A Survey of Dynamical Billiards

Markus Himmelstrand, mhim@kth.se
Victor Wilén, vwilen@kth.se

Department of Mathematics, Analysis
Royal Institute of Technology (KTH)
Supervisor: Maria Saprykina
Abstract

In this report we will present the basic concepts and results of the theory of dynamical billiards which idealizes the concept of a volumeless ball reflecting against the inside of a billiard table without friction. This motion will continue indefinitely and it is of interest to study its behaviour.

We will show that the study of a billiard system can be reduced to the study of an associated map called the billiard map defined on a cylindrical phase space. Using this formalism the specific systems where the billiard table is given by a circle, right isosceles triangle and ellipse will be studied in some detail along with the existence of periodic points through Birkhoff’s famous theorem and some more novel results such as an instance of Benford’s law regarding the distribution of first digits in real-life data. We will also define the concept of a caustic and investigate their existence and non-existence which will lead us to the concept of circle homeomorphisms and will also provide the opportunity to illustrate the systems with some simulations and yield some more informal and practical insight into the behaviour of these systems.
1 The dynamical billiard

We will here introduce the general concept of a dynamical billiard flow in a bounded region of the plane. It will be seen how the billiard flow can be described both as a trajectory in continuous time as well as by considering the points of reflection at the boundary. The latter option will give a discrete representation of the flow that will be a precursor to the billiard map introduced in section 2.

1.1 Definition

Begin by considering a bounded region Ω in the plane. Denoting its boundary by ∂Ω it will be assumed that there exists a continuously differentiable curve

\[ \gamma : [0, 1] \rightarrow \mathbb{R}^2 \]

such that \( \gamma([0, 1]) = \partial \Omega \). We also assume that this curve is closed (i.e. that \( \gamma(0) = \gamma(1) \)) and that it does not intersect itself anywhere on \([0, 1]\). Introducing coordinate functions \( x \) and \( y \) by \( \gamma(t) = (x(t), y(t)) \) for \( t \in [0, 1] \) the arc length parameter \( s \) can be defined as

\[ s(t) := \int_0^t |\gamma'(t')| \, dt' = \int_0^t \sqrt{x'(t')^2 + y'(t')^2} \, dt'. \]

Defining the total arc length as \( L := s(1) \) the boundary ∂Ω can now be parametrized with the interval \([0, L] \) instead of \([0, 1] \). To avoid introducing confusing notation the boundary curve will still be denoted by \( \gamma \). We can now prove the following useful

**Lemma 1.1.** Denoting by \( \frac{dy}{dx} \) the derivative of \( \gamma \) with respect to arc length it holds that \( |\frac{dy}{dx}| = 1 \) (noting that \( \frac{dx}{ds} \) is tangent to the curve this means that \( \frac{dy}{dx} \) is a unit tangent to \( \gamma \)).

**Proof.**

\[
\left| \frac{d\gamma}{ds} \right| = \sqrt{\left( \frac{dx}{ds} \right)^2 + \left( \frac{dy}{ds} \right)^2} = \sqrt{\left( \frac{dx}{dt} \frac{dt}{ds} \right)^2 + \left( \frac{dy}{dt} \frac{dt}{ds} \right)^2} =
\]

\[
= \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \left| \frac{dt}{ds} \right| = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \frac{1}{\left| \gamma'(t) \right|} = 1
\]

We are now ready to define the concept of a dynamical billiard. Consider a continuous curve \( r(t), t \in [0, \infty) \), in \( \Omega \) with the following properties

1. \( r(0) \in \partial\Omega \)

2. For every \( t > 0 \) such that \( r(t) \notin \partial\Omega \), \( r \) is twice differentiable at \( t \). More specifically, for any such \( t \) we assume that \( r''(t) = 0 \) and \( |r'(t)| = 1 \). Note that this implies the existence of a unit vector \( u \) such that \( r'(t') = u \) for all \( t' \) in some neighborhood of \( t \). Geometrically this means that \( r \) consists of linear segments where each segment has its endpoints on \( \gamma \).

3. If \( t > 0 \) and \( r(t) \in \partial\Omega \) then there is an \( \epsilon > 0 \) such that for any \( t' \in (t - \epsilon, t + \epsilon) \) we have \( r(t') \notin \partial\Omega \). Denote, in accordance with the previous point, the unit tangents of \( r \) in \((t - \epsilon, t)\) and \((t, t + \epsilon)\) by \( u \) and \( v \) respectively and the unit tangent to \( \gamma \) at \( r(t) \) by \( w \). Then \( u \) will make the same angle with \( w \) as \( v \) makes with \( w \). This is referred to as the law of reflection.
Any \( r \) fulfilling the requirements above will be referred to as a billiard trajectory and the set of all such trajectories constitutes the dynamical billiard system of the region \( \Omega \).

The above definition of billiards gives a continuous description but it will be more convenient to work with a discrete model. Therefore, let \( r \) be a billiard trajectory and define \( s_0 \in [0, L] \) such that \( r(0) = \gamma(s_0) \). By the definition of \( r \) it will follow a linear path until it intersects \( \gamma \) at a new point, say \( \gamma(s_1) \) and then continue along a new linear segment. Continuing in this fashion we get a sequence of points \( \{s_i\}_{i=0}^{\infty} \) that completely determines the billiard trajectory (the linear segments of \( r \) can be deduced from the law of reflection). Hence it is shown that the billiard trajectory is completely determined by its points of reflection at the boundary.

\section{The billiard map}

The purpose of this section is to introduce the billiard map as a means of describing the billiard system introduced in section 1. We will also prove some important properties about this map, especially for when the billiard is convex. For this purpose the concept of a generating function for the billiard system will be introduced. The relationship between the generating function and periodical orbits will also be investigated.

\subsection{Definition}

Assume as in section 1 that \( \Omega \) is a bounded region and let \( \gamma \) be its continuously differentiable boundary curve parametrized by its arc length. We will define a map \( T \) on the set \( M := [0, L] \times [0, \pi] \) (\( L \) denotes the total arc length of \( \gamma \) as before) by taking \((s, \theta) \in M \). Then let \( \gamma(s') \) be the point of intersection with a straight line segment from the point \( \gamma(s) \) making the angle \( \theta \) with \( \gamma \) at this point (if more than one such intersection exists we just take the one closest to but not equal to \( \gamma(s) \)). Denote by \( \theta' \) the angle that this line segment makes with \( \gamma \) at \( \gamma(s') \). Then we can define \( T(s, \theta) := (s', \theta') \). The map \( T \) is called the billiard map. Note that the billiard trajectory starting at \( \gamma(s) \) with initial angle \( \theta \) can be obtained from the sequence \( \{T^k(s, \theta)\}_{k=0}^{\infty} \). By what was proved in section 1 any billiard trajectory can be obtained in this manner.

It will be convenient in many situations to visualize the behaviour of the billiard in the so called phase space. The phase space consists of all possible states of the billiard system, i.e. of every point \((s, \theta) \in [0, 1] \times [0, \pi] \) which represents a point on the boundary and an angle. A trajectory of a billiard can then be thought of as a point set in the phase space which is invariant under the billiard map \( T \). Note also that if we consider the interval \([0, 1]\) as the set \( S^1 = \mathbb{R}/\mathbb{Z} \), i.e. the reals modulo the integers, then the phase space can be viewed as a cylinder.

\subsection{The generating function}

Before we can start proving any properties of the billiard map introduced in section 2.1 we must first introduce the important concept of a generating function. This is contained in the following

\textbf{Definition 2.1.} Let \( d \) denote the function \( d(x, y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \) for \( x, y \in \mathbb{R}^2 \). Then the generating function \( H \) for \( \gamma \) is defined for \( s, s' \in [0, L] \) as \( H(s, s') := -d(\gamma(s), \gamma(s')) \).

From now on it will be assumed that the region \( \Omega \) bounded by \( \gamma \) is convex. We can then prove the following

\textbf{Theorem 2.1.} Let \( s, s' \in [0, L] \) and denote by \( u \) the unit vector from \( \gamma(s) \) to \( \gamma(s') \). If \( \theta \) is the angle that \( u \) makes with \( \gamma \) at \( s \) and \( \theta' \) is the corresponding angle at \( s' \) (see figure 1) then
Proof.

\[
\frac{\partial H}{\partial s} = \frac{1}{2(2(x(s) - x(s'))^2 + (y(s) - y(s'))^2}) (2(x(s) - x(s')) \frac{dx}{ds} + 2(y(s) - y(s')) \frac{dy}{ds}) = \frac{1}{\gamma(s) - \gamma(s')} (\gamma(s) - \gamma(s')) \cdot \frac{d\gamma}{ds} = \frac{d\gamma}{ds} \cos \theta = \cos \theta
\]

This proves the first half of the theorem. The second half is proven by performing a similar calculation by differentiating with respect to \(s'\) instead. □

3 Properties of the billiard map

We start by introducing the change of variables \(r := -\cos \theta\) and from here on consider the billiard map \(T\) as a function on \(M := [0, L] \times [-1, 1]\) instead (the notation will be unchanged to avoid confusion). The conclusion of theorem 2.1 for convex billiards can then be restated as

\[
\frac{\partial H}{\partial s} = -r,
\frac{\partial H}{\partial s'} = r'.
\]

The following definition will be crucial to the next theorem.
Definition 3.1. Let $f : M \to M$ be a homeomorphism such that $f(s, r) = (S(s, r), R(s, r))$. Then $f$ is called a twist-map if $S(s, r)$ is strictly increasing in $r$ for any fixed $s$. If $f$ is a diffeomorphism this requirement can be stated as

$$\frac{\partial S}{\partial r} > 0.$$ 

This will be referred to as the twist property.

We can now prove the following

Theorem 3.1. The billiard map $T$ is area-preserving and has the twist-property for convex billiards.

Proof. Introduce the coordinate functions $S$ and $R$ for $(s, r) \in M$ by $T(s, r) := (S(s, r), R(s, r))$. To prove that $T$ is area-preserving define the function $\tilde{H}(s, r) := H(s, S(s, r))$. Differentiating $\tilde{H}$ with respect to $s$ then gives

$$\frac{\partial \tilde{H}}{\partial s} = \frac{\partial H}{\partial s} + \frac{\partial H}{\partial S} \frac{\partial S}{\partial s} = -r + R \frac{\partial S}{\partial s}.$$ 

Differentiating with respect to $r$ gives

$$\frac{\partial \tilde{H}}{\partial r} = \frac{\partial H}{\partial s} \frac{\partial S}{\partial r} = R \frac{\partial S}{\partial r}.$$ 

By now calculating $\frac{\partial^2 \tilde{H}}{\partial s \partial r}$ from the two expressions above we get

$$-1 + \frac{\partial^2 H}{\partial s \partial r} \frac{\partial S}{\partial s} + \frac{\partial H}{\partial S} \frac{\partial^2 S}{\partial s \partial r} = \frac{\partial R}{\partial s} \frac{\partial S}{\partial r} + R \frac{\partial^2 S}{\partial s \partial r} \Rightarrow$$

$$-1 + \frac{\partial R}{\partial s} \frac{\partial S}{\partial r} + R \frac{\partial^2 S}{\partial s \partial r} = \frac{\partial R}{\partial s} \frac{\partial S}{\partial r} + R \frac{\partial^2 S}{\partial s \partial r} \Rightarrow$$

$$|JT| = \frac{\partial S}{\partial r} \frac{\partial R}{\partial s} - \frac{\partial S}{\partial s} \frac{\partial R}{\partial r} = 1.$$ 

This proves that $T$ is area-preserving. To prove that $T$ has the twist property we must prove that $\frac{\partial S}{\partial r} > 0$. According to the chain rule

$$\frac{\partial S}{\partial r} = \frac{\partial \theta}{\partial \theta} \frac{\partial S}{\partial \theta} = \frac{1}{\sin \theta} \frac{\partial S}{\partial \theta}.$$ 

Also, by the definition of the partial derivative, we have

$$\frac{\partial S}{\partial \theta} := \lim_{\Delta \theta \to 0} \frac{S(s, \theta + \Delta \theta) - S(s, \theta)}{\Delta \theta} = \lim_{\Delta \theta \to 0} \Delta s' \frac{1}{\sin \Delta \theta}.$$ 

Now consider the situation in figure 2. Applying the sine theorem to the triangle made by the three marked corners gives

$$\frac{|\gamma(s') - \gamma(s)|}{\sin \alpha} = \frac{|\gamma(s') + \Delta s' - \gamma(s')|}{\sin \Delta \theta}.$$ 

Observe that by applying Taylor’s theorem we can write

$$\sin \Delta \theta = \Delta \theta + O(\Delta \theta^3)$$
and that
\[ |\gamma(s' + \Delta s') - \gamma(s')| = |\Delta s'\gamma'(s') + O(\Delta s'^2)| = \Delta s' + O(\Delta s'^2). \]
This finally gives
\[ \frac{|\gamma(s') - \gamma(s)|}{\sin \alpha} = \frac{\Delta s' + O(\Delta s'^2)}{\Delta \theta + O(\Delta \theta^3)} \to \frac{\partial S}{\partial r} \text{ as } \Delta \theta \to 0. \]
This proves that \( \frac{\partial S}{\partial r} = \frac{|\gamma(s') - \gamma(s)|}{\sin \alpha} > 0 \) and finishes the proof.

