Optimal initial perturbations in streamwise corner-flow

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Localised optimal initial perturbations are studied to gain an understanding of the global stability properties of streamwise corner-flow. A self-similar and a modified base-flow are considered. The latter mimics a characteristic deviation from the self-similar solution, commonly observed in experiment. Power-iterations in terms of subsequent direct and adjoint linearised Navier-Stokes solution sweeps are employed to converge optimal solutions for two optimisation times. The optimal response manifests as a wave packet that initially gains energy through the Orr mechanism and continues growing exponentially thereafter. The study at hand represents the first global stability analysis of streamwise corner-flow and confirms key observations made in theoretical and/or experimental work on the subject. Namely, the presence of an inviscid instability mechanism in the near-corner region and a destabilising effect of the characteristic mean-flow deformation found in experiment.

Corner flow, optimal initial condition, power iteration, adjoint

1. Introduction

Optimal initial conditions for the flow in an axial corner are considered. The classical corner-flow problem, as sketched in figure 1, consists of two right-angled, semi-infinite flat plates with the potential flow $u_\infty$ aligned with the intersection line. Here, $x$ denotes the streamwise direction while the two spanwise coordinates $y$ and $z$ span the transversal plane. Additionally, a 45°-rotated, auxiliary coordinate system is introduced with its ordinate axis $s$ along the corner bisector. A set of equations based on the Blasius similarity transformation and matched asymptotic expansions, hence called the corner-flow equations, was derived by Rubin (1966). They describe the laminar, incompressible flow in the transversal corner plane and were first solved by Rubin & Grossman (1971). As far-field boundary conditions, the authors relied on the solution of an additional set of equations governing the asymptotic secondary cross-flow induced
Figure 1. Sketch of the flow in a streamwise corner (Schmidt & Rist 2011).

by the superposition of the displacement effects of the two plates (Pal & Rubin 1971). Detailed overviews of varied numerical corner-flow studies including the effects of compressibility (Weinberg & Rubin 1972), non-zero streamwise pressure gradient (Ridha 1992) or arbitrary angles (Barclay & Ridha 1980), only to mention a few, can be found in the works by Ridha (2003), Galionis & Hall (2005) or Schmidt & Rist (2011).

Numerous linear stability analyses based on the self-similar corner-flow solutions mentioned above have been conducted in the past, starting with studies of one-dimensional velocity profiles of the blending boundary layer between the corner region and the asymptotic far-field solution by Lakin & Hussaini (1984), Dhanak (1992, 1993) and Duck & Dhanak (1996). The two-dimensional linear stability problem of the transversal plane was first addressed by Balachandar & Malik (1993) within an inviscid framework. The effect of a streamwise pressure gradient was included in the first viscous corner-flow stability study by Parker & Balachandar (1999), the influence of compressibility in Schmidt & Rist (2011) and the spatial non-parallel stability problem was examined Galionis & Hall (2005) and Alizard et al. (2009) by solving the parabolised stability equations.

However, one of the main questions in the corner flow problem remained unanswered. There exists an obvious gap between numerical studies which consistently predict a critical Reynolds number of $\approx 9 \times 10^4$ (Balachandar & Malik 1993; Parker & Balachandar 1999; Galionis & Hall 2005; Schmidt & Rist 2011) and experimental evidence that suggests a transitional Reynolds number as low as $\approx 10^4$ even for a small favourable pressure gradient. The experimental work of numerous authors over two decades starting with early measurements by Nomura (1962) was collected and compared to each other and to theory by Zamir (1981). The reader is referred to this comprehensive aggregation of data for an insight into the experimental evidence.

Only recently, Alizard et al. (2010) computed the sensitivity of the self-similar corner-flow solution with respect to base-flow variations and suggested
a transient growth mechanism to explain the low transitional Reynolds number. In this paper, we follow that route and assess the question of how much kinetic energy a localized perturbation can maximally gain from the mean flow while being advected downstream over some given time $\tau$. This is done by computing global optimal initial conditions restricted to a finite streamwise extent using a time-stepper based optimisation technique. The use of optimisation techniques to calculate optimal perturbations in fluid dynamic applications dates back to Farrell (1988), who calculated the optimal excitation in constant shear flow, first in two and later in three dimensions (Butler & Farrell 1992; Farrell & Ioannou 1993).

