KTH Engineering Sciences

Statistical energy analysis and variational principles for the prediction of sound transmission in multilayered structures

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Doctoral Thesis

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Abstract

Multilayered structures have many application in industry and society: they have peculiar properties and serve a variety of purposes, like structural support, thermal insulation, vibrational and acoustic isolation. This thesis concerns the prediction of sound transmission in multilayered structures. Two problems are herein investigated: the transmission of energy through structures and the transmission of energy along structures. The focus of the analysis is on the mid to high frequency range. To predict sound transmission in these structures, statistical energy analysis (SEA) is used.

SEA models are devised for the prediction of the sound reduction index for two kinds of multilayered structures, double-walls used in buildings and trim-panels in vehicles; the double-walls comprise an air cavity in between flat plasterboard or glass plates, whereas the trim-panels a porous layer in between curved aluminium and rubber layers. The SEA models are based upon the wave-types carrying energy. The novelty in these SEAs is an element describing the waves in the air cavity, or in the porous layer, fully coupled to the mass-impeded external layers. Compared to measurements, the proposed SEA performs well: for double-walls, it performs better than previous models; for trim-panels, it is an original result. The parameters of the new SEA element, such as modal density, are derived from the coupling equations describing the fully coupled waves. For double-walls, these equations are derived via Newton's laws. For trim-panels, a variational approach based upon a modified Hamilton's principle valid for non-conservative systems is preferred, because it is a powerful machinery for deriving equations of motion and coupling conditions of a medium as complex as the porous layer. The modified Hamilton's principle for non-conservative systems is based upon a self-adjoint functional analogous to the Lagrangian, inspired by Morse and Feshbach's construction. A self-adjoint variational principle for Biot's equations in the displacement formulation is devised. An equivalent mixed formulation is obtained changing the coordinates of the displacement formulation via Lagrange multipliers. From this mixed formulation, the Lagrangian for a porous material with a limp frame is derived, which yields the continuity of the total displacement of the porous layer. Lagrange multipliers help to obtain the correct coupling functionals between a porous material and a solid. The Lagrange multipliers introducing the continuity of the frame and the solid displacements equal the traction of the in-vacuo frame, thus disappearing if the latter is limp. Measurements to gather material parameters for a Biot model of the porous layer have been conducted.

The effects of spatial energy decay in the transmission along structures predicted by SEA is studied: a major effect is the increased relevance of indirect coupling loss factors between SEA elements. This may jeopardize the usefulness of SEA at higher frequencies.

Papers contained in the Doctoral thesis

This thesis consists of the following papers:

1. S. Finnveden and M. Barbagallo. A cavity-wall element for the statistical energy analysis of the sound transmission through double walls, *to be submitted to Applied Acoustics*.

2. M. Barbagallo and S. Finnveden. A self-adjoint variational principle for anisotropic viscoelastic Biot's equations. *International Journal of Engineering Science* 2013; 63, pp 71-83.

3. M. Barbagallo and S. Finnveden. Statistical energy analysis of the sound transmission through layered panels using a variational formulation of the porous material, *to be submitted*.

4. S. Finnveden, N.-E. Hörlin and M. Barbagallo. Characterization of the in vacuo viscoelastic material properties of porous foams used in vehicles, *submitted to Journal of the Acoustical Society of America*.

5. M. Barbagallo and S. Finnveden. Characterisation of a generic trim-panel: sound reduction index and material parameters.

6. M. Barbagallo and S. Finnveden. Spatial energy decay and indirect couplings in statistical energy analysis, *Proceedings of ISMA 2010*.

Division of work between the authors of the papers

1. Finnveden developed the theory and wrote the paper with Barbagallo.

2. Barbagallo performed the analysis and wrote the paper, supervised by Finnveden.

3. Barbagallo performed the analysis and wrote the paper, supervised by Finnveden; Liu provided plots of the dispersion curves.

4. Barbagallo did the measurements with the first rig and participated in the setting up of the second rig, developed and used by Hörlin; Barbagallo wrote a preliminary Mid-mod report for the measurements; Finnveden wrote the paper and did the inverse estimation.

5. Barbagallo did the measurements and wrote the report, supervised by Finnveden.

6. Barbagallo performed the analysis and wrote the paper, supervised by Finnveden.

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Contents

Part I

Overview and Summary

Chapter 1

Overview

Multilayered structures have many applications in industry and society. The fuselage of an airplane, the floor of a truck, the door of a car, the wall of a building are examples of such structures. These structures have peculiar properties and serve a variety of purposes, such as structural support, thermal insulation, vibrational and acoustic isolation. The characterisation of their properties is an on-going challenge for engineers: the physics of these structures is usually complex to understand and to model.

As their name suggests, these structures are made of layers of different materials. The combination of the properties of each layer and the way they interact with each other define what a multilayered structures is useful for. *How* the interaction between layers functions and *which* consequences this interaction has on the wished use of the panel are the interrogatives motivating and driving a lot of research, including this thesis.

Common materials used as layers are wood, chipboard, plaster, rubber, metal, cloth, porous materials, mineral wools, honeycomb structures. Some of the layers may often be shaped with particular geometries, such as curved plates and extruded metal profiles. Porous materials often serve as layers because of their appealing properties: they are light, they dissipate well vibrations and they are thermal isolators.

This thesis is concerned with sound transmission in multilayered structures employed in buildings and vehicles. Two kinds of multilayered structures are studied: *double-walls* used in buildings and *trim-panels* used in vehicles. The studied double-walls comprise flat plasterboards separated by an air cavity and a double-glass window, while the trim-panels a curved aluminium layer and a rubber layer separated by a layer of porous material.

Two kinds of problems are investigated: first, the transmission of energy through the structures - e.g., the airborne sound insulation capacity of a double-wall; second, the transmission of energy along *long* structures - e.g., the propagation of waves along the fuselage of an airplane.

The focus of the analysis is on the mid-frequency and high-frequency ranges. The

distinction into high, mid or low frequency ranges may be explained in two fashions, related to each other. First, if the wavelength of the acoustic disturbance is small (large) compared to the dimension of the structure, one speaks of high (low) frequency. Second, whether the statistics of the problem should be included or not in the analysis. At low frequencies, nominally identical structures, e.g. the doors of a car from the same production line, behave similarly, having similar vibroacoustic responses: statistics is not necessary. On the contrary, at higher frequencies the responses change dramatically and statistics must be included in the analysis. Thus, depending on frequency, a deterministic or a statistical approach should be preferred and, consequently, different mathematical methods may be employed. For instance, standard finite elements can be used at low frequencies for a deterministic analysis, while energy methods can be used at high frequencies for a statistical analysis. The mid-frequency range is particularly challenging because it is too high for deterministic approaches and too low for statistical ones; as a number of recently funded EU-projects shows, the research interest in this frequency range is high. Most of the work in this thesis has, in fact, been funded by one of these projects, Mid-Mod.

The title of the thesis is briefly explained. Statistical energy analysis is perhaps the most common high frequency and energetic approach and is the principal method of choice in this work. Variational principles enter in the work because they are the theoretical foundation of part of the analysis: a modified Hamilton's principle also valid for non-conservative systems is used to study the trim-panel.

1.1 Transmission through multilayered structures

Multilayered structures may be employed to acoustically isolate a target environment from a sound source. Think of a wooden double-wall separating a bedroom and a room with a stereo playing. The power emitted by the stereo impinges on the external layer of the structure: some power is reflected and some is transferred to the other layers. The layers are typically constituted by different materials with different geometries. The energy flows through them via a number of *channels*. The energy transported by these channels may be structure-borne or air-borne, may be due to resonant or non-resonant motion, may exist at low frequencies or at high frequencies only. The resonant-energy channels are associated to different types of propagating waves. These wave-types depend on the excitation, on the layers and on their boundary and coupling conditions. Once the energy has flown through the whole structure, it is radiated into the target environment. To achieve a good sound insulation, most of the energy should be dissipated within the double-wall, reflected back into the room with the stereo and little radiated into the bedroom. There is a lot of interesting physics happening here.

The airborne sound transmission of a multilayered structure can be quantified via the sound reduction index (SRI). The SRI of a structure depends on its vibroacoustic

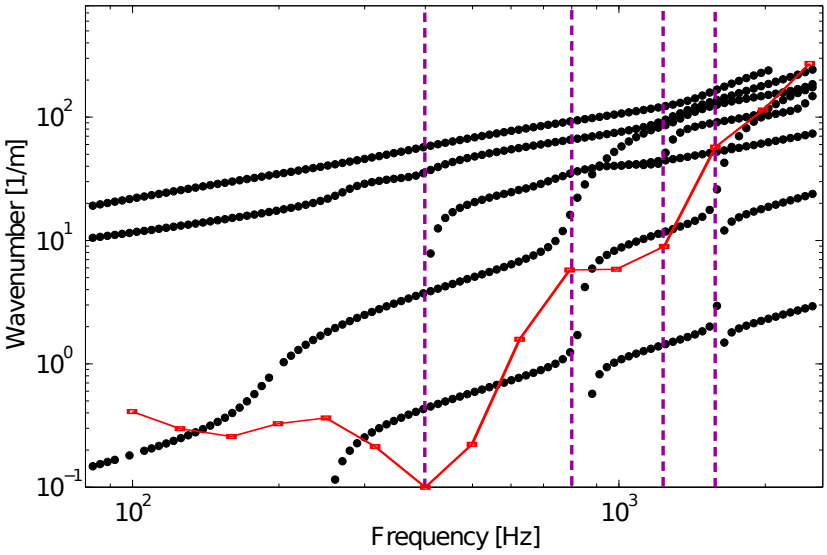

Figure 1.1: Dispersion curves of a multilayered panel, described in Ref. [57]. Numerical dispersion curves from waveguide finite element method [42] (black dots). Measured SRI of a multilayered panel taken from Ref. [57] plotted over the dispersion curves (full curve with squares). The frequencies at which new waves start propagating correspond to changes in the SRI, as evidenced by the vertical dashed lines.

response, which in turn depends on the waves propagating within it: as illustrated in Fig. 1.1, the frequencies at which new waves cut-on, i.e. start propagating, correspond to changes in the SRI. The wave motion of multilayered structures can be complex: each layer may support many waves, the coupled substructures may support other waves, waves may cut-on at specific frequencies. Propagating waves transport energy across the structure and, eventually, the SRI depends on them. The understanding of this energy flow is salient and fascinating. Papers 1 and 3 deal with this.

In the literature there are many works on techniques to predict the SRI of multilayered structures: Paper 1 and 3 respectively give an overview for double-wall panels and for multilayered panels containing a porous layer. It is wished that any mathematical approach calculating the SRI of a structure has the following properties, besides giving a correct prediction. First, it should provide insights into the physics of the structure. Second, it should be able to identify and rate the contributions of the various energy channels to the SRI. Third, the statistical variability of real-life structures should be taken into account. Fourth, it should provide all this in a short calculation time.

Standard deterministic methods may not easily meet all four properties: they may not help understanding the physics, unless post-processing routines are implemented; the rating of the various contribution to the SRI is not immediate; their usefulness is questioned by the statistical variation of, e.g., the properties of the layers and the inexact knowledge of boundary and coupling conditions; their computational cost is high, especially when frequency dependant parameters and porous materials are involved [32]. On the contrary, statistical energy analysis (SEA) may, in principle, comfortably meet all four properties.