It is also worth noting that the billiard map is a homeomorphism on \( M \) (i.e., continuous with a continuous inverse). That the billiard map is onto \( M \) is easily verified by taking a point \( (s, \theta) \in M \) and letting \( (s', \theta') = T(s, \pi - \theta) \). By construction of \( T \) it follows that \( T(s', \theta') = (s, \theta) \) (see left part of figure 3). That \( T \) is one-to-one follows by assuming that \( T(s, \theta) = T(s', \theta') \) and observing that \( \Delta \theta := |\Theta(s', \theta') - \Theta(s, \theta)| \) is nonzero if \( s \neq s' \) (see the geometric construction in the right part of figure 3). Hence \( s = s' \) and by the twist property of the billiard map it now follows that \( \theta = \theta' \), proving that \( T \) is one-to-one. Since the billiard map is continuous and the inverse of any continuous surjective function defined on a compact set is also continuous it follows that the billiard map is a homeomorphism on \( M \).

\[ Figure 3: \text{ Proof that the billiard map is one-to-one and onto.} \]

3.1 Periodic orbits

We will now start to look at another important application of the generating function: periodic orbits for convex billiards. If \( T \) is the billiard map, then a point \( (s, \theta) \) is periodic with period
\( N \) if \( T^N(s, \theta) = (s, \theta) \) and for any \( 0 < n < N \) it holds that \( T^n(s, \theta) \neq (s, \theta) \). It will be shown that for convex billiards the periodic orbits can be described as extreme points of the generating function. But first we prove the following

**Lemma 3.1.** Let \( s_0, s_1, s_2 \in [0, L] \) be distinct points for a convex billiard. Define for \( s \in [0, L] \)

\[
\tilde{H}(s_1) := H(s_0, s_1) + H(s_1, s_2)
\]

where \( H \) denotes the generating function as usual. Then the points \( s_0, s_1 \) and \( s_2 \) are consecutive points of the same billiard trajectory if and only if \( s_1 \) is an extreme point to \( \tilde{H} \).

![Figure 4: Three consecutive points on the boundary of the billiard region.](image)

**Proof.** The situation is illustrated in figure 4. Differentiating \( \tilde{H} \) with respect to \( s \) and using theorem 2.1 gives

\[
\frac{d\tilde{H}}{ds} = \frac{\partial H}{\partial s'}(s_0, s_1) + \frac{\partial H}{\partial s}(s_1, s_2) = -\cos \alpha + \cos \alpha'.
\]

This together with the law of reflection proves the lemma.

Now consider a sequence of \( k \) distinct points \( \{s_i\}_{i=0}^{k-1} \). Define the generating function \( \tilde{H} \) on this sequence as

\[
\tilde{H}(s_0, \ldots, s_{k-1}) := \sum_{i=0}^{k-2} H(s_i, s_{i+1}) + H(s_{k-1}, s_0).
\]

Then we can prove the following

**Theorem 3.2.** The sequence \( \{s_i\}_{i=0}^{k-1} \) is a \( k \)-periodic orbit in a convex billiard if and only if it is an extreme point for \( \tilde{H} \).

**Proof.** As in the proof of lemma 3.1 differentiate \( \tilde{H} \) with respect to each of the variables \( s_i \). This gives for \( i = 1 \)

\[
\frac{\partial \tilde{H}}{\partial s_1} = \frac{\partial H}{\partial s'}(s_0, s_1) + \frac{\partial H}{\partial s}(s_1, s_2)
\]

and similarly for any other \( i \). These \( k \) equations are all zero if and only if the sequence is an extreme point for \( \tilde{H} \) and by lemma 3.1 this happens if and only if all the points lie on the same billiard trajectory hence proving the theorem.
4 Billiard in a circular domain

4.1 Properties of billiard in a circle

One of the simplest examples of a billiard system is the one that exhibits the greatest degree of symmetry, that is when the domain of the system is a circle $C_r = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = r^2\}$. In this very special case many of the general properties of convex billiards such as existence of periodic orbits and caustics which we will discuss later in the article can be deduced with relative ease on account of the corresponding billiard map being quite easy to express and analyse explicitly which often is not the case for more exotic tables. The reason we will begin with this system is threefold: it serves to illustrate the sort of questions one may ask about a billiard system, certain orbits in general billiards are in fact isomorphic to that on a circle, and it also has direct application outside the specific subject of billiards.

The first step to analysing this system is appropriately to reduce the dimension of the corresponding phase space that a given orbit occupies which can be done by observing from Figure 5 or explicitly from the billiard map $T : (s, \theta) \mapsto (s + 2R\theta, \theta)$ for a circle of radius $R$ that the angle of reflection at the boundary, our coordinate $\theta$, remains constant under the billiard flow. This implies that for any orbit containing a point $(s, \theta)$ the remaining orbit will be constrained to the sub circle $S^1 \times \{\theta\}$ on the cylindrical phase space, and as such the billiard map $T$ on $C$ can be replaced with a new map $R_\alpha$ on a circle $S^1$ which moves points a constant distance $\alpha = 2R\theta$ along the boundary at every iteration. We will without loss of generality consider the case where the circle is of circumference 1, $R = 1/2\pi$. Or to summarize the dynamics can be studied on the basis of the function:

$$R_\alpha : S^1 \to S^1, \quad R_\alpha(x) = x + \alpha \quad (\text{mod} \ 1).$$

If $\alpha$ is rational such that $p/q$ is its reduced fraction it is immediately realized that the corresponding orbit will be periodic with period $q$ and be carried $p$ revolutions around the circle before returning to the starting point. There are thus infinitely many periodic orbits of this kind but what will occupy our primary attention is not this case but rather what happens when the rotation number is irrational (i.e the most ‘realistic’ case), seeing as the orbit that such a map generates cannot be periodic and hence is more interesting. The three properties of such orbits that will be derived and discussed below are that they will be dense, uniformly distributed, and in fact ergodic.

**Theorem 4.1.** If the rotation number $\alpha \notin \mathbb{Q}$ then every positive semi-orbit is dense.

**Remark:** Geometrically this simply states that if the billiard ball never returns to the same point it will pass arbitrarily close to every other point on the boundary.

**Proof.** Fix $\varepsilon > 0$, and partition $S^1$ into $k = \lfloor 1/\varepsilon \rfloor + 1$ intervals $\{\Delta_i\}$ of equal length $1/k < \varepsilon$ numbered in sequential order where interval $\Delta_1$ has the starting point $x$ and at the border. Now considering the partial orbit of the first $k + 1$ points $\{R_\alpha^i(x)\}_{i=0}^k$ at least two points $R_\alpha^i(x)$ and $R_\alpha^{i+n}$ must (by the pigeon hole principle) lie in the same interval and therefore be at a distance
where the final equality follows from the fact that $\bigcup \Delta_i = S^1$ and so we see that $F(\Delta) \leq 1/(k-1)$.

Now take any arc $\Delta$ and cover it by arcs of length $1/k$, in total $n = \lceil \ell(\Delta)/k \rceil + 1$ from which we have $n/k - \ell(\Delta) \leq 1/k$. Now

$$F(\Delta) \leq \sum_{i=1}^{n} F(\Delta_i) \leq \frac{n}{k-1} \leq \ell(\Delta) \frac{k}{k-1} + \frac{1}{k-1}.$$
So letting \( k \to \infty \) yields \( F(\Delta) \leq \ell(\Delta) \). Now it remains only to establish \( F(\Delta) \geq \ell(\Delta) \) which actually follows from the above argument since from \( F_\Delta(x,n) = n - F(\Delta^c) \) we see \( F(\Delta) = 1 - F(\Delta^c) \) so \( F(\Delta) = 1 - F(\Delta^c) \geq \ell(\Delta) \) which completes the proof.

This we refer to as the uniformity property as the orbit is uniformly distributed along the boundary. Though interesting in its own right this can also straight forwardly be extended to show that the system is ergodic, that is the mean of any Riemann integrable function \( g : S^1 \to S^1 \) taken over any non periodic orbit is the same as its average over all of \( S^1 \) whenever the rotation number is irrational.

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} g(R^i_\alpha(x)) = \int_{S^1} g(x) \, dx \quad (\alpha \notin \mathbb{Q}, \forall x \in S^1)
\]

which follows from the fact that every such function can be approximated arbitrarily well by step functions and Theorem 4.2 can be cast in step function form

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi_\Delta(R^i_\alpha(x)) = \int_{S^1} \chi_\Delta(x) \, dx \quad (\alpha \notin \mathbb{Q}, \forall x \in S^1)
\]

where \( \chi_\Delta \) is the characteristic function that is 1 on the set \( \Delta \) and 0 elsewhere, whereupon the details of proof are relatively straightforward [1, p.103,104].

Remark An alternate approach to this result which is less elementary but which can be used to extend this equidistribution theorem into higher dimensional circles (tori) \( T^n = S^1 \times \cdots \times S^1 \) is the Kroenecker-Weyl method [1, p.103,104] but it is outside the scope of this text.

### 4.2 Billiard in an isosceles triangle

In this article we review primarily results concerning convex billiards with smooth boundaries as reflections becomes undefined at points where the tangent is discontinuous and though this issue can be corrected by defining that the particle terminate or reflect according to some auxiliary law the billiard map would not remain continuous under such a correction rendering many of the standard arguments inapplicable.

Nevertheless polygonal billiards with certain symmetries are still quite tractable to study directly and as an example we here review the dynamics of the billiard inside a right isosceles triangle which illustrates the useful methods of mirroring and folding. Note that the operation of reflection at a boundary is geometrically the same as continuing the trajectory over the boundary and folding it back over it. Consequently the motion can be studied by instead mirroring the billiard table itself such that the trajectory remains a straight line in a plane covered by triangles produced by folding the triangles in all possible ways. In general such a folding procedure would cause folded images to overlap but in a few limited cases such as this one and as can be seen in Figure 6 the pattern of unfolding repeats in terms of square blocks containing 8 distinct orientations of the original triangle meaning that the billiard

![Figure 6: Folding of triangle](image-url)
in a triangle is isomorphic to straight line flow on a torus (that is a surface periodic along two axis).

In terms of continuous time flow the orbit on the torus can be written as
\[
\phi^t : S^1 \times S^1 \to S^1 \times S^1
\]
\[
\phi^t(x, y) = (x + \alpha t, y + \beta t) \pmod{2}
\]
where \((\alpha, \beta)\) is the initial velocity if the particle in the billiard.

**Lemma 4.1.** The return maps to a given boundary segment on the isosceles triangle is isomorphic to a rotation on a circle and the corresponding rotation is irrational if and only if the fraction of the components of the initial velocity is irrational.

**Proof.**

**Case 1.** Considering first the vertical line segments on the triangle we see from the geometry that the linear flow on the torus will lie on the two lines corresponding to this state for all times \(t = (\beta/\alpha)k\) where \(k\) is any integer. Note that the orientation of the two segments \(x = 0\) and \(x = 1\) are different but moving from one of these lines to the other is done at time increments of \(t = 1/\alpha\) where \(\phi^{1/\alpha}\) acts as translation along the y-axis of the circle \(\{x = 1\} \times S^1\) with a distance of \(\beta/\alpha\). Or to summarize \(\phi^{1/\alpha} \simeq R_{\beta/\alpha}\).

