magnetic version.

In chapter 3 we first define an appropriate magnetic field for the Grushin operator by

$$\vec{\mathcal{A}} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4) = \Big( -\frac{\partial_y d}{d}, \frac{\partial_x d}{d}, -2y\,\frac{\partial_t d}{d}, 2x\,\frac{\partial_t d}{d} \Big),$$

where $d$ is the Kaplan distance function defined in (1.2.3). The natural $\mathrm{curl}_G$ operator could be defined by the vector field

$$\mathrm{curl}_G\ = (-\partial_y, \partial_x, -2y\,\partial_t, 2x\,\partial_t).$$

The respective magnetic field defined by

$$B(x,y,t) = \mathrm{curl}_G\ \cdot \vec{\mathcal{A}}$$

has a "right" homogeneity for contributing to the Hardy inequality. However, its support is not concentrated at the origin as in (1.3.2). It is easy to compute that

$$B(x,y,t) = 2\,\frac{|z|^2}{d^4}, \qquad z = (x,y).$$

Then we define the magnetic Grushin operator with the magnetic field $\beta\, B$ as

$$G_{\mathcal{A}} = -(\nabla_G + i\beta\mathcal{A})^2. \tag{1.3.7}$$

The main result of chapter 3 is:

**Theorem 1.3.2.** *Assume that* $-1/2 \leq \beta \leq 1/2$*. Then for the quadratic form of the magnetic Grushin operator* (1.3.7) *we have the following Hardy inequality*

$$\int_{\mathbb{R}^3} |(\nabla_{G_0} + i\beta\mathcal{A})u|^2 \, dzdt \geq (1+\beta^2) \int_{\mathbb{R}^3} \frac{|z|^2}{d^4} |u|^2 \, dzdt. \qquad (1.3.8)$$

This Theorem shows that if $\beta \neq 0$, then the inequality (1.2.7) can be improved.

In chapter 4 this result is generalized to the $L^p$-spaces and the main result of this chapter is:

**Theorem 1.3.3.** *Let* $1 < p < \infty$ *and let us assume that* $-1/2 \leq \beta \leq 1/2$*. Then for the quadratic form of the magnetic Grushin operator* (1.3.7) *we have the following Hardy inequality*

$$\begin{aligned} &\left\| \left(\nabla_G + i\beta\,\vec{\mathcal{A}}\right) u \right\|_p + \left\| \left(\nabla_G + i\beta\,\vec{\mathcal{A}}\right) \bar{u} \right\|_p \\ &\qquad \geq \frac{2}{p} \sqrt{(4-p)^2 + p^2\beta^2} \int_{\mathbb{R}^3} |u|^p \frac{|z|^p}{d^{2p}} \, dzdt. \qquad (1.3.9) \end{aligned}$$

The aim of this result is to show that introduction of the magnetic field improves the constant in the respective Hardy inequality. Note that if $\beta = 0$, then this inequality coincides with (1.2.10) and if $p = 2$, then the statement of the Theorem 1.3.3 is "almost" the same as (1.3.8).

## 1.4 Lieb-Thirring inequalities for harmonic Grushin operator

Finally, in chapter 5 we study Lieb-Thirring inequalities for a version of harmonic Grushin operator.
The Weyl-type asymptotics for the number of bound states gave rise to the question, whether there is a semi-classical bound for the moments of the negative eigenvalues of operators of the Schrödinger class $P := -\Delta + V$ in $L^2(\mathbb{R}^d)$:

$$\sum_{\lambda<0} |\lambda|^\gamma = tr(-\Delta + V)_-^\gamma$$

of the form

$$\operatorname{Tr}(-\Delta + V)_-^\gamma \leq \frac{C_{\gamma,d}}{(2\pi)^d} \int\int (|\xi|^2 + V(x))_-^\gamma d\xi dx,$$

or equivalently

$$\sum_{\lambda<0} |\lambda|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}}(x)dx,$$

where $L_{\gamma,d} = C_{\gamma,d} L_{\gamma,d}^{cl}$ is the Lieb-Thirring constant. $L_{\gamma,d}^{cl}$ is defined as

$$L_{\gamma,d}^{cl} = \frac{1}{(2\pi)^d} \int (|\xi|^2 - 1)_+^\gamma d\xi = \frac{\Gamma(\gamma+1)}{2^d \pi^{\frac{d}{2}} \Gamma(\gamma+1+\frac{d}{2})}.$$

If the potential $V$ is a function growing at infinity, then the spectrum of the Schrödinger operator $-\Delta + V$ is discrete and one is usually interested in the inequality

$$\sum (\lambda - \lambda_k)_+^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} (\lambda - V(x))_+^{\gamma+\frac{d}{2}} dx. \qquad (1.4.1)$$

Such inequalities are naturally related to Weyl's asymptotic formula as $\lambda \to \infty$ that are known for a large class of potentials. However, the question of uniform estimates with respect to $\lambda$ and the potential function $V$ is still a challenging problem. In particular, the sharp constant in (1.4.1) was not known even for the multidimensional harmonic oscillator ($V = |x|^2$) until the paper of R. de la Bretéche [dlB], where the author obtained the following result:

Let $H = -\Delta + |x|^2$ be the multidimensional harmonic oscillator acting in $L^2(\mathbb{R}^d)$. Its spectrum is discrete and its eigenvalues are

$$\{\lambda_k\} = \{2|k| + d\}, \quad k = (k_1, ..., k_d), \quad k_j \in \mathbb{Z}, \quad |k| = \sum_{j=0}^{d} k_j.$$

In particular, in [dlB] the author justifed the Lieb-Thirring conjecture for any $\gamma \geq 1$:

$$\sum (\lambda - \lambda_k)^\gamma \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\lambda - \xi^2 - |x|^2)_+^\gamma dx d\xi, \qquad (1.4.2)$$

see also [L2] for some generalizations.

In chapter 5 we consider a version of the harmonic oscillator for the Grushin operator

$$G_0 = -\Delta_z - 4|z|^2 \partial_t^2.$$

It is well known that $G_0$ appears as the "radial" part of the sub-elliptic Heisenberg-Hörmander Laplacian. Our main result concerns the operator

$$G = G_0 + |z|^2 t^2, \qquad (1.4.3)$$

which could be considered as a version of harmonic oscillator for the sub-elliptic Grushin operator $G_0$.

In order to formulate our result, we need the following notations:
Let us introduce the Euler-Mascheroni constant $\gamma$ (Euler 1735), see [E] and [Con],

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right) = \int_1^\infty \left(\frac{1}{[x]} - \frac{1}{x}\right) dx.$$

Its numerical value is $\gamma = 0.57721\ldots$. (It is not known if $\gamma$ is rational or irrational). Then we can define the harmonic number $\mathcal{H}(n)$ by

$$\mathcal{H}(n) := \sum_{k=1}^{n}\frac{1}{k} = \gamma + \psi(n+1),$$

where $\psi(t)$ is known as the Gauss digamma function defined by

$$\psi(t) = \frac{\Gamma'(t)}{\Gamma(t)}.$$

We can also introduce the value of $\mathcal{H}(n+1/2)$ as

$$\mathcal{H}(n+1/2) = \gamma + \psi(n+3/2).$$

Then, using the properties of the $\Gamma$-function, we can find that

$$\sum_{k=0}^{n}\frac{1}{2k+1} = \ln 2 + \frac{1}{2}\,\mathcal{H}\left(n+\frac{1}{2}\right).$$

Thus

$$\sum_{k\le\frac{\lambda^2}{32}-\frac{1}{2}}\frac{1}{2k+1} = \ln 2 + \frac{1}{2}\,\mathcal{H}\left(\left[\frac{\lambda^2}{32}-\frac{1}{2}\right]+\frac{1}{2}\right)$$

with

$$\psi(n+1/2) = -\gamma H(n-1/2).$$

**Theorem 1.4.1.** *The spectrum of the operator* (1.4.3) *is discrete and its eigenvalues* $\{\lambda_j\}$ *satisfy uniformly with respect to* $\lambda < 0$ *the following sharp inequality*

$$\sum_{j=0}^{\infty}(\lambda-\lambda_j)_+ \le \frac{1}{(2\pi)^2}\lambda^3\left(\ln 2 + \frac{1}{2}\,\mathcal{H}\left(\left[\frac{\lambda^2}{32}-\frac{1}{2}\right]+\frac{1}{2}\right)\right)$$
$$\times\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(1-(|\xi|^2+4|z|^2))_+\,d\xi dz. \quad (1.4.4)$$

The inequality (1.4.4) is sharp and this is confirmed by the Weyl-type asymptotic formula as $\lambda \to \infty$.
It is also known that

$$\mathcal{H}(n) \sim \ln(n) + \gamma \quad \text{as} \quad n \to \infty. \tag{1.4.5}$$

Therefore

$$\sum_{k \le \frac{\lambda^2}{32} - \frac{1}{2}} \frac{1}{2k+1} \sim \ln \lambda + O(1) \quad \text{as} \quad \lambda \to \infty.$$

Using (1.4.5), we also find that

$$\sum_{j=0}^{\infty} (\lambda - \lambda_j)_+ \sim \frac{1}{(2\pi)^2} \lambda^3 \ln \lambda \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (1 - (|\xi|^2 + 4|z|^2))_+ \, d\xi dz.$$

# 2. $L^p$-Hardy type inequalities for magnetic Laplacians

*In this chapter we establish an $L^p$-Hardy inequality related to Laplacians with magnetic fields with Aharonov-Bohm vector potentials.*

## 2.1 Introduction

The classical multidimentional $L^p$-Hardy inequality in $\mathbb{R}^n$ reads as follows:

$$\int_{\mathbb{R}^n} |\nabla u|^p dx \geq C_{n,p} \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx, \tag{2.1.1}$$

where $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}),\ p > 1$ and the constant

$$C_{n,p} = \left| \frac{n-p}{p} \right|^p$$

is the best possible yet not achieved, see [MMP], [M] and [KO]. Note that if $n = p$, then the left hand side of (2.1.1) equals zero.

In [LW] the authors considered the case $n = p = 2$ and pointed out that introducing an Aharonov-Bohm-type magnetic field makes the inequality (2.1.1) non-trivial. In particular, it has been shown that if

$$\vec{A} = \beta \frac{(-x_2, x_1)}{|x|^2}, \qquad |x|^2 = x_1^2 + x_2^2, \tag{2.1.2}$$

then

$$\int_{\mathbb{R}^2} |(\nabla + i\vec{A})\, u(x)|^2 \, dx \geq \min_{k \in \mathbb{Z}} (k - \beta)^2 \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2} \, dx, \tag{2.1.3}$$

where $u \in C_0^\infty(\mathbb{R}^2 \setminus 0)$.

Symbolically this inequality could be rewritten as

$$-(\nabla + i\vec{A})^2 - \frac{\min_{k \in \mathbb{Z}} (k-\beta)^2}{|x|^2} \geq 0. \tag{2.1.4}$$

It is well known that the standard Laplacian $-\Delta$ in $L^2(\mathbb{R}^2)$ has a resonance state at the spectral point zero. Namely, any perturbation by a non-positive electric potential $V$ generates at least one negative eigenvalue. The inequality (2.1.4) shows that this is not the case for the Aharonov-Bohm magnetic Laplacian $-(\nabla + i\vec{A})^2$.

In particular, this fact allowed A. Balinsky, W. D. Evans and R. T. Lewis [BEL] to show that if the potential $V$ depends only on $|x|$, $V(x) = V(|x|)$, then for the Friedrichs extentions of the operator

$$H = -(\nabla + i\vec{A})^2 - V, \qquad V \geq 0$$

defined on $C_0^\infty(\mathbb{R}^2 \setminus 0)$, there is a Cwikel-Lieb-Rozenblum inequality ([C], [Lieb] and [Roz]). That is, if $N(H)$ is the number of the negative eigenvalues $\{\lambda_k\}$ of the operator $H$, then

$$N(H) = \#\{k : \ \lambda_k < 0\} \leq C_\beta \int_{\mathbb{R}^2} V(|x|)\, dx, \qquad C_\beta > 0.$$

The sharp constant $C_\beta$ in the latter inequality was obtained in the paper of A. Laptev [L1] and it equals

$$C_\beta = \frac{1}{4\pi} \sup_k \Big\{ \nu^{-1/2} \cdot \big( \#\{k : \ -\nu + (k-\beta)^2 < 0, \, k \in \mathbb{Z}\} \big) \Big\}.$$

A possibility of obtaining such a sharp constant follows by simple argument already shown in the paper [L3], where the author considered the CLR inequality for the operators

$$-\Delta + \frac{b}{|x|^2} - V, \qquad b > 0,$$

where $V(x) = V(|x|)$, see also [LN], [LSo1l] and [LSol2].

It turned out that the inequality (2.1.1) could be generalized. The main result of this chapter is the following Theorem:

**Theorem 2.1.1.** *Let the vector field $\vec{A}$ be defined by* (2.1.2) *and let $-1/2 \leq \beta \leq 1/2$. Then for any $u \in C_0^\infty(\mathbb{R}^2 \setminus 0)$ we have*

$$||(\nabla + i\vec{A})u||_{L^p} + ||(\nabla + i\vec{A})\bar{u}||_{L^p} \geq 2\, \frac{\left((2-p)^2 + p^2\,\beta^2\right)^{1/2}}{p} \, \left( \int_{\mathbb{R}^2} \frac{|u|^p}{|x|^p}\, dx \right)^{\frac{1}{p}}. \quad (2.1.5)$$

**Remark 2.1.2.** *Note that if* $\beta = 0$ *in* (2.1.5), *then this inequality coincides with the classical* $L^p$*-Hardy inequality which in this case takes the form*

$$2\left(\int_{\mathbb{R}^2} |\nabla u|^p dx\right)^{1/p} \geq 2\,\frac{|2-p|}{p}\,\left(\int_{\mathbb{R}^2} \frac{|u|^p}{|x|^p} dx\right)^{1/p}.$$

**Remark 2.1.3.** *If* $-1/2 \leq \beta \leq 1/2$ *and* $p = 2$, *then* (2.1.5) *implies*

$$||(\nabla + i\vec{A})u||_{L^2} + ||(\nabla + i\vec{A})\bar{u}||_{L^2} \geq 2\,|\beta|\,\left(\int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\,dx\right)^{\frac{1}{2}}. \qquad (2.1.6)$$

*We claim that this inequality is sharp. Indeed, from the paper [LW] we obtain*

$$\left(\int_{\mathbb{R}^2} |(\nabla + i\vec{A})u|^2\,dx\right)^{\frac{1}{2}} \geq |\beta|\,\left(\int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\,dx\right)^{\frac{1}{2}}$$

*and also*

$$\left(\int_{\mathbb{R}^2} |(\nabla + i\vec{A})\bar{u}|^2\,dx\right)^{1/2} \geq |\beta|\,\left(\int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\,dx\right)^{\frac{1}{2}}.$$

*Both these inequalities are sharp. Adding them up gives us* (2.1.6).