SEA is a statistical method for predicting the vibroacoustic response of built-up structures at higher frequencies, intrinsically accounting for their statistical variation; SEA's rationale and its underlaying assumptions are presented in Ref. [53]; criteria for stating the applicability of SEA to complex structures are still object of researches, such as Ref. [11]. SEA is based upon a set of elements describing the vibroacoustic response of the structure and satisfying the statement of energy conservation; the statement is expressed via a set of linear equations, which is solved for the energies of the elements in a blink of an eye on any computer. The elements may correspond to wave-types, rather than to substructures: this sometimes improves the performance of SEA, in terms of the predicted result and of the first two properties above. The identification of the elements governing the response is immediate. More details on SEA are given in Chapter 2. For now it suffices to say that very important parameters for SEA are the wavenumbers of these wave-types. The relation between wavenumbers and frequencies is termed dispersion relation; for a coupled structure an example is given by Fig. 1.1. The importance of dispersion relations is twofold. First, their understanding may suggest a set of SEA elements describing the vibroacoustic response. Second, the wavenumber of a wave-type is related to its modal density, which is a parameter important for SEA. The dispersion relations may be obtained in at least three ways. First, they may be found in textbooks for uncoupled wave-types, such as bending waves in plates or longitudinal waves in rods (e.g., Ch. 3 of Ref. [38]). Second, they may be calculated from numerical deterministic methods, such as the waveguide finite element method (WFEM), Ref. [25]. Third, they may be obtained from analytical analyses, perhaps simplifying part of the problem to obtain simple results (e.g., see Section 3.5.1). In this thesis, all these method are employed (Papers 1 and 3).

The WFEM is a deterministic tool convenient for describing structures with a complex cross-section and constant properties along the other direction(s). Its main outputs are the dispersion relations of the propagating waves, see Fig. 1.1. In Papers 1 and 3, the WFEM is used for diagnostic analyses of the dispersion curves of double-walls and trim panels. Upon the understanding of their dispersion curves, the wave-types carrying energy through the structure are identified, thus setting the basis for an SEA.

Despite its appealing features, the potentialities of SEA with multilayered structures have not been fully explored. In this thesis, original SEAs for this class of structures are derived. In Refs. [13, 14, 12] Craik and Smith use SEA for calculating the SRI of double-walls with and without added absorption. The cavity with added absorption is modelled as an air cavity with a higher damping loss factor; the damping loss factor is estimated via a reverberation-time measurement, which, however, decays too quickly An alternative to Craik and Smith's work for double-walls without added absorption is shown in Fig. 1.2. The figure presents the set of SEA elements devised in Paper 1, for the same structures as those used by Craik *et al.*. The novelty is the cavity-wall element, modelling the waves in the air cavity fully coupled to the walls, when the motion of the walls is mass-impeded only. In Paper 3, a similar approach is used for trim panels

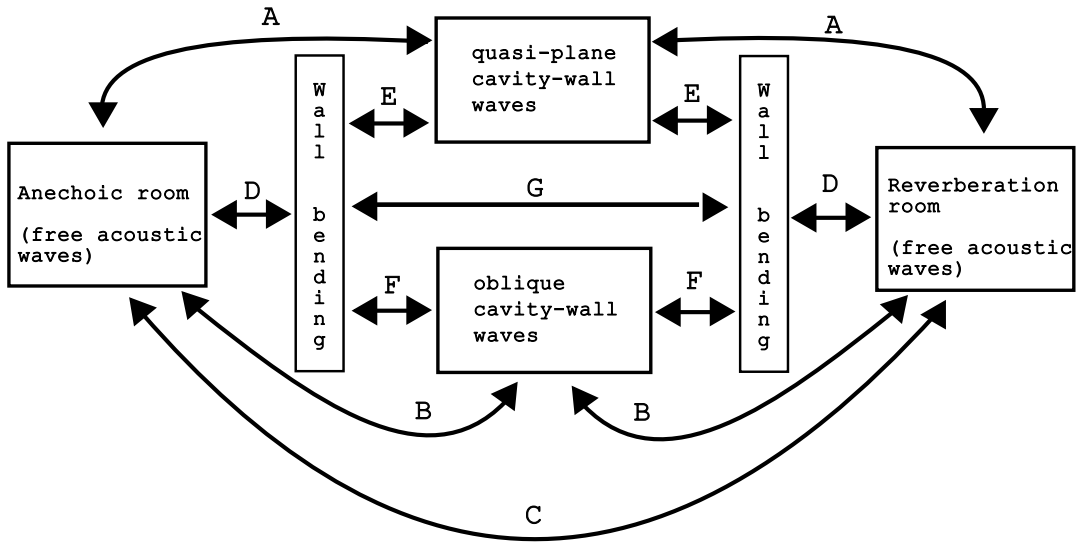

Figure 1.2: SEA for a double-wall filled with air, see Paper 1. The arrow define different channels for energy flow: **A**, quasi-plane cavity-wall waves ↔ free acoustic waves; **B**, oblique cavity-wall waves ↔ free acoustic waves; **C**, free acoustic waves ↔ free acoustic waves (mass law, non-resonant); **D**, free acoustic waves ↔ wall bending waves; **E**, quasi-plane cavity-wall waves ↔ bendin waves; **F**, oblique cavity-wall waves ↔ wall bending waves; **G**, wall bending waves ↔ wall bending waves.

comprising porous layers: SEA elements modelling the waves in the porous layer fully coupled to the mass-impeded external layers are sought; these waves are named *trim-coupled waves*. The porous material is described by an equivalent fluid theory, derived from Biot theory assuming a limp frame; the input parameters to these theories can be measured, as shown in Ref. [37] and in Paper 5. The author thinks this is the first SEA based upon either Biot's or equivalent fluid theories for modelling porous layers in multilayered structures.

An additional incentive for more research in this field comes from Ref. [36]: after having compared many existing models for sound transmission in double-walls, Hongisto concludes that the versatility of most of these models is very limited. Paper 1 is relevant also in this regard: the proposed SEA performs well with three types of building-construction double-walls, two aluminium double-walls and one double-glass window.

The SEA model in Fig. 1.2 needs parameters for the wave-types in each SEA element. All elements but the two cavity-wall wave elements are standard SEA elements and the expressions of their SEA parameters are known. The equations describing the cavity-wall waves in double-walls are derived in Paper 1 via Newton's law and the calculation of the SEA parameters follows. The equations describing the trim-coupled waves in the trim panel are more difficult to derive than the cavity-wall waves, because a porous material is generally more complex to model than air. The trim-coupled waves are described by the coupling equations between the external layers and the porous layer; these equations are derived in Paper 3 and Chapter 2. To derive them, a more powerful machinery than Newton's law is preferred, due to the complexity of the involved media: Hamilton's variational principle, a sound machinery to derive equations of motions and coupling

conditions for complex structures (see, e.g., Ref. [51] on the usefulness of variational principles for dealing with complex couplings).

1.2 The variational approach

In this thesis a variational approach is used to derive the equations describing the trim-coupled waves in the trim-panel. This is done within the framework of Hamilton's principle: the Lagrangians of the various layers, and any constraint functional, are summed: this sum is required to be stationary for the true coupled motion of the trim-panel. Hence, the equations of motion of the layers and their coupling equations follow. Once the trim-coupled wave equations are obtained, their wavenumber, their modal density and their wave form are calculated and used in an SEA for sound transmission (Chapters 3, 4 and Paper 3). In this work the variational approach also helps: to obtain equivalent formulations of a medium when its variable(s) of response is (are) changed; to better understand complicated couplings, such as those of porous materials; to manage coupling conditions when simplified theories are derived from more complex ones: e.g., the coupling conditions of a porous material, first described by the three Biot's waves then by one equivalent fluid wave.

The layers in the trim-panel are non-conservative, because they are constituted either of viscoelastic materials, or of intrinsically lossy materials, like porous materials. Hamilton's principle can, however, be used with conservative systems only; hence, alternative solutions are needed to cope with the non-conservative layers of the trim-panel.

The literature offers some solutions [54, 47]; one particularly interesting is sometimes called Morse and Feshbach's construction [48]. The original system under study is considered together with its *mirror* system. The mirror system is in all aspects identical to the original, except that it has negative losses. Intuitively, the losses of the original system *go to* the mirror system, thus making the global system "conservative". Therefore, Hamilton's principle can be applied to the global system and used to derive the equations of motion of the (non-conservative) original system and its boundary/coupling conditions. Mathematically speaking, the mirror system is the adjoint of the original and the global system is self-adjoint.

Morse and Feshbach's construction has already been used in vibroacoustics [30, 49, 20, 25] and this work continues this tradition. Before using the construction for the trim-panel, a self-adjoint variational principle for the porous layer described by viscoelastic Biot's equations is defined, since such principle was missing in the literature. The principle is presented in Paper 2 and summarised in Chapter 3.

1.3 Measurements

Various measurements on the trim-panel and its components are taken. This is a necessary step to obtain data, both to input to numerical models and, then, to check their performance, e.g. calculating the SRI. The measurements taken on the trim-panel are: 1) SRI of the trim-panel following ISO 15186 - 1:2000 (Paper 3); 2) static flow resistivity of the porous materials following ISO 9053:991 (Paper 5); 3) frequency dependant viscoelastic parameters of the in-vacuo frame of the porous materials (Paper 4); 4) static Young's modulus and Poisson's ratio of the frame of the porous material, not in-vacuo (Paper 5); 5) shear modulus of the rubber plate (Paper 5).

The measurements follow ISO standards or published procedures [26]. The exception is measurement 3, which presents original experimental rigs to estimate the viscoelastic properties of porous materials, using an inverse estimate procedure. The reader can refer to Papers 3, 4 and 5 for more details on these measurements: Part I of this thesis does not specifically summarise them.

1.4 Transmission along long structures

The motivation of the work on SEA for the vibroacoustic transmission along structures many wavelengths long comes from two observations in the literature, Refs. [22, 27]. These works contain problems that can be handled by SEA. Under certain conditions, however, SEA fails where it is most expected to work: at higher frequencies. An explanation may be the violation of one SEA hypothesis, i.e., the requirement that the energy density in an SEA element remains constant. If an element is many-wavelengths long, however, spatial energy decay may occur and this SEA hypothesis is violated. Then, the neat form of the SEA equations stating energy conservation has to be modified to account for the so-called indirect couplings between non-adjacent elements. Consequently, the convenience and the applicability of SEA are questioned. Paper 6 investigates this problem and Chapter 2 summarises some of its conclusions.

1.5 Organisation of the thesis

The next chapters summarise the most relevant results of this thesis. Chapter 2 presents SEA, introducing its rationale and its founding hypotheses; it is related to Papers 1, 3 and 6. Chapter 3 introduces Hamilton's principle, illustrates a self-adjoint variational principle for porous materials, and shows how to derive the equations of the trim-coupled waves; it is related to Papers 2 and 3. The calculated modal density is used in Chapter 4 in the SEA of the whole trim-panel, which calculates its SRI; it is related to Paper 3 (and uses the data collected in the measurements in Papers 4 and 5). Chapter 5 draws conclusions. Part 2 contains the appended papers.

Chapter 2

Statistical energy analysis

In this chapter statistical energy analysis (SEA) is presented. The first section illustrates its rationale and its underlying assumptions. The second section investigates the consequences of energy decay in a structure, by studying the form of SEA equations and its variables, relating them to power injection measurements and deterministic analyses.