**Case 2.** For the horizontal boundary segment the analysis is exactly the same where the translation is instead \(\alpha/\beta\).

**Case 3** Similarly as in the above argument the diagonal segment on the billiard corresponds to the diagonal segments \(x - y = 0\) and \(y + x = 1\) correspond to where we want to find the smallest \(t\) that brings the system back to the same line. We only consider return to \(x - y = 0\) and realize that this corresponds to solving \((y + \beta t) - (x + \alpha t) = 1\) which assuming we start on \(y - x = 0\) results in \(t = 1/(\alpha - \beta)\). Since the diagonal has irrational length \(2\sqrt{2}\) we need to check that the resulting translation (rotation) is irrational with respect to this length. Let \(\gamma = \alpha/\beta\) then both translations are a distance \(\gamma' = \gamma \pmod{2}\) along each axis and the total distance traversed along the diagonal is \(\sqrt{2}\gamma'\) which is irrational with respect to \(2\sqrt{2}\) if and only if \(\gamma\) is irrational. (Note \(\gamma \in \mathbb{Q} \Rightarrow \gamma' \in \mathbb{Q}\) since modulo leaves the decimal sequence invariant and the decimal of an irrational is irrational).

This makes it possible to make an immediate categorization

**Theorem 4.3.** Given an initial velocity \((\alpha, \beta)\) in an isosceles billiard the trajectory is periodic if \(\gamma = \alpha/\beta\) is rational and aperiodic, dense, and uniformly distributed on each boundary segment if \(\gamma\) is irrational.

**Proof.** Follows immediately from Lemma 4.1 and applying the result to Theorems 4.1 and 4.2 respectively.

This argument has of course assumed that the trajectory does not pass through a corner where reflection might be considered undefined. For an introduction to the subject of polygonal billiards see chapter 7 in [2].

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1where \(\phi^t(x) = \phi(x, t)\) is a common notation in dynamics to distinguish between the dynamical variable \(t\) and the initial condition \(x\).
4.3 Application of the uniform distribution property of the circle billiard

We will here discuss an interesting property regarding the distribution of the first integer of powers of two. The result can be stated as

**Theorem 4.4.** Let $n$ be a nonnegative integer and let $k \in \{1,2,\ldots,9\}$. The probability that $k$ is the first digit of $2^n$ can then be written as

$$\log_{10}(1 + \frac{1}{k}).$$

**Proof.** First we observe that $k$ is the first digit of $2^n$ if and only if there exists a nonnegative integer $q$ such that $k10^q \leq 2^n < (k+1)10^q$. Taking the logarithm of this expression gives

$$\log_{10} k + q \leq n \log_{10} 2 < \log_{10} (k+1).$$

Let $\{x\}$ denote the fractional part of any real number $x$. Then the above inequality implies that $\{n \log_{10} 2\} \in [\log_{10} k, \log_{10}(k+1)) := I$. Now note that $\log_{10} 2$ is an irrational number. Then the equidistribution of the circle billiard implies that the sequence $\{n \log_{10} k\}$ is equidistributed and has a uniform probability distribution. This implies that the probability for a number in the series to be in $I$ is proportional to the length of the interval, i.e. proportional to $\log_{10}(k+1) - \log_{10} k = \log_{10}(1 + \frac{1}{k})$ which proves the theorem. \qed

4.3.1 Benfords law

The distribution that we derived in section 4.3 is called the Benford distribution and occurs in an empirical law called Benfords law regarding the frequency of first digits in real-life data. The name comes from F. Benford who provided many examples of this phenomenon. The law was not however discovered by Benford but by S. Newcomb in 1881. It states that the first digits of many real-life data are Benford distributed. An example of this phenomenon can be found in figure 7 where the relative frequency of first digits in the populations of the world divided by countries has been plotted against the Benford distribution (the data is taken from [3] for the year 2011).

5 Existence of Birkhoff periodic orbits

We will here prove the existence of at least two periodic orbits of an arbitrary period in convex billiards. The method of proof relies on the generating function introduced in section 2.2 and the fact that (in convex billiards) periodic orbits are extreme points of this function. Indeed the existence of the first periodic orbit will be found from a straightforward application of this and the fact that continuous real-valued functions assume a minimum on compact sets. The other orbit will be harder to find and relies on the so called mini-max principle which loosely stated says that between two minima there must be a saddle point. This saddle point will be proven to be the second orbit.

5.1 Birkhoff periodic orbits

We will first introduce the notion of a Birkhoff periodic orbit. Let $\gamma$ be a convex billiard curve parametrized by the unit interval $[0,1]$. Fix a positive integer $n$ and consider an ordered sequence of distinct vertices $n$ in $[0,1]$ denoted by $(x_1,\ldots,x_n)$. We will view the numbers $x_i$ as
Figure 7: Relative frequency of first digits in the world’s population by countries plotted against the function $\log_{10}(1 + \frac{1}{x})$.

Figure 8: The green orbit corresponds to $(x_1, x_2, x_3, x_4, x_5)$ while the red orbit corresponds to $(x_1, x_3, x_5, x_2, x_4)$. Real numbers modulo the integers. These correspond to polygonal paths in the billiard with $n$ distinct vertices. Note that the ordering of the vertices is relevant when determining the orbit corresponding to it (see figure 8). We call this an orbit of period $n$.

We will now distinguish between different orbits consisting of the same set of vertices by introducing the notion of rotation number for these configurations. Fixing the orientation for $\gamma$ to be counterclockwise we can find the smallest numbers $t_i \in (0, 1)$ such that $x_i = x_{i-1} + t_i$ for $i \neq 1$ and $x_1 = x_n + t_n$ where addition is taken modulo 1 (observe that the numbers $t_i$ are uniquely defined). Since the configuration corresponds to a closed polygonal path we have that $t_1 + \ldots + t_n \in \mathbb{Z}$. This number will be denoted by $\rho$ and is called the rotation number of the configuration. Calculating $\rho$ for the configurations in figure 8 we see that the dashed path has
rotational number 2 while the dotted path has rotational number 1.

We can now make the following

**Definition 5.1.** Let $p$ and $q$ be positive integers and let $(x_1, \ldots, x_q)$ be an ordered set of distinct vertices in $[0, 1]$. If the orbit corresponding to these vertices has rotational number $p$ and is a $q$-periodic orbit of the billiard $\gamma$ we call it a Birkhoff periodic orbit of class $(p, q)$.

Note that in the case that $p$ and $q$ are not relatively prime we can find an $n$-periodic orbit that is a multiple of an orbit with period less than $n$. Hence we will always assume that $p$ and $q$ are relatively prime.

We can now state the main result of this section concerning the existence of Birkhoff periodic orbits.

**Theorem 5.1.** Let $\gamma$ be a strictly convex billiard that is $C^2$ and fix $p$ and $q$ as positive integers that are relatively prime. Then there exists at least two distinct Birkhoff periodic orbits of class $(p, q)$.

We will refer to this as Birkhoff’s theorem.

### 5.2 Proof of Birkhoff’s theorem

We begin by denoting the space of all $q$-periodic polygonal paths with rotation number $p$ inscribed in $\gamma$ by $M_{p,q}$, i.e. $(s_0, \ldots, s_{q-1}) \in M_{p,q}$ if and only if the path given by connecting $s_i$ with $s_{i+1}$ for $0 \leq i \leq q-2$ and $s_{q-1}$ with $s_0$ is rotation number $p$. We will consider its closure $M := M_{p,q}$. Note that $M$ also contains degenerate paths for which two vertices are equal and hence have a period less than $q$. An important part of our proof will be to assure that the orbits we find are indeed not degenerate.

Now define the generating function for these configurations as (for notation see section 2.2)

$$\tilde{H}(s_0, \ldots, s_{q-1}) := \sum_{i=0}^{q-2} H(s_i, s_{i+1}) + H(s_{q-1}, s_0).$$

![Figure 9: Creating a longer $q$-periodic polygon from a $(q - 1)$-periodic polygon.](image)

We recall that $q$-periodic orbits correspond to extreme points of this functional and we will find our first orbit by finding a minimum of $\tilde{H}$. To do this we observe that $M$ is closed and
bonded in \( R^n \) and hence \( M \) is a compact set by the Heine-Borel theorem. This implies, since \( \tilde{H} \) is a continuous functional and continuous functions assume extreme values on compact sets, that \( \tilde{H} \) assumes a minimum on \( M \). Denote the vertices of this orbit by \((x^*_1, \ldots, x^*_q)\). If we can prove that this orbit is not degenerate we have found our first orbit of class \((p,q)\). Observe that the generating function is, by definition, the negative of the length of the orbit. Now assume that the orbit is degenerate and let \( x^*_1 = x^*_2 \) and that the other vertices are distinct. Then we can choose a new vertex \( x' \) between \( x_2 \) and \( x^*_1 \) and create a new polygonal path \((x', x^*_2, \ldots, x^*_q)\) \( \in M \) (see figure 9). Then, by the triangle inequality, this new polygonal path is strictly longer than the original orbit and hence this orbit is not a minimum for the generating function. This contradicts the orbit was chosen and hence proves that it is non-degenerate. This is our first orbit and we will continue to refer to it as \( x^* = (x^*_1, \ldots, x^*_q) \).

To find the second orbit we will look for another extreme point of the generating function in \( M \). This point will be a saddle point between two minimum. To get a second minimum of the generating function we make a cyclic permutation of the indices of the first orbit, say \( x'' = (x'_2, \ldots, x'_q, x'_1) \). This corresponds geometrically to the same orbit but is a distinct point in \( M \).

Now define the set \( N := \{(x_1, \ldots, x_q) \mid x_i \leq x_{i+1}, i \neq q; x_q \leq x_1 (mod 1) \} \) (note that \( N \subset M \)). Consider the space \( C \) of all continuous functions

\[
f : [0,1] \rightarrow N
\]

such that \( f(0) = x^* \) and \( f(1) = x'' \). Since \( \tilde{H} \circ f \) is continuous for any \( f \in C \) it assumes a maximum on the closed interval \([0,1]\). We can assume that this maximum is greater than \( \tilde{H}(x^*) = \tilde{H}(x'') \) for any \( f \in C \) since otherwise \( \tilde{H} \) is constant on \( f \) and any point along \( f \) would correspond to a new orbit of class \((p,q)\). Having assumed this define

\[
L_{\text{min max}} = \inf_{f \in C} \max_{t \in [0,1]} \tilde{H}(f(t)).
\]

We will now show that \( \tilde{H} \) assumes this value at some point in \( N \). To prove this choose a sequence \( \{f_n\} \) in \( C \) such that

\[
L_{\text{min max}} \leq \max_{t \in [0,1]} \tilde{H}(f_n(t)) < L_{\text{min max}} + \frac{1}{n}
\]

(this can be done by definition of the infimum) and choose points \( t_n \in [0,1] \) such that \( \max_{t \in [0,1]} \tilde{H}(f_n(t)) = \tilde{H}(f_n(t_n)) \). Now consider the sequence \( \{f_n(t_n)\} \) in \( N \) and observe that it is bounded since \( N \) is bounded. Hence there exists a convergent subsequence \( \{f_{n'}(t_{n'})\} \) that converges to some \( u \in N \) since \( N \) is closed. Then the following inequality holds

\[
|\tilde{H}(u) - L_{\text{min max}}| \leq |\tilde{H}(u) - \tilde{H}(f_{n'}(t_{n'}))| + |\tilde{H}(f_{n'}(t_{n'})) - L_{\text{min max}}|.
\]

Take some \( \epsilon > 0 \) and choose a positive integer \( K \) such that

\[
|\tilde{H}(u) - \tilde{H}(f_K(t_K))| < \epsilon.
\]

This can be done since \( \tilde{H} \) is continuous and the sequence converges to \( u \). Also observe that the second term can be approximated by

\[
|\tilde{H}(f_{n'}(t_{n'})) - L_{\text{min max}}| < \frac{1}{n'}
\]

which follows by construction of the sequence \( \{f_n(t_n)\} \). Hence if \( K' \) is chosen such that \( \frac{1}{K'} < \epsilon \) then for \( K'' = \max(K, K') \) it follows that

\[
|\tilde{H}(u) - L_{\text{min max}}| \leq |\tilde{H}(u) - \tilde{H}(f_{K''}(t_{K''}))| + |\tilde{H}(f_{K''}(t_{K''})) - L_{\text{min max}}| < 2\epsilon.
\]
Since $\epsilon > 0$ was arbitrary it follows that $\tilde{H}(u) = L_{\text{minmax}}$. Hence the set $\tilde{H}^{-1}(L_{\text{minmax}})$ is nonempty and it will be shown that this set must contain another $(p, q)$ orbit.