**Remark 2.1.4.** *In [LW] the authors proved a more general result. Namely, let*

$$\vec{A}(x) = \Psi(x/|x|)\,\frac{(-x_2, x_1)}{|x|^2}.$$

*Then the value*

$$\overline{\Psi} = \frac{1}{2\pi}\int_0^{2\pi} p\Psi(\theta)d\theta$$

*is interpreted as the magnetic flux and we have*

$$\int_{\mathbb{R}^2} |(\nabla + i\vec{A})|\,|u(x)|^2\,dx \geq \min_{k\in\mathbb{Z}}(k - \overline{\Psi})^2 \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2}\,dx. \qquad (2.1.7)$$

*It would be interesting to obtain an* $L^p$*-version of this more general case with* $p \neq 0$.

**Remark 2.1.5.** *Note that in the paper [BE] the authors considered an $L^p$-Hardy inequality for the Aharonov-Bohm magnetic gradient with*

$$\vec{A}(x) = \Psi(x/|x|)\,\frac{(-x_2, x_1)}{|x|^2},$$

*where $\Psi$ is not necessarily a constant function. They studied mainly the circular part of the operator. Gauging away the magnetic field, the authors reduced the problem to a problem for the p-Laplacian on the interval $(0, 2\pi)$ with some boundary conditions. The sharp constant in the respective inequality is still unknown.*
*In Theorem 2.1.1 we consider a special case where $\Psi \equiv \text{constant}$, but obtain a result which gives sharp constants and moreover shows the interplay between the classical $L^p$-type Hardy inequality and its magnetic version.*

## 2.2 Auxiliary statements

Let $\vec{F}$ be a vector-function with values in $\mathbb{C}^n$ such that
$\vec{F} \in C^\infty(\mathbb{R}^n \setminus 0, \mathbb{C}^n)$. Therefore we have:

**Lemma 2.2.1.** *Let $u \in C_0^\infty(\mathbb{R}^n \setminus 0)$ and $p > 1$. Then the following inequality holds:*

$$\int_{\mathbb{R}^n} |\nabla u|^p dx \geq \frac{1}{p^p}\,\frac{\left|\int_{\mathbb{R}^n} \operatorname{div}\vec{F}\,|u(x)|^p dx\right|^p}{\left(\int_{\mathbb{R}^n} |\vec{F}(x)|^{\frac{p}{p-1}} \cdot |u|^p\right)}.$$

*Proof.* Let $\vec{F}(x) = (F_1(x), F_2(x), \ldots, F_n(x))$, $F_j \in C^\infty(\mathbb{R}^n \setminus 0)$ be a vector-function, $u \in C_0^\infty(\mathbb{R}^n \setminus 0)$, $x = (x_1, x_2, \ldots, x_n)$. Then by integrating by parts, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \operatorname{div}\vec{F}\,|u(x)|^p dx\right| \\
&\quad = \left|\int_{\mathbb{R}^n} \vec{F}(x)\cdot\nabla(|u(x)|^p)dx\right| = \left|\int_{\mathbb{R}^n} \vec{F}(x)\cdot(\nabla(u^{p/2}\bar{u}^{p/2}))\,dx\right| \\
&\quad = \frac{p}{2}\left|\int_{\mathbb{R}^n} \vec{F}(x)\cdot(u^{p/2-1}\cdot\bar{u}^{p/2}\cdot\nabla u + u^{p/2}\bar{u}^{p/2-1}\nabla\bar{u})dx\right| \\
&\quad\quad \leq p\int_{\mathbb{R}^n} |\vec{F}(x)|\,|u|^{p-1}|\nabla u|\,dx.
\end{aligned}$$

By using Hölder's inequality, we find that

$$p\int_{\mathbb{R}^n}|\vec{F}(x)||u|^{p-1}\cdot|\nabla u|dx \le p\left(\int_{\mathbb{R}^n}|\vec{F}(x)|^{\frac{p}{p-1}}|u|^p dx\right)^{\frac{p-1}{p}}\cdot\left(\int_{\mathbb{R}^n}|\nabla u|^p dx\right)^{1/p}$$

and consequently

$$\left|\int_{\mathbb{R}^n}\operatorname{div}\vec{F}\,|u(x)|^p\,dx\right| \le p\left(\int_{\mathbb{R}^n}|\vec{F}(x)|^{\frac{p}{p-1}}|u|^p\,dx\right)^{\frac{p-1}{p}}\cdot\left(\int_{\mathbb{R}^n}|\nabla u|^p\,dx\right)^{1/p}.$$

Raising both sides to the power of $p$ and rearranging the inequality, we finally arrive at

$$\int_{\mathbb{R}^n}|\nabla u|^p\,dx \ge \frac{1}{p^p}\frac{\left|\int_{\mathbb{R}^n}(\operatorname{div}\vec{F})\,|u(x)|^p dx\right|^p}{\left(\int_{\mathbb{R}^n}|\vec{F}(x)|^{\frac{p}{p-1}}\cdot|u|^p\right)^{p-1}}\,.$$

□

Let now $A=\left(\int_{\mathbb{R}^2}\operatorname{div}\vec{F}\,|u(x)|^p\,dx\right)$ and $B=\left(\int_{\mathbb{R}^n}|\vec{F}(x)|^{\frac{p}{p-1}}\cdot|u|^p\,dx\right)$. Since

$$\frac{A^p}{B^{p-1}} \ge pA-(p-1)B,$$

we immediately obtain the following Corollary:

**Corollary 2.2.2.** *For any $u\in C_0^\infty(\mathbb{R}^n\setminus 0)$ and $p>1$ we have*

$$\int_{\mathbb{R}^n}|\nabla u|^p\,dx \ge \frac{1}{p^p}\left[p\cdot\left|\int_{\mathbb{R}^n}\operatorname{div}\vec{F}\,|u(x)|^p\,dx\right|-(p-1)\int_{\mathbb{R}^n}|\vec{F}(x)|^{\frac{p}{p-1}}\,|u|^p\,dx\right]. \tag{2.2.1}$$

## 2.3 Classical $L^p$-Hardy inequality

We first consider the case $p\neq 2$. Let $C\in\mathbb{R}$ be a real constant and assume that

$$\vec{F}=C\,\nabla_x\left(|x|^{2-p}\right).$$

Direct computations lead to

$$\vec{F}=(2-p)\,C\,\frac{x}{|x|^p}$$

and taking divergency of it gives

$$\operatorname{div} \vec{F} = (2-p)\, C \,\left[ \frac{n}{|x|^p} - p \sum_{j=1}^{n} \left( \frac{x_j^2}{|x|^{p+2}} \right) \right] = (2-p)\, C\,(n-p) \frac{1}{|x|^p}.$$

Note that

$$|\vec{F}| = |C| \left( \sum_{j=1}^{n} \frac{x_j^2}{|x|^{2p}} \right)^{1/2} = |C|\, |2-p|\, \frac{1}{|x|^{p-1}}.$$

Then by using (2.2.1), we find

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla u|^p \, dx \\
&\geq \frac{1}{p^p} \left[ p\, |(n-p)(2-p)|\, |C| \int_{\mathbb{R}^2} \frac{|u|^p}{|x|^p} dx - (p-1)\, |(2-p)\, C|^{\frac{p}{p-1}} \int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} dx \right] \\
&\quad = \frac{1}{p^p} \left( |C|\, p\, |(n-p)(2-p)| - (p-1)\, |(2-p)\, C|^{\frac{p}{p-1}} \right) \int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} \, dx.
\end{aligned}$$

Minimizing with respect to the constant $C$, we find that

$$C = \frac{|n-p|^{p-1}}{|2-p|}$$

and therefore

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla u|^p \, dx \geq \frac{1}{p^p} \Big( & \frac{|n-p|^{p-1}}{|2-p|}\, p\, |(n-p)(2-p)| \\
& - (p-1) \left| (2-p)\, \frac{|n-p|^{p-1}}{|2-p|} \right|^{\frac{p}{p-1}} \Big) \int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} \, dx \\
& = \left| \frac{n-p}{p} \right|^p \int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} \, dx.
\end{aligned}$$

This proves the well-known classical $L^p$-Hardy inequality in $\mathbb{R}^n$ for $p \neq 2$:

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dx \geq \left| \frac{n-p}{p} \right|^p \int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} \, dx.$$

The case $p = 2$ can be proved similarly by defining

$$\vec{F} = C\, \nabla_x \, \log[x|.$$

Therefore, we obtain:

**Theorem 2.3.1.** *For any $u \in C_0^\infty(\mathbb{R}^n \setminus 0)$ and $0 < p < \infty$ we have*

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dx \geq \left| \frac{n-p}{p} \right|^p \int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} \, dx.$$

## 2.4 $L^2$-type Hardy inequality with magnetic fields

Here we shall reproduce the result obtained in [LW] for Laplacians with Aharonov-Bohm magnetic fields for the case (2.1.2):

$$\vec{A} = \beta \, \frac{(-x_2, x_1)}{|x|^2}, \qquad |x|^2 = x_1^2 + x_2^2.$$

**Theorem 2.4.1.** *Let $\beta \in \mathbb{R}$ and let $u \in C_0^\infty(\mathbb{R}^2 \setminus 0)$. Then*

$$\int_{\mathbb{R}^2} |(\nabla + i\vec{A})u|^2 \, dx \geq \min_{k \in \mathbb{Z}} (k - \beta)^2 \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2} \, dx. \tag{2.4.1}$$

*Proof.* Indeed, by using polar coordinates $(r, \theta)$, we have

$$u(x) = \frac{1}{\sqrt{2\pi}} \sum_k u_k(r) e^{ik\theta}.$$

Therefore

$$\begin{aligned}
\int_{\mathbb{R}^2} |(\nabla + i\beta\vec{A})u|^2 \, dx &= \int_0^\infty \int_0^{2\pi} \left( |u_r'|^2 + \left| \frac{u_\theta' + i\beta u}{r} \right|^2 \right) r \, d\theta \, dr \\
&\geq \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \left| \sum_k \frac{k+\beta}{r} \, u_k e^{ik\theta} \right|^2 r \, d\theta \, dr \\
&= \int_0^\infty \sum_k \left| \frac{k+\beta}{r} \, u_k \right|^2 r \, d\theta \, dr \\
&\geq \min_{k \in \mathbb{Z}} (k+\beta)^2 \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2} \, dx.
\end{aligned}$$

The proof is complete. □

**Remark 2.4.2.** *It is enough to prove the inequality* (2.4.1) *for*

$$-\frac{1}{2} \leq \beta \leq \frac{1}{2}.$$

*In this case it reduces* (2.4.1) *to*

$$\int_{\mathbb{R}^2} |(\nabla + i\vec{A})u|^2 \, dx \geq \min_{k \in \mathbb{Z}} \beta^2 \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2} \, dx.$$

*This is possible due to the standard procedure of gauging away the integer part of the magnetic field by simply substituting the function $u(x)$ in the quadratic form*

$$\int_{\mathbb{R}^2} |(\nabla + i\vec{A})u|^2 \, dx$$

*by $u(x)$ $\exp ik\theta$, where $\theta = x/|x|$, $k \in \mathbb{Z}$.*

## 2.5 $L^p$-type Hardy inequality with magnetic fields

We first prove an auxiliary result similar to Lemma 2.2.1:
Let $\vec{F}$, as before, be a vector-function with values in $\mathbb{C}^n$ such that

$$\vec{F}(x) = (F_1(x), F_2(x), \dots, F_n(x)), \qquad F_j \in C_0^\infty(\mathbb{R}^n \setminus 0)$$

and let

$$\vec{A}(x) = (A_1(x), A_2(x), \dots, A_n(x)), \qquad A_j \in C_0^\infty(\mathbb{R}^n \setminus 0),$$

$j = 1, 2, \dots, n$.