2.1 The rationale of SEA

SEA is an approximate method to estimate steady state ensemble, space and frequency band averages of vibroacoustic energy in built-up structures at somewhat higher frequencies [53]. Today, there seems to be an accord, or at least a hope, in vehicle industry and other parties interested in vibroacoustic predictions that SEA works, if only the frequency is high enough - as is evidenced by the public homepages of the ongoing EC-funded projects Mid-Mod and Mid-Frequency.

The governing equations in an SEA express the law of energy conservation for elements, each describing in a frequency band one kind of response in a substructure of the whole structure. The formulation relies on two postulates. First, the rate of energy dissipated in an element is assumed proportional to the energy in that element. Second, the rate of energy flow between two directly coupled elements is assumed proportional to the difference of the elements' modal power potentials, being the energies times the expected frequency spacing between their natural frequencies [53]. This is the law of coupling power proportionality (CPP). It is a very powerful potential flow model for the conduction of vibroacoustic energy, similar to Fourier's law for heat conduction and Ohm's law for electrical conduction.

The constant of proportionality in the CPP law, the so-called conductivity, is often calculated by a one way method, where the action of one element onto the other elements is investigated. In doing so, field characteristics of the first element are defined, such as the amplitudes of vibroacoustic waves impinging on a junction; the amplitude in a diffuse field or the amplitudes of modes proper to the uncoupled element. Often the

calculations are simplified by considering elements that are extended away from the junctions towards infinity. The conductivity evaluated by a one way method describes a potential flow and can be used in a proper SEA model only if it is a symmetric function of the elements' properties in such a way that it is equally evaluated when field characteristics in the first and in the second element are prescribed. This symmetry relation is termed SEA reciprocity [53] or consistency [45] relation. It is demonstrated for numerous one way calculations. Similar to one way methods in nature, but not in mathematical formulation, are perturbation methods [23, 41]

The first postulate is an established engineering approximation; however, the second postulate, the CPP law, is useful only sometimes. To date it has been exactly demonstrated for the rate of energy flow between two conservatively coupled, randomly excited, oscillators [56] and, also, for the ensemble averaged energy flow between two one-dimensional structures, with random properties, in the limit of vanishing spatial damping decay within the elements, [19], see also [44]. In contrast, for three oscillators [58] and for the ensemble averaged of three one-dimensional structures [19], the CPP law is true for weak coupling only. For general structures, the range of usefulness of the CPP law is not established, though, slowly appears a common understanding of the criteria for a valuable SEA model. One example is provided in a article by Le Bot and Cotoni [11], stating that, for each element, each junction and each frequency band, SEA is valid if: i) The number of natural frequencies is large; ii) The modal overlap factor is large (it is the ratio of resonances' 3-dB bandwidth to their expected frequency spacing); iii) The strength of coupling is small; iv) The spatial decay of energy in a wave travelling across an element is small. The first three criteria are for many engineering structures better fulfilled as frequency increases while the opposite is true for the fourth criterion.

In this thesis, the work related to the transmission through multilayered panels is founded upon the identification of a good set of SEA elements, which catches well the physics of the structure and which can be used to obtain a good prediction of the SRI at mid-frequencies (Chapters 4 and Papers 1, 3); e.g, Fig. 1.2 shows the set of elements used for double-walls, which is similar to that used for trim-panels except for the cavity-wall waves element, substituted by the trim-coupled waves element. The work related to transmission along long structures investigates what happens to SEA if the fourth assumption fails (next section 2.2 and Paper 6).

2.2 SEA and indirect couplings

The fundamental equation in SEA expresses the law of energy conservation. This law can also be expressed based on calculated or measured response using the power injection method (PIM). Thus, for a given excitation, the input power is evaluated and the energies of the elements are linearly related to this input power. The procedure is repeated for excitation in all elements and an energy distribution (ED) model is

devised [34]. Then, when the ED model is inverted, an *SEA-like* model results. Like SEA models, an SEA-like model relates the energy stored in the elements to the input powers. Though, as stressed by Fredö in Ref. [28], the term SEA-like signifies that the energy balance is applied for the individual and deterministic case, not for the ensemble averaged energy balance for random structures, as SEA attempts. Nevertheless, the dependant variables in an SEA-like and a proper SEA model are equal when the frequency- spatial- ensemble-averaged responses of the single specimen and of the ensemble are similar, which is most likely when the modal overlap factors are large, see e.g. [55].

An SEA-like model is, with respect to the precision of the measurements or calculations upon which it is based, an exact statement of energy conservation. Comparing the equations defining such a model with those that would apply for a proper SEA model, two major differences are apparent. First, the consistency relation is not fulfilled, though there are evidences for that it approximately applies for higher frequencies, when there are many resonances in the frequency bands and the elements' modal overlap factors are large. Specifically, the consistency relation is fulfilled for a homogeneous structure if the frequency and space averaged point mobilities of the elements equal those for the corresponding infinite elements [40] (and later, equally, [45]). The second difference is that the rate of energy flow to an element is not governed by the CPP law. Thus, in this mathematical clothing, there appears energy flows between elements that are not physically connected; this artefact is termed indirect coupling [45, 46], or tunnelling [35, 27, 17]. In the literature, there are different names for models that include indirect couplings, e.g., advanced SEA [35], extended SEA [27] or quasi-SEA [45]. In some cases such models might be useful. They cannot, however, be used in general predictive SEA software, as it is not practical to calculate all the necessary coupling routines that consider all the possible arrangements of connections in large structures.

The presence of indirect couplings does not, of course, describe an energy flow between two elements that are not physically connected but signifies that the energy flow between two elements depends on the response in other elements and, consequently, that CPP does not properly describe the flow of energy in the system. In the literature tunnelling is demonstrated to happen in the regime of strong coupling and small modal overlap factors [19, 27]. In particular, if a structure's response is defined by one global mode indirect couplings cannot be neglected [45].

In Ref. [46] Mace investigates the importance of indirect couplings; he notes that for increasing modal overlap factors, the indirect coupling loss factors asymptote to zero and concludes that a proper SEA model can be formulated. The study in Paper 6 confirms that with increasing modal overlap factors the indirect coupling loss factors decreases but so does also the direct coupling loss factors and the net effect is, in fact, an increasing tunnelling. Consequently, proper SEA models are not useful, for the investigated structures, at higher frequencies. Previously, Fredö found increasing tunnelling in a three-plate structure with increasing frequency ([27], paper IV) and

Finnveden found a consistent overestimation of the energy flow in a three-pipe structure at very high frequencies [22]; these observations are the basis for the study on the high frequency limitation of standard SEA done in Paper 6.

In Paper 6, the vibrations of long structures with many elements in series are described by SEA-like and proper SEA models. The vibrations considered in the deterministic SEA-like model are excited by a large number of uncorrelated point forces, thus simulating rain-on-the-roof excitation. Most of the calculations are made in the high modal overlap regime where the frequency averaged response of an ensemble and an individual structure should be similar. This surmise is supported by a few Monte Carlo runs considering elements with random properties. The investigated structures are very long compared to the wavelength and are well connected, meaning that a substantial part of the energy coming into a junction is transmitted. The applications in mind are structures such as ventilation ducts, pipelines and rib-stiffened shell structures such as found in trains, aircraft and ships. Besides of being long and well connected these systems often have piece wise constant properties along the structure and support a numerable set of waves propagating in this direction. Therefore, each section is as a set of parallel one-dimensional structures, being independent channels of energy flow. Not to obscure the general picture and to facilitate extensive averaging based on Monte Carlo simulations, simple one-dimensional structures consisting of rods in series are considered. Thus, it is possible to highlight the findings on the effects of spatial energy decay on tunnelling and the resulting failure of predictive standard SEA at high frequencies.

The investigations in Paper 6 are restricted to one-dimensional systems. Two considerations, however, can be asserted; first, it is argued that two- and three-dimensional systems with long mean free paths [11] might probably suffer the same consequences of spatial energy decay; second, one-dimensional systems are actually relevant: structures such ventilation ducts, pipelines, rib-stiffened panels that are common in trains, ships and aircrafts, are long and show piece-wise constant properties along one direction, thus supporting a numerable set of waves propagating in this direction [22]; therefore, each section can be modelled as a set of parallel one-dimensional structures, each being an independent channel of energy flow.

Chapter 3

Variational principles

Variational principles are a fascinating topic that has occupied a lot time in the writing of this thesis. The first two sections of this chapter are dedicated to explain Hamilton's principle to a reader not accustomed to it; they also explain the technique to adapt it to non-conservative systems. The following three sections present the variational principles defined in this thesis. The last section uses the equations of the trim-coupled waves, derived from a simplified analysis based upon variational principles, to derive their wave form and modal density. This chapter is related to Papers 2 and 3.

3.1 Premise

During the last football European championship, Italy won against England at penalties; one of the finest ones was Pirlo's chip penalty. Very few people would be able to score like that, but many would be able to correctly guess which trajectory took the ball, choosing between line A and line B in Fig. 3.1 (it is line A[1]!). What matters now is that line A intuitively illustrates a principle of nature, i.e. Hamilton's variational principle: line A is the trajectory that minimises the difference of the potential and kinetic energies of the ball, from the moment Pirlo kicks it to the moment it ends behind the back of the English goalkeeper. On the contrary, line B does not minimise that quantity (a similar and more formal example is in Ch. 19 of Ref. [18]).

[1]Pirlo probably exploited the Magnus effect, neglected here; however, the ball used in EURO 2012 was presumably engineered to be less sensitive to it after the complaints in the 2010 World Cup.

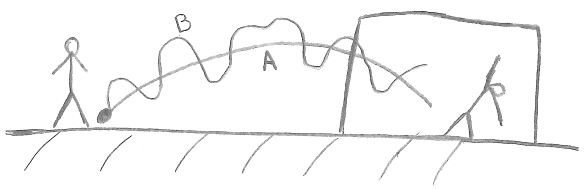

Figure 3.1: Pirlo (left) beating the English goalkeeper. The trajectory of the ball is A.

3.2 Hamilton's principle

The motion of a body is described by its boundary value problem, i.e., by a partial differential equation of motion in its domain and by conditions on its boundary; if time is a variable, initial values should also be prescribed.

One way to get equations of motion is Newton's second law. This is often done in text books, as in Cremer and Heckl for waves in solids an in Pierce for waves in air (Ch. 3 in Ref. [38] and Ch. 1 in Ref. [50]). The advantage of this approach is that for simple problems it is straightforward to apply with "basic" mathematic skills. The limitations of this approach are the following. First, this approach does not directly yield the boundary conditions of the medium, but only the equation in the domain, thus leading to an incomplete description. Second, the dimensions of the domain do not explicitly appear in the equation of motion. Third, if the motion of the medium is given by various wave-types, or if various media are coupled, it may become cumbersome or unfeasible. Fourth, it uses vectors, thus depending on the system of coordinates[2].

Another way to describe the motion of a body is Hamilton's variational principle. This principle is based upon a scalar quantity called Lagrangian and upon the machinery of calculus of variations. The Lagrangian is defined as the difference of strain and kinetic energies; mathematically, it is a functional, i.e., a *function of functions*: energy is a function of displacement, which itself is a function. The Lagrangian of a system is expressed using variables of response at choice, e.g., particle displacement or pressure in a fluid. Hamilton's principle requires that the integral between the times t_1 and t_2 of the Lagrangian, i.e. the action, is stationary for the true motion of the system: the functions for which the first variation of the action equals zero are sought or, equivalently, the functions that make the action stationary are sought (in fact, it is not relevant whether the action has a minimum, a maximum or a saddle, see e.g. Ch. 19 [18] or p 83 [59]). Searching for these functions is the field of calculus of variation[3]. The functions that make the action stationary in its domain are the Euler-Lagrange equations, or equations of motion. The functions that make the action stationary on in its boundary are the boundary conditions, usually distinguished in essential and natural conditions. The essential conditions involve the variable of response itself, the natural conditions its higher spatial derivatives.