To prove this we will first show that the gradient of $\tilde{H}$ is nonzero on the boundary of $N$ except at the points $x^*$ and $x^{**}$; indeed, we will show that it points in the outward direction (see figure 10). This follows from the following simple argument. Remember that the first orbit is $x^* = (x_1^*, \ldots, x_q^*)$ and let $(s_1, \ldots, s_q)$ be an element of $N$. Now suppose that $s_k = x_k^*$ for some $k$ (this corresponds to being on the boundary of $N$). We will calculate the directional derivative $\frac{\partial \tilde{H}}{\partial s_k}$. Using theorem 2.1 this can be calculated as (here $\theta_{xy}$ denotes the angle that the line segment from $x$ to $y$ makes with the billiard table at $x$, see figure 11)

$$\frac{\partial \tilde{H}}{\partial s_k} = \frac{\partial H(s_{k-1}, s_k)}{\partial s_k} + \frac{\partial H(s_k, s_{k+1})}{\partial s_k} = \cos \theta_{s_k s_{k+1}} - \cos \theta_{s_{k-1} s_k}.$$ 

Now observe that by the strict convexity of the billiard table it follows that

$$\cos \theta_{x_{k+1}^* x_k^*} > \cos \theta_{x_{k-1}^* x_k^*} = \cos \theta_{s_k s_{k-1}}.$$ 

From the twist-property of the billiard map $T = (S, R)$ it follows that

$$S(s_k, -\cos \theta_{s_k s_{k+1}}) = s_{k+1} < x_{k+2}^* = S(x_{k+1}^*, -\cos \theta_{x_{k+1}^* x_{k+2}^*}),$$

which implies the conclusion. This analysis can be performed on the rest of the boundaries in a similar manner and proves that the gradient of $\tilde{H}$ is outward pointing.

Now we are ready to show that $\tilde{H}^{-1}(L_{\text{minmax}})$ contains an extreme point of $\tilde{H}$. Assuming that this is false it follows that there exists an $\epsilon > 0$ such that $\tilde{H}^{-1}(L_{\text{minmax}} - \epsilon, L_{\text{minmax}} + \epsilon)$ contains no extreme points of $\tilde{H}$. Observe that if this did not hold then there would exist a sequence $\{u_n\}$ of extreme points of $\tilde{H}$ such that

$$\tilde{H}(u_n) \in (L_{\text{minmax}} - \epsilon, L_{\text{minmax}} + \epsilon).$$
Since the sequence is bounded there would exist a convergent subsequence $\{u_n^i\}$ that converges to some $u$. The continuity of $\tilde{H}$ and its derivative would then imply that $\tilde{H}(u) = L_{\min\max}$ and $\nabla \tilde{H}(u) = 0$ which would contradict the assumption that $\tilde{H}^{-1}(L_{\min\max})$ contains no extreme points.

Choosing $\epsilon > 0$ such that $\tilde{H}^{-1}(L_{\min\max} - \epsilon, L_{\min\max} + \epsilon)$ contains no extreme points it follows that there exist $\delta > 0$ such that $\|\nabla \tilde{H}\| > \delta$ on $\tilde{H}^{-1}(L_{\min\max} - \epsilon, L_{\min\max} + \epsilon)$. Now take $u \in \tilde{H}^{-1}(-\infty, L_{\min\max} + \epsilon)$ and fix some $t > 0$. Then from the definition of gradient of $\tilde{H}$ it follows that

$$
\tilde{H} \left( u - t \frac{\nabla \tilde{H}}{\|\nabla \tilde{H}\|} \right) = \tilde{H}(u) - t \nabla \tilde{H} \cdot \frac{\nabla \tilde{H}}{\|\nabla \tilde{H}\|} + O(t^2) = 
$$

$$= \tilde{H}(u) - t \|\nabla \tilde{H}\| + O(t^2).
$$

Now assume that $u \in \tilde{H}^{-1}(L_{\min\max} - \epsilon, L_{\min\max} + \epsilon)$. It follows that

$$\tilde{H} \left( u - t \frac{\nabla \tilde{H}}{\|\nabla \tilde{H}\|} \right) = \tilde{H}(u) - t \|\nabla \tilde{H}\| + O(t^2) < \tilde{H}(u) - t \delta + O(t^2).
$$

By the definition of the big-O notation it follows that

$$
\lim_{t \to 0} \frac{|O(t^2)|}{t} = 0
$$

hence $t$ can be chosen such that $O(t^2) < \frac{t \delta}{2}$ which implies that

$$\tilde{H} \left( u - t \frac{\nabla \tilde{H}}{\|\nabla \tilde{H}\|} \right) \leq \tilde{H}(u) - \frac{t \delta}{2} < L_{\min\max} + \epsilon - \frac{t \delta}{2}.
$$

This proves that if we take $\epsilon = \frac{t \delta}{2}$ then $\tilde{H}(u) < L_{\min\max}$ for $u \in \tilde{H}^{-1}(L_{\min\max} - \epsilon, L_{\min\max} + \epsilon)$. Now taking $u \in \tilde{H}^{-1}(-\infty, L_{\min\max} - \epsilon)$ it follows that

$$\tilde{H} \left( u - t \frac{\nabla \tilde{H}}{\|\nabla \tilde{H}\|} \right) \leq \tilde{H}(u) + O(t^2) \leq L_{\min\max} - \epsilon + O(t^2) < L_{\min\max} - \epsilon + \frac{t \delta}{2} = L_{\min\max}.
$$

Now take $f \in C$ such that $\max_{t \in [0,1]} \tilde{H}(f(t)) < L_{\min\max} + \epsilon$ and define $\tilde{f}(s) = f(s) - t \frac{\nabla \tilde{H}(f(s))}{\|\nabla \tilde{H}(f(s))\|}$. Then $\tilde{f} \in C$ and by the previous calculation it follows that $\max_{t \in [0,1]} \tilde{H}(\tilde{f}(t)) < L_{\min\max}$ contrary to the definition of $L_{\min\max}$. Hence the set $\tilde{H}^{-1}(L_{\min\max})$ contains an extreme point $u$ of $\tilde{H}$. Since the gradient of $\tilde{H}$ is nonzero this extreme point cannot lie on the boundary of $N$ hence proving that $u$ is a proper $(p,q)$ orbit which finishes the proof.
6 Caustics

6.1 Introduction

One property of a class of billiard which can be used to analyse the dynamics of a billiard map is if there exists caustics. A caustic, geometrically, is a curve such that if the trajectory of the billiard tangents the curve at a single point then all subsequent trajectories are also be tangents the same curve. Note that caustic need not be confined inside the billiard table and the point of tangency too may lie outside it were the trajectory extended to a complete line stretching to infinity. The existence of caustics is not a general property, it is easy to show that \( x^4 + y^4 = 1 \) has no convex caustics but a class of tables which do have an infinite number of caustics is the billiard inside an ellipse \( x^2/a^2 + y^2/b^2 = 1 \) and will therefore demand some special attention.

As is hinted by Figure 12 every orbit in an ellipse either has a confocal ellipse or a hyperbola as its caustic and the figure also displays how the orbits in phase space correspondingly occupy invariant sets of curves instead of spreading out more uniformly. It is common that caustics also directly refers to these curves themselves. Before continuing with a more general discussion we show formally the existence of caustics in the case of an ellipse, which is special in the sense that every single orbit has a corresponding caustics and fall into two primary characteristic cases: ellipse and hyperbola. There does exist a third degenerate case where the trajectory passes though the foci but it is not strictly speaking tangency in that case. For this discussion we will require a simple lemma.

**Lemma 6.1.** Any trajectory that passes though one foci is reflected though the second foci. That is the lines corresponding to the string construction make equal angles with the tangent of the

![Figure 12](image-url)
ellipses

Proof. The proof is straightforward. Let $f_1$ and $f_2$ be the foci of the ellipse. Then given $f(x) = |x - f_1| + |x - f_2|$ we have that $\nabla f$ is normal to the ellipse along it and evaluating the gradient we see that it is the sum of the unit vectors from the foci to $x$ meaning their sum is normal the ellipse which corresponds to them making equal angles with the tangent.

![Figure 13: Geometric construction of tangential a confocal ellipse (left) and hyperbola (right)](image)

We will use this to show the three cases where an ellipse

**Theorem 6.1.** If a trajectory passes outside the two foci of the ellipse in one iteration it will remain tangent to a confocal ellipse.

**Proof.** Let $p_1p_0p_2$ be a billiard trajectory not passing between the two foci $f_1$ and $f_2$ of the ellipse (see the left figure in Figure 13. We are going to construct two ellipses such that the line segments $p_1p_0$ and $p_0p_2$ are corresponding tangents and finally that the two ellipses are equal.

For the first ellipse mirror $f_1$ in $p_1p_0$ to form a new point $f'_1$, and then denote by $a$ the point at the intersection of the lines $p_1p_0$ and $f'_1f_2$. Now $a$ lies on the ellipse formed by a string of length $\ell(f'_1f_2)$ attached to the two original foci and due to the fact that $f'_1$ was constructed by mirroring $f$ in $a$ will also be a point of tangency.

Now if the exact same construction of a point $f'_2$ is performed by mirroring $f_2$ in $p_2p_0$ and taking $b$ to be the similar point of intersection it can too is seen that $b$ lies at a point of tangency of an ellipse formed by a string of length $\ell(f'_2f_1)$.

$\ell(f'_1f_2) = \ell(f'_2f_1)$ by the following argument: since $f_1p_0f_2$ also make equal angles with the boundary of the ellipse (Lemma 6.1) the angles $\angle p_1p_0f_1$ and $\angle p_2p_0f_2$ are equal. From this we see that the triangles $\triangle f'_1f_2p_0$ and $\triangle f'_2f_1p_0$ have have at least two sides of equal length and one angle in common and hence they are equivalent and the ellipses are the same. By induction the orbit thus remains tangent to the same confocal ellipse.

**Theorem 6.2.** If the trajectory passes between the two foci then the trajectory is and will remain tangent to a confocal hyperbola.

**Proof.** The construction is nearly exactly the same as in the case of the ellipse, see the right figure in Figure 13. The points of intersection, $a$ of $p_1p_0$ and $f'_1f_2$ and $b$ of $p_2p_0$ and $f'_2f_1$ now lie on a hyperbola defined such that the difference between the distances between a point on the hyperbola to the foci is $\ell(f'_1f_2) - \ell(f'_2f_1)$.
This leaves a third possibility not illustrated in 12 and that is if the trajectory passes though one of the foci. The basic behaviour is already present in Lemma 6.1, the trajectory is bound to always pass though the foci. The first such orbit is the two periodic orbit along the major axis of the ellipse passing though both foci but as we show below the other orbits that do not lie on the axis still converge to it.

**Theorem 6.3.** If the trajectory passes through one of the foci while not lying on the major axis it will converge to the major axis.

*Proof.* The proof utilizes the geometric construction in Figure 14 where we can associate with each point $p_i$ and triangle with corners at the foci and our result will follow from the fact that the triangle becomes flattened at every iteration. Consider the related sequences $\{\alpha_i\}_{i=0}^{\infty}$, $\{\beta_i\}_{i=0}^{\infty}$, and $\{\gamma_i\}_{i=0}^{\infty}$. By basic geometry we have $\alpha_{i+1} + \beta_{i+1} < \pi$ and since $\beta_i < \pi - \alpha_i = \beta_{i+1}$ we see that $\{\beta_i\}_{i=0}^{\infty}$ is a monotonically increasing sequence and since it is geometrically bounded by $\pi$ it must converge, and it remains to show that the limit is $\pi$.