**Lemma 2.5.1.** *For any $u \in C_0^\infty(\mathbb{R}^2)$ it holds that*

$$||(\nabla + i\vec{A})u||_{L^p} + ||(\nabla + i\vec{A})\bar{u}||_{L^p} \geq \frac{2}{p} \frac{\left|\int_{\mathbb{R}^n} ((\nabla + ip\,\vec{A}) \cdot \vec{F})\,|u|^p\right|}{\left(\int_{\mathbb{R}^n} |\vec{F}|^{\frac{p}{p-1}}\,|u|^p\right)^{\frac{p-1}{p}}}.$$

*Proof.* The proof is very similar to the proof of Lemma 2.2.1. Indeed, by integrating by parts, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} ((\nabla + ip\,\vec{A}) \cdot \vec{F})\,|u(x)|^p dx\right| \\
&\qquad = \left|\int_{\mathbb{R}^n} \vec{F}(x) \cdot ((\nabla + ip\,\vec{A})|u(x)|^p)\,dx\right| \\
&\qquad = \left|\int_{\mathbb{R}^n} \vec{F}(x) \cdot ((\nabla + ip\,\vec{A})(u^{p/2}\bar{u}^{p/2}))\,dx\right| \\
&= \frac{p}{2}\left|\int_{\mathbb{R}^n} \vec{F}(x) \cdot (u^{p/2-1}\,\bar{u}^{p/2}(\nabla + i\vec{A})u + u^{p/2}\bar{u}^{p/2-1}(\nabla + i\vec{A})\bar{u})dx\right| \\
&\qquad \leq \frac{p}{2}\int_{\mathbb{R}^n} |\vec{F}(x)|\,|u|^{p-1}\left(|(\nabla + i\vec{A})u| + |(\nabla + i\vec{A})\bar{u}|\right)\,dx. \qquad (2.5.1)
\end{aligned}$$

By using Hölder's inequality, we find that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\vec{F}(x)|\,|u|^{p-1}\,|(\nabla + i\vec{A})u|\,dx \\
&\qquad \leq \left(\int_{\mathbb{R}^n} |\vec{F}(x)|^{\frac{p}{p-1}}|u|^p dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |(\nabla + i\vec{A})u|^p\,dx\right)^{1/p}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\vec{F}(x)|\,|u|^{p-1}\,|(\nabla + i\vec{A})\bar{u}|\,dx \\
&\qquad \leq \left(\int_{\mathbb{R}^n} |\vec{F}(x)|^{\frac{p}{p-1}}|u|^p dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |(\nabla + i\vec{A})\bar{u}|^p\,dx\right)^{1/p}.
\end{aligned}$$

Using (2.5.1), we obtain

$$\frac{1}{p}\frac{\left|\int_{\mathbb{R}^n}((\nabla+i\vec{A})\cdot\vec{F})\,|u(x)|^p dx\right|}{\left(\int_{\mathbb{R}^n}|\vec{F}(x)|^{\frac{p}{p-1}}|u|^p dx\right)^{\frac{p-1}{p}}}\le\left(\int_{\mathbb{R}^n}|(\nabla+i\vec{A})u|^p\,dx\right)^{1/p}$$

and respectively

$$\frac{1}{p}\frac{\left|\int_{\mathbb{R}^n}(\nabla+i\vec{A})\cdot\vec{F}\,|u(x)|^p dx\right|}{\left(\int_{\mathbb{R}^n}|\vec{F}(x)|^{\frac{p}{p-1}}|u|^p dx\right)^{\frac{p-1}{p}}}\le\left(\int_{\mathbb{R}^n}|(\nabla+i\vec{A})\bar{u}|^p\,dx\right)^{1/p}.$$

Adding the last two inequalities, we obtain the statement of the Lemma.

□

## 2.6 Proof of Theorem 2.1.1

Let $-1/2\le\beta\le 1/2$. In order to prove Theorem 2.1.1, we apply Lemma 2.5.1 with $n=2$ and

$$\vec{A}=\beta\,\frac{(-x_2,x_1)}{|x|^2},\qquad |x|^2=x_1^2+x_2^2.$$

Let us choose $\vec{F}=\vec{\mathcal{F}}_1+\vec{\mathcal{F}}_2$ such that

$$\vec{\mathcal{F}}_1(x)=c\,\frac{(x_1,x_2)}{|x|^p}\quad\text{and}\quad\vec{\mathcal{F}}_2(x)=-i\,\beta\,\frac{(-x_2,x_1)}{|x|^p},$$

where $c\in\mathbb{R}$ is a real constant. Then clearly we have the following properties:

$$\vec{\mathcal{F}}_1\cdot\vec{\mathcal{F}}_2=0,$$

$$\nabla\cdot\vec{\mathcal{F}}_1(x)=c\,\frac{2-p}{|x|^p}\quad\text{and}\quad\nabla\cdot\vec{\mathcal{F}}_2=0,$$

as well as

$$|F|=|\vec{\mathcal{F}}_1+\vec{\mathcal{F}}_2|=\frac{\sqrt{c^2+\beta^2}}{|x|^{p-1}}.$$

Moreover, we also have

$$\vec{A}\cdot\vec{\mathcal{F}}_1=0\quad\text{and}\quad i\vec{A}\cdot\vec{\mathcal{F}}_2=\beta^2\,\frac{1}{|x|^p}.$$

Applying Lemma 2.5.1, we obtain

$$\left[\left(\int_{\mathbb{R}^2} |(\nabla + i\vec{A})u|^p dx\right)^{1/p} + \left(\int_{\mathbb{R}^2} |(\nabla + i\vec{A})\bar{u}|^p dx\right)^{1/p}\right]$$

$$\geq \frac{2}{p}\,\frac{\left|\int_{\mathbb{R}^2} (\nabla + ip\vec{A})\cdot(\vec{\mathcal{F}}_1 + \vec{\mathcal{F}}_2)\,|u|^p\,dx\right|}{\left(\int_{\mathbb{R}^2} |\vec{\mathcal{F}}_1 + \vec{\mathcal{F}}_2|^{\frac{p}{p-1}}\,|u|^p dx\right)^{\frac{p-1}{p}}}$$

$$= \frac{2}{p}\,\frac{\left|\int_{\mathbb{R}^2} (c\,(2-p) + p\,\beta^2)\frac{|u|^p}{|x|^p}\,dx\right|}{\left(\int_{\mathbb{R}^2} (c^2+\beta^2)^{\frac{p}{2(p-1)}}\,\frac{|u|^p}{|x|^p}\,dx\right)^{\frac{p-1}{p}}}$$

$$= \frac{2}{p}\,\frac{|c(2-p) + p\,\beta^2|}{(c^2+\beta^2)^{1/2}}\,\left(\int_{\mathbb{R}^2} \frac{|u|^p}{|x|^p}\,dx\right)^{\frac{1}{p}}.$$

Maximizing the right hand side with respect to $c$, we find that

$$c = \frac{2-p}{p}$$

and therefore we finally obtain

$$||(\nabla + i\vec{A})u||_{L^p} + ||(\nabla + i\vec{A})\bar{u}||_{L^p}$$
$$\geq \frac{2}{p}\,\left((2-p)^2 + p^2\,\beta^2\right)^{1/2}\,\left(\int_{\mathbb{R}^2} \frac{|u|^p}{|x|^p}\,dx\right)^{\frac{1}{p}}. \quad (2.6.1)$$

The proof is complete.

# 3. Hardy inequalities for a magnetic Grushin operator

*We introduce a magnetic field for a Grushin sub-elliptic operator and then show that its quadratic form satisfies an improved Hardy inequality.*

## 3.1 Introduction

The classical Hardy inequality states that if $d \geq 3$, then for any function $u$ such that $\nabla u \in L^2(\mathbb{R}^d)$ it holds that

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx \geq \Big(\frac{d-2}{2}\Big)^2 \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} \, dx. \tag{3.1.1}$$

It is well known that the constant $(d-2)^2/4$ in (3.1.1) is sharp but not achieved. The literature concerning different versions of Hardy inequalities and their applications is extensive and we are not able to cover it in this chapter. We just mention the classical paper of M. Sh. Birman [B], the article of E. B. Davies [D] and the book of V. Maz'ya [M].

Among many applications of the inequality (3.1.1) we would like to mention that this inequality together with the Schwarz inequality implies

$$\Big(\int_{\mathbb{R}^d} |x|^2 |u(x)|^2 \, dx\Big)\Big(\int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx\Big) \geq \Big(\frac{d-2}{2}\Big)^2 \Big(\int_{\mathbb{R}^d} |u(x)|^2 \, dx\Big)^2.$$

The latter takes particular symmetrical form, if in the second integral of the left hand side we use Parseval's formula for the Fourier transform $\hat{u}$ of the function $u$:

$$(2\pi)^d \Big(\int_{\mathbb{R}^d} |x|^2 |u(x)|^2 \, dx\Big)\Big(\int_{\mathbb{R}^d} [\xi|^2 |\hat{u}(\xi)|^2 \, d\xi\Big) \geq \Big(\frac{d-2}{2}\Big)^2 \Big(\int_{\mathbb{R}^d} |u(x)|^2 \, dx\Big)^2.$$

This inequality expresses the Heisenberg uncertainty principle which states that a non-trivial $L^2$-function and its Fourier transform cannot simultaneously be very small near the origin.

The Hardy inequalities were also studied for some sub-elliptic operators, see for example papers [G], [GL], [A1], [A2], [DGN], [NCH] and [K], and in particular for the sub-Laplacian on the Heisenberg group $\mathbb{H}$. The latter is the prime example of the non-commutative harmonic analysis and we refer to [Ste] for the background material.
Let us consider $\mathbb{H}$ as $\mathbb{R}^3$ with coordinates $(x, y, t)$ and the (non-commutative) multiplication $(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' - 2(xy' - yx'))$. The vector fields

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}\,, \qquad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$$

are left-invariant and the sub-Laplacian on $\mathbb{H}$ is given by

$$H = -X^2 - Y^2 = -\left(\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}\right)^2 . \tag{3.1.2}$$

The quadratic form $h$ of the operator $H$ is defined by the equality

$$h[u] = \int_{\mathbb{R}^3} (|Xu|^2 + |Yu|^2)\, dzdt. \tag{3.1.3}$$

Let $z = (x, y)$, $|z| = \sqrt{x^2 + y^2}$ and let us consider the so-called Kaplan distance function from $(z, t)$ to the origin, defined by

$$d(z, t) = (|z|^4 + t^2)^{1/4}.$$

The function $d$ is positively homogeneous with the property

$$d(\lambda z, \lambda^2 t) = \lambda\, d(z, t), \qquad \lambda > 0$$

and it has a singularity at zero.
The Grushin operator (see [Gr]),

$$G = -\Delta_z - 4|z|^2\partial_t^2, \tag{3.1.4}$$

gives another example of a sub-elliptic operator. Its quadratic form $g$ respectively equals

$$g[u] = \int_{\mathbb{R}^3} (|\nabla_z u|^2 + 4|z|^2|\partial_t u|^2)\, dzdt. \tag{3.1.5}$$

For the forms (3.1.3) and (3.1.5) the following sharp Hardy inequalities were discussed in details in [G] and [GL]:

$$h[u] = \int_{\mathbb{R}^3} (|Xu|^2 + |Yu|^2)\, dzdt \geq \int_{\mathbb{R}^3} \frac{|z|^2}{d^4}\, |u|^2\, dzdt \tag{3.1.6}$$

and

$$g[u] = \int_{\mathbb{R}^3} (|\nabla_z u|^2 + 4|z|^2|\partial_t u|^2)\, dzdt \geq \int_{\mathbb{R}^3} \frac{|z|^2}{d^4}\, |u|^2\, dzdt. \tag{3.1.7}$$

Inequalities (3.1.6) and (3.1.7) are related. Indeed, the operator $H$ defined in (3.1.2) could be rewritten in the form

$$Hu = -\Delta_z u - 4|z|^2\partial_t^2 - 4\partial_t\, Tu = Gu - 4\partial_t\, Tu, \tag{3.1.8}$$

where $T = y\,\partial_x - x\,\partial_y$. In particular, if $u(z,t) = u(|z|,t)$, then $Tu = 0$ and on this subclass of functions the inequalities (3.1.6) and (3.1.7) coincide.

The classical Hardy inequality (3.1.1) becomes trivial for the two-dimensional case. In [LW] the authors have noticed that for some magnetic forms in two dimensions the Hardy inequality holds in its classical form. For example, if $\beta A$ is the Aharonov-Bohm magnetic field

$$\beta\, A = \beta\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \quad \beta \in \mathbb{R},$$

then

$$\int_{\mathbb{R}^2} |(\nabla + i\beta\, A)u|^2\, dxdy \geq \min_k |k-\beta|^2 \int_{\mathbb{R}^2} \frac{|u|^2}{x^2+y^2}\, dxdy.$$

Here the form in the left hand side is considered on the class of functions obtained by the closure from the class $C_0^\infty(\mathbb{R}^2 \setminus 0)$ with respect to the metric defined by the form

$$\int_{\mathbb{R}^2} (|\nabla u|^2 + |x|^{-2}|u|^2)\, dx.$$

In this chapter we introduce a vector field for the Grushin operator $G$ defined in (3.1.4) and obtain an improvement of the Hardy inequality (3.1.7).

Let us first define the "Grushin vector field" as

$$\nabla_G = (\partial_x, \partial_y, 2x\partial_t, 2y\partial_t).$$

Clearly,

$$G = -|\nabla_G|^2.$$

We now introduce a magnetic field as

$$\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4) = \Big(-\frac{\partial_y d}{d}, \frac{\partial_x d}{d}, -2y\,\frac{\partial_t d}{d}, 2x\,\frac{\partial_t d}{d}\Big).$$

Then the magnetic Grushin operator with the magnetic field $\mathcal{A}$ and $\beta \in \mathbb{R}$ could be defined as

$$G_{\mathcal{A}} = -(\nabla_G + i\beta\mathcal{A})^2. \tag{3.1.9}$$

Our main result is the following Theorem:

**Theorem 3.1.1.** *Assume that $-1/2 \leq \beta \leq 1/2$. Then for the quadratic form of the magnetic Grushin operator* (3.1.9) *we have the following Hardy inequality:*

$$\int_{\mathbb{R}^3} |(\nabla_{G_0} + i\beta\mathcal{A})u|^2 \, dzdt \geq (1+\beta^2) \int_{\mathbb{R}^3} \frac{|z|^2}{d^4} |u|^2 \, dzdt. \tag{3.1.10}$$

Concluding the text of the introduction, we would like to make some remarks concerning open questions related to sub-elliptic operators.