This variational approach is often used when the medium to be described is complicated. For instance, the equations of motion of shells and of porous materials are derived with this approach [4, 8]. A difficulty of this approach is that it requires more advanced mathematical skills, i.e. calculus of variations. Another difficulty is due to an intrinsic limitation of Hamilton's principle: it can only be applied to conservative systems, i.e., systems in which the forces can be obtained from scalar potentials: e.g., the stresses

[2]Fifth, the author has always had a hard time using it!

[3]The search for the functions making a functional stationary is conceptually similar to the search for the point of stationarity of a function, i.e., to the point in which its first derivative equals zero.

in a body are derived from the strain energy that is a potential. The possibility to derive a force from a scalar potential is granted by the self-adjointness of the operators describing the system under study. Conservative systems are indeed described by self-adjoint operators, but non-conservative systems are not: e.g., the operator describing viscoelastic systems is not self-adjoint.

However, there exist workarounds to use a variational approach with non-conservative systems too. Internal losses, e.g. viscoelasticity, are dealt as follows: Hamilton's principle is applied to the conservative elastic system, its equations of motion are derived and then viscoelastic forces are directly inserted into the equations of motion. For elastic and viscoelastic theories, this is called by Biot isomorphism [9, 10]: the elastic operators are simply substituted by the viscoelastic ones. From Newton's laws point of view, this is perfectly fine: mass times acceleration equals *any* force, conservative or not (see, e.g., p 19-7 Ch. 19 Ref. [18]). Viscous losses need a different workaround: a pseudo-potential dissipation functional is used. This functional is originally introduced by Rayleigh [52] and, among others, used by Biot to derive his equations for porous materials [8]. So, Hamilton's principle is applied to the non-viscous conservative medium; then, the viscous losses are added ad-hoc via the pseudo-potential (see Eqs. (15) in Paper 2). Notable is Maximov's approach in Ref. [47]: considering interacting mechanical and heat fields, he can establish energy conservation in dissipative media and, thus, define a generalised variational principle; see also Ref. [3].

An even more interesting solution, at least from an engineering point of view, is Morse and Feshbach's construction [48]. It is beautiful because it allows the use of the machinery of Hamilton's principle also with non-conservative systems, by building a self-adjoint functional analogue to the Lagrangian that can be the basis of a self-adjoint variational principle. The functional is said to be an *analogy* to the Lagrangian because a Lagrangian refers to a *real* system, related to *nature*, while the Morse and Feshbach's construction is rather *artificial*. The functional constructed via this procedure is, in fact, a bilinear functional of the generalised coordinates of the original system and its adjoint system; the adjoint system is in all aspects identical to the original, except that it has negative losses, i.e., operators adjoint to the original ones (more on this later). It is an elegant procedure, but comes at the sacrifice of certain amount of reality in some of the *intermediate* results. The *final* results are, in fact, realistic and useful: the first variation of this constructed functional in respect to the adjoint coordinates yields the equations making the original system stationary; hence, it is an analogy to the Lagrangian, but for brevity, the term "Lagrangian" is herein used.

Self-adjoint variational principles that are stationary for true motion have appealing properties. To quote Fung and Tong on p 313 in Ref. [29]: "basic minimum principles exist, which are among the most beautiful of theoretical physics". Ref. [31] on p 370 considers them as starting points for new formulations of mechanical structures. They are excellent concepts for deriving equations of motions and natural coupling conditions, see e.g. [8, 43, 15, 2, 25, 20, 51]. They provide the theoretical foundation to

derive simplified theories for: (visco)elastic media, e.g. condensing a three-dimensional medium to two- or one-dimension (see p 45 of Ref. [33]); complex structures, e.g. describing a specific kind of motion only, like the radial-axial vibrations of fluid-filled pipes [21]. On p 313 in Ref. [29] it is added that "the computational methods of solution of variational problems are the most powerful tools for obtaining numerical results in practical problems of engineering importance"; e.g., variational principles constitute a sound basis for new finite element formulations, as indicated by Refs. [25] and [59] in pp. 76-83. Finally, in a variational formulation, the variables of response are chosen at will and there are systematic methods for coordinate transformations; such powerful methods are used in the present work, specifically in Papers 2, 3 and in the rest of this chapter.

The next two subsections comment on two interrogatives on Hamilton's principle and on the mathematical definition of adjoint operators.

3.2.1 Two interrogatives on Hamilton's principles

On p 43 of Ref. [7], Berdichevsky answers to two fascinating interrogatives[4] raised by Hertz: is Hamilton's principle supposing intentions in nature, i.e., to make a certain functional stationary? And about the functional: does it have any physical meaning at all? Berdichevsky provides neat explanations to both interrogatives on pp 43-44 of Ref. [7] (see also Ch. 19 of Ref. [18]). The answer to the first question relies on quantum mechanics, is a bit complicated and the reader may refer to the textbook. The answer to the second question follows from relativity theory and is simpler to grasp: the Lagrangian is the total energy in the Newtonian limit of the observer's time frame, a limit that is valid for velocities much lower that the speed of light. The Lagrangian has a clear meaning, it is not a "meaningless" quantity that happens to be stationary.

3.2.2 Self-adjoint operators

The operator $\mathfrak{D}^{\mathrm{A}}$ that is adjoint to $\mathfrak{D}$ satisfies the following relation

$$\int u\,\mathfrak{D}\,(v)\,\mathrm{d}t = \int v\,\mathfrak{D}^{\mathrm{A}}\,(u)\,\mathrm{d}t, \tag{3.1}$$

for any function u and v. If $\mathfrak{D} \equiv \mathfrak{D}^{\mathrm{A}}$, then $\mathfrak{D}$ is self-adjoint. For instance, if u and v describe displacement, viscous losses have $\mathfrak{D}_{\mathrm{v}} = \mathrm{d}/\mathrm{d}t$, which is an operator that is not self-adjoint, i.e.,

$$\int u\,\mathfrak{D}_{\mathrm{v}}\,(v)\,\mathrm{d}t \neq \int v\,\mathfrak{D}_{\mathrm{v}}\,(u)\,\mathrm{d}t. \tag{3.2}$$

[4]The field of variational principle may likely trigger in a student "philosophical" thoughts, which typically have difficult answers, if any: the author was happy to find fairly understandable explanations to two such interesting interrogatives, although beyond the scope of this thesis.

Its adjoint operator is $\mathfrak{D}^{\mathrm{A}} = -\mathrm{d}/\mathrm{d}t$, which characterises a system with negative losses, as expected from the discussion above.

Self-adjoint operators are very important because they are related to three fundamental properties in acoustics. First, the principle of reciprocity is demonstrated starting from the self-adjointess of the operators: e.g., the reciprocity of the Green's functions is based upon the assumption that the Green's function is self-adjoint (see the note on p 199 in Ref. [50]). Second, if an operator is self-adjoint, then it is symmetric. Third, if an operator is symmetric, then the matrixes used in computer implementation can be solved with efficient schemes.

3.3 Fluid formulations

In this section Hamilton's principle and the variational machinery are applied to a fluid. First, the Lagrangian in the displacement formulation is defined; second, via a change of coordinates the Lagrangian is expressed in the pressure formulation. This analysis and the obtained results may facilitate the understanding of next section, where the same machinery is applied to a porous material.

Time-harmonic motion at angular frequency ω is considered; accordingly, all variables in this chapter are expressed in this form,

$$v = \frac{1}{2}\left(\tilde{v}e^{\mathrm{i}\omega t} + \tilde{v}^{*}e^{-\mathrm{i}\omega t}\right), \tag{3.3a}$$

$$v^{\mathrm{A}} = \frac{1}{2}\left(\tilde{v}^{\mathrm{A}}e^{\mathrm{i}\omega t} + \tilde{v}^{\mathrm{a}}e^{-\mathrm{i}\omega t}\right), \tag{3.3b}$$

where v is a variable of response, $\tilde{v}$ is its complex amplitude, $\tilde{v}^{*}$ is its complex conjugate and the upper index $^{\mathrm{a}}$ is introduced so that $\tilde{v}^{\mathrm{a}} = \left(\tilde{v}^{\mathrm{A}}\right)^{*}$. Below, cartesian tensor notation is used, a comma in the subscript indicates a partial derivation.

The strain and kinetic potential[5] densities of a fluid described by its displacement are[6]

$$\overline{W}^{\mathrm{e}} = \tilde{U}^{\mathrm{a,e}}_{i,i}\,\tilde{R}^{\mathrm{e}}\,\tilde{U}^{\mathrm{e}}_{i,i}, \qquad \overline{K}^{\mathrm{e}} = \tilde{U}^{\mathrm{a,e}}_{i}\,\omega^{2}\tilde{\rho}^{\mathrm{e}}\,\tilde{U}^{\mathrm{e}}_{i}, \tag{3.4}$$

the superscript $^{\mathrm{e}}$ indicates that these quantities describe a fluid: $\tilde{U}^{\mathrm{e}}$ is the fluid displacement, $\tilde{R}^{\mathrm{e}}$ is the fluid bulk modulus and $\tilde{\rho}^{\mathrm{e}}$ the fluid density. The Lagrangian is defined as the difference between these two potentials:

$$\overline{L}^{\mathrm{e}} = \int_{V}\left(\overline{W}^{\mathrm{e}} - \overline{K}^{\mathrm{e}}\right)\mathrm{d}V. \tag{3.5}$$

[5]They are termed "potentials" and not simply "energies" for the same reason that the Lagrangian is in reality an analogy to the Lagrangian: they are artificial constructed bilinear functionals, from which stresses and inertia forces can be derived.

[6]Notice that only the original variables and the complex conjugate of the adjoint ones appear in $\overline{W}$ and $\overline{K}$: if a variational principle for Eq. (3.5) is true, so it is for the conjugate variables.