Now since convergent sequences are Cauchy there exist for every $\varepsilon > 0$ some $N$ such that such that $\beta_{N+1} - \beta_N < \varepsilon$ and reversing the argument from before $\beta_{N+1} = \beta_N - \beta_{N} - \alpha_{N} = \gamma_{N} < \varepsilon$ and we have that $\gamma_i \to 0$ which means $\beta_i \to \pi$ and $\alpha_i \to 0$.

In the phase space these results correspond to every orbit being constrained on invariant sets of curves. In the case of the ellipse it is possible to recover these curves explicitly as level curves a subset of which can be seen in Figure 15. This image is commonly used to illustrate how perturbing a circle into an ellipse preserves the presence of invariant curves but yields a significantly more complicated appearance. One method of explicitly deriving the function in question is provided in the appendix, A.1.

---

**Figure 14:** Convergence to greater axis

**Figure 15:**

We now return to one of the implications of Figure 12 and Figure 15 where one may note that the caustic corresponding to an ellipse yields a closed non self intersecting curve in the phase
space. It is relatively straightforward to deduce that this is always the case when the caustic is convex which we summarize as a lemma

**Lemma 6.2.** Every convex caustic corresponds to an invariant circle in the phase space.

*Proof (sketch).* Assuming there exists a convex caustic then for every points $s$ along the table boundary there are only two reflection angles $\theta_1$ and $\theta_2$ that result in tangency with the caustic, and choosing a specific orientation every $s$ has a unique $\theta(s)$. Upon varying $s$, $\theta(s)$ varies continuously if the caustic is smooth forming a closed curve $(s, \theta(s))$.

If we start with an invariant circle $(s, \theta)$ we may recreate the geometric caustic from the envelope of a family of rays starting from the boundary in the direction of the reflection angles $\theta$.

A famous result by V.F Lazutkin, later improved by R. Douady proves that a large range of tables have convex caustics [4] [5]

**Theorem 6.4.** If the border of the table is $C^6$ and strictly convex then there exists a collection of smooth convex caustics close to the border of the table of the table whose union has positive area.

These specific caustics are also sometimes referred to as whisper galleries in references to the acoustic phenomenon where two people standing close to a wall in an oval room may communicate without shouting as the soundwave becomes concentrated close to the wall much like the billiard ball according to this theorem.

Furthermore, this fact that convex caustics are circles and invariant under the billiard map has important local implication as the fact that the billiard map is homeomorphic on all of the phase plane means that orbits corresponding to caustics can be studied using the machinery of circle homeomorphisms.

The basic concept is similar to the methods employed to the circle billiard in the previous section as one isolates the fields of study to a sub circle (Figure 5 which indeed corresponded to a geometric caustic.

The existence of invariant closed curves in general also has important global implication for the whole billiard system as they partition the phase space into invariant disjoint subsets which prevents any orbit, the implications of which are treated in more closely in section 6.6 while we begin to review the local implications now.

### 6.2 Homeomorphism of the circle

Much of the theory we shall review briefly in this section is due to the work of mathematician Henri Poincaré who studied circle homeomorphisms in the later half of the 19th century through a powerful characterizing number, which while it might not be trivial to compute, can be used to classify different behaviours and show what dynamics may coexist and what may not. [1, p.125-133].

In order to formulate the results of interest with some precision we review some classical definitions useful for describing functions on circles in terms of functions on the line. Similarly to how a periodic function $F: \mathbb{R} \to \mathbb{R}$ can be projected onto the circle forming an function $f: S^1 \to S^1$, by for example $f(x) = F(\lfloor x \rfloor)$, it is possible to *lift* certain functions defined on a circle to functions defined on $\mathbb{R}$.

A homeomorphism is a continuous function that has a continuous inverse.
**Proposition 6.1.** If $f : S^1 \to S^1$ is continuous then there exists a function $F : \mathbb{R} \to \mathbb{R}$, unique up to the addition of an integer, called a lift such that

$$f \circ \pi = \pi \circ F$$

where $\pi : \mathbb{R} \to S^1$ projects the real line onto the circle. $\deg(f) = F(x + 1) - F(x)$ is called the degree of $f$ and is independent of the lift and $x$. If $f$ is a homeomorphism then $|\deg(f)| = 1$.

**Definition 6.1.** If $\deg(f) = 1$ then we say that the map is orientation preserving, and if $\deg(f) = -1$ we say that it is orientation reversing.

The notion of degree corresponds to how the map stretches and orients the circle.

**Theorem 6.5.** Let $f : S^1 \to S^1$ be a circle homeomorphism and $F : \mathbb{R} \to \mathbb{R}$ a lift. The number

$$\rho(F) = \lim_{n \to \infty} \frac{F^n(x) - x}{n}$$

exists and is independent of $x$. The number depends on the lift only to the extent $\rho(F) - \rho(\tilde{F}) = F - \tilde{F} \in \mathbb{Z}$.

Since this number is well defined up to an integer one usually takes the fractional part $\rho(f) = \{\rho(F)\}$, a number which is called the rotation number a name which hints to the fact that in the case of a circle rotation the rotation number reduces to the rotation number $\alpha$. As one notes immediately the limit could equally well be taken without $-x$ but is taken for historical reasons of how proofs are best constructed, and to better illustrate $\rho$ as the mean distance a map $f$ translates any given point.

Three things the rotation number directly implies are

1. If $\rho(f) = 0$ then $f$ has a fixed point
2. If $\rho(f)$ is rational if and only if $f$ has periodic points.
3. If $\rho(f) = p/q \in \mathbb{Q}$ then every periodic orbit has the same period $q$ and $F^q(x) = x + p$, that is it moves $x$ $p$ revolutions around the circle.

Note that these three properties already hold for rotations on a circle, the third of which implies for billiards that all periodic orbits that tangent the same caustic must have the same period and similar geometric structure. Unlike for circle rotations rational rotation number does not generally exclude the possibility of aperiodic orbits but it does allow the following classification theorem

**Theorem 6.6.** If $f : S^1 \to S^1$ is a circle homeomorphism such that $\rho(f) = p/q \in \mathbb{Q}$ then there are two possible orbits

1. If $f$ has exactly one periodic point then every other point on the circle is heteroclinic to two points under $f^q$.
2. If $f$ has more than one periodic orbit then each non periodic orbit is heteroclinic to two points on two different periodic orbits.

It is perhaps the possibility of the type 2 behaviour that holds the highest interest as an orbit converging a periodic point might be considered unexpected. This convergent behaviour has already been encountered in the ellipse with the orbit converging to the major axis and the two 2-periodic points along it but this did not strictly a consequence of the above theorem as
the corresponding curve is not a circle or caustic in the strictest sense. However one may still wish to investigate if there are other orbits along caustics with periodic points that show this convergent behaviour. For the elliptical caustics however the answer is in the negative as if one trajectory is tangent to a given ellipse then all the orbits tangent to it are periodic with the same period, a consequence of the geometrical Poncelet closure theorem reviewed in the next section.

6.3 Poncelet Porism

The Poncelet Porism is a purely geometrical result with greater generality than the fact that all orbits in an elliptic billiard table that stay tangent to an elliptic caustic have the same period. However it has this fact as an immediate consequence when considered together with theorem 6.1 [6, p.397-399]

**Theorem 6.7 (Poncelet porism).** Let \( \Gamma \) and \( \gamma \) be nested ellipses. If \( \gamma \) can be circumscribed by an \( n \)-gon with corners in \( \Gamma \), then every point on \( \Gamma \) is a vertex of such an \( n \)-gon.

**Proof.** Let \( \Gamma \) be the outer ellipse, and let \( \gamma \) the inner ellipse. First we define two functions \( R_{\gamma, \Gamma}(x) \) and \( L_{\gamma, \Gamma}(x) \) which are the distances to tangency, ergo the length of the line segments from \( x \in \Gamma \) to \( \gamma \) along the two left and right lines that tangent \( \gamma \) seen in Figure 16 (a).

Now we let \( \Gamma(s) \) be parametrized with curve length and let \( T(s) \) be coordinate corresponding to the point \( \Gamma(T(s)) \) defined so that the line \( \Gamma(s)\Gamma(T) \) tangents \( \gamma \) in the right hand orientation. If \( s \in \mathbb{R} \) corresponds to a point that generates a \( n \)-sided polygon this corresponds to \( T^n(s) = s \) (mod \( L \)) where \( L \) is the total length of \( \Gamma \).

Our goal will be to find a monotone function \( g \) which rescales the parametrization \( s \) so that \((g\circ T)(s) = g(s) + C \). If such a function exists then we may define a new function \( G(x) = x + C \). Now from \( G^n(x) = (g \circ T^n \circ g^{-1})(x) = x + nC \) we see that if \( g^{-1}(x) \) is a \( n \)-periodic point of \( T \) then \( x \) is an \( n \)-periodic point of \( G \) but since \( G \) is just translation then all coordinates \( x \in \mathbb{R} \) are \( n \)-periodic points of \( G \), and inverting to get \( T = g^{-1} \circ G \circ g \) we have that every point \( s \in \mathbb{R} \) is a periodic point of \( T \).

Now we simply show the existence of such function \( g \) but before continuing note that we may always assume that the outer ellipse is a circle, for if it was not we could apply an affine transformation rescaling and rotating the configuration as to convert the outer ellipse to a circle. Such a transformation maps parallel lines into parallel lines so if we could inscribe a polygon in the original configuration the image of the polygon would still be properly inscribed in the images of the ellipses.

Working with a circle has the consequence that any line connecting to points along \( \Gamma \) will make equal angles with the boundary. Now, if we start with \( s \) and choose a coordinate \( s' \) close to \( s \), and let \( T \) and \( T' \) be the image coordinates construct two triangles such as in Figure 16 where in the limit where \( s' \) is infinitesimally close to \( s \) we have from the law of sines

\[
\frac{|\Gamma(T') - \Gamma(T)|}{\sin \varepsilon} = \frac{L_{\gamma, \Gamma}(\Gamma(T))}{\sin (\pi - \alpha)}, \quad \frac{|\Gamma(s') - \Gamma(s')|}{\sin \varepsilon} = \frac{R_{\gamma, \Gamma}(\Gamma(s))}{\sin \alpha}.
\]

which in differential notation can be cast as

\[
\frac{dT}{L_{\gamma, \Gamma}(\Gamma(T))} = \frac{ds}{R_{\gamma, \Gamma}(\Gamma(s))} \quad \text{or} \quad \frac{dT}{ds} = \frac{L_{\gamma, \Gamma}(\Gamma(T))}{R_{\gamma, \Gamma}(\Gamma(s))}.
\]

Now we once again remind ourselves that affine transforms preserve the ratio of two line segments along the same line (as they are stretched equally). This means that the right hand fraction of
the rightmost expression is invariant under an affine transform $\Lambda$ which transforms the inner ellipse $\gamma$ into a circle. If it already is a circle we may just apply the identity, and so we have

\[
\frac{dT}{ds} = \frac{L_{\Lambda \gamma, \Lambda \Gamma}(\Lambda \Gamma(T))}{R_{\Lambda \gamma, \Lambda \Gamma}(\Lambda \Gamma(s))}
\]

but by symmetry and the definition of the distances to tangency $L$ and $R$ we have that $L_{\Lambda \gamma, \Lambda \Gamma} \equiv R_{\Lambda \gamma, \Lambda \Gamma}$ when $\Lambda \gamma$ is a circle and we may denote this common function $D_{\Lambda \gamma, \Lambda \Gamma}$, and we can construct the monotone function

\[
g(s) = \int_0^s \frac{dy}{D_{\Lambda \gamma, \Lambda \Gamma}(\Lambda \Gamma(y))}
\]

where we see that (1) is equivalent to

\[
\frac{d[g(T(s))]}{d[g(s)]} = 1
\]

and the proof is complete. \qed

6.4 String construction of convex caustics

In this section we will prove that to any strictly convex curve $\gamma$ in the plane (i.e. the curve is convex and $|\gamma''(t)| > 0$ everywhere) we can find another curve $\Gamma$ such that it has $\gamma$ as caustic. This will be done with the so called string construction. Intuitively, the method of proof can be described as taking a string of some length $L'$, where $L'$ is greater than the length $L$ of $\gamma$, and a pen. Then we put the string around $\gamma$ and pull it tight at one point with the pen. Now, keeping the string tight, we move the pen around $\gamma$, thus tracing out a new curve $\Gamma$. This new curve will have $\gamma$ as caustic.