**Remark 3.1.2.** *It would be interesting to prove a similar result for the Heisenberg quadratic form. To us it is not clear which would be a suitable version of the magnetic field for this case.*

**Remark 3.1.3.** *To our knowledge, the definitions of the Grushin and Heisenberg Laplacians with constant magnetic fields are unknown. It would be interesting to define such operators and to study their spectrum, possibly identifying the notion of Landau-type levels.*

**Remark 3.1.4.** *For a multi-dimensional harmonic oscillator we have natural creation and annihilation operators. It would be interesting to define "harmonic oscillators" with Heisenberg and Grushin operators and respectively related creation and annihilation operators.*

**Remark 3.1.5.** *The results of this chapter were published in [AerL].*

## 3.2 Simple proofs of Hardy inequalities for Heisenberg and Grushin operators

For the sake of completeness we present here simple proofs of the inequalities (3.1.6) and (3.1.7).

**Proposition 3.2.1.** *For any function $u$ for which $h[u] < \infty$ the following inequality holds true:*

$$\int_{\mathbb{R}^3} (|Xu|^2 + |Yu|^2) \, dzdt \geq \int_{\mathbb{R}^3} \frac{|z|^2}{d^4} |u|^2 \, dzdt. \tag{3.2.1}$$

*Proof.* It is enough to prove (3.2.1) for functions $u \in C_0^\infty(\mathbb{R}^3 \setminus 0)$. Let us consider the following non-negative expression

$$I = \int_{\mathbb{R}^3} \left| \left( X + \alpha \frac{Xd}{d} \right) u \right|^2 dzdt + \int_{\mathbb{R}^3} \left| \left( Y + \alpha \, \frac{Yd}{d} \right) u \right|^2 dzdt,$$

where $\alpha \in \mathbb{R}$.
Clearly,

$$d(z,t)^{-1} X \, d(z,t) = \frac{x|z|^2 + yt}{d^4(z,t)}, \quad d(z,t)^{-1} Y \, d(z,t) = \frac{y|z|^2 - xt}{d^4(z,t)}.$$

We look at

$$\begin{aligned} 0 \leq I = \int_{\mathbb{R}^3} \Bigg[ \left( X + \alpha \frac{Xd}{d} \right) u \left( X + \alpha \frac{Xd}{d} \right) \bar{u} \\ + \left( Y + \alpha \frac{Yd}{d} \right) u \left( Y + \alpha \frac{Yd}{d} \right) \bar{u} \Bigg] \, dxdydt. \end{aligned}$$

Opening brackets, we find that

$$\begin{aligned} I = \int_{\mathbb{R}^3} \left( |Xu|^2 + |Yu|^2 \right) dxdydt \\ + \int_{\mathbb{R}^3} Xu \, \alpha \, \frac{Xd}{d} \bar{u} \, dxdydt + \int_{\mathbb{R}^3} \alpha \frac{Xd}{d} \, uX\bar{u} \, dxdydt \\ + \int_{\mathbb{R}^3} Yu \, \alpha \, \frac{Yd}{d} \bar{u} \, dxdydt + \int_{\mathbb{R}^3} \alpha \frac{Yd}{d} uY\bar{u} \, dxdydt \\ + \alpha^2 \int_{\mathbb{R}^3} \left| \frac{Xd}{d} u \right|^2 \, dxdydt + \alpha^2 \int_{\mathbb{R}^3} \left| \frac{Yd}{d} u \right|^2 \, dxdydt. \end{aligned} \tag{3.2.2}$$

Integrating by parts leads to

$$\begin{aligned} \int_{\mathbb{R}^3} Xu \, \alpha \, \frac{Xd}{d} \bar{u} \, dxdydt + \int_{\mathbb{R}^3} \alpha \frac{Xd}{d} uX\bar{u} \, dxdydt \\ + \int_{\mathbb{R}^3} Yu \, \alpha \, \frac{Yd}{d} \bar{u} \, dxdydt + \int_{\mathbb{R}^3} \alpha \frac{Yd}{d} uY\bar{u} \, dxdydt \\ = -\alpha \int_{\mathbb{R}^3} \left[ u \left( X \left( \frac{Xd}{d} \right) \right) \bar{u} + u \frac{Xd}{d} X\bar{u} \right] dxdydt + \alpha \int_{\mathbb{R}^3} \frac{Xd}{d} uX\bar{u} \, dxdydt \\ + \alpha \int_{\mathbb{R}^3} \left[ u \left( Y \left( \frac{Yd}{d} \right) \right) \bar{u} + u \frac{Yd}{d} Y\bar{u} \right] dxdydt + \alpha \int_{\mathbb{R}^3} \frac{Yd}{d} uY\bar{u} \, dxdydt \\ = -\alpha \int_{\mathbb{R}^3} X \left( \frac{Xd}{d} \right) |u|^2 \, dxdydt - \alpha \int_{\mathbb{R}^3} Y \left( \frac{Yd}{d} \right) |u|^2 \, dxdydt \end{aligned}$$

and therefore (3.2.2) becomes

$$I = \int_{\mathbb{R}^3} (|Xu|^2 + |Yu|^2)\, dzdt - \alpha \int_{\mathbb{R}^3} \left( X\, \frac{Xd}{d} + Y\, \frac{Yd}{d} \right) |u|^2\, dzdt$$
$$+ \alpha^2 \int_{\mathbb{R}^3} \left( \left( \frac{Xd}{d} \right)^2 + \left( \frac{Yd}{d} \right)^2 \right) |u|^2\, dzdt \geq 0.$$

Splitting the computation into three parts, gives at first that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left( \left| \frac{Xd}{d} u \right|^2 + \left| \frac{Yd}{d} u \right|^2 \right) dxdydt \\
&= \int_{\mathbb{R}^3} \left( \left( \frac{((x^2+y^2)x + yt)^2 + ((x^2+y^2)y - xt)^2}{((x^2+y^2)^2 + t^2)^2} \right) |u|^2 \right) dxdydt \\
&= \int_{\mathbb{R}^3} \left( \left( \frac{(x^2+y^2)^2(x^2+y^2) + t^2(x^2+y^2)}{((x^2+y^2)^2 + t^2)^2} \right) |u|^2 \right) dxdydt \\
&= \int_{\mathbb{R}^3} \left( \frac{(x^2+y^2)((x^2+y^2)^2 + t^2)}{((x^2+y^2)^2 + t^2)^2} |u|^2 \right) dxdydt \\
&= \int_{\mathbb{R}^3} \left( \frac{x^2+y^2}{(x^2+y^2)^2 + t^2} |u|^2 \right) dxdydt.
\end{aligned}$$

Next we have that

$$\begin{aligned}
&- \alpha \int_{\mathbb{R}^3} \left( X\, \frac{Xd}{d} + Y\, \frac{Yd}{d} \right) |u|^2\, dzdt \\
&\qquad = -\alpha \int \left( (\partial_x + 2y\partial_t) \left( \frac{(x^2+y^2)x + yt}{(x^2+y^2)^2 + t^2} \right) |u|^2 \right) dxdydt \\
&\qquad\qquad - \alpha \int \left( (\partial_y - 2x\partial_t) \left( \frac{(x^2+y^2)y - xt}{(x^2+y^2)^2 + t^2} \right) |u|^2 \right) dxdydt,
\end{aligned}$$

where

$$\begin{aligned}
&- \partial_x \left( \frac{(x^2+y^2)x + yt}{(x^2+y^2)^2 + t^2} \right) - \partial_y \left( \frac{(x^2+y^2)y - xt}{(x^2+y^2)^2 + t^2} \right) \\
&\qquad = - \frac{t^2(x^2 + 3y^2 + y^2 + 3x^2)}{((x^2+y^2)^2 + t^2)^2} = - \frac{t^2(4x^2 + 4y^2)}{((x^2+y^2)^2 + t^2)^2} \\
&\qquad\qquad = \frac{-4(x^2+y^2)}{((x^2+y^2)^2 + t^2)^2},
\end{aligned}$$

and

$$
\begin{aligned}
&-2y\partial_t\left(\frac{(x^2+y^2)x+yt}{(x^2+y^2)^2+t^2}\right)+2x\partial_t\left(\frac{(x^2+y^2)y-xt}{(x^2+y^2)^2+t^2}\right)\\
&\qquad=\frac{-2y^2((x^2+y^2)^2+t^2)}{((x^2+y^2)^2+t^2)^2}+\frac{2y\cdot 2t((x^2+y^2)x+yt)}{((x^2+y^2)^2+t^2)^2}\\
&\qquad+\frac{(-2x^2)((x^2+y^2)^2+t^2)}{((x^2+y^2)^2+t^2)^2}-\frac{2x\cdot 2t((x^2+y^2)y-xt)}{((x^2+y^2)^2+t^2)^2}\\
&=\frac{-2y^2(x^2+y^2)^2-2y^2t^2+4y^2t^2-2x^2(x^2+y^2)-2x^2t^2+4x^2t^2}{((x^2+y^2)^2+t^2)^2}\\
&\qquad\qquad\frac{-2(x^2+y^2)(x^2+y^2)^2+2t^2(x^2+y^2)}{((x^2+y^2)^2+t^2)^2}.
\end{aligned}
$$

It holds then that

$$
\begin{aligned}
&-\alpha\int(\partial_x+2y\partial_t)\left(\frac{(x^2+y^2)x+yt}{(x^2+y^2)^2+t^2}\right)\\
&\qquad+(\partial_y-2x\partial_t)\left(\frac{(x^2+y^2)y-xt}{(x^2+y^2)^2+t^2}\right)|u|^2dxdydt\\
&=-\alpha\int\frac{2(x^2+y^2)(x^2+y^2)^2-2t^2(x^2+y^2)+4t^2(x^2+y^2)}{((x^2+y^2)^2+t^2)^2}|u|^2dxdydt\\
&\qquad\qquad=-2\alpha\int\frac{x^2+y^2}{(x^2+y^2)^2+t^2}|u|^2dxdydt.
\end{aligned}
$$

In conclusion we get that

$$
\begin{aligned}
0\le&\int\left[\left(X+\alpha\frac{Xd}{dh}\right)u\left(X+\alpha\frac{Xd}{dh}\right)\bar{u}\right.\\
&\qquad+\left.\left(Y+\alpha\frac{Yd}{dh}\right)u\left(Y+\alpha\frac{Yd}{dh}\right)\bar{u}\right]dxdydt\\
=&\int\left(|Xu|^2+|Yu|^2+\alpha^2\frac{x^2+y^2}{(x^2+y^2)^2+t^2}|u|^2-2\alpha\frac{x^2+y^2}{(x^2+y^2)^2+t^2}|u|^2\right)dxdydt
\end{aligned}
$$

and by choosing $\alpha=1$, we get that

$$
\begin{aligned}
&\int\left(|Xu|^2+|Yu|^2-\frac{x^2+y^2}{(x^2+y^2)^2+t^2}|u|^2\right)dxdydt\\
&\qquad\qquad=\int\left(|Xu|^2+|Yu|^2-\frac{|z|^2}{d^4}|u|^2\right)dxdydt\ge 0.
\end{aligned}
$$

Hence, we finally obtain

$$h[u] = \int_{\mathbb{R}^3} \left(|Xu|^2 + |Yu|^2\right) dzdt \geq \int_{\mathbb{R}^3} \frac{|z|^2}{d^4} |u|^2 dzdt.$$

$\square$

**Proposition 3.2.2.** *For any function $u$ such that $g[u] < \infty$ we have*

$$\int_{\mathbb{R}^3} (|\nabla_z u|^2 + 4|z|^2 |\partial_t u|^2)\, dzdt \geq \int_{\mathbb{R}^3} \frac{|z|^2}{d^4}\, |u(z,t)|^2\, dzdt. \tag{3.2.3}$$

*Proof.* Let us first notice that by introducing polar coordinates $x = r\cos\varphi,\ y = r\sin\varphi$ and $r = |z|$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} (|\nabla_z u|^2 + 4|z|^2|\partial_t u|^2)\, dzdt = \\
&\qquad \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} (|\partial_r u|^2 + r^{-2}|\partial_\varphi u|^2 + 4r^2|\partial_t u|^2)\, r\, dr\, d\varphi\, dt \\
&\qquad\qquad \geq \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} (|\partial_r u|^2 + 4r^2|\partial_t u|^2)\, r\, dr\, d\varphi\, dt.
\end{aligned}$$

So the proof is reduced to the inequality

$$\int_{-\infty}^{\infty}\int_0^{\infty} (|\partial_r u|^2 + 4r^2|\partial_t u|^2)\, r\, drdt \geq \int_{-\infty}^{\infty}\int_0^{\infty} \frac{r^2}{r^4+t^2} |u|^2\, r\, drdt.$$

Let $d = d(r,t) = (r^4 + t^2)^{1/4}$. Then simple computation, in which we integrate by parts, gives

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_0^{\infty} \left(\left|\left(\partial_r + \alpha\frac{\partial_r d}{d}\right) u\right|^2 + 4r^2\left|\left(\partial_t + \alpha\frac{\partial_t d}{d}\right) u\right|^2\right) r\, drdt \\
&\qquad = \int_{-\infty}^{\infty}\int_0^{\infty} (|\partial_r u|^2 + 4r^2|\partial_t u|^2)\, r\, drdt \\
&\quad - \int_{-\infty}^{\infty}\int_0^{\infty} \left(6\alpha\frac{r^2}{d^4} - 4\alpha\frac{r^6 + r^2t^2}{d^8} - \alpha^2\frac{r^6 + r^2t^2}{d^8}\right) |u|^2 r\, drdt \\
&= \int_{-\infty}^{\infty}\int_0^{\infty} (|\partial_r u|^2 + 4r^2|\partial_t u|^2) r\, drdt - \int_{-\infty}^{\infty}\int_0^{\infty} (2\alpha - \alpha^2)\frac{r^2}{d^4}\, |u|^2 r\, drdt.
\end{aligned}$$