Its first variation is stationary for the true motion; varying the adjoint displacement, the equations of motions and boundary integrals of the original system are obtained:

$$\tilde{R}^{\mathrm{e}}\tilde{U}^{\mathrm{e}}_{i,ii} + \omega^2\tilde{\rho}^{\mathrm{e}}\tilde{U}^{\mathrm{e}}_i = 0; \tag{3.6a}$$

$$\overline{B}^{\mathrm{e}} = \int_S n_i\delta\tilde{U}^{\mathrm{a,e}}_i\tilde{R}^{\mathrm{e}}\tilde{U}^{\mathrm{e}}_{i,i}\mathrm{d}S = -\int_S n_i\delta\tilde{U}^{\mathrm{a,e}}_i\tilde{p}^{\mathrm{e}}\mathrm{d}S = 0. \tag{3.6b}$$

The derivation of the pressure formulation is based upon the invariance of the Lagrangian to coordinate transformations. In order to switch from the displacement coordinate to the pressure coordinate, Lagrange multipliers are employed. The Lagrange multipliers λ^{e} and $\lambda^{\mathrm{a,e}}$ introduce the fluid dilatation $\tilde{e}^{\mathrm{e}} = \tilde{U}_{i,i}$ as an additional variational parameter to eliminate the fluid displacement from $\overline{L}^{\mathrm{e}}$. A new functional is defined:

$$\breve{L}^{\mathrm{e}} = \overline{L}^{\mathrm{e}} - \int_V \Big(\lambda^{\mathrm{a,e}}(\tilde{e}^{\mathrm{e}} - \tilde{U}^{\mathrm{e}}_{i,i}) + \lambda^{\mathrm{e}}(\tilde{e}^{\mathrm{a,e}} - \tilde{U}^{\mathrm{a,e}}_{i,i})\Big)\mathrm{d}V. \tag{3.7}$$

The original and adjoint fluid displacements and dilatations are varied independently, obtaining the following relations:

$$\lambda^{\mathrm{e}} = \tilde{R}^{\mathrm{e}}\,\tilde{e}^{\mathrm{e}}; \qquad \lambda^{\mathrm{a,e}} = \tilde{R}^{\mathrm{e}}\,\tilde{e}^{\mathrm{a,e}}; \tag{3.8}$$

$$\lambda^{\mathrm{e}}_{,i} = -\omega^2\tilde{\rho}^{\mathrm{e}}\,\tilde{U}^{\mathrm{e}}_i, \qquad \lambda^{\mathrm{a,e}}_{,i} = -\omega^2\tilde{\rho}^{\mathrm{e}}\,\tilde{U}^{\mathrm{a,e}}_i. \tag{3.9}$$

It follows that λ^{e} equals minus the pressure of the fluid,

$$\lambda^{\mathrm{e}} \equiv -\tilde{p}^{\mathrm{e}}. \tag{3.10}$$

Using Eqs. (3.8)-(3.10) the fluid displacements, dilatations and Lagrange multipliers are eliminated from $\breve{L}^{\mathrm{e}}$; integrating by parts all the terms featuring a second-order spatial derivative, the functional $\breve{L}^{\mathrm{e}}$ is rewritten as

$$\breve{L}^{\mathrm{e}} = \int_V \Big(-\breve{W}^{\mathrm{e}} + \breve{K}^{\mathrm{e}}\Big)\mathrm{d}V + \breve{F}^{\mathrm{e}} = \int_V \Big(-\tilde{p}^{\mathrm{a,e}}\frac{1}{\tilde{R}^{\mathrm{e}}}\tilde{p}^{\mathrm{e}} + \tilde{p}^{\mathrm{a,e}}_{,i}\frac{1}{\omega^2\tilde{\rho}^{\mathrm{e}}}\tilde{p}^{\mathrm{e}}_{,i}\Big)\mathrm{d}V + \breve{F}^{\mathrm{e}}. \tag{3.11}$$

where

$$\breve{F}^{\mathrm{e}} = -\int_S n_i\,\tilde{p}^{\mathrm{a,e}}\frac{1}{\omega^2\tilde{\rho}^{\mathrm{e}}}\tilde{p}^{\mathrm{e}}_{,i}\,\mathrm{d}S - \int_S n_i\,\tilde{p}^{\mathrm{a,e}}_{,i}\frac{1}{\omega^2\tilde{\rho}^{\mathrm{e}}}\tilde{p}^{\mathrm{e}}\,\mathrm{d}S, \tag{3.12}$$

where n_i is the normal pointing outwards. Varying the adjoint pressure in $\breve{L}^{\mathrm{e}}$, the equations of motions and boundary integrals follow:

$$\frac{1}{\tilde{R}^{\mathrm{e}}}\,\tilde{p}^{\mathrm{e}} + \frac{1}{\omega^2\tilde{\rho}^{\mathrm{e}}}\,\tilde{p}^{e}_{,ii} = 0, \tag{3.13a}$$

$$\breve{B}^{\mathrm{e}} = -\int_S n_i\,\delta\tilde{p}^{\mathrm{a,e}}_{,i}\frac{1}{\omega^2\tilde{\rho}^{\mathrm{e}}}\tilde{p}^{\mathrm{e}}\,\mathrm{d}S = -\int_S n_i\delta\tilde{U}^{\mathrm{a,e}}_i\tilde{p}^{\mathrm{e}}\mathrm{d}S = 0. \tag{3.13b}$$

Eqs. (3.6b) and (3.13b) are equal. Thus, a fluid described by the Lagrangian in Eq. (3.11) does not need the fluid-structure coupling functional to coupled to solids or, rather, this functional is not added ad-hoc, but results from the change of coordinates:

it is $\breve{F}^{\mathrm{e}}$ in Eq. (3.12). The fluid, therefore, looks at the surrounding world in the same fashion regardless of the chosen variable of response.

The functional $\breve{F}^{\mathrm{e}}$ is further defined: it is an analogy to twice the work at the boundary. Integration by parts shows that

$$\breve{F}^{\mathrm{e}} = 2\int_V \left(\widetilde{W}^{\mathrm{e}} - \breve{K}^{\mathrm{e}}\right)\mathrm{d}V. \tag{3.14}$$

Substituting this result into Eq. (3.11), it follows that

$$\breve{L}^{\mathrm{e}} = \int_V \left(\widetilde{W}^{\mathrm{e}} - \breve{K}^{\mathrm{e}}\right)\mathrm{d}V = \overline{L}^{\mathrm{e}}. \tag{3.15}$$

Eqs. (3.14) and (3.15) explain the change of sign of the strain and kinetic potentials in Eq. (3.11) compared to Eq. (3.5).

3.4 A self-adjoint variational principle for Biot's equations

In a seminal work Biot formulates a theory for poro-elastic materials that models their macroscopical behavior by considering them as homogenized two-phase media, i.e., the solid elastic frame and the compressible fluid in the pores [8]. For porous materials without dissipation, Biot gives expressions of the strain and the kinetic energies; their difference defines the Lagrangian. Biot also introduces a Rayleigh-like dissipation pseudo-potential modeling the viscous drag [52], p 315 (see Eqs. (15) in Paper 2). Consequently, via variational calculus he devises the Euler–Lagrange–Rayleigh equations as functions of the frame and the fluid displacements. These equations are also referred to as the displacement (u, U)-formulation. The employment of a dissipation pseudo-potential is a common technique to handle viscous losses. Paper 2 presents a short overview on notable articles using it. Herein, Morse and Feshbach's construction is instead used. In the following, the main results for Biot's theory are reported; the complete derivation and analysis is found in Paper 2.

A variational principle needs potentials; the postulate of a strain energy functional is a standard step in the derivation of variational principles, see e.g. pp 331 and 380 in Ref. [29]. Besides, demonstrations of Hamilton's principle rely, at least implicitly, on the fact that the strain energy density is a potential. The variational principle for harmonic motion, a *modified* Hamilton's principle also valid for non-conservative media, reads

$$\delta\overline{L} = \delta\int_V \left(\overline{W} - \overline{K}\right)\mathrm{d}V = 0; \tag{3.16}$$

$\overline{L}$ is called Lagrangian, $\overline{W}$ and $\overline{K}$ strain and kinetic potential densities. These potentials depend on the frame and fluid displacements, respectively $\tilde{u}_i$ and $\tilde{U}_i$; accordingly, the

frame strain $\tilde{\epsilon}_{ij}$ and the fluid dilatation $\tilde{e}$ are defined as

$$\tilde{\epsilon}_{ij} = \frac{1}{2}\left(\tilde{u}_{i,j} + \tilde{u}_{j,i}\right); \qquad \tilde{e} = \tilde{U}_{i,i}. \tag{3.17}$$

$\overline{W}$ and $\overline{K}$ are, respectively,

$$\overline{W} = \tilde{\epsilon}^{\mathrm{a}}_{ij}\,\tilde{D}_{ijkl}\,\tilde{\epsilon}_{kl} + \tilde{\epsilon}^{\mathrm{a}}_{ij}\,\tilde{Q}_{ij}\,\tilde{e} + \tilde{e}^{\mathrm{a}}\,\tilde{R}\,\tilde{e} + \tilde{e}^{\mathrm{a}}\,\tilde{Q}_{ij}\,\tilde{\epsilon}_{ij}. \tag{3.18}$$

and

$$\overline{K} = \tilde{u}^{\mathrm{a}}_i\,\omega^2\tilde{\rho}^{\mathrm{sc}}_{ij}\,\tilde{u}_j + \tilde{u}^{\mathrm{a}}_i\,\omega^2\tilde{\rho}^{\mathrm{c}}_{ij}\,\tilde{U}_j + \tilde{U}^{\mathrm{a}}_i\,\omega^2\tilde{\rho}^{\mathrm{fc}}_{ij}\,\tilde{U}_j + \tilde{U}^{\mathrm{a}}_i\,\omega^2\tilde{\rho}^{\mathrm{c}}_{ij}\,\tilde{u}_j; \tag{3.19}$$

$\tilde{D}_{ijkl}$ is the viscoelastic modulus tensor of the frame, Q_{ij} the potential coupling tensor, R the bulk modulus of the fluid in the pores, $\tilde{\rho}^{\mathrm{sc}}_{ij}$, $\tilde{\rho}^{\mathrm{fc}}_{ij}$ and $\tilde{\rho}^{\mathrm{c}}_{ij}$ tensors of equivalent densities for the frame, fluid and the coupling between them, accounting for mass, viscous and inertial effects; details on the Biot material parameters and on the functionals can be found in Paper 2. The potentials in Eqs. (3.18) and (3.19) are clearly self-adjoint. Applying Eq. (3.16) Biot's equations are retrieved, together with the boundary integrals.

Biot's equations were originally expressed using the frame and the fluid displacements. Refs. [6] and [5] derive a *mixed* (u,p)-formulation, employing the frame displacement and the pore fluid pressure as field variables, thereby reducing the number of variables from six to four. This formulation is derived via a variable substitution in Biot's equations and a subsequent weak formulation of these equations. Herein, the derivation of the mixed formulation is based upon the invariance of the Lagrangian to coordinate transformations. In order to switch from (u,U) coordinates to (u,p) coordinates, Lagrange multipliers are employed. Specifically, to eliminate the dependance of $\overline{L}$ on the fluid displacement, the fluid dilatation $\tilde{e}$ is introduced as an additional variational parameter. A new functional is defined:

$$\breve{L} \equiv \overline{L} - \overline{C}, \tag{3.20}$$

where $\overline{C}$ expresses a constraint via the Lagrange multipliers λ and λ^{a}

$$\overline{C} = \int_V \left(\lambda^{\mathrm{a}}\left(\tilde{e} - \tilde{U}_{i,i}\right) + \lambda\left(\tilde{e}^{\mathrm{a}} - \tilde{U}^{\mathrm{a}}_{i,i}\right)\right)\mathrm{d}V. \tag{3.21}$$

Similarly to Eqs. (3.8), in Paper 2 it is shown that

$$\lambda \equiv \tilde{s} = -h\,\tilde{p}, \tag{3.22}$$

where $\tilde{s}$ is the fluid stress, h the porosity and $\tilde{p}$ the fluid pressure. The fluid displacements, the fluid dilatations and the Lagrange multipliers can be eliminated from