6.5 Proof of the string construction

Let $\gamma$ be parametrized by the interval $[0, 1]$ and choose $L$ to be greater than the arc length of $\gamma$. Fix some point $t \in [0, 1]$ and let $u_t$ be the unit tangent to $\gamma$ at $\gamma(t)$. Now consider the line
\( l(r) := \gamma(t) + ru \) for \( r \geq 0 \). For each \( r \geq 0 \) we can choose a point \( \gamma(t') \) such that \( \gamma(t') - l(r) \) is tangent to \( \gamma \) at \( \gamma(t') \) (see figure 17). The uniqueness of this \( t' \) is guaranteed by the strict convexity of \( \gamma \).

Now define a function \( L(r) \) as

\[
L(r) := r + |l(r) - \gamma(t')| + L(t, t').
\]

Figure 17: String construction of curve with caustic \( \gamma \).

Here \( L(t, t') \) denotes the arc length from \( \gamma(t) \) to \( \gamma(t') \) going counterclockwise in figure 17. We will prove that \( L(r) \) is strictly increasing and continuous. To prove that it is strictly increasing we must show that the section \( r_1t'_1t'_2 \) is shorter than \( r_1r_2t'_2 \). Denoting the intersection of the two dashed lines in figure 17 by \( p \) we can note that it suffices to prove that the section \( t'_1t'_2 \) is shorter than \( t'_1pt'_2 \). Now consider the geometrical construction in figure 18. The dashed curve \( s_1s_3s_2 \) is constructed by intersecting the segment \( s_1s_2 \) of \( \gamma \) at the midpoint, denoted by \( s_3 \). This curve is clearly shorter than the segment \( s_1ps_2 \). In the same manner we construct the dotted curve by bisecting each of the two new segments. This dotted curve is also shorter than \( s_1ps_2 \). Continuing in this fashion we construct a sequence of polygonal curves denoted by \( \{q_n\} \) that are all strictly shorter than the segment \( s_1ps_2 \).

Figure 18: Approximation of \( \gamma \) with polygonal curves to prove that \( L(r) \) is strictly increasing.

We now want to show that this sequence of polygonal curves has the section \( s_1s_2 \) of \( \gamma \) as limit. To do this we note that \( \gamma \) is uniformly continuous. This implies that for any \( \epsilon > 0 \) we can
choose a $\delta > 0$ such that for any $|t - t'| < \delta$ we have $|\gamma(t) - \gamma(t')| < \epsilon$. Hence if we choose a positive integer $N$ big enough the polygonal curve $q_n$ for $n \geq N$ will fulfill (see figure 19)

$$\max_{s_1 \leq t \leq s_2} |\gamma(t) - q_n(t)| < \epsilon.$$ 

This proves that the sequence $\{q_n\}$ is uniformly convergent to $\gamma$ and hence if we denote by $L_n$ the length of the polygonal curves and by $L_{s_1s_2}$ the length of the segment of $\gamma$ between $s_1$ and $s_2$ then we have

$$\lim_{n \to \infty} L_n = L_{s_1s_2}.$$ 

But this implies that the segment $s_1s_2$ of $\gamma$ is strictly less than the segment $s_1ps_2$ which proves that $L(r)$ is strictly increasing.

Now to show that $L(r)$ is continuous we look at its value for $r + \Delta r$. Let $t' + \Delta t'$ be the corresponding point on $\gamma$. Then we can write

$$L(r + \Delta r) = r + \Delta r + |l(r + \Delta r) - \gamma(t' + \Delta t')| + L(t, t' + \Delta t').$$

To prove that $L(r)$ is continuous we must show that

$$\lim_{\Delta r \to 0} L(r + \Delta r) = L(r).$$

Since $L(r)$ is a sum of continuous functions it suffices to prove that $\Delta t' \to 0$ as $\Delta r \to 0$. To prove this we assume that $\Delta s' > \epsilon$ always for some $\epsilon > 0$ to prove a contradiction. This implies, since $\gamma$ is strictly convex, that $|u_{t'} - u_{t' + \Delta t'}| > \delta$ for some $\delta > 0$. But this implies (see the construction in figure 17) that $\Delta r$ cannot go to zero, hence we have a contradiction. This implies that $L(r)$ is continuous.

Now fix $L' > L$ where $L$ is the length of $\gamma$. Having proved that $L(r)$ is a continuous and strictly increasing function such that $L(0) = L$ we can choose $r_1 > 0$ such that $L(r_1) = L'$. Now define $\Gamma(t) := l(r_1) = \gamma(t) + r_1 u_t$. Defining $\Gamma(t)$ for each $t \in [0, 1]$ we get a parametrized set in the plane. We will show that this set is a continuously differentiable curve such that it has $\gamma$ as its caustic.

It is obvious by construction that $\Gamma(t) \neq \Gamma(t')$ for any $t \neq t'$ except for $\Gamma(0) = \Gamma(1)$, hence $\Gamma$ is closed and does not intersect itself. To prove that $\Gamma$ is differentiable we must evaluate the expression $\Gamma(t + \Delta t) - \Gamma(t)$. By the definition of $\Gamma$ we have

$$\Gamma(t + \Delta t) - \Gamma(t) = (\gamma(t + \Delta t) - \gamma(t)) + (r_{t+\Delta t} u_{t+\Delta t} - r_t u_t).$$
Since $\gamma$ is assumed to be continuously differentiable we can write

$$\gamma(t + \Delta t) - \gamma(t) = \Delta t \gamma'(t) + O(\Delta t^2).$$

We also have

$$r_{t+\Delta t} u_{t+\Delta t} - r_t u_t = r_{t+\Delta t} (u_t + \Delta t u'_t + O(\Delta t^2)) - r_t u_t = (r_{t+\Delta t} - r_t) u_t + r_{t+\Delta t} \Delta t u'_t + O(\Delta t^2).$$

**Figure 20:** Geometrical proof that $\Gamma$ is differentiable.

This shows that it suffices to prove that $r_{t+\Delta t} \to r_t$ if $\Delta t \to 0$. But this follows from the fact that if $\Delta t \to 0$ then $u_{t+\Delta t} \to u_t$ which in turn implies that $u_{t+\Delta t} \to u_t$ (see figure 20). This gives the desired conclusion by the construction of the curve. Hence $\Gamma$ is a differentiable curve.

**Figure 21:** Proof that $\gamma$ is a caustic of $\Gamma$.

To prove that $\Gamma$ has $\gamma$ as caustic we will follow the procedure in [2, p.73,74]. Take a point $x \in \Gamma$ and let $y$ be a reference point on $\gamma$. Now denote by $f(x)$ and $g(x)$ respectively the distances from $x$ to $y$ by going around $\gamma$ in a counterclockwise and clockwise manner respectively.
(see figure 21). Then $\Gamma$ is by construction a level curve of $f + g$ and hence $\nabla (f + g) = 0$. We now want to prove that $\nabla f$ is the unit vector along $ax$. To do this we consider a level curve of $f$, say $f = c$ for some constant $c$, through the point $x$. Then if $a = \gamma(t)$ for some $t$ we can write $x = \gamma(t) + (c - t)\gamma'(t)$. This implies that $x' = (c - t)\gamma''(t)$. But then, since $t$ is an arc length parameter, we have that $\gamma'$ and $\gamma''$ are orthogonal (this follows by differentiating the expression $1 = |\gamma'(t)| = \gamma'(t) \cdot \gamma'(t)$ which in turn implies that $x'$ is orthogonal to $ax$ since $\gamma'$ is parallel to $ax$. Hence we have proven that the level curve of $f$ is perpendicular to $ax$ at $x$ which must imply that $\nabla f$ is parallel to $ax$ since we are working in the plane. That the directional derivative of $f$ is one follows from that $\Delta f = \Delta t$, the change in the arc length parameter as we move $x$ along $\Gamma$. Hence it follows that $\nabla f$ is a unit tangent along $ax$. Similarly one proves that $\nabla g$ is a unit tangent along $bx$. But this together with $\nabla f + \nabla g = 0$ implies that $\nabla f$ and $\nabla g$ must make the same angle with the tangent at $x$ (see figure 22). By definition this proves that $\gamma$ is a caustic of $\Gamma$.

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (2,2) -- (4,0) -- (2,-2) -- cycle;
\draw (0,0) -- (1,1);
\draw (2,2) -- (1,1);
\draw (4,0) -- (1,1);
\draw (2,-2) -- (1,1);
\draw (1,1) circle (0.5cm);
\node at (1,1) {\textbullet};
\node at (1,1) {$\nabla f$};
\node at (2.5,1) {$\nabla g$};
\end{tikzpicture}
\end{center}

Figure 22: $\nabla f$ and $\nabla g$ make equal angles with $\Gamma$ at $x$.

### 6.6 Non existence of Caustics and global implications

The existence of convex caustics is not generally guaranteed. For most tables a given orbit will not remain tangent to a single curve and in fact a long standing conjecture by Birkhoff has been that the billiard in an ellipse is the only table where almost every orbit belongs to an invariant circle supporting which at least partial results have been made [2, p.95].

In this section we illustrate how caustics not only determine the dynamics of the trajectories tangent to them but also have consequences for orbits far away from the corresponding curves in phase space. The implication itself comes from the fact that invariant curves partition the phase space and prevents orbits from exploring more than a limit subset which immediately excludes ergodicity. We will investigate this by considering sets with invariant curves as boundaries and therefore will have need of reminding ourselves of the following result.

**Lemma 6.3.** Homeomorphisms preserve boundaries

*Proof*. Use the definition of the boundary $\partial A = \overline{A} \setminus A^\circ$ and the fact that the order of applying a homeomorphism and operations of intersection, closure, and interior are interchangeable.

$$f(\partial A) = f(\overline{A}) \setminus f(A^\circ) = \overline{f(A)} \setminus f(A)^\circ = \partial f(A).$$

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And so if a set has invariant curves as it’s boundary we may immediately derive our partitioning result

**Theorem 6.8.** Let $A$ be region of phase between two invariant circles $\Gamma_1$ and $\Gamma_2$. Then given any point in $A$ its entire orbit is confined in $A$. That is $T(A) = A$.

*Proof.* Make the identification of phase space with the annulus where the region in question is a sub-annulus, the intersection of the set bounded by the outer boundary and the unbounded part of the inner boundary. We have from the Jordan Curve theorem that circles divide the plane into two connected sets and so the set bounded by two circles is connected. Since $T$ is continuous the image $T(A)$ has to be connected but there is only one connected set with $\Gamma_1$ and $\Gamma_2$ as boundaries so the set must be invariant under $T$. $\square$

**Remark** In the case of convex caustics this means that a trajectory which at one instant is contained between the boundary and the interior caustic will remain in this region forever, and any trajectory passing though the caustic will continue to pass though it. Of course there

What we have considered so far have been the fact that caustics tend to make the dynamic more well ordered and possible to characterize, but we will now take a step in the opposite direction showing how regions of the phase space without caustics can exhibit a great deal of complexity. Let a billiard map have two invariant circles with different rotational numbers $\rho$ such that the sub cylinder bounded by these curves is free of invariant circles. We call such a region a *region of instability* and it can be shown that it contains at least one orbit that moves arbitrarily close to both the upper and lower circle. For the discussion we require a relatively strong theorem by Birkhoff [7, p.430] concerning functions defined on semi-unbounded cylinder $A = S^1 \times [0, \infty)$ but which is also applicable to our bounded phase space $A$

**Theorem 6.9.** Let $f : A \to A$ be an orientation preserving twist homeomorphism and $U \subset NW(f)$ an $f$-invariant open relatively compact set containing $S^1 \times \{0\}$ with connected boundary. Then $\partial U$ is the graph of a continuous function $\psi : S^1 \to (0, \infty)$

Where $NW(f)$ is the set of so called *nonwandering points*, points such that every neighbourhood of a point $U$ of it eventually self intersects after a finite number of iterations $\phi^n(U) \cap U \neq \emptyset$. For the billiard map $T$, this is always satisfied for any subset since $NW(T) = A$ as area preservation guarantees this by the Poincaré recurrence theorem, and also compactness is simple to assume since the closed cylinder is compact.