2. Governing equations

The motion of an infinitesimal perturbation $\{u'(x, t), p'(x, t)\}$ in a steady base flow $U(x)$ is governed by the linearised Navier-Stokes equations

$$\frac{\partial u'}{\partial t} = -\left(U \cdot \nabla\right) u' - (u' \cdot \nabla) U - \nabla p' + \frac{1}{Re} \nabla^2 u', \quad \nabla \cdot u' = 0 \quad \text{in } \Omega. \quad (1a)$$

where $u'$, $U$ are the perturbation and base-flow velocities and $p'$ the perturbation pressure. $Re = u_\infty \delta^*/\nu$ is the Reynolds number based on the displacement thickness $\delta^* = \int_0^\infty \left(1 - \frac{U_x}{u_\infty}\right) dy$ of the asymptotic far-field solution which coincides with the classical Blasius boundary layer solution for the streamwise and wall normal velocity component. Unit density is assumed and all quantities are non-dimensionalised by their respective far-field values. In particular, we take the displacement thickness at the inlet of the computational domain as characteristic length scale and for non-dimensionalisation.

2.1. Optimal initial conditions

Within the framework of optimal initial conditions we are interested in finding an initial solution to (1), possibly subject to other constraints, that maximises some measure of energy while evolving over a finite time span $t \in [0, \tau]$. To do so, we first introduce the concept of a linear evolution operator $A(t)$ that maps an initial solution $u'(t_0)$ to $u'(t + t_0)$ at some other time instant, i.e. $u'(t + t_0) = A(t)u'(t_0)$. Further, we use the standard $L_2$ inner product $\langle u, v \rangle = \int_\Omega u \cdot v \, d\Omega$, which conveniently defines the norm associated with the kinetic energy of the perturbation $\|u'\|_2 = \langle u', u' \rangle$. As we are interested in the transient development of an initial perturbation, we first have to define a suitable measure of such, i.e. transient growth. The term refers to the ratio of the perturbation kinetic energy at some finite time $\tau$ normalized by its initial condition (Trefethen et al. 1993; Reddy & Henningson 1993) and can be recast in terms of the linear evolution operator as follows:

$$\frac{E(\tau)}{E(0)} = \frac{\|u'(\tau)\|_2}{\|u'(0)\|_2} = \frac{\langle A(\tau)u'(0), A(\tau)u'(0) \rangle}{\langle u'(0), u'(0) \rangle} = \frac{\langle u'(0), A^\dagger(\tau)A(\tau)u'(0) \rangle}{\langle u'(0), u'(0) \rangle}. \quad (2)$$

Here, the action of the adjoint operator $A^\dagger$ is given by $\langle A(\tau)\mathbf{u}', \mathbf{u}' \rangle = \langle \mathbf{u}', A^\dagger \mathbf{u}' \rangle$. By looking for the initial solution that maximises the transient
growth we define the maximum growth $G(\tau)$ as

$$G(\tau) = \max_{||u'(0)|| \neq 0} \frac{E(\tau)}{E(0)} = \max_{||u'(0)|| \neq 0} \frac{\langle u'(0), A^\dagger(\tau) A(\tau) u'(0) \rangle}{\langle u'(0), u'(0) \rangle}.$$  

From (3), note that the last term in the equality corresponds to the induced matrix norm of $A^\dagger(\tau) A(\tau)$. Hence, the dominant eigenvalue of $A^\dagger(\tau) A(\tau)$ (or singular value of $A(\tau)$, respectively) is the solution to the optimisation problem (3).

The action of the evolution operator on an initial perturbation can be obtained from integrating the underlying initial value problem. From that perspective, the linear operator defined by equations (1) acts as the infinitesimal generator of the evolution operator $A(\tau)$ (Edwards et al. 1994). The action of the adjoint evolution operator $A^\dagger(\tau)$ can be approximated just like that by integration of the adjoint equations corresponding to the linearised Navier-Stokes equations (1)

$$-\frac{\partial u^\dagger}{\partial t} = -(U \cdot \nabla) u^\dagger + (\nabla U)^T \cdot u^\dagger - \nabla p^\dagger + \frac{1}{Re} \nabla^2 u^\dagger, \quad \nabla \cdot u^\dagger = 0 \quad \text{in } \Omega. \quad (4)$$

As stated by Corbett & Bottaro (2000), the easiest way to calculate the optimal initial perturbation is to apply power iterations of the form $u'(0)^{k+1} = A^\dagger(\tau) A(\tau) u'(0)^k$ to calculate the eigenvector corresponding to the largest eigenvalue of $A^\dagger(\tau) A(\tau)$. We define a convergence criterion through the residual of two consecutive iteration steps in terms of the energy norm $||u'(0)^k - u'(0)^{k-1}||_2 \leq \epsilon$, where $\epsilon$ is the convergence tolerance and superscript $k$ denotes the iteration level. Upon convergence, $u'(0)^k$ approximates the optimal initial condition sought.