$\breve{L}$, as done going from Eq. (3.7) to Eq. (3.11). Eventually, $\breve{L}$ reads

$$\begin{aligned}\breve{L} &= \int_V \Big(\tilde{u}^{\mathrm{a}}_{i,j}\, \tilde{D}_{ijkl}\, \tilde{u}_{k,l} - \tilde{u}^{\mathrm{a}}_{i,j}\, \frac{\tilde{Q}_{ij}\tilde{Q}_{kl}}{\tilde{R}}\, \tilde{u}_{k,l} - \tilde{u}^{\mathrm{a}}_{i}\, \omega^2 \tilde{\rho}_{ij}\, \tilde{u}_j - \tilde{p}^{\mathrm{a}}\, \frac{h^2}{\tilde{R}}\, \tilde{p} + \tilde{p}^{\mathrm{a}}_{,i}\, \frac{h^2 \hat{\rho}^{\mathrm{fc}}_{ij}}{\omega^2}\, \tilde{p}_{,j} \\ &+ \tilde{u}^{\mathrm{a}}_{i,j} \tilde{\gamma}_{ij} \tilde{p} + \tilde{p}^{\mathrm{a}} \tilde{\gamma}_{ij} \tilde{u}_{i,j} \Big) \mathrm{d}V + \breve{F}, \end{aligned} \tag{3.23}$$

where

$$\tilde{\rho}_{ij} = \tilde{\rho}^{\mathrm{sc}}_{ij} - \tilde{\rho}^{\mathrm{c}}_{ik} \hat{\rho}^{\mathrm{fc}}_{kl} \tilde{\rho}^{\mathrm{c}}_{lj}; \qquad \tilde{\gamma}_{ij} = h \left(\tilde{\rho}^{\mathrm{c}}_{ik} \hat{\rho}^{\mathrm{fc}}_{kj} - \frac{\tilde{Q}_{ij}}{\tilde{R}} \right); \qquad \tilde{\rho}^{\mathrm{fc}}_{ik}\, \hat{\rho}^{\mathrm{fc}}_{kj} = \delta_{ij}. \tag{3.24}$$

The surface functional $\breve{F}$ in Eq. (3.23) is given by

$$\breve{F} = -\int_S n_i\, \tilde{p}^{\mathrm{a}} \frac{h^2 \hat{\rho}^{\mathrm{fc}}_{ij}}{\omega^2}\, \tilde{p}_{,j}\, \mathrm{d}S - \int_S n_i\, \tilde{p}^{\mathrm{a}}_{,j} \frac{h^2 \hat{\rho}^{\mathrm{fc}}_{ij}}{\omega^2}\, \tilde{p}\, \mathrm{d}S. \tag{3.25}$$

$\breve{L}$ is self-adjoint and defines a variational principle:

$$\delta \breve{L} = 0, \tag{3.26}$$

from which the equations of motion in the mixed formulation and their boundary integrals follow.

3.4.1 Couplings to other media

Interesting insights to the results obtained for a porous material are gained looking at the boundary integrals resulting from the first variations of the Lagrangians.

The boundary integrals for the original system obtained from Eq. (3.16) are

$$\int_S n_i\, \delta \tilde{u}^{\mathrm{a}}_j\, \tilde{\sigma}_{ij}\, \mathrm{d}S + \int_S n_i\, \delta \tilde{U}^{\mathrm{a}}_i\, \tilde{s}\, \mathrm{d}S = 0, \tag{3.27}$$

where $\tilde{\sigma}_{ij}$ is the stress of the frame and $\tilde{s}$ is the stress of the fluid. This equation is satisfied either by prescribing the displacements, or by having free stresses, or by a combination of the two. The boundary integrals obtained from the Lagrangian in the mixed formulation in Eq. (3.26) are identical, see Appendix B of Paper 2. This result is expected because the Lagrangian is invariant to changes of coordinates. In other words, the way the porous material looks at the surrounding world does not depend on the choice of coordinates. A specific coupling can thus be studied in any of the formulations, the one that is most convenient (see e.g. Section 3.5).
The boundary integrals of the popular weak form by Atalla in Ref. [5] differ from Eq. (3.27). That weak form, of course, satisfies the known coupling conditions to solids, fluids and other porous materials, but it does so in a different fashion than the displacement formulation. Atalla's weak form couples naturally to solids and porous materials, i.e.,

without the need of any coupling integral; it needs additional fluid-structure coupling integral to couple to fluids. It might be argued that on the boundary Atalla's weak form behaves more like a solid, than a porous material.

Two explanations to the difference between the self-adjoint variational principle and the weak form are given. First, many weak forms may correspond to a set of equations of motion, whereas it exists only one functional that is stationary for that set. Second, the weak form is often derived from a set of equations of motion, thus containing information only on the domain of the medium. The variational principle, instead, contains information both on the domain and on its boundary. This difference is relevant if the variational principle has boundary terms, like $\breve{F}$ in Eq. (3.25), which may not easily obtained via a weak formulation.

3.4.2 Equivalent fluid theory

Besides Biot's theory, another common theory for porous materials is equivalent fluid theory: the porous material is modelled as a fluid with complex bulk modulus and density. The porous material is thus described by one compressional wave, instead of two compressional waves and one shear wave as in Biot's theory.

Two equivalent fluid theories exist, one for rigid porous frames, one for limp porous frames (e.g., see p 251 Ref. [1]); this thesis employs the latter because it better suits porous layers attached to vibrating solid layers, like in the trim panels under study (p 253 Ref. [1]). Equivalent fluid theory for limp frame is conveniently derived from the mixed formulation: the first two terms in Eq. (3.23) are the strain potential relative to the stress of the frame in-vacuo, which equals zero for a limp frame. Thus, the functional for such a limp porous material is[7]:

$$L_{\mathrm{por}} = \int_V \Big(- u_i^{\mathrm{a}}\, \omega^2 \rho\, u_i \; - \, p^{\mathrm{a}}\, \frac{h^2}{R}\, p \; + \, p^{\mathrm{a}}_{,i}\, \frac{h^2}{\omega^2\, \rho^{\mathrm{fc}}}\, p_{,i} + \varepsilon^{\mathrm{a}} \gamma\, p + p^{\mathrm{a}} \gamma\, \varepsilon \Big) \mathrm{d}V + F \qquad (3.28)$$

where ε is the divergence of the frame displacement. The first variation of the Lagrangian L_{por} in Eq. (3.28) is stationary for the true motion of the system, yielding the equation of motion and the boundary integral of the equivalent fluid. Taking the variation of the adjoint coordinates, it follows

$$\begin{aligned} &\int_V \delta u_i^{\mathrm{a}} \left(-\omega^2 \rho u_i - \gamma p_{,i} \right) \mathrm{d}V + \int_V \delta p^{\mathrm{a}} \left(-\frac{h^2}{R} p - \frac{h^2}{\rho^{\mathrm{fc}} \omega^2}\, p_{,ii} + \gamma\, u_{i,i} \right) \mathrm{d}V + \\ &\int_S \delta u_i^{\mathrm{a}}\, n_i\, \gamma\, p\, \mathrm{d}S - \int_S n_i\, \delta p^{\mathrm{a}}_{,i} \frac{h^2}{\rho^{\mathrm{fc}} \omega^2}\, p\, \mathrm{d}S = 0. \end{aligned} \qquad (3.29)$$

From the fundamental lemma of calculus of variations, the equations of motion of the limp frame and of the fluid follow from the first two functionals in Eq. (3.29),

[7] For the slightly different notation in the remaining of this chapter, see Paper 3.

respectively:

$$- \quad \omega^2 \rho u_i - \gamma p_{,i} = 0, \tag{3.30a}$$

$$- \quad \frac{h^2}{R} p - \frac{h^2}{\rho^{\mathrm{fc}} \omega^2} \, p_{,ii} + \gamma \, u_{i,i} = 0. \tag{3.30b}$$

The divergence of the frame is eliminated from the fluid equation of motion, obtaining the equation of motion of an equivalent fluid, Eq. (3.31):

$$\frac{R}{h^2} \, p_{,ii} + \omega^2 \, \rho^{\mathrm{L}} \, p = 0; \qquad \rho^{\mathrm{L}} = \frac{\rho \, \rho^{\mathrm{fc}}}{\gamma^2 \rho^{\mathrm{fc}} + h^2 \, \rho}, \tag{3.31}$$

The two boundary integrals in Eq. (3.29) may be collected and rewritten:

$$- \int_S n_i \delta u_i^{\mathrm{t,a}} \, p \, \mathrm{d}S = 0, \tag{3.32}$$

u_i^{t} is the total displacement defined as

$$u_i^{\mathrm{t}} = (1 - h) u_i + h U_i. \tag{3.33}$$

This boundary integral contains the (variation of the) normal total displacement, which is the continuous quantity when the equivalent fluid couples to solids, fluids or other porous materials; notice that neither the normal frame displacement nor the normal fluid displacement equals the normal solid displacement, which might be counterintuitive. The author could not obtain the same boundary integral calculating the weak forms of Eqs. (3.30) or Eq. (3.31).

3.5 Variational principle for a trim panel

The trim panels studied in this thesis are made of two external layers, one of aluminium and one of rubber, with a porous layer in between them; its cross-section is shown in Fig. 3.2. To start, a modified Hamilton's variational principle for the trim panel is defined. Then, in Subsection 3.5.1 the model of the trim panel is simplified assuming that its solid parts are limp. Upon variational calculus, the equations of motion of the (limp) porous layer and of the (limp) external layers follow, together with the coupling equations between them. These coupling equations are the sought result: they describe the trim-coupled waves. This wave-type constitutes an SEA element, as expounded in next chapter.

The Lagrangian of the coupled trim-panel is defined summing the Lagrangians of the three layers and the functionals defining the constraints on the displacements of the three layers. These constraints are introduced in a variational form via Lagrange multipliers; their actual expressions are identified at the end of this section. The Lagrangian of the

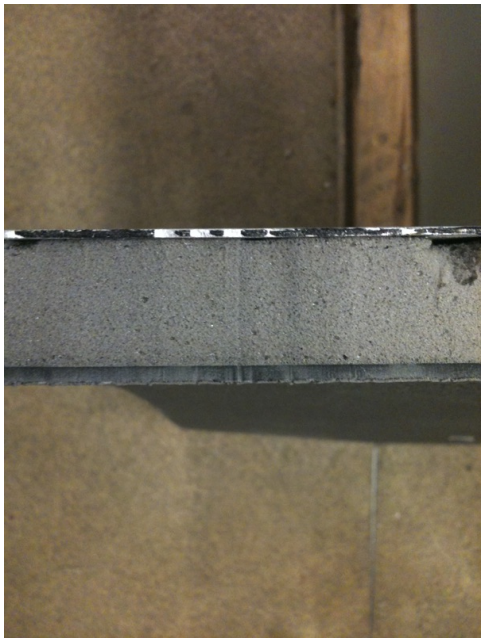

Figure 3.2: From top to bottom, the aluminium plate, the porous layer and the rubber plate.

coupled trim-panel, L_{tot}, reads

$$L_{\text{tot}} = L_{\text{por}} + L_{\text{p1}} + L_{\text{p2}} + T + V, \tag{3.34}$$

where L_{por} is the Lagrangian of the porous material in any of the three formulations presented above, L_{p1} and L_{p2} are the Lagrangians of the external layers, T is the functional expressing the continuity of the solid displacements, V is the functional expressing the continuity of the normal total displacement in the porous material and the normal displacements in the external layers. The variational principle based upon L_{tot} is:

$$\delta L_{\text{tot}} = 0, \tag{3.35}$$

where δ indicates the first variation of L_{tot}. A sketch of the trim-panel of Fig. 3.2 in a cartesian system of coordinates is given in Fig. 3.3. L_{p1} and L_{p2} are detailed in Paper 3 and the reader may refer to that; in the following, the focus is on the constraint functionals.