And by this one can state the following theorem[7, p.433]

**Theorem 6.10.** Suppose $C_1$ and $C_2$ bound a region of instability of an area preserving twist map $T$ and let $\varepsilon > 0$. Then there exists a point $p$ in the $\varepsilon$-neighborhood $W_{1,\varepsilon}$ of $C_1$ with an iterate in the $\varepsilon$-neighborhood of $W_{2,\varepsilon}$ of $C_2$.

*Proof.* If not then there would exist some $\varepsilon$ such that $V_0 = \bigcup_{n \in \mathbb{N}} f^n(W_{1,\varepsilon})$ is disjoint from $W_{2,\varepsilon}$. Our aim will be to show that if this is the case then the upper boundary of $V_0$, $\Gamma$, is the graph of a continuous function of $S^1$ and hence isomorphic to a circle. Since $V_0$ is dynamically invariant under $T$ by definition, $T(V_0) = V_0$ and homeomorphism preserve boundaries we would have $T(\partial V) = \partial T(V) = \partial V$, and the boundary segment would hence be an invariant circle which would be a contradiction since the region of instability was supposed to be free of invariant circles.

To show that upper boundary of $V_0$ is a graph we will apply theorem 6.9 but it is not certain that $V_0$ would satisfy all the necessary properties. For example it is quite possible that $V_0$ might contain some holes such that its whole boundary would not be connected. For this reason we
construct another set with \( \Gamma \) as its boundary which is certain to have a connected boundary (in a modified sense).

Let \( W \) be the connected component between \( C_2 \) and \( \Gamma \) which exists since \( V_0 \) is disjoint from \( W_{2,\varepsilon} \). We are certain that \( W \) does not contain any holes since \( V_0 \) must be connected on account of the fact that every subimage \( f^n(W_{1,\varepsilon}) \) contains \( C_1 \). \( W \) is therefore an annulus, and to remove the second part of the boundary \( C_2 \) from the discussion we take the area between \( C_1 \) and \( C_2 \) consider it to be an annulus in the \( \mathbb{R}^2 \)-plane with \( C_2 \) as its inner circle whereupon we deform \( C_2 \) to a single point and complete \( W \) with a point such that the resulting set is a simply connected set in the plane surrounded by an an annulus \( V_0 \).

This configuration where \( \partial W \) is embedded in \( \partial V_0 \) is special in the sense that it immediately implies that \( \partial W = \Gamma \) is connected by the following Lemma A.7.8 in [7, p.737].

**Lemma 6.4.** Suppose \( O_1, O_2 \subset \mathbb{R}^2 \) are disjoint open sets such that \( \partial O_1 \subset \partial O_2 \). Then \( \partial O_1 \) is connected.

Now we take the part of \( V = (\overline{W})^c \) contained below \( C_2 \) on the ellipse and and see that it is an invariant set with a connected boundary, also containing \( S^1 \times \{0\} \) and so by Birkhoff's theorem its (upper) boundary is an invariant circle which completes the necessary contradiction.

This means that there exists at least one orbit which has a relatively complicated behaviour moving from the lower circle to the upper circle meaning one can expect some complexity in the phase space, and has an immediate corollary

**Corollary 6.1.** If a table has no convex caustic then there exists an orbit such that the angular coordinate \( \theta \) takes values in the whole range \((0, \pi)\)

**Proof.** If there are no caustics then there are no invariant circles except for the boundary circles \( S^1 \times \{0\} \) and \( S^1 \times \{1\} \) and these have different rotation numbers 0 and 1 so all of phase space is a region of instability. Then one may take \( \varepsilon \) to be arbitrarily small whereupon the statement follows from Theorem 6.10.

In Figure 23 we see an example of an orbit in a table that has no convex caustics where the freedom in angle of reflection is immediately evident. This is one motivation for relying another simple result due to Mather which provides a very simple condition under which a billiard table has no convex caustics and thus 6.10 is applicable.

With this as a partial motivation we now review a result which establishes that the curvature of the billiard table determines whether the existence of convex caustics can exist.

### 6.7 Nonexistence of caustics at locally flat boundaries

We will here prove another important result about the existence of caustics in convex billiards. In section 6.4 we proved that we could always create a billiard table that had a given strictly convex curve as caustic. Here we will prove that if we flatten the boundary of the billiard table, i.e. we set the curvature \( \kappa := |\Gamma''(t)| = 0 \) at some point \( t \), then the table has no convex caustics. This result is a direct consequence of an important equation from geometrical optics which can be shown to hold in billiards as well. We state it here without proof.

![Figure 23: Simulation of 50 iterations inside the billiard \( x^4 + y^2 = 1 \) which has no convex caustics](image-url)
Figure 24: Disappearance of caustics at zero curvature.

**Theorem 6.11.** Suppose that $\Gamma$ has a caustic $\gamma$ completely contained within $\Gamma$. Take a point $\gamma(t)$ on $\gamma$ and let $u_t$ be the unit tangent to $\gamma$ at this point. Reflect this tangent in $\Gamma$, say $\Gamma(s)$ according to the law of reflection and let $\gamma(t')$ be the point of tangency with $\gamma$ of this reflected tangent. Denote by $\alpha$ the reflection angle and let $f := |\gamma(t) - \Gamma(s)|$ and $\overline{f} := |\Gamma(s) - \gamma(t')|$. Then the following relation holds between $\alpha$, $f$ and $\overline{f}$:

$$\frac{1}{f} + \frac{1}{\overline{f}} = \frac{2\kappa}{\sin \alpha}.$$ 

Here $\kappa$ denotes the curvature of $\Gamma$ at $\Gamma(s)$.

This equation is often called the mirror equation and we will use it to prove

**Theorem 6.12.** Let $\Gamma$ be a convex billiard and suppose $\kappa = 0$ at some point $\Gamma(s)$. Then $\Gamma$ has no convex caustics $\gamma$.

**Proof.** Suppose that a convex caustic $\gamma$ exists and let $\Gamma(s)$ be the point for which $\kappa = 0$. Then, since both curves are convex I can find a point $\gamma(t)$ such that the straight line from $\gamma(t)$ to $\Gamma(s)$ is tangent to $\gamma$ (see figure 24). Denoting the length of this line segment by $f$ we then reflect this tangent at $\Gamma(s)$ according to the law of reflection to get another tangent to $\gamma$ at some point $\gamma(t')$. Denoting the respective length by $\overline{f}$ the mirror equation gives at $\Gamma(s)$ that

$$\kappa = \frac{\sin \alpha}{2} \left( \frac{1}{f} + \frac{1}{\overline{f}} \right) > 0.$$ 

This contradicts our assumption on $\kappa$ and hence $\Gamma$ can have no convex caustics. \qed

### 6.7.1 Examples and test for tables without caustics

We may apply this to prove a straightforward result for a class of tables given by algebraic curves

**Proposition 6.2.** Of all the tables given by the implicit curves

$$x^{2n} + y^{2n} = 1$$

where $n \geq 1$, only $n = 1$ has convex caustics
Proof. The curvature of an implicit curve $f(x, y) = C$ is given by

$$\kappa = -\frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_x^2f_{yy}}{(f_x^2 + f_y^2)^{3/2}}$$

which is derived in appendix A.2. Evaluating the numerator for $f(x, y) = x^{2n} + y^{2n}$ we see that it is

$$-2^2n^2(2n-1)(x^{2(n-1)}y^{2(2n-1)} + x^{2(2n-1)}y^{2(n-1)})$$

which is 0 at the points of intersection with the coordinate axis $x = 0$ and $y = 0$ if $n \geq 1$.

This argument can of course be extended to all polynomial algebraic curves where there are no mixed terms $x^iy^j$ to complicate it. The nonexistence of caustics of course does not exclude the existence of non convex caustics and the associated invariant curves.

We now take the opportunity to in a novel way illustrate how the phase space may look in the absence of convex caustics, though it should be stressed that the phase space of any given table has no reason to resemble another one.

Take the orbit starting at the rightmost point of $x^4 + y^4 = 1$ such that the initial ray makes the initial angle $\pi/6$ with the boundary then the coordinates in phase space of the first 10 000 iteration is displayed in Figure 25 which wanders close to both the upper and lower circles but stays out of a number of circular regions which other simulations indicate correspond to non-convex caustics encircling periodic points.

7 Outer billiards

In this section we will make a short introduction to the subject of outer billiards. This system is closely related to the ordinary billiards but instead of letting the ball reflect at the inside of the table we will now consider the case where the billiard ball reflects on the outside of the table in a manner which will be specified. A new question that arises in context of these billiards is wheter the orbit will be bounded or not. We will make no attempt to any rigorous treatment of these systems but will instead discuss some interesting results regarding them and sketch the proofs of some of them. This presentation will follow the one given in [2, p.147-160].

7.1 Definition

Consider a continuous closed simple convex curve in the plane and denote it by $\gamma$. Now take a point $u$ in the plane outside of $\gamma$. The outer billiard reflection of this point is defined by considering the two tangent lines from $u$ to $\gamma$ (assuming they exist). By convention we choose the tangent line to the right of $\gamma$ as seen from $u$ (see figure 26). Assuming that this tangent line is tangent to $\gamma$ at exactly one point on $\gamma$ denoted by $v$ consider the length from $L := |u - v|$. By extending the tangent line from $v$ a distance $L$ a new point $w$ outside of $\gamma$ is reached. Define $T(u) = w$ where $T$ is called the outer billiard map. The map $T$ is well-defined everywhere that there exists a tangent line from $u$ to $\gamma$ to the right of $\gamma$ that intersects the curve at a unique point. It is easily seen that if for example the billiard table contains straight line segments then the outer billiard map will not be defined at points that lie at extensions of these segments. One can compare this to the situation in ordinary billiards when the ball hits a corner of the curve and the motion becomes undefined.
10000 iterations inside the table $x^4 + y^4 = 1$ in the coordinates reflection angle over a curve parametrization.

Figure 26: Construction of the outer billiard map

7.2 Properties of the outer billiard map

In this section we will present some properties of the outer billiard map along with some sketches of the proofs. First we note that like the ordinary billiard map the outer billiard map is area-preserving for smooth boundaries.

**Theorem 7.1.** Assume that $\gamma$ is smooth (continuously differentiable) and let $T$ be the outer
Let $X$ and $Y$ be two “close” points on $\gamma$ and extend the tangents of these points by some positive real number $r$ (see figure 27). Denote by $A, A', B$ and $B'$ the endpoints of these tangents. Note that the billiard map $T$ maps the segment $AA'$ to $BB'$. Now take $\epsilon > 0$ and repeat this construction for $r - \epsilon$ and denote the new endpoints of the resulting tangents by $C, C', D$ and $D'$ respectively. The billiard map $T$ now maps $AA'C'C$ to $BDD'B'$ and we want to show that this is done in an area-preserving fashion in the limit $X \to X'$. Denoting by $\delta$ the angle between $AB$ and $A'B'$ we can calculate the area of $A'AY$ and $C'CY$ which gives

$$|A'AY| \approx \frac{r^2 \delta}{2},$$

and

$$|C'CY| \approx \frac{(r - \epsilon)^2 \delta}{2} = \frac{r^2 \delta}{2} + \frac{\epsilon^2 \delta}{2} - \epsilon \delta.$$

Using this the area of $AA'C'C$ is given by

$$|AA'C'C| \approx \frac{r \epsilon \delta}{2} \approx \epsilon \delta.$$

A similar calculation gives the same for $BDD'B'$ and hence the area-preservation is proven. \qed

**Remark:** By using an argument similar to the one in theorem 7.1 a result similar to the string construction for inner billiards can be proven. In the case of outer billiards the theorem states that given an outer billiard curve $\gamma$ and a convex curve $\Gamma$ that is invariant under the outer billiard map $T$ one can recover the curve $\gamma$ from $\Gamma$. The idea is to fix a number $c > 0$ and then consider all line segments that intersects $\Gamma$ by cutting out sections of area $c$ from $\Gamma$. The curve $\gamma$ is then taken to be the envelope of all these segments (see figure 28).