We are interested in an optimal initial perturbation that is restricted to some streamwise area $\Lambda \subset \Omega$ of the computational domain. This adds an additional constraint to the optimisation problem (3). A detailed derivation of the constrained optimisation problem using a Lagrange multiplier technique can be found in the article by Monokrousos et al. (2010).

The overall optimal initial perturbation algorithm looks as follows. Starting from some initial perturbation $u'(0)^1$, the $k$th iteration reads: (a) Integrate the linearised Navier-Stokes system (1) with $u'(0)^{k-1}$ as initial condition to obtain $u'(\tau)^k$; (b) Integrate the adjoint Navier-Stokes system (4) with $u'(\tau)^k$ as initial condition to obtain $u'(0)^k$; (c) Localise the perturbation to the subspace $\Lambda$ and apply scaling $||u'(0)^k||_2 = 1$; (d) Check the convergence criterion and go back to (a) with $k \rightarrow k + 1$ if it is not met.

Successive initial velocity field iterates can be scaled to unity perturbation kinetic energy as done in (c) without loss of generality within the linear framework.
Figure 2. Steady, self-similar base flow: (a) isolines of the streamwise velocity $U(y,z)$; (b) streamwise velocity profiles: (—) far-field, $U(y,z \to \infty)$; (---) along the corner bisector, $U(y = z)$; (c) second partial derivatives of streamwise velocity profiles: (—) far-field, $U_{yy}(y,z \to \infty)$; (---) along the corner bisector, $U_{ss}(y = z)$; (d) isolines of the crossflow velocity $V(y,z)$: (---) negative; (—) positive; (e) crossflow velocity far-field profiles: (—) wall normal direction, $V(y,z \to \infty)$; (---) tangential direction, $W(y,z \to \infty)$; (---) along the corner bisector, $V(y = z) = W(y = z)$.

3. Numerical methods

3.1. Base flow calculation

The steady base flow $\mathbf{U}(x)$ is calculated with the method described and validated in detail in Schmidt & Rist (2011).

Figure 2 shows the self-similar base flow calculated on $\Omega = [0, L_y] \times [0, L_z]$ with a polynomial order of $N = 50$. A large domain size of $L_y = L_z = 75$ guarantees validity of the asymptotic flow assumptions. The figure depicts the near-corner region $[0, 15] \times [0, 15]$. The streamwise velocity as seen in part (a) approaches its asymptotic limit at only a short distance from the opposing wall. From (d), this is obviously not the case for the crossflow velocity field. The slow (algebraic) decay of the corner influence poses the main difficulty in corner flow calculations. The crossflow velocity field results from the superposition of the displacement effects of the two flat plates. Velocity profiles for all three components along the bisector and for the asymptotic limit $z \to \infty$ are depicted in part (b) and (e), respectively. A closer look at second derivative of the streamwise velocity profile along the bisector in (c) reveals an inflexion point at
Additionally, we calculate a modified base-flow that is meant to resemble the departure of the streamwise velocity field from the self-similar (non-modified) solution as commonly found in corner-flow experiments, e.g. by Zamir & Young (1970). Their, and all following experimental studies, revealed a fast departure from the concave, self-similar solution close to the leading edge in form of an outward, convex bulge in the corner region. The typical shape of the deformation can be seen by comparison of the modified and the self-similar case as in figure 3a. The deformation was realised by circular, Gaussian-distributed volume forcing of the streamwise momentum equation. Kornilov & Kharitonov (1982) show in their experiments that the deformation predominantly results from the local pressure gradient induced by the deceleration of the flow at the leading edge where both walls intersect. Here, we restrict ourselves to a qualitative reproduction of this most prominent experimental feature. Direct matching with experimental data is not possible because leading edge effects cannot be considered within our framework.

The force amplitude is adjusted to approximately reproduce the mean-flow deviation seen in the data by Zamir & Young (1970). We assume validity of the asymptotic far-field solution nevertheless. A comparison with figure 4 in Zamir & Young (1970, p. 319) shows considerable similarity between our solution

\[ s \approx 2 \] that can give raise to an inviscid instability as addressed by Balachandar & Malik (1993).
and their hot-wire measurements for the streamwise velocity in the transversal plane. We hence assume that the modified solution shares the basic stability characteristics with the experimentally observed flow. Figure 3d and 3e give a comparison between the modified and the non-modified solution in terms of the streamwise velocity profile and its second derivative along the bisector coordinate s. The second derivative of the modified flow exhibits three inflexion points as in comparison to the unmodified flow with just one. A closer look on the velocity profiles and their curvature along the q-axis perpendicular to the bisector, as given in 3b and 3c, reveals two additional inflexion points not present in the unmodified case. Due to its highly inflexional nature, the modified base flow is likely to facilitate even stronger instabilities than those expected for the self-similar solution.