The constraints are introduced in a variational form, using Lagrange multipliers. The functional T reads

$$T = -\sum_{l=1}^{2}(-1)^l \int_{S_l} \left(\lambda_i^{\text{a,p}l}\left(u_i - w_i^{\text{p}l}\right) + \lambda^{\text{p}l}\left(u_i^{\text{a}} - w_i^{\text{a,p}l}\right)\right) \mathrm{d}S_l, \tag{3.36}$$

where $\lambda_i^{\text{p}l}$ is a Lagrange multiplier and $\lambda_i^{\text{a,p}l}$ its adjoint; these Lagrange multipliers are

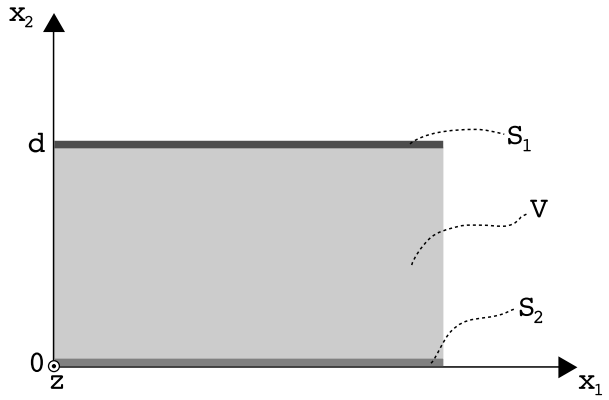

Figure 3.3: Coordinate systems used in the article. Volume V is occupied by the porous layer, which is connected to the rubber layer at surface S_1, and to the aluminium layer at surface S_2. The local normal vectors point outwards of the structures. The global normal vector is aligned with the positive x_2 axis.

vectors because T is (an analogy to) work on the boundary S_l. The functional V reads

$$V = -\sum_{l=1}^{2}(-1)^l \int_{S_l} \left(\beta^{\mathrm{a},\mathrm{p}_l}\left(n_i\, u_i^{\mathrm{t}} - n_i\, w_i^{\mathrm{p}_l}\right) + \beta^{\mathrm{p}_l}\left(n_i\, u_i^{\mathrm{t,a}} - n_i\, w_i^{\mathrm{a},\mathrm{p}_l}\right)\right) \mathrm{d}S_l, \tag{3.37}$$

where β^{p_l} is a Lagrange multiplier and $\beta^{\mathrm{a},\mathrm{p}_l}$ its adjoint.

In Paper 3, the actual values of the Lagrange multipliers are evaluated. The values of the Lagrange multipliers in Eqs. (3.36) and (3.37) are identified by calculating the variation in Eq. (3.35), with the multipliers as independent parameters. For plate p_1 it follows:

$$\beta^{\mathrm{p}_1} \equiv p; \qquad \lambda_i^{\mathrm{p}_1} \equiv -\left(\sigma_{ij}^{\mathrm{t}} + p\delta_{ij}\right)\, n_j = -n_j\, \hat{D}_{ijkl}\, \tilde{\epsilon}_{kl}, \text{ on } S_1; \tag{3.38}$$

where $\hat{D}_{ijkl}$ is the stiffness of the frame in vacuo and, thus, $\lambda_i^{\mathrm{p}_1}$ is identified as minus the traction of the frame in-vacuo: if the frame of the porous material is limp, the constraint introduced by the functional T in Eq. (3.36) disappears. Similar results are obtained for β^{p_2} and $\lambda_i^{\mathrm{p}_2}$ for plate p_2, and also apply to the adjoint Lagrange multipliers.

Thus, when the frame is limp, the continuity of the solid displacements disappears, accommodating the decrease of the number of waves in the model of the porous materials, from three Biot waves to one equivalent fluid wave.

In Paper 3 these results on the Lagrange multipliers are derived using the displacement formulation, because it allows for simpler calculations. The results are valid also for the mixed formulation, in light of the comments in Section 3.4.1; they are also valid for a limp porous material and they are used in the next subsection.

3.5.1 The trim-coupled waves

In the SEA for the sound transmission through the trim-panel presented in the next chapter and derived in Paper 3, two elements describe the bending waves in the external

layers modelled as plates (Fig. 1.2). In this and in the next section, these bending waves are neglected and the focus is on the elements accounting for the trim-coupled waves, substituting the cavity-wall waves in Fig. 1.2.

The equations describing the trim-coupled waves are devised for a porous material that has a limp frame, described by the Lagrangian in Eqs. (3.28). The trim-coupled waves propagate in the equivalent fluid and are fully coupled to the external plates. If the wavelength of the trim-coupled wave is much larger than the free wavelength of the bending wave in a plate, then the motion of the plate is mass-impeded and, equivalently, its material is assumed limp. The viscoelastic forces of plate p_l are thus neglected and its Lagrangian only comprises kinetic energy. The functional T in Eq. (3.36) disappears because the Lagrange multiplier λ_i equals zero for a limp porous material.

The equations of motion and the coupling equations for the original system are retrieved from L_{tot} by taking the variations with respect to the adjoint generalised coordinates; the full analysis is in Paper 3. The equations resulting from the fundamental lemma of calculus of variations are: the equation of motion of the equivalent fluid, Eq. (3.31), the equations of motion of the forced limp plates, the coupling equations of the trim-coupled waves,

$$p = -d\,\mu^{\mathrm{p}1}\,p_{,2},\ \text{on } S_1; \qquad p = d\,\mu^{\mathrm{p}2}\,p_{,2},\ \text{on } S_2, \tag{3.39}$$

where

$$\mu^{\mathrm{p}1} = \frac{m^{\mathrm{p}1}}{d\,\rho^{\mathrm{L}}}; \qquad \mu^{\mathrm{p}2} = \frac{m^{\mathrm{p}2}}{d\,\rho^{\mathrm{L}}}, \tag{3.40}$$

and d is the distance between the two plates, $m^{\mathrm{p}1}$ and $m^{\mathrm{p}2}$ are the mass per unit areas of the plates. Eqs. (3.39) satisfy the continuity of the normal total displacement of the porous layer and of the normal displacements of the external layers.

3.5.2 Comment on the variational statements

In this thesis, it is shown that if the variational statement of the coupling between two or more media is *properly* set-up, no additional, ad-hoc coupling functional is needed. By "properly" it is meant that: first, the Lagrangians of all media are included; second, the functional expressing constraints are also included; third, the Lagrangians have proven to be invariant to coordinate transformations both in the domain and on its boundary (see discussion in Section 3.4.1).

The essential conditions may be either imposed in the statement, as in finite elements procedures and in Paper 2, or introduced in a variational form using Lagrange multipliers, as above and in Paper 3 (see p 43 in Ref. [39]). In this work, the latter method shows its usefulness by revealing what happens to the condition on the continuity of the solid displacements when equivalent fluid theory for limp frame is derived from Biot's theory, Eq. (3.38).

3.6 Wave form and modal density of the trim-coupled waves

Eqs. (3.39) are the sought equations describing the trim-coupled waves. The solution $p(x, y, z)$ to Eq. (3.31) can be decomposed as ($x = x_1$, $y = x_2$, $z = x_3$)[8]:

$$p(x, y, z) = g(x, z)f(y) \tag{3.41}$$

where the function $g(x, z)$ describes a wave field along $x - z$, while the function $f(y)$ a modal field along y. It follows that

$$\frac{\partial^2 g(x,z)}{\partial x^2} + \frac{\partial^2 g(x,z)}{\partial z^2} + \kappa^2\, g(x,z) = 0; \tag{3.42a}$$

$$\frac{\partial^2 f(y)}{\partial y^2} + \alpha^2\, f(y) = 0, \tag{3.42b}$$

where

$$k_{\mathrm{L}}^2 = \kappa^2 + \alpha^2; \qquad k_{\mathrm{L}} = \frac{\omega}{c_{\mathrm{L}}}; \qquad c_{\mathrm{L}} = \sqrt{\frac{R}{h^2 \rho_{\mathrm{L}}}}, \tag{3.43}$$

k_{L} is the wavenumber of the equivalent fluid wave and c_{L} its wavespeed. A wave solution to $f(y)$ in Eq. (3.42b) is a linear combination of sines and cosines. Solving the boundary value problem given by the equation of motion in Eq. (3.42b) and the coupling conditions in Eqs. (3.39), the following transcendental eigenvalue problem is obtained:

$$\left(1 - \mu^{\mathrm{p}1}\mu^{\mathrm{p}2}(\alpha\, d)^2\right) \sin(\alpha\, d) + \left(\mu^{\mathrm{p}1} + \mu^{\mathrm{p}2}\right)\, \alpha\, d\, \cos(\alpha\, d) = 0. \tag{3.44}$$

Eq. (3.44) can be solved for αd at each angular frequency ω, since $\mu^{\mathrm{p}1}$ and $\mu^{\mathrm{p}2}$ are frequency dependant. The eigenvalue problem in Eq. (3.44) is approximately solved by:

$$\alpha_0 = \frac{2\pi f_{\mathrm{dw}}}{c_{\mathrm{L}}}; \qquad \alpha_r = \frac{2\pi f_r}{c_{\mathrm{L}}}. \tag{3.45}$$

The frequency f_{dw} corresponds to the typical double-wall resonance, which can be expressed as

$$f_{\mathrm{dw}} = \frac{c_{\mathrm{L}}}{2\pi d}\sqrt{\frac{1}{\mu^{\mathrm{p}1}} + \frac{1}{\mu^{\mathrm{p}2}}}; \tag{3.46}$$

the frequencies f_r have half-wavelengths that are integer fractions of the plate distance,

$$f_r = \frac{r\, c_{\mathrm{L}}}{2\pi d}; \qquad r \in \mathbb{N}. \tag{3.47}$$

When Eq. (3.44) is solved in MATLAB using the fzero function, the real parts of α_0 and α_r are used as initial values.

Two groups of waves are identified: one corresponds to the *quasi-plane* waves starting

[8]The remaining of this section uses scalars instead of zeroth-order tensors, as in Paper 3 where this section is taken from.

to propagate a f_{dw} ($r = 0$); one corresponds to the *oblique* coupled waves starting to propagate at f_r ($r \in \mathbb{N}$). To each mode r described by α_r, corresponds a set of waves with wavenumber κ_r propagating in the $x - z$ plane; κ_r is obtained via Eq. (3.43). Finally, the pressure $p_r(x, y, z)$ for each mode r reads:

$$p_r(x, y, z) = \tilde{p} \left(\sin(\alpha_r\, y) + d\mu^{\mathrm{p2}} \alpha_r \cos(\alpha_r\, y)\right) e^{-\mathrm{i}\kappa_r (x \cos\theta + z \sin\theta)}, \tag{3.48}$$

where $\tilde{p}$ is the complex amplitude of the wave and θ is the angle of propagation of the wave. The trigonometric functions in the parentheses describe the modes along y, while the exponential functions the waves travelling along $x - z$. The function fzero yields as output the real part of κ_r,

$$\hat{\kappa}_r = \mathrm{Re}\left[\kappa_r\right]. \tag{3.49}$$

For all $\hat{\kappa}_r$, the number of modes in a large two-dimensional structure below the angular frequency ω is asymptotically given by (p 302 in Ref. [38])

$$N_r(\omega) = \frac{\hat{\kappa}_r^2 S}{4\pi}. \tag{3.50}$$

The asymptotic modal density in $x - z$ is calculated as [38]:

$$n_r = \frac{\left(\hat{\kappa}_r^2(\omega_{\mathrm{u}}) - \hat{\kappa}_r^2(\omega_{\mathrm{l}})\right) S}{4\pi \left(\omega_{\mathrm{u}} - \omega_{\mathrm{l}}\right)}, \tag{3.51}$$

where ω_{u} and ω_{l} are the upper and lower frequency of a frequency band having bandwidth $\omega_{\mathrm{u}} - \omega_{\mathrm{l}}$.