We now complete the proof by substituting $\alpha = 1$. $\square$

## 3.3 Proof of Theorem 3.1.1

*Proof.* Let us now consider

$$\begin{aligned}\int_{\mathbb{R}^3} &|(\nabla_{G_0} + i\beta\mathcal{A})u|^2\, dzdt \\ &= \int \left( \left| \left( \partial_x - i\beta \frac{\partial_y d}{d} \right) u \right|^2 + \left| \left( \partial_y + i\beta \frac{\partial_x d}{d} \right) u \right|^2 \right) dxdydt \\ + &\int \left( \left| \left( 2x\partial_t - 2i\beta y \frac{\partial_t d}{d} \right) u \right|^2 + \left| \left( 2y\partial_t + 2i\beta x \frac{\partial_t d}{d} \right) u \right|^2 \right) dxdydt.\end{aligned}$$

We introduce polar coordinates for the $z$-plane:

$$r = \sqrt{x^2 + y^2}, \quad \frac{x}{r} = \cos\varphi \quad \text{and} \quad \frac{y}{r} = \sin\varphi$$

so that

$$\frac{\partial\varphi}{\partial x} = -\frac{y}{r^2}, \quad \frac{\partial\varphi}{\partial y} = \frac{x}{r^2}, \quad \partial_x = \cos\varphi \frac{\partial}{\partial r} - \frac{y}{r^2}\frac{\partial}{\partial\varphi} \quad \text{and} \quad \partial_y = \sin\varphi \frac{\partial}{\partial r} + \frac{x}{r^2}\frac{\partial}{\partial\varphi}.$$

As before, the distance function is the Kaplan function defined by $d = (r^4 + t^2)^{1/4}$. We also have

$$\frac{\partial_y d}{d} = \frac{r^3 \sin\varphi}{r^4 + t^2}, \qquad \frac{\partial_x d}{d} = \frac{r^3 \cos\varphi}{r^4 + t^2}$$

and

$$2y\frac{\partial_t d}{d} = \frac{yt}{r^4 + t^2}, \qquad 2x\frac{\partial_t d}{d} = \frac{xt}{r^4 + t^2}.$$

Let us split the quadratic form into two integrals:

$$\int_{\mathbb{R}^3} |(\nabla_{G_0} + i\beta\mathcal{A})u|^2\, dzdt = I_1 + I_2,$$

where

$$\begin{aligned} I_1 = \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} &\left( \left| \left( \cos\varphi\, \partial_r - \frac{\sin\varphi}{r}\, \partial_\varphi - i\beta\, \frac{r^3 \sin\varphi}{r^4 + t^2} \right) u \right|^2 \right. \\ &+ \left. \left| \left( \sin\varphi\, \partial_r + \frac{\cos\varphi}{r}\, \partial_\varphi + i\beta\, \frac{r^3 \cos\varphi}{r^4 + t^2} \right) u \right|^2 \right) r\, drd\varphi dt, \quad (3.3.1)\end{aligned}$$

and

$$I_2 = \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} \Big(\Big|\Big(2\cos\varphi\, r\,\partial_t - i\beta\sin\varphi\frac{rt}{r^4+t^2}\Big)u\Big|^2 + \Big|\Big(2\sin\varphi\, r\,\partial_t + i\beta\cos\varphi\frac{rt}{r^4+t^2}\Big)u\Big|^2\Big)\, r\, drd\varphi dt. \quad (3.3.2)$$

Computation of (3.3.1) gives

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} \left(\left|\left(\cos\varphi\partial_r - \frac{\sin\varphi}{r}\cdot\left(\partial_\varphi + i\beta\frac{r^4}{r^4+t^2}\right)\right)u\right|^2\right. \\ &\cdot \left.\left|\left(-\cos\varphi\partial_r + \frac{\sin\varphi}{r}\left(\partial_\varphi + i\beta\frac{r^4}{r^4+t^2}\right)\right)\bar{u}\right|^2\right) \\ &+ \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} \left(\left|\left(\sin\varphi\partial_r + \frac{\cos\varphi}{r}\left(\partial_\varphi + i\beta\frac{r^4}{r^4+-t^2}\right)\right)u\right|^2\right. \\ &\cdot \left.\left|\left(-\sin\varphi\partial_r - \frac{\cos\varphi}{r}\left(\partial_\varphi + i\beta\frac{r^4}{r^4+t^2}\right)\right)\bar{u}\right|^2\right) rdrd\varphi dt \\ &= \int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} \left(|\partial_r u|^2 + \frac{1}{r^2}\left|\partial_\varphi u + i\beta\frac{r^4}{r^4+t^2}u\right|^2\right) rdrd\varphi dt. \end{aligned}$$

Let us represent $u$ via Fourier series

$$u(r,\varphi,t) = \sum_{k=-\infty}^{\infty} u_k(r,t)\frac{e^{ik\varphi}}{\sqrt{2\pi}}$$

and thus

$$\partial_\varphi u(r,\varphi,t) = \sum_{k=-\infty}^{\infty} iku_k(r,t)\frac{e^{ik\varphi}}{\sqrt{2\pi}}.$$

Then, since $-1/2 \le \beta \le 1/2$, we find that

$$\begin{aligned} \frac{1}{r^2}\int_0^{2\pi}\left|\partial_\varphi u + i\beta\,\frac{r^4}{r^4+t^2}\,u\right|^2 d\varphi &= \frac{2\pi}{r^2}\sum_k\left(k+\beta\,\frac{r^4}{r^4+t^2}\right)^2|u_k|^2 \\ &\ge \frac{2\pi}{r^2}\min_k\left(k+\beta\,\frac{r^4}{r^4+t^2}\right)^2\sum_k|u_k|^2 \\ &= \frac{1}{r^2}\min_k\left(k+\beta\,\frac{r^4}{r^4+t^2}\right)^2\int_0^{2\pi}|u|^2\,d\varphi \\ &= \beta^2\,\frac{r^6}{(r^4+t^2)^2}\int_0^{2\pi}|u|^2\,d\varphi, \end{aligned}$$

because the minimum is reached when $k = 0$. Hence

$$I_1 \geq \beta^2 \frac{r^6}{(r^4+t^2)^2} \int_0^{2\pi} |u|^2 d\varphi.$$

Computing (3.3.2) gives that

$$I_2 = \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \left( 4r^2 |\partial_t u|^2 + \beta^2 \frac{r^2 t^2}{(r^4+t^2)^2} |u|^2 \right) r dr d\varphi dt.$$

Putting $I_1$ and $I_2$ together gives

$$\begin{aligned} &\int_{\mathbb{R}^3} \left( |(\nabla_{G_0} + i\beta\mathcal{A})u|^2 \right) dz dt \\ &\qquad \geq \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \left( |\partial_r u|^2 + 4r^2 |\partial_t u|^2 \right) r dr d\varphi dt \\ &\qquad\qquad + \beta^2 \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \frac{r^2(r^4+t^2)}{(r^4+t^2)^2} |u|^2 r dr d\varphi dt, \end{aligned}$$

which then yields

$$\begin{aligned} &\int_{\mathbb{R}^3} \left( |(\nabla_{G_0} + i\beta\mathcal{A})u|^2 \right) dz dt \\ &\qquad \geq \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \left( |\partial_r u|^2 + 4r^2 |\partial_t u|^2 \right) r dr d\varphi dt \\ &\qquad\qquad + \beta^2 \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \frac{r^2}{r^4+t^2} |u|^2 r dr d\varphi dt. \end{aligned}$$

Applying Proposition 3.2.2 to the first integral of the right hand side gives

$$\int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \left( |\partial_r u|^2 + 4r^2 |\partial_t u|^2 \right) r dr d\varphi dt \geq \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \frac{r^2 |u|^2}{r^4+t^2} r dr d\varphi dt,$$

which leads to the final conclusion

$$\int_{\mathbb{R}^3} \left( |(\nabla_{G_0} + i\beta\mathcal{A})u|^2 \right) dz dt \geq (1+\beta^2) \int_{\mathbb{R}^3} \frac{|z|^2}{z^4+t^2} dz dt$$

and that completes the proof.

□

# 4. $L^p$-Hardy inequalities for sub-elliptic operators

*In this chapter we establish an $L^p$-Hardy inequality related to Grushin operators with a magnetic field introduced in the previous chapter.*

## 4.1 Introduction

In this chapter we shall extend the results obtained in chapter 3 to $L^p$ classes of functions. As in the previous chapter, we consider the Heisenberg-Hörmander Laplacian

$$H = -X^2 - Y^2 = -\left(\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}\right)^2 \tag{4.1.1}$$

with $X$ and $Y$ defined by

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}.$$

Let us define the Grushin vector-field as

$$\nabla_G = (\partial_x, \partial_y, 2x\partial_t, 2y\partial_t). \tag{4.1.2}$$

Then we have

$$G = -|\nabla_G|^2,$$

where $G$ is the Grushin operator

$$G = -\Delta_z - 4|z|^2\partial_t^2. \tag{4.1.3}$$

Here we denote $z$ by $z = (x, y)$ and $|z| = \sqrt{x^2 + y^2}$.
Let $1 < p < \infty$. Then the $L^p$-quadratic forms for the operators $H$ and $G$ are

$$h_p[u] = \int_{\mathbb{R}^3} |(Xu, Yu)|^p \, dzdt, \tag{4.1.4}$$

where $|(Xu, Yu)| = \sqrt{|Xu|^2 + |Yu|^2}$, and

$$g_p[u] = \int_{\mathbb{R}^3} |\nabla_G u|^p \, dzdt \tag{4.1.5}$$

respectively.
Let us introduce the Kaplan distance function from $(z,t)$ to the origin

$$d(z,t) = (|z|^4 + t^2)^{1/4}.$$

The function $d$ is positively homogeneous with the property

$$d(\lambda z, \lambda^2 t) = \lambda \, d(z,t), \qquad \lambda > 0$$

and it has a singularity at zero.
For the $p$-forms $h_p[u]$ and $g_p[u]$ the following Hardy-type inequalities hold true for functions $u \in C_0^\infty(\mathbb{R}^3 \setminus 0)$:

$$h_p[u] \geq \frac{|4-p|^p}{p^p} \int_{\mathbb{R}^3} \frac{|z|^p}{d^{2p}} \, |u|^p \, dzdt \tag{4.1.6}$$

and

$$g_p[u] \geq \frac{|4-p|^p}{p^p} \int_{\mathbb{R}^3} \frac{|z|^2}{d^{2p}} \, |u(z,t)|^p \, dzdt. \tag{4.1.7}$$

**Remark 4.1.1.** *The Hardy constant $|4-p|^p \, p^{-p}$ for the Heisenberg-Hörmander Laplacian and for the Grushin operators is the same. This could be explained by the fact that the operator $H$ defined in* (4.1.1) *could be rewritten in the form*

$$Hu = -\Delta_z u - 4|z|^2 \partial_t^2 - 4\partial_t \, Tu = Gu - 4\partial_t \, Tu,$$

*where $T = y \, \partial_x - x \, \partial_y$. In particular, if $u(z,t) = u(|z|,t)$, then $Tu = 0$ and on this subclass of functions the inequalities* (4.1.6) *and* (4.1.7) *coincide.*

**Remark 4.1.2.** *One can show that the constant $|4-p|^p \, p^{-p}$ is sharp but not achieved.*

The main result of this chapter is an $L^p$-version of Theorem 3.1.1 from chapter 3. Let us define

$$\vec{\mathcal{A}} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4) = \Big( -\frac{\partial_y d}{d}, \frac{\partial_x d}{d}, -2y \, \frac{\partial_t d}{d}, 2x \, \frac{\partial_t d}{d} \Big). \tag{4.1.8}$$

Then the magnetic Grushin operator with the magnetic field $\mathcal{A}$ and with the "flux" $\beta \in \mathbb{R}$ could be defined as

$$G_{\mathcal{A}} = -(\nabla_G + i\beta \mathcal{A})^2. \tag{4.1.9}$$

We shall prove the following statement:

**Theorem 4.1.3.** *Let $1 < p < \infty$ and let us assume that $-1/2 \leq \beta \leq 1/2$. Then for the quadratic form of the magnetic Grushin operator* (4.1.9) *we have the following Hardy inequality:*

$$\frac{2}{p}\sqrt{(4-p)^2+p^2\beta^2}\int_{\mathbb{R}^3}|u|^p\frac{|z|^p}{d^{2p}}\,dzdt \\ \leq \left\|\left(\nabla_G + i\beta\,\vec{\mathcal{A}}\right)u\right\|_p + \left\|\left(\nabla_G + i\beta\,\vec{\mathcal{A}}\right)\bar{u}\right\|_p.$$

**Remark 4.1.4.** *The main aim of this result is to show that by introducing the magnetic field* (4.1.8)*, we improve the constant in the respective Hardy inequality. Note that if $\beta = 0$, then this inequality coincides with* (4.1.7) *and if $p = 2$, then we obtain the statement of Theorem 3.1.1 in chapter 3.*

## 4.2 $L^p$-Hardy inequalities for the Heisenberg-Hörmander Laplacian

The following result has been obtained in L. D'Ambrosio in [A1]. We shall present its proof for the sake of completeness. Note that if $p = 2$, then this result coincides with Proposition 3.2.1 from chapter 3. Its proof is given with a slightly different techniques.

**Proposition 4.2.1.** *For any function $u \in C_0^\infty(\mathbb{R}^3 \setminus 0)$ we have*

$$h_p[u] \geq \frac{|4-p|^p}{p^p}\int_{\mathbb{R}^3}\frac{|z|^p}{d^{2p}}|u|^p\,dzdt. \tag{4.2.1}$$

In order to prove this result we introduce the matrix

$$\sigma = \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \end{pmatrix}.$$

Then

$$\sigma^T\sigma = \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \\ 2y & -2x & 4x^2+4y^2 \end{pmatrix}$$

and we define

$$\mathrm{div}_H\vec{F} = \mathrm{div}\,\cdot\sigma^T\sigma\vec{F},$$

where div is the standard divergency in $\mathbb{R}^3$:

$$\operatorname{div} \vec{L} = \partial_x L_1 + \partial_y L_2 + \partial_t L_3, \vec{L} = (L_1, L_2, L_3).$$

Note that

$$\nabla_H = (X, Y)^T = \sigma \nabla = \begin{pmatrix} \partial_x + 2y\partial_t \\ \partial_y - 2x\partial_t \end{pmatrix},$$

where $\nabla = (\partial_x, \partial_y, \partial_t)^T$.