Next we turn to periodic orbits for strictly convex and smooth outer billiards. A periodic orbit for an outer billiard table is a polygon circumscribed around the billiard curve $\gamma$ such that each side is bisection by its point of tangency at $\gamma$ (see figure 29). Like for inner billiards, periodic orbits for convex outer billiards can be found to be equivalent to extreme points of a functional defined on the space of all circumscribing polygons. In this case the functional is the area of the polygon. Note that for inner billiards it was the perimeter length of the inscribed polygon that was used.
7.3 Bounded orbits

In this section we will briefly account for an interesting question that arises in the context of outer billiards; for which tables are the orbits bounded? This distinguishes the outer billiard systems from the inner billiards systems since in the latter case the orbits are trivially bounded by the billiard curve itself. This question has still not been resolved in the general case but solutions have been given for two large classes of billiard curves: $C^5$ curves and rational polygons. The latter is defined as all polygons that can be described as affine images of polygons whose vertices have integer coordinates (remember that an affine transformation is a transformation of the form $x \mapsto ax + b$ where $a$ and $b$ are real numbers). We state the result for these classes of tables without proof.

**Theorem 7.2.** If the billiard curve $\gamma$ is $C^5$ and strictly convex then every orbit of the outer billiard is bounded. This conclusion also holds if $\gamma$ is a rational polygon.

**Remark:** The result for rational polygons can actually be made stronger by defining another class of polygons called quasi-rational polygons. We refer to [2, p.158-160] for a further discussion of these polygons. As an example of a table for which associated outer billiard has unbounded orbits one can take the semi-circle. In [8] it is proved that for this particular system there exists orbits escaping to infinity.
7.4 Outer billiards and planetary motion

One of the reasons that outer billiards became popular was due to an interpretation by J. Moser of these systems as a model for planetary motion. To motivate this we consider a billiard curve $\gamma$ and choose a starting point $u$ “far away” from $\gamma$. Now taking the second iterate of the billiard map $T$, $T^2$, of this point we can notice that the point seems to move in a centrally symmetric way with $\gamma$ as center of motion (see figure 30). More precisely one can prove that this motion satisfies Kepler’s second law; that the area per time unit that the positional vector of the moving point sweeps over is constant. For a further discussion of this topic see [2, p.155–157].

Figure 30: Illustration of outer billiard movement as planetary motion.
A Calculations

A.1 Notes for constructing level curves

Figure 31

There exists a well known integral of the billiard inside an elliptic table

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is essentially a function of the cartesian position \((x, y)\) of the billiard ball, and its cartesian velocity \((u, v)\) that is constant over every orbit. This integral is

$$b^2 xu + a^2 yv$$

and is relatively straightforward to derive, see [2, p. 54].

The basic idea is to transform the billiard integral \(xu/a^2 + yv/b^2\) into the normal variables of the billiard map of arclength \(s\), and angle relative to tangent \(\theta\). Instead of using \(s\) directly we use the traditional parametrizing variable \(\varphi\) such that the curve of the ellipse is spanned by

$$\gamma(\varphi) = (x, y) = (a \cos \varphi, b \sin \varphi)$$

where both variables \(x\) and \(y\) in the integral are reduced to sines and cosines. The relative difficulty lies in expressing the velocity \((u, v)\) in terms of \(\varphi\) and \(\theta\). To do this we propose the following constructions expressing \((u, v)\) in the form \((\cos \alpha, \sin \alpha)\) where \(\alpha = \alpha(\varphi, \theta)\) defined as the sums of three different \(h(\varphi), g(\varphi)\) and \(\theta\) as understood to have the meaning in Figure 31

\(h\) is by construction the argument of \(\gamma\)

$$h(\varphi) = \text{Arg}(\gamma(\varphi))$$

where for the details of defining it mathematically one may consult any text on complex analysis or analytical geometry. \(g\) demands some additional attention where we use \(\gamma\) to derive a normalized position vector \(\hat{\gamma}\) and the tangent \(t\) at \(\gamma(\varphi)\)

$$\hat{\gamma} = \frac{(a \cos \theta, b \sin \theta)}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}, \quad t(\theta) = \frac{(-a \sin \theta, b \cos \theta)}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}}$$
where \( \cos(g(\varphi)) = \gamma(\varphi) \cdot t(\varphi) \) and basic algebra will yield

\[
g(\theta) = \arccos \left( \frac{(b^2 - a^2) \tan \theta}{\sqrt{(a^2 + b^2 \tan^2 \theta)(b^2 + a^2 \tan^2 \theta)}} \right).
\]

The integral may therefore be written in the form

\[
b \cos \varphi \cos(h(\varphi) + g(\varphi) + \theta) + a \sin \varphi \sin(h(\varphi) + g(\varphi) + \theta) = \text{constant}.
\]

In case it is desired to have the actual curve length variable instead \( \theta \) it could in principle be recovered.

### A.2 Calculation of the Curvature of an Implicit planar curve

One of the Frenet equations for three dimensional curves is \([9][10]\)

\[
\frac{d\mathbf{N}}{ds} = -\kappa \mathbf{T} + \tau \mathbf{B}
\]

where \( \kappa \) is the curvature, \( \tau \) is the torsion, and \( \{\mathbf{N}, \mathbf{T}, \mathbf{B}\} \) are the local normal, tangential, and binormal directions associated with the curve. From this equation it is possible to recover the curvature

\[
\kappa = -\frac{d\mathbf{N}}{ds} \cdot \mathbf{T}
\]

using the fact that the vectors are orthogonal to each other. In the case of a planar curve, \( f(x, y) = 0 \), the normal direction is given by the gradient and the tangential direction can be recovered by rotating the normal by \( \pi/2 \) radians in the plane

\[
\mathbf{N} = \frac{(f_x, f_y)}{|\nabla f|}, \quad \mathbf{T} = \frac{(-f_y, f_x)}{|\nabla f|}.
\]

Furthermore

\[
\frac{d}{ds} = \frac{dx}{ds} \frac{\partial}{\partial x} + \frac{dy}{ds} \frac{\partial}{\partial y} = \frac{1}{\sqrt{f_x^2 + f_y^2}} \left( -f_y \frac{\partial}{\partial x} + f_x \frac{\partial}{\partial y} \right) = \mathbf{T} \cdot \nabla
\]

where \( \gamma(s) = (x(s), y(s)) \) is the curve parametrized by curve length hence

\[
\kappa = -T_T |\nabla| T
\]

\[
= \frac{1}{|\nabla f|^3} \left( f_{xy} f_x - f_{xx} f_y, f_{yy} f_x - f_{xx} f_y, -f_y \right) \left( -f_y \right) \\
= -\frac{1}{|\nabla f|^3} \left( f_{xy} f_x - f_{xx} f_y, f_{yy} f_x - f_{xx} f_y, -f_y \right) \left( -f_y \right) \\
= -\frac{f_{xx} f_y^2 - 2f_{xy} f_x f_y + f_{yy} f_x^2}{|\nabla f|^3}.
\]

A slightly more symbolic approach may be found in \([10, p.637]\)

### B Basic notions in analysis

We will here give a brief survey of some elementary concepts in analysis which we use throughout the text. For further reading and proofs of these facts consult for example \([11]\).
### B.1 Metric spaces

Let $M$ be any set and assume we can define a function $d : M \times M \to \mathbb{R}$ such that $d$ satisfies the following axioms:

- $d(x, y) \geq 0$ for all $x, y \in M$
- $d(x, y) = d(y, x)$ for all $x, y \in M$
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in M$.

Then $d$ is called a metric or distance function on $M$ and $M$ is referred to as a metric space.

**Example:** Letting $M = \mathbb{R}^n$ for some natural number $n > 0$ we can define for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $M$

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}.$$  

It is easily shown that this $d$ satisfies the axioms above and hence that $\mathbb{R}^n$ is a metric space with this metric. This metric is usually referred to as the Euclidean metric.

Now we can define some topological concepts. Hence let $M$ be any metric space with metric $d$. For any point $x \in M$ and real number $r > 0$ we define

$$B(x, r) := \{y \in M | d(x, y) < r\}$$

and call this the open ball of $x$ with radius $r$. We can now make the following crucial definitions.

**Definition B.1.** Let $K$ be a subset of $M$. Then $K$ is called open if for any $x \in K$ there exists $\epsilon > 0$ such that the open ball $B(x, \epsilon)$ is contained in $K$. $K$ is called closed if its complement is open.

**Remark:** Note that by this definition it follows that $M$ and $\emptyset$ are both open sets. The following theorem gives an important property of open sets.

**Theorem B.1.** Let $M$ be a metric space.

- If $\{V_n\}$ are open sets of $M$ then their union is also an open set.
- If $\{V_i\}_{i=0}^{N}$ are open sets for some finite number $N$ then their intersection is also open.

The next theorem will give a useful characterization of when a subset is closed but first we need two more definitions.

**Definition B.2.** Let $K \subset M$ and take $x \in K$. Then $x \in M$ is called a limit point of $K$ if for any $\epsilon > 0$ there exists $y \in K$ such that $y \in B(x, \epsilon)$. A point $x \in K$ is called isolated if there exists $\epsilon > 0$ such that $B(x, \epsilon) \cap K = \emptyset$.

**Definition B.3.** Let $K \subset M$. Then we define $\overline{K}$ as the smallest closed set containing $K$ (this is well-defined since there is at least one closed set containing $K$, namely $M$). This set is called the closure of $K$. We also defined $K^c$ to be the largest open set containing $K$.

**Theorem B.2.** A set $K \subset M$ is closed if and only if every limit point of $K$ lies in $K$.

Finally, we call a set $K \subset M$ bounded if there exists an open ball $B(x, R)$ such that $K \subset B(x, R)$.
B.2 Compactness

We now turn to the concept of compactness. Let as in section B.1 $M$ be a metric space with metric $d$ and let $K$ be a subset of $M$. We define an open covering of $K$ to be any collection $\{V_\alpha\}$ of open sets of $M$ such that $K \subset \bigcup_\alpha V_\alpha$. We can now define the concept of compactness.

**Definition B.4.** A subset $K \subset M$ is called compact if for any open covering $\{V_\alpha\}$ of $K$ there exists a finite subcovering, i.e. there exists a finite subset $\{V_{\alpha_i}\}$ such that $K \subset \bigcup_{\alpha_i} V_{\alpha_i}$.

Any student that has taken a course in elementary analysis will probably have heard the definition of a compact set as a set that is closed and bounded and even though it can be shown that any compact set in a metric space is indeed closed and bounded the converse need not be true in general. In $\mathbb{R}^n$ however this result does hold and it is contained in a famous theorem by Heine and Borel.

**Theorem B.3.** Let $K \subset \mathbb{R}^n$ where $\mathbb{R}^n$ is equipped with the Euclidean metric. Then $K$ is compact if and only if it is closed and bounded.

B.3 Continuity

Let $M$ and $N$ be metric spaces equipped with the metrics $d_M$ and $d_N$ respectively and let $f : M \to N$ be a function. Then we can make an important

**Definition B.5.** The function $f$ is called continuous at $x \in M$ if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $d_M(x, y) < \delta$ for $y \in M$ then $d_N(f(x), f(y)) < \epsilon$. The function is called continuous on $M$ if it is continuous at every $x \in M$.

**Remark:** Another useful and equivalent definition of continuity is that $f$ is continuous on $M$ if for any open set $K \subset N$ the set $f^{-1}(K)$ is open in $M$.

Note that in the definition B.5 we assume that the $\delta$ we find may depend on both the point $x \in M$ and the $\epsilon$ we choose. If we assume that the $\delta$ we find only depends on the $\epsilon$ we have chosen then we call $f$ uniformly continuous. The next theorem gives a useful condition for when a function is uniformly continuous.

**Theorem B.4.** Let $M$ be a compact metric space and let $N$ be metric space. Assume that $f : M \to N$ is a continuous function. Then $f$ is uniformly continuous.

It will be of particular interest for us to study real-valued functions on compact metric spaces and the next theorem is crucial in this aspect.

**Theorem B.5.** Let $M$ be a compact metric space and assume that $f : M \to \mathbb{R}$ is a continuous function. Then $f$ assumes a minimum and a maximum on $M$.

**Definition B.6.** A function is said to be a homeomorphism if it is continuous and has a continuous inverse.

**Definition B.7.** A function is said to be a diffeomorphism if it is differentiable and has a differentiable inverse.
References


