Lie Algebras in Braided Monoidal Categories

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Abstract

We begin by recalling some basic definitions from Lie algebra theory to motivate our subsequent transition to the more general setting of category theory. Next, we develop a relatively self-contained introduction to those areas of category theory needed for an understanding of what follows. Here we also motivate and introduce the graphical calculus notations. We then state the definitions of a braided commutator algebra, a braided Lie algebra, and a braided commutator Lie algebra. We proceed to show that color Lie algebras and Lie superalgebras are examples of braided Lie algebras. Thus, we are interested in examining color Lie algebras and Lie superalgebras in the generalized setting of