**Lemma 4.2.2.** *Let $\vec{F} = (F_1, F_2, F_3)$ be a vector-function such that $\vec{F} \in C^\infty(\mathbb{R}^3 \setminus 0)$ and let $u \in C_0^\infty(\mathbb{R}^3 \setminus 0)$. If $p > 1$ and if $\operatorname{div}_H \vec{F}$ is either non-negative or non-positive, then*

$$\frac{1}{p^p} \int_{\mathbb{R}^3} |u|^p \left|\operatorname{div}_H \vec{F}\right| dzdt \le \int_{\mathbb{R}^3} |\sigma \vec{F}|^p \, |\operatorname{div}_H \vec{F}|^{-(p-1)} |(Xu, Yu)|^p \, dzdt.$$

*Proof.* The proof follows from the following simple series of inequalities including the Hölder inequality:

$$\begin{aligned}
\left| \int_{\mathbb{R}^3} |u|^p \operatorname{div}_H \vec{F} \, dzdt \right| &= \left| \int_{\mathbb{R}^3} |u|^p \operatorname{div} (\sigma^T \sigma \, \vec{F}) \, dzdt \right| \\
&= \left| \int_{\mathbb{R}^3} (\sigma \nabla |u|^p) \, (\sigma \vec{F}) \, dzdt \right| \le p \int_{\mathbb{R}^3} |u|^{p-1} \, |\nabla_H u| \, |\sigma \vec{F}| \, dzdt \\
&= p \int_{\mathbb{R}^3} |u|^{p-1} \, |\operatorname{div}_H \vec{F}|^{(p-1)/p} \, \frac{1}{|\operatorname{div}_H \vec{F}|^{(p-1)/p}} \, |\nabla_H u| \, |\sigma \vec{F}| \, dzdt \\
&\le p \left( \int_{\mathbb{R}^3} |u|^p |\operatorname{div}_H \vec{F}| \, dzdt \right)^{(p-1)/p} \left( \int_{\mathbb{R}^3} \frac{|\sigma \vec{F}|^p}{|\operatorname{div}_H \vec{F}|^{p-1}} \, |\nabla_H u|^p \, dzdt \right)^{1/p}.
\end{aligned}$$

□

We are now ready to prove Proposition 4.2.1:

*Proof.* Let us introduce the vector-function

$$\vec{F} = \frac{1}{d^{2p}} \, (x|z|^p, y|z|^p, \, t|z|^{p-2}/2)^T,$$

where $d = ((x^2 + y^2)^2 + t^2)^{1/4}$ is the Kaplan distance to the origin and $|z| = \sqrt{x^2 + y^2}$. Then

$$\operatorname{div} \sigma^T \, \sigma \vec{F} = \operatorname{div} \frac{1}{d^{2p}} \begin{pmatrix} x \, |z|^p + yt \, |z|^{p-2} \\ y \, |z|^p - xt \, |z|^{p-2} \\ 2 \, t \, |z|^p \end{pmatrix}.$$

We compute the latter expression by careful computation of each term:

$$\partial_x \left( \frac{1}{d^{2p}} \left(x\,|z|^p + yt\,|z|^{p-2}\right)\right)$$
$$= \frac{1}{d^{2p+4}} \big(-2\,p\,x\,|z|^2\,\left(x\,|z|^p + yt|z|^{p-2}\right)$$
$$+(|z|^4+t^2)(|z|^p + px^2|z|^{p-2} + yt(p-2)x|z|^{p-2})\big)\,,$$

$$\partial_y \left( \frac{1}{d^{2p}} \left(x\,|z|^p - xt\,|z|^{p-2}\right)\right)$$
$$= \frac{1}{d^{2p+4}} \big(-2\,p\,y\,|z|^2\,\left(y\,|z|^p - xt|z|^{p-2}\right)$$
$$+(|z|^4+t^2)(|z|^p + py^2|z|^{p-2} - xt(p-2)y|z|^{p-2})\big)$$

and finally

$$\partial_t \left( \frac{1}{d^{2p}} 2t|z|^p \right) = \frac{1}{d^{2p+4}}\,\left(-pt\,2t\,|z|^p + 2|z|^p(|z|^4+t^2)\right).$$

Adding all these derivatives together, we obtain

$$\operatorname{div}_H \vec{F} = \operatorname{div}\sigma^T\,\sigma\vec{F} = \frac{(4-p)}{d^{2p+4}}\,(|z|^4+t^2)|z|^p = (4-p)\frac{|z|^p}{d^{2p}}.$$

Moreover,

$$\sigma\vec{F} = \frac{1}{d^{2p}} \begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \end{pmatrix} \begin{pmatrix} x|z|^p \\ y|z|^p \\ t|z|^{p-2}/2 \end{pmatrix} = \frac{1}{d^{2p}} \begin{pmatrix} x|z|^p + yt|z|^{p-2} \\ y|z|^p - xt|z|^{p-2} \end{pmatrix}.$$

Computing $|\sigma\vec{F}|^p$, we find that

$$|\sigma\vec{F}|^2 = \frac{1}{d^{4p}}\,\left((x|z|^p + yt|z|^{p-2})^2 + (y|z|^p - xt|z|^{p-2})\right)$$
$$= \frac{1}{d^{4p}}\,\left(|z|^{2p+2} + t^2\,|z|^{2p-2}\right) = \frac{|z|^{2p-2}}{d^{4p-4}}. \qquad (4.2.2)$$

Therefore miraculously we have

$$|\sigma\vec{F}|^p\,|\operatorname{div}_H \vec{F}|^{-(p-1)} = \left(\frac{|z|^{2p-2}}{d^{4p-4}}\right)^{p/2} \left(|4-p|\,\frac{|z|^p}{d^{2p}}\right)^{-(p-1)} = |4-p|^{-(p-1)}. \qquad (4.2.3)$$

This finally gives the necessary statement, if we substitute (4.2.2) and (4.2.3) into the inequality stated in Lemma 4.2.2 . □

## 4.3 $L^p$-Hardy inequalities for the Grushin operator

Similarly to Proposition 4.2.1 we show (see also [A2]):

**Proposition 4.3.1.** *For any function $u \in \mathbb{C}_0^\infty(\mathbb{R}^3 \setminus 0)$ we have*

$$g_p[u] \geq \frac{|4-p|^p}{p^p} \int_{\mathbb{R}^3} \frac{|z|^2}{d^{2p}} |u(z,t)|^p \, dzdt.$$

In order to prove this statement we again need an auxiliary Lemma.

**Lemma 4.3.2.** *Let $\vec{F} = (F_1, F_2, F_3, F_4)$ be a vector-function such that $\vec{F} \in C^\infty(\mathbb{R}^3 \setminus 0)$ and let $u \in C_0^\infty(\mathbb{R}^3 \setminus 0)$. If $p > 1$ and if $\nabla_G \cdot \vec{F}$ is either non-negative or non-positive, then*

$$\frac{1}{p^p} \int_{\mathbb{R}^3} |u|^p \left| \nabla_G \cdot \vec{F} \right| dzdt \leq \int_{\mathbb{R}^3} |\vec{F}|^p \, |\nabla_G \vec{F}|^{-(p-1)} |\nabla_G u|^p \, dzdt, \quad (4.3.1)$$

*where the Grushin gradient $\nabla_G$ has been introduced in* (4.1.2).

*Proof.* The proof follows from the following simple inequalities

$$\begin{aligned}
\left| \int_{\mathbb{R}^3} |u|^p \, \nabla_G \cdot \vec{F} \, dzdt \right| &= \left| \int_{\mathbb{R}^3} (\nabla_G |u|^p) \cdot \vec{F} \, dzdt \right| \\
&\leq p \int_{\mathbb{R}^3} |u|^{p-1} \, |\nabla_G u| \, |\vec{F}| \, dzdt \\
= p \int_{\mathbb{R}^3} |u|^{p-1} \, |\nabla_G \cdot \vec{F}|^{(p-1)/p} & \frac{1}{|\nabla_G \cdot \vec{F}|^{(p-1)/p}} \, |\nabla_H u| \, |\vec{F}| \, dzdt \\
\leq p \left( \int_{\mathbb{R}^3} |u|^p |\nabla_G \cdot \vec{F}| \, dzdt \right)^{(p-1)/p} & \left( \int_{\mathbb{R}^3} \frac{|\vec{F}|^p}{|\nabla_G \cdot \vec{F}|^{p-1}} \, |\nabla_G u|^p \, dzdt \right)^{1/p} .
\end{aligned}$$

□

We continue now to prove Proposition 4.3.1:

*Proof.* We define $\vec{F}$ as follows:

$$\vec{F} = \frac{1}{d^{2p}} \left( x|z|^p, y|z|^p, tx|z|^{p-2}, ty|z|^{p-2} \right).$$

Then

$$\partial_x \left( \frac{x|z|^p}{d^{2p}} \right) = \frac{1}{d^{2p+4}} \left( -2px^2|z|^{p+2} + (|z|^4 + t^2)(|z|^p + x^2|z|^{p-2}) \right),$$

$$\partial_y\left(\frac{y|z|^p}{d^{2p}}\right)=\frac{1}{d^{2p+4}}\left(-2py^2|z|^{p+2}+(|z|^4+t^2)(|z|^p+y^2|z|^{p-2})\right),$$
$$2x\partial_t\left(\frac{xt|z|^{p-2}}{d^{2p}}\right)=\frac{1}{d^{2p+4}}\left(-2px^2t^2|z|^{p-2}+2(|z|^4+t^2)x^2|z|^{p-2}\right)$$

and

$$2y\partial_t\left(\frac{yt|z|^{p-2}}{d^{2p}}\right)=\frac{1}{d^{2p+4}}\left(-2py^2t^2|z|^{p-2}+2(|z|^4+t^2)y^2|z|^{p-2}\right).$$

This implies

$$\begin{aligned}\nabla_G\cdot\vec{F}=&\frac{1}{d^{2p+4}}(-2p\left(|z|^4+t^2\right)+(|z|^4+t^2)(2+p))\,|z|^p\\ &=(4-p)\,\frac{|z|^p}{d^{2p}}.\end{aligned}\tag{4.3.2}$$

Besides, we have that

$$\begin{aligned}|\vec{F}|^2=&\frac{1}{d^{4p}}\left(x^2|z|^{2p}+y^2|z|^{2p}+t^2x^2|z|^{2p-2}+t^2y^2|z|^{2p-2}\right)\\ =&\frac{|z|^{2(p-1)}}{d^{4(p-1)}}.\end{aligned}\tag{4.3.3}$$

Substituting (4.3.2) and (4.3.3) into the inequality (4.3.1), we complete the proof. □

## 4.4 $L^p$-Hardy inequality for the magnetic Grushin operator

Let $\vec{F}=\vec{\mathcal{F}}_1-i\vec{\mathcal{F}}_2$, where

$$\vec{\mathcal{F}}_1=\frac{c}{d^{2p}}\left(x|z|^p,y|z|^p,tx|z|^{p-2},ty|z|^{p-2}\right),$$

$c\in\mathbb{R}$ and

$$\vec{\mathcal{F}}_2=\beta\frac{1}{d^{2p}}\left(-y|z|^p,x|z|^p,-ty|z|^{p-2},tx|z|^{p-2}\right).$$

We also have that

$$\begin{aligned}\vec{\mathcal{A}}=(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4)=\Big(-\frac{\partial_y d}{d},\frac{\partial_x d}{d},-2y\,\frac{\partial_t d}{d},2x\,\frac{\partial_t d}{d}\Big)\\ =\frac{1}{d^4}\left(-y|z|^2,x|z|^2,-yt,+xt\right).\end{aligned}\tag{4.4.1}$$

Then the vector-functions $\vec{\mathcal{F}}_1$, $\vec{\mathcal{F}}_2$, $\vec{\mathcal{A}}$ satisfy the following properties:

$$\vec{\mathcal{F}}_1 \cdot \vec{\mathcal{F}}_2 = \vec{\mathcal{F}}_1 \cdot \vec{\mathcal{A}} = 0. \tag{4.4.2}$$

Moreover,

$$\begin{aligned}\nabla_G \cdot \vec{\mathcal{A}} = -\frac{1}{d^8}\big(&-4yx|z|^4 + 4xy|z|^4 - 4xyt^2 + 4yxt^2 \\ &+d^4(-2yx + 2xy - 2xy + 2yx)\big) = 0\end{aligned} \tag{4.4.3}$$

and

$$\begin{aligned}\beta\,\vec{\mathcal{A}} \cdot \vec{\mathcal{F}}_2 &= \beta^2\,\frac{1}{d^{2p+4}}\,\left(y^2|z|^{p+2} + x^2|z|^{p+2} + t^2y^2|z|^{p-2} + t^2x^2|z|^{p-2}\right) \\ &= \beta^2\,\frac{1}{d^{2p+4}}\,\left(|z|^{p+4} + t^2|z|^p\right) = \beta^2\,\frac{|z|^p}{d^{2p}}.\end{aligned}$$