3.2. Direct and adjoint perturbation field calculation

The direct (1) and adjoint (4) perturbation equations are solved with the spectral element code Nek5000 (Fischer et al. 2008). The code uses semi-implicit timestepping with a weighted residual spectral element method (Patera 1984) for the spatial discretisation. For this study, we choose to combine 2nd-order-accurate timestepping with 7th-order-accurate spectral elements (N = 7) and solve the governing equations in $P_N - P_{N-2}$ formulation.

A computational domain of size $\Omega = [337.7, 661.9][734.7] \times [0, 40] \times [0, 40]$ is resolved by $110\{130\} \times 40 \times 40$ spectral elements. The streamwise extend corresponds to a Reynolds number regime of $1000 \leq Re_\delta \leq 1400\{1475\}$. While uniform grid-spacing is applied along the x-axis, grid points are clustered near the wall for proper resolution of the boundary layer in the y-z-plane. Numbers in curly brackets $\{\}$ refer to an extended computational domain which is necessitated by the modified base-flow calculation for $\tau = 400$ to contain the response wave packet. As initial perturbation $u'(0)^1$, we chose a wave packet generated by a spherical volume force acting on the streamwise momentum component in the time interval $t \in [0, 30]$ and centred in space about $(340.65, 2.01, 2.01)$. Note that any initial perturbation that has a non-zero projection on the optimal solution would suffice. The streamwise extent of the initial perturbation is restricted by the localisation procedure described in §2 to $\Lambda = [365, 425] \times [0, 40] \times [0, 40]$. The perturbations are smoothly ramped to zero in the outmost 10% of both sides of the localisation region, following a fifth-order polynomial distribution.

4. Results

Optimal initial conditions are calculated for the unmodified and the modified base-flow for two optimisation times $\tau = 200$ and $\tau = 400$. In the following, we first compare the amplification rates for all parameter combinations. Second, we discuss the underlying flow phenomena by examination of the coherent structures representing the optimal initial condition and the optimal responses.
Figure 4. Global perturbation kinetic energy for direct (a) and adjoint (b) sweep; (——) unmodified base-flow, $\tau = 400$; ($--$) modified base-flow, $\tau = 400$; ($\cdot\cdot\cdot$) unmodified base-flow, $\tau = 200$; ($\bullet\bullet\bullet$) modified base-flow, $\tau = 200$.

<table>
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<th>$\tau$</th>
<th>$|u'(\tau)|_2$</th>
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Table 1. Values of the perturbation kinetic energy of the optimal response $\|u'\|_2$ and temporal amplification rate $\omega_i$ at final time $t = \tau$.

The temporal evolution of the perturbation kinetic energy is depicted in figure 4. All four cases exhibit similar direct (4a) and adjoint (4b) evolution behaviour. From 4a, rapid transient growth at $0 < t \lesssim 150$ is followed by constant exponential growth as indicated by the constant slope in the logarithmic scale. The exponential energy increase indicates modal growth, even for the smaller optimisation time. This contrasts findings by Monokrousos et al. (2010) for the self-similar flat plate boundary layer. Here, streamwise vortices that induce non-modally growing, elongated streamwise streaks were identified as optimal initial conditions for an optimisation time $\tau = 720$ for the Blasius boundary layer solution. Seemingly, this scenario does not apply to corner flow. Also from 4, it becomes apparent that the perturbation energy development for the modified base-flow exceeds that of the self-similar base flow while no substantial difference can be seen when comparing results for different optimisation times but for the same base flow. As reasoned from the highly inflexional nature of the modified base flow profile (§3.1), the outward bulge in the streamwise velocity isolines significantly destabilises the base-flow. Table 1 summarises the findings in terms of perturbation kinetic energy of the optimal response and the corresponding amplification rate. Under the assumption of exponential growth, we define the temporal amplification rate in terms of the
rate of change of the kinetic energy as $\omega_t = \frac{\partial}{\partial t} \ln \|u\|_2$ to allow direct comparison to linear theory. It can be seen that the amplification rates for the modified base flow case are significantly higher than for the self-similar solution. This again suggests that the deformation as seen in experiment destabilises the flow. The absolute numbers for the perturbation kinetic energy at $t = \tau$ are given in the same table.