The most important results are the modal density in Eq. (3.51) and the pressure wave in Eq. (3.48): both are used in the SEA of the trim-panel in the next chapter, where the two groups of waves are used in the SEA of trim-panel as two separate SEA elements.

Results consistent to Eqs. (3.39) and (3.51) are obtained in Paper 1 for the coupled waves in the cavity of a building construction double-wall, suggesting that these two kinds of structures may be studied in a similar fashion. In that paper, the analysis is done starting from Newton second law. A variational approach is, however, preferred with the trim-panel to deal with a more complex medium - the porous layer.

Chapter 4

SEA of a trim panel

The first step in an SEA is the identification of a set of elements. As anticipated in Section 1.1, the SEA of the sound transmission through double-walls in Fig. 1.2 derived in Paper 1 sets the basis for the SEA of the trim panel[1]. To calculate the SRI of the trim-panel, the SEA mimics a SRI measurement: the trim-panel is placed in a rigid wall between a reverberant room and an anechoic room; a diffuse sound field is generated in the reverberation room via a distributed source. Thus, the SEA elements corresponding to the wave-types carrying the energies from the reverberation room to the anechoic room are: 1, an element describing the acoustic waves in the reverberation room; 2, an element describing the bending waves in the aluminium plate; 3, an element describing the quasi-plane coupled trim-plane wave cutting-on at f_{dw}, Eq. (3.46); 4, an element describing the oblique trim-coupled waves, each cutting-on at f_r, Eq. (3.47); 5, an element describing the bending waves in the rubber plate; 6, an element describing the acoustic waves in the anechoic room. The wave form of the trim-coupled waves and their modal density are calculated in the previous chapter; Paper 3 shows how to derive the SEA parameters describing internal and coupling losses of all elements.

4.1 Results

In Ref. [16], Doutres et al. derive a criteria for assessing the validity of equivalent fluid theory for limp frames, based on a parameter called frame stiffness influence (FSI). The FSI expresses the ratio of two wave speeds, i.e., the wave speed of the wave in frame in-vacuo and the wave speed of the equivalent fluid wave. A porous material can be considered limp if the FSI is below a threshold, which in Ref. [16] is determined simulating 256 porous materials.

The SEA model derived in the previous section is used to calculated the SRI of two trim-panels, a car-floor and a truck-floor; the material parameters and details on the

[1]The SEA elements of the trim panel can also be identified from the dispersion curves of the waves propagating along the $x - z$ plane: this is done in Paper 3 but omitted in this introduction.

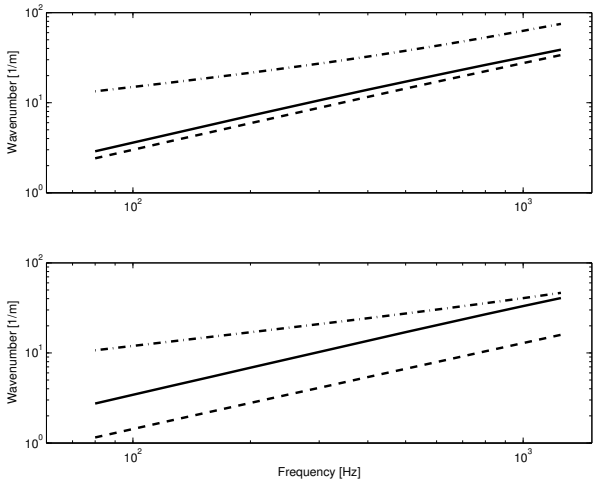

Figure 4.1: Real parts of the wavenumbers of the uncoupled porous layer in the car-floor (top) and in the truck-floor (bottom). Fast wave (dashed line), slow wave (dot-dashed line), equivalent fluid wave (full line).

SRI measurement are in Papers 4 and 5. The porous layer in the car-floor is limp according to the FSI, whereas that in the truck-floor is not.

In Fig. 4.1, the dispersion curves of the uncoupled porous layers in the two floors are shown; the real wavenumbers of the fast and slow waves are calculated using Allard's expressions on p 121 in Ref. [1], while the real wavenumber k_{L} of the uncoupled equivalent fluid wave using Eqs. (3.43). The two figures show that when equivalent fluid theory works, its wave has a wavenumber close to the fast Biot waves.

The SEA defined in the previous section is applied to the car-floor and the truck-floor. In Figs. 4.2-4.3, the SRI predicted by SEA is plotted and compared to SRI measurements.

In Fig. 4.2, SEA shows to perform generally well with the car-floor. SEA catches fairly well the double-wall resonance frequency, around 260 Hz. The trend of the SRI between 260 Hz and 1000 Hz is also well caught. At frequencies below 260 Hz, SEA prediction is poor: this problem is also encountered with building construction double-walls in Ref. [24]. Fig. 4.2 also shows a reduced SEA that only employs three elements, i.e., the acoustic waves in the two rooms and the quasi-plane trim-coupled waves. In the frequency range of interest, these three elements are enough to calculate the SRI, thus highlighting their relevance.

In Fig. 4.3, SEA shows to give a rather incorrect prediction of the SRI of the truck-floor, but for the trend between 500 Hz and 1000 Hz which is quite correct. It seems that if the resonance frequency in the 500 Hz third-octave frequency band was caught and not underestimated, the SEA prediction would give the correct level of the SRI between 500 Hz and 1000 Hz. It might be argued that the equivalent double-wall resonance frequency corresponding to the dip in the SRI curve also depends on the frame stiffness, which is neglected in the equivalent fluid model. The dispersion curves in Fig. 4.1 show that the equivalent fluid wave likely has a lower stiffness than the fast wave: accordingly, in Fig. 4.3 the resonance frequency around 500 Hz is underestimated.

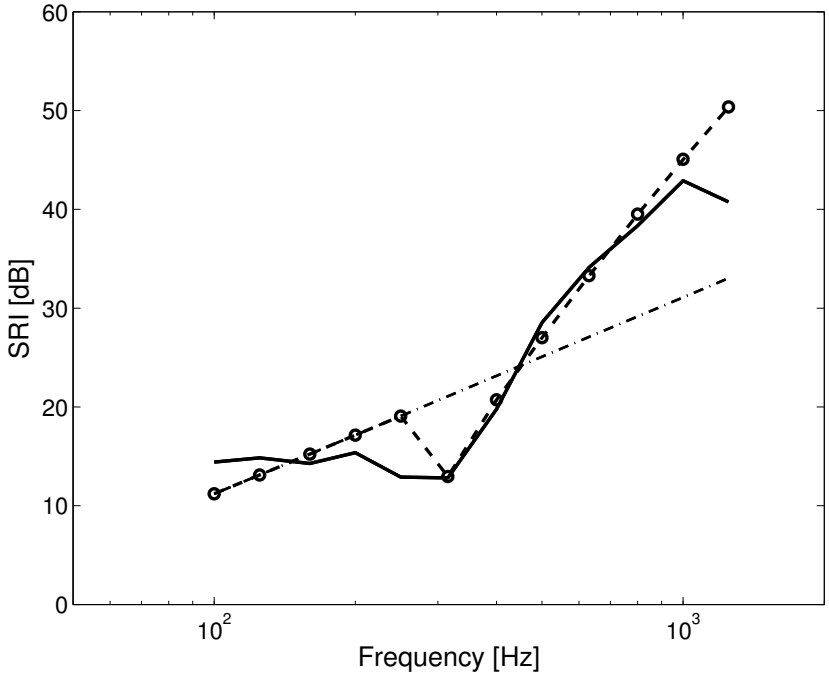

Figure 4.2: SRI curves, car-floor. Measurement (full line), SEA prediction (dashed line), reduced SEA prediction (circles), mass-law prediction (dash-dotted line). Note that the SEA and reduced SEA predictions are almost identical.

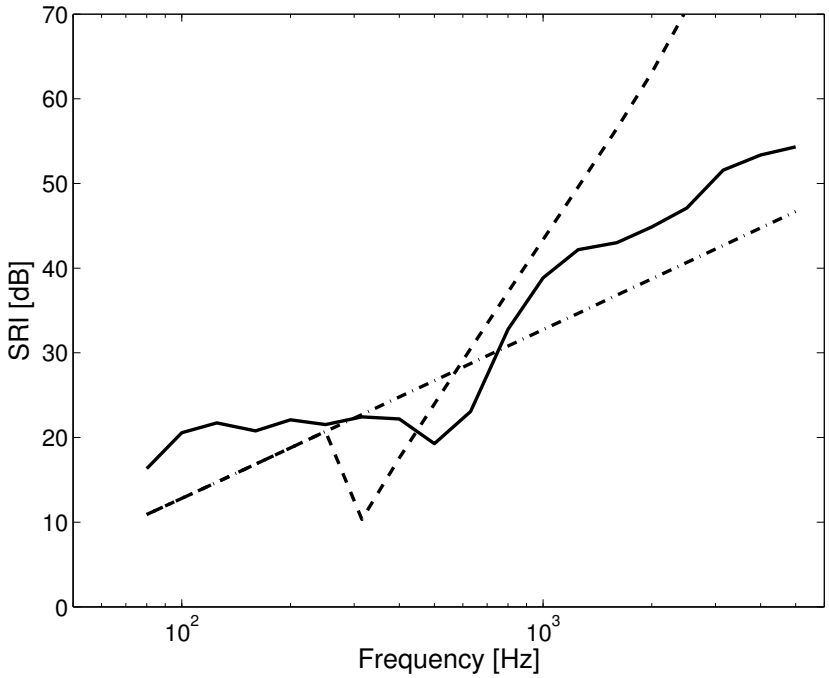

Figure 4.3: SRI curves, truck-floor. Measurement (full line), SEA prediction (dashed line), mass-law prediction (dash-dotted line).

Chapter 5

Conclusions

This thesis investigates sound transmission in multilayered structures in the mid to high frequency range, using statistical energy analysis and variational principles. The multilayered structures of interest are building construction double-walls, double-glass windows and vehicle trim-panels with a porous layer. The main scientific contributions of this thesis are now summarised.

The problem of transmission through double-walls and trim-panels is throughly investigated, from theory to simulations to experiments. The SEAs devised for predicting their SRI are novel. All SEA elements refer to wave-types, permitting a deeper understanding of the physics. Regarding double-walls, the new SEA performs better than older ones [13, 14, 12] and proves to be versatile. Regarding trim-panels, the proposed SEA is the first one that models the porous layer as an equivalent fluid derived from Biot theory (to the knowledge of the author). One element of the SEA for the trim-panel corresponds to the waves in the trim fully coupled to the external layers; the equations describing these waves and their SEA parameters are derived. These equations are derived using a variational principle, being a convenient machinery for media as complex as the porous layer. The variational approach shows how to reduce coupling conditions when going from Biot theory to equivalent fluid theory. The non-conservative layers in the trim panel require the definition of self-adjoint variational principles valid for dissipative systems: this is achieved following Morse and Feshbach's construction. The self-adjoint variational principle for Biot's equations is novel and may be seen as a continuation to Biot's original approach, based upon Hamilton's principle. Despite skepticism towards Morse and Feshbach's construction [3, 7, 54], this thesis shows its advantages by deriving original, sound and useful results for porous materials, fluids and multilayered panels. The original study on energy transmission along long structures questions the applicability of standard SEA, setting the basis for future research: at high frequencies the statement of energy conservation as interpreted by SEA is incomplete because indirect couplings become relevant.

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Part II

Appended Papers