Note that $\vec{\mathcal{F}}_1$ coincides with the vector-function $\vec{F}$ from the previous section and therefore by (4.3.2) we have

$$\begin{aligned}\nabla_G \cdot \vec{\mathcal{F}}_1 =& \frac{c}{d^{2p+4}}\left(-2p\,(|z|^4 + t^2) + (|z|^4 + t^2)(2+p)\right)\,|z|^p \\ =& \,c\,(4-p)\,\frac{|z|^p}{d^{2p}}\end{aligned} \tag{4.4.4}$$

and

$$|\vec{\mathcal{F}}_1 + i\vec{\mathcal{F}}_2| = \sqrt{c^2+\beta^2}\,\frac{|z|^{(p-1)}}{d^{2(p-1)}}. \tag{4.4.5}$$

Besides, using (4.4.2), (4.4.3) and (4.4.4), we also find that

$$\begin{aligned}\left(\nabla_G + i\beta\,\vec{\mathcal{A}}\right) \cdot \vec{F} = \left(\nabla_G + i\beta\,\vec{\mathcal{A}}\right) &\cdot (\vec{\mathcal{F}}_1 - i\,\vec{\mathcal{F}}_2) \\ &= (c\,(4-p) + \beta^2)\,\frac{|z|^p}{d^{2p}}.\end{aligned} \tag{4.4.6}$$

**Lemma 4.4.1.** *Let $\vec{F} = (F_1, F_2, F_3, F_4)$ be a vector-function such that $\vec{F} \in C^\infty(\mathbb{R}^3 \setminus 0, \mathbb{C}^3)$ and let $u \in C_0^\infty(\mathbb{R}^3 \setminus 0)$. If $p > 1$ and if*

$\left(\nabla_G + i\beta\, p\, \vec{\mathcal{A}}\right) \cdot \vec{F}$ *is either non-negative or non-positive, then*

$$\frac{2}{p}\left(\int_{\mathbb{R}^3} |u|^p \left|\left(\nabla_G + i\beta\, p\, \vec{\mathcal{A}}\right)\cdot \vec{F}\right| \, dzdt\right)^{1/p}$$
$$\leq \left[\left(\int_{\mathbb{R}^3} \frac{|\vec{F}|^p}{\left|\left(\nabla_G + i\beta\, p\, \vec{\mathcal{A}}\right)\cdot\vec{F}\right|^{p-1}} \left|\left(\nabla_G + i\beta\, \vec{\mathcal{A}}\right) u\right|^p \, dzdt\right)^{1/p}\right.$$
$$\left. + \left(\int_{\mathbb{R}^3} \frac{|\vec{F}|^p}{\left|\left(\nabla_G + i\beta\, \vec{\mathcal{A}}\right)\cdot\vec{F}\right|^{p-1}} \left|\left(\nabla_G + i\beta\, \vec{\mathcal{A}}\right) \bar{u}\right|^p \, dzdt\right)^{1/p}\right]. \quad (4.4.7)$$

*Proof.* The proof is similar to the proof of Lemma 4.3.2, thus

$$\left|\int_{\mathbb{R}^3} |u|^p \left(\nabla_G + i\beta\, p\, \vec{\mathcal{A}}\right)\cdot \vec{F}\, dzdt\right| = \left|\int_{\mathbb{R}^3} \left(\left(\nabla_G + i\beta\, p\, \vec{\mathcal{A}}\right) u^{p/2}\bar{u}^{p/2}\right)\cdot \vec{F}\, dzdt\right|$$
$$\leq \frac{p}{2}\left|\int_{\mathbb{R}^3} |u|^{p-2}\, u \left(\left(\nabla_G + i\beta\, \vec{\mathcal{A}}\right) u\right)\cdot \vec{F}\, dzdt\right|$$
$$+ \frac{p}{2}\left|\int_{\mathbb{R}^3} |u|^{p-2}\, \bar{u} \left(\left(\nabla_G + i\beta\, \vec{\mathcal{A}}\right) \bar{u}\right)\cdot \vec{F}\, dzdt\right|$$
$$\leq \frac{p}{2}\int_{\mathbb{R}^3} |u|^{p-1} \left|\left(\nabla_G + i\beta\, \vec{\mathcal{A}}\right) u\right| |\vec{F}|\, dzdt$$
$$+ \frac{p}{2}\int_{\mathbb{R}^3} |u|^{p-1} \left|\left(\nabla_G + i\beta\, \vec{\mathcal{A}}\right) \bar{u}\right| |\vec{F}|\, dzdt$$
$$\leq \frac{p}{2}\left(\int_{\mathbb{R}^3} |u|^p \left|\left(\nabla_G + i\beta\, p\, \vec{\mathcal{A}}\right)\cdot\vec{F}\right| \, dzdt\right)^{(p-1)/p}$$
$$\times \left[\left(\int_{\mathbb{R}^3} \frac{|\vec{F}|^p}{\left|\left(\nabla_G + i\beta\, p\, \vec{\mathcal{A}}\right)\cdot\vec{F}\right|^{p-1}} \left|\left(\nabla_G + i\beta\, \vec{\mathcal{A}}\right) u\right|^p \, dzdt\right)^{1/p}\right.$$
$$\left. + \left(\int_{\mathbb{R}^3} \frac{|\vec{F}|^p}{\left|\left(\nabla_G + i\beta\, \vec{\mathcal{A}}\right)\cdot\vec{F}\right|^{p-1}} \left|\left(\nabla_G + i\beta\, \vec{\mathcal{A}}\right) \bar{u}\right|^p \, dzdt\right)^{1/p}\right]. \quad (4.4.8)$$

Rearranging the terms, we obtain the statement of the Lemma. □

We are now able to finish the proof of Theorem 4.1.3.

*Proof.* Using the properties (4.4.4) and (4.4.6), we find that

$$|\vec{F}|^p = (c^2+\beta^2)^{p/2}\ \frac{|z|^{p(p-1)}}{d^{2p(p-1)}}$$

and

$$\left|\left(\nabla_G + i\beta\, p\, \vec{\mathcal{A}}\right)\cdot \vec{F}\right|^{-(p-1)} = \frac{1}{|c\,(4-p)+p\,\beta^2|^{(p-1)}}.$$

Therefore the inequality (4.4.7) becomes

$$\begin{aligned}
&\frac{2}{p}\left|c\,(4-p)+p\beta^2\right|^{1/p}\left(\int_{\mathbb{R}^3}|u|^p\,\frac{|z|^p}{d^{2p}}\,dzdt\right)^{1/p}\\
&\quad\le \frac{(c^2+\beta^2)^{1/2}}{|c\,(4-p)+p\,\beta^2|^{(p-1)/p}}\ \left[\left(\int_{\mathbb{R}^3}\left|\left(\nabla_G+i\beta\,\vec{\mathcal{A}}\right)u\right|^p\,dzdt\right)^{1/p}\right.\\
&\qquad\qquad\left.+\left(\int_{\mathbb{R}^3}\left|\left(\nabla_G+i\beta\,\vec{\mathcal{A}}\right)\bar{u}\right|^p\,dzdt\right)^{1/p}\right]
\end{aligned}$$

and we finally obtain

$$\begin{aligned}
\frac{2}{p}\,\frac{\left|c\,(4-p)+p\,\beta^2\right|}{(c^2+\beta^2)^{1/2}}\ \left(\int_{\mathbb{R}^3}|u|^p\,\frac{|z|^p}{d^{2p}}\,dzdt\right)^{1/p} &\le \left\|\left(\nabla_G+i\beta\,\vec{\mathcal{A}}\right)u\right\|_p\\
&\quad+\left\|\left(\nabla_G+i\beta\,\vec{\mathcal{A}}\right)\bar{u}\right\|_p.
\end{aligned}$$

Minimizing with respect to $c$, we find that

$$c=\frac{4-p}{p}$$

and therefore

$$\begin{aligned}
&\frac{2}{p}\sqrt{(4-p)^2+p^2\beta^2}\ \int_{\mathbb{R}^3}|u|^p\,\frac{|z|^p}{d^{2p}}\,dzdt\\
&\qquad\le \left\|\left(\nabla_G+i\beta\,\vec{\mathcal{A}}\right)u\right\|_p+\left\|\left(\nabla_G+i\beta\,\vec{\mathcal{A}}\right)\bar{u}\right\|_p.
\end{aligned}$$

With this we complete the proof of our main result. □

# 5. On some spectral inequalities for a class of Grushin operators

*The aim of this chapter is to find CLR and Lieb-Thirring inequalities for a class of Grushin operators. Since this operator is non-elliptic, these inequalities will not take their classical form.*

## 5.1 Introduction

The Weyl-type asymptotics for the number of bound states gave rise to the question, whether there is a semi-classical bound for the moments of the negative eigenvalues of operators of the Schrödinger class $P := -\Delta - V$ in $L^2(\mathbb{R}^d)$

$$\sum_{\lambda<0} |\lambda|^\gamma = tr(-\Delta - V)_-^\gamma$$

of the form

$$\mathrm{Tr}\,(-\Delta - V)_-^\gamma \leq \frac{C_{\gamma,d}}{(2\pi)^d} \int\int (|\xi|^2 - V(x))_-^\gamma d\xi dx,$$

or equivalently

$$\sum_{\lambda<0} |\lambda|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma+\frac{d}{2}}(x)dx,$$

where $L_{\gamma,d} = C_{\gamma,d} L_{\gamma,d}^{cl}$ is the Lieb-Thirring constant and $L_{\gamma,d}^{cl}$ is defined as

$$L_{\gamma,d}^{cl} = \frac{1}{(2\pi)^d} \int (1 - |\xi|^2)_+^\gamma d\xi = \frac{\Gamma(\gamma+1)}{2^d \pi^{\frac{d}{2}} \Gamma(\gamma + 1 + \frac{d}{2})}.$$

Note that $L_{\gamma,0}^{cl} = \dfrac{\omega_d}{(2\pi)^d}$, where $\omega_d$ is the volume of the unit ball in $\mathbb{R}^d$.

The Lieb-Thirring inequalities give mean estimates for the moments of the negative eigenvalues of Schrödingier operators in terms of external fields. The most interesting cases are when $\gamma = 0$, which gives the counting function for the number of bound states, and $\gamma = 1$ that gives a bound for the total energy of the system. The original result from

[LTh] attained in 1975 states that if $\gamma > \max(0, 1 - d/2)$, then there is a universal constant $L_{\gamma,d}$ depending only on $\gamma$ and $d$ such that

$$\sum_{\lambda<0} |\lambda|^{\gamma} \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma+\frac{d}{2}}(x)dx, \tag{5.1.1}$$

where $\lambda_1 \leq \lambda_2 \leq \ldots$ are the negative eigenvalues of the Schrödinger operator.
The Cwikel-Lieb-Rozenblum inequality ( [C], [Li], [Roz]) refers to the critical case where $\gamma = 0$:

$$N(V) = \#\{k: \ \lambda_k < 0\} \leq \int_{\mathbb{R}^d} V_+^{\frac{d}{2}}(x)dx, \quad d \geq 3. \tag{5.1.2}$$

If $\lambda_k \to \infty$, then the analog of the Lieb-Thirring inequalities is the uniform with respect to $\lambda > 0$ inequality for the value of $\sum(\lambda - \lambda_k)_+^{\gamma}$. In this case we consider the spectral problem for the operator

$$(-\Delta + V)u = \lambda u.$$

Here one can interpret the Lieb-Thirring inequality as the inequality for negative eigenvalues for the operator $-\Delta + V - \lambda$.
For such operators, after interchanging integrals and traces, one usually considers the counting function

$$\sum_j (\lambda - \lambda_k)_+^{\gamma} = \gamma \int_0^{\infty} \operatorname{Tr}(-\Delta + V - \lambda + t)_-^0 t^{\gamma-1} dt. \tag{5.1.3}$$

The Weyl-type asymptotics usually establish the following asymptotic formula for a large class of potential functions $V$ :

$$\sum_j (\lambda - \lambda_k)_+^{\gamma} \sim \gamma \int_0^{\infty} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (|\xi|^2 + V(x) - \lambda + t)_-^0 t^{\gamma-1} dt dx d\xi$$
$$\text{as} \quad \lambda \to \infty.$$

Note that the expression $(|\xi|^2 + V(x) - \lambda)$ appearing in the integral in (5.1.3) is the classical symbol of the operator $-\Delta + V - \lambda$.
Applying the Fubini-Tonelli Theorem and scaling allows us to represent the "volume" in phase space as

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (|\xi|^2 + V(x) - \lambda + t)_-^0 dx d\xi$$
$$= \int \int_{(|\xi|^2+V-\lambda+t)<0} 1 \, dx d\xi = \omega_d \int_{\mathbb{R}^d} (V(x) - \lambda + t)_-^{\frac{d}{2}} dx$$

and therefore finally we have

$$\sum_j (\lambda - \lambda_k)_+^\gamma \sim \mathrm{L}^{cl}_{\gamma,d} \int_0^\infty (V(x) - \lambda)_-^{\gamma+\frac{d}{2}} dx \quad \text{as} \quad \lambda \to \infty.$$

If $V$ tends to infinity, then the Weyl formula could be written as

$$N(V-\lambda) \sim \left(\frac{1}{2\pi}\right)^d \int_{(|\xi|^2+V(x))\leq\lambda} dx d\xi = \frac{\omega_d}{(2\pi)^d} \int (V-\lambda)_-^{\frac{d}{2}} dx$$
$$= \frac{1}{(2\pi)^d} \mathrm{Vol}[\{(x,\xi) : |\xi|^2 + V(x) < \lambda\}] \qquad \text{as} \quad \lambda \to \infty, \quad (5.1.4)$$

where the right hand side is finite for any $\lambda > 0$.