We now seek for the causes of the observations made above by examining the flow structures representing the optimal initial condition and response.

Isosurfaces of the streamwise perturbation velocity of the optimal initial conditions and their corresponding responses are shown in figure 5. The contour levels are separately set to 20% of the global maximum since the perturbation amplitudes differ by multiple orders of magnitude, as seen in table 1. The first observation is, that both the optimal initial conditions as well as the responses are even-symmetric with respect to the corner bisector. Also, the structures are active in the corner region only. This confirms the presumption of a dominant, inviscid instability mechanism active in the near-corner region. Note that the initial condition is artificially localised in the streamwise direction, only. Both, the optimal initial conditions as well as the optimal responses appear in form of modulated wave packets. When comparing flow structures of the localised optimal initial conditions with (5b,f,d,h) and without (5a,e, c,g) base-flow modification, we observe that the streamwise wave number has approximately doubled in the latter case. Also, from comparison of the streamwise responses’ wave packet locations, the group velocity of the optimal response has increased.
Figure 6. Streamwise perturbation velocity isocontours for the self-similar (filled contours: (yellow/bright) positive, (blue/dark) negative) and the modified (lines: (——) positive, (−−−) negative) base-flow for optimisation time $\tau = 200$; (a,b) localised optimal initial conditions (transversal plane at $x = 400$ and bisectorial plane); (c,d) optimal responses (transversal plane at the respective global maximums’ streamwise positions and bisectorial plane).

The transient growth mechanism becomes apparent in figure 6. Here, lines of equal streamwise perturbation velocity in the transversal and bisectinal planes of the optimal initial conditions as well as the corresponding responses are shown for $\tau = 200$ and for both base-flows. It can be seen that the contour lines of the optimal initial conditions on the bisector in 6b are tilted backwards, i.e. against the mean shear. A forward tilt is observed for the optimal responses in 6d. This behaviour is typically found in transient growth and related studies and termed the Orr mechanism. It allows the initial perturbations to gain maximum energy from the mean flow over a short period of time as indicated by the initial perturbation amplification seen in figure 4a. The flow structures of the optimal responses depicted in 6c closely resemble the inviscid corner mode emerging in viscous linear stability calculations (Parker & Balachandar 1999) in the unmodified case.

For the modified base-flow case, the optimal perturbation and the response follow the base-flow deformation contours closely in form the characteristic outward bulge (compare 6a,c with 3a) and are therefore located further away from the wall intersection in 6b,d.
5. Summary and conclusions

Localised optimal initial perturbations for the corner-flow problem were calculated in a computational domain starting a distance from the leading edge corresponding to a local displacement-thickness-based Reynolds number of $Re_{\delta^*} = 1000$. Two optimisation times, $\tau = 200$ and $\tau = 400$, were considered for two different base-flows, i.e. the well-established self-similar corner-flow solution and a modified base-flow that mimics characteristics found in experiment. A power-iteration-based optimisation technique consisting of direct and adjoint linearised Navier-Stokes solution sweeps was used to converge the optimal initial conditions.

In all cases, the optimal initial condition manifests as a wave packet, symmetric with respect to the bisector and localised in the near-corner region. The perturbation is inherently of inviscid nature, similar to the corner-mode found in linear stability investigations. Exponential growth of the optimal response is observed after an initial time period of substantial transient growth caused by the Orr mechanism. The basic flow manipulation leads to increased amplification rates that go along with a higher group velocity and streamwise wave number.

The study at hand is the first global stability analysis of streamwise corner-flow known to the authors and confirms three hypotheses, previously postulated in literature. First, the presence of a dominating inviscid instability mechanism in the corner region (Balachandar & Malik 1993). Second, a destabilising effect through the mean-flow deformation as observed experimentally (Kornilov & Kharitonov 1982; Desai & Mangler 1974). Third, a high sensitivity to mean-flow deformations as found by Alizard et al. (2010), although this aspect is closely related to the second mentioned.

However, the survey at hand underlies certain restrictions. All possible non-linear interactions are neglected as we consider linear perturbations only. The spatial growth of the wave packet and the base-flow itself restricts us to relatively high Reynolds numbers and short optimisation times to keep the computational effort reasonable. Also, the modified base-flow is artificially created in a qualitative manner, thereby neglecting its genuine origin.

A survey on the global, linear and non-linear stability properties of corner-flow at low Reynolds numbers as well as laminar-turbulent transition is work in progress.
References


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