Although Weyl's asymptotic formula is known for a large class of potentials, the question of uniform estimates with respect to $\lambda$ and the potential function $V$

$$\sum (\lambda - \lambda_k)^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} (V-\lambda)_-^{\gamma+\frac{d}{2}} dx \tag{5.1.5}$$

is still a challenging problem. In particular, the sharp constant in (5.1.5) was not known even for the multidimensional harmonic oscillator ($V = |x|^2$) until the paper of R. de la Bretéche [dlB], where the author obtained the following result:

Let $H = -\Delta + |x|^2$ be the multidimensional harmonic oscillator acting in $L^2(\mathbb{R}^d)$. The spectrum of such operators is discrete and its eigenvalues are

$$\{\lambda_k\} = \{2|k| + d\}, \quad k = (k_1, ..., k_d), \quad k_j \in \mathbb{Z}, \quad |k| = \sum_{j=0}^d k_j.$$

In particular, in [dlB] the author justified the Lieb-Thirring conjecture for any $\gamma \geq 1$ such that

$$\sum (\lambda - \lambda_k)^\gamma \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\lambda - \xi^2 - |x|^2)_+^\gamma dx d\xi. \tag{5.1.6}$$

In this paper we consider a version of harmonic oscillator for the Grushin operator, see [Gr]:

$$G_0 = -\Delta_z - 4|z|^2\partial_t^2.$$

It is well known that $G_0$ appears as the "radial" part of the sub-elliptic Heisenberg-Hörmander Laplacian.

Let us consider $\mathbb{H}$ as $\mathbb{R}^3$ with coordinates $(x,y,t)$ and the (non-commutative) multiplication $(x,y,t)\circ(x',y',t') = (x+x', y+y', t+t'-2(xy'-yx'))$. The vector fields

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}\,, \qquad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$$

are left-invariant and the the Heisenberg-Hörmander Laplacian on $\mathbb{H}$ is given by

$$H = -X^2 - Y^2 = -\left(\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}\right)^2 . \tag{5.1.7}$$

The quadratic form $h$ of the operator $H$ is defined by the equality

$$h[u] = \int_{\mathbb{R}^3} (|Xu|^2 + |Yu|^2)\, dzdt. \tag{5.1.8}$$

The Grushin operator

$$G = -\Delta_z - 4|z|^2\partial_t^2 \tag{5.1.9}$$

gives another example of a sub-elliptic operator. Its quadratic form $g$ respectively equals

$$g[u] = \int_{\mathbb{R}^3} (|\nabla_z u|^2 + 4|z|^2|\partial_t u|^2)\, dzdt. \tag{5.1.10}$$

The forms (5.1.8) and (5.1.10) are related. Indeed, the operator $H$ defined in (5.1.7) could be rewritten in the form

$$Hu = -\Delta_z u - 4|z|^2\partial_t^2 - 4\partial_t\, Tu = Gu - 4\partial_t\, Tu, \tag{5.1.11}$$

where $T = y\,\partial_x - x\,\partial_y$. In particular, if $u(z,t) = u(|z|,t)$, then $Tu = 0$ and on this subclass of functions the inequalities (5.1.8) and (5.1.10) coincide.

Our main result concerns the operator

$$G = G_0 + |z|^2 t^2, \tag{5.1.12}$$

which could be considered as a version of harmonic oscillator for the sub-elliptic Grushin operator $G_0$.

In order to formulate our main result we need the following notations: Let us introduce the Euler-Mascheroni constant $\gamma$ (Euler 1735, see [E] and [Con]):

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} - \ln(n)\right) = \int_1^\infty \left(\frac{1}{[x]} - \frac{1}{x}\right)\, dx.$$

Its numerical value is $\gamma = 0.57721\ldots$. (It is not known if $\gamma$ is rational or irrational). Then we can define the harmonic number $\mathcal{H}(n)$ by

$$\mathcal{H}(n) := \sum_{k=1}^{n} \frac{1}{k} = \gamma + \psi(n+1),$$

where $\psi(t)$ is known as the Gauss digamma function defined by

$$\psi(t) = \frac{\Gamma'(t)}{\Gamma(t)}.$$

We can also introduce the value of $\mathcal{H}(n+1/2)$ as

$$\mathcal{H}(n+1/2) = \gamma + \psi(n+3/2).$$

Then, by using the properties of the $\Gamma$-function, we can find that

$$\sum_{k=0}^{n} \frac{1}{2k+1} = \ln 2 + \frac{1}{2}\mathcal{H}\left(n + \frac{1}{2}\right).$$

Thus

$$\sum_{k \leq \frac{\lambda^2}{32} - \frac{1}{2}} \frac{1}{2k+1} = \ln 2 + \frac{1}{2}\mathcal{H}\left(\left[\frac{\lambda^2}{32} - \frac{1}{2}\right] + \frac{1}{2}\right) \tag{5.1.13}$$

with

$$\psi(n+1/2) = -\gamma H(n-1/2).$$

**Theorem 5.1.1.** *The spectrum of the Grushin operator* (5.1.12) *is discrete and its eigenvalues* $\{\lambda_j\}$ *satisfy uniformly the following sharp inequality with respect to* $\lambda < 0$*:*

$$\sum_j (\lambda - \lambda_j)_+ \leq \frac{1}{96}\lambda^3 \left(\ln 2 + \frac{1}{2}\mathcal{H}\left(\left[\frac{\lambda^2}{32} - \frac{1}{2}\right] + \frac{1}{2}\right)\right). \tag{5.1.14}$$

**Theorem 5.1.2.** *The spectrum of the Grushin operator* (5.1.12) *is discrete and its counting function of the eigenvalues* $\{\lambda_j\}$ *of the Grushin operator* (5.1.12) *satisfy uniformly the following inequality with respect to* $\lambda > 0$*:*

$$\#\{j : \lambda_j < \lambda\} \leq \frac{1}{32}\lambda^2 \left(\ln 2 + \frac{1}{2}\mathcal{H}\left(\left[\frac{\lambda^2}{32} - \frac{1}{2}\right] + \frac{1}{2}\right)\right) + \frac{\lambda^2}{32}. \tag{5.1.15}$$

**Remark 5.1.3.** *It is known that*

$$\ln\left(n+\frac{3}{2}\right) < \mathcal{H}\left(n+\frac{1}{2}\right) < 1+\ln\left(n+\frac{1}{2}\right).$$

*Therefore we can also conclude that*

$$\sum_j (\lambda-\lambda_j)_+ \le \frac{1}{96}\lambda^3\left(1+\ln 2+\frac{1}{2}\ln\left(\left[\frac{\lambda^2}{32}-\frac{1}{2}\right]+\frac{1}{2}\right)\right).$$

**Remark 5.1.4.** *It is also known that*

$$\mathcal{H}(n) \sim \ln(n)+\gamma \quad \text{as} \quad n\to\infty. \tag{5.1.16}$$

*Therefore*

$$\sum_{k\le\frac{\lambda^2}{32}-\frac{1}{2}} \frac{1}{2k+1} \sim \ln\lambda + O(1) \quad \text{as} \quad \lambda\to\infty.$$

*Apart from estimates from above, one can easily obtain a similar inequalities from below and using* (5.1.16)*, we find that*

$$\sum_j (\lambda-\lambda_j)_+ = \frac{1}{96}\lambda^3\ln\lambda + O(\lambda^3)$$

*and*

$$\#\{j:\ \lambda_j<\lambda\} = \frac{1}{32}\lambda^2\ln\lambda + O(\lambda^2).$$

**Remark 5.1.5.** *Note that some inequalities for sub-elliptic operators are obtained in the paper by G. Rozenblum and M. Solomyak [RozSol].*

## 5.2 Explicit computation of the spectrum of the Grushin harmonic oscillator

In this section we prove the following Proposition:

**Proposition 5.2.1.** *For the Grushin harmonic oscillator*

$$-\Delta_z - 4|z|^2\partial_t + 4|z|^2t^2, \quad z=(x,y)$$

*acting in $L^2(\mathbb{R}^3)$, the spectrum is discrete and its eigenvalues equal*

$$\lambda_{n_1,n_2,k} = 2\sqrt{2k+1}\cdot 2(n_1+n_2+1).$$

*Proof.* Note that the equation

$$(-\Delta_z - 4|z|^2\partial_t + 4|z|^2t^2)\, u(z,t) = \lambda\, u(z,t) \tag{5.2.1}$$

admits a separation of variables. Namely, making the substitution $u(x,y,t) = u(z,t) = v(z)w(t)$ turns (5.2.1) into

$$-\Delta_z v(z) \cdot w(t) + 4|z|^2 v(z)(-\partial_t^2 w + t^2 w) = \lambda v w.$$

The eigenvalue problem for the operator $-\partial_t^2 + t^2$ is the standard eigenvalue problem for the one-dimensional harmonic oscillator

$$-\partial_t^2 w_k + t^2 w_k = \mu_k w_k,$$

whose eigenvalues equal $\mu_k = (2k+1)$ and whose eigenfunctions are

$$w_k = H_k(t)\, e^{-t^2/2}, \quad k = 0, 1, 2, ...,$$

where $H_k$ are the Hermite polynomials

$$H_k(t) = (-1)^k\, e^{t/2} \frac{d^k}{dt^k}\, e^{-t/2}.$$

Now the equation (5.2.1) can be reduced to

$$\left(-\Delta_z v + 4 \cdot (2k+1)\, |z|^2 v\right)\; w_k = \lambda\, v\, w_k, \tag{5.2.2}$$

where $-\Delta_z + 4(2k+1)|z|^2$ is the harmonic oscillator in two dimensions. Introducing the change of valiables $x' = \sqrt{\alpha\, x}'$ , $y' = \sqrt{\alpha\, y}$ in the equation

$$-\Delta_{xy} v + \alpha^2 |z|^2\, v = \nu\, v,$$

enables us to obtain

$$-\alpha\, \Delta_{x'y'} v + \alpha\, |z'|^2\, v = \nu\, v.$$

This implies that

$$\nu_n = \alpha\, 2\, (n_1 + n_2 + 1) \quad \text{and} \quad v_n = H_{n_1}(x') e^{-x'^2/2} \cdot H_{n_2}(y') e^{-y'^2/2},$$

where $\alpha = 2\sqrt{2k+1}$. Hence

$$\lambda_{n_1,n_2,k} = 4\sqrt{2k+1} \cdot (n_1 + n_2 + 1).$$

$\square$

## 5.3 Proofs of Theorems 5.1.1 and 5.1.2

*Proof of Theorem 5.1.1.* In order prove Theorem 5.1.1, we apply the result obtained by de la Bretéche [dlB] (see also [L2]) for the eigenvalues $\lambda_{n_1,n_2,k}$ of the operator $G_0 + 4\,|z|^2\,t^2$ in dimension two. Then

$$\sum_{n_1,n_2,k=0}^{\infty} (\lambda - \lambda_{n_1,n_2,k})_+ \leq \sum_{n_1,n_2,k=0}^{\infty} (\lambda - 4(n_1+n_2+1)\sqrt{2k+1})_+$$
$$\leq \frac{1}{(2\pi)^2} \sum_k \int_{\mathbb{R}^4} (\lambda - |\xi|^2 - 4(2k+1)\,|z|^2)_+\, d\xi dz. \quad (5.3.1)$$

In the expression

$$\sum_{n_1,n_2,k=0}^{\infty} (\lambda - 4(n_1+n_2+1)\sqrt{2k+1})_+$$

the values $n_1$ and $n_2$ satisfy the inequality

$$1 \leq n_1 + n_2 + 1 \leq \frac{\lambda}{4\sqrt{2k+1}}, \quad (5.3.2)$$

and therefore the integer $k$ cannot be very large. Note that the values of $|\xi|^2$ and $|z|^2$ take all the values between $[0,\infty)$. This implies that the sum on the right hand side of (5.3.1) is not finite, if we do not use the fact that due to (5.3.2) we have that

$$k \leq \frac{1}{2}\frac{\lambda^2}{16} - \frac{1}{2}.$$

Substituting $z' = \sqrt{2k+1}\,z$ in the integral (5.3.1), we find

$$\frac{1}{(2\pi)^2} \sum_k \int_{\mathbb{R}^4} (\lambda - |\xi|^2 - 4(2k+1)\,|z|^2)_+\, d\xi dz$$
$$= \frac{1}{(2\pi)^2} \sum_{k \leq \frac{\lambda^2}{32} - \frac{1}{2}} \frac{1}{2k+1} \cdot \int_{\mathbb{R}^4} (\lambda - |\xi|^2 - 4\,|z'|^2)_+\, d\xi dz'$$
$$= \lambda^3 \frac{1}{(2\pi)^2} \sum_{k \leq \frac{\lambda^2}{32} - \frac{1}{2}} \frac{1}{2k+1} \cdot \int_{\mathbb{R}^4} (\lambda - |\xi|^2 - 4\,|z'|^2)_+\, d\xi dz'$$
$$= \lambda^3 \frac{1}{96} \sum_{k \leq \frac{\lambda^2}{32} - \frac{1}{2}} \frac{1}{2k+1}.$$

By using (5.1.13), we complete the proof of Theorem 5.1.1.

□

*Proof of Theorem 5.1.2.* The proof of Theorem 5.1.2 is very similar. Indeed,

$$\begin{aligned}
\#\{j : \lambda_j < \lambda\} &= \sum_{n_1,n_2,k:4(n_1+n_2+1)\sqrt{2k+1}<\lambda} 1 \\
&= \sum_{n_1,n_2,k:(n_1+n_2+1)<\lambda/4\sqrt{2k+1}} 1 \\
&\leq \sum_{n_2,k=0}^{\infty} \left( \frac{\lambda}{4\sqrt{2k+1}} - n_2 \right)_+ \\
&\leq \frac{1}{2} \sum_{k\leq\frac{1}{2}\frac{\lambda^2}{16}-\frac{1}{2}} \frac{\lambda}{4\sqrt{2k+1}} \left( \frac{\lambda}{4\sqrt{2k+1}} + 1 \right) \\
&\leq \frac{1}{32}\lambda^2 \left( \ln 2 + \frac{1}{2}\mathcal{H}\left( \left[ \frac{\lambda^2}{32} - \frac{1}{2} \right] + \frac{1}{2} \right) \right) + \frac{\lambda^2}{32}.
\end{aligned}$$

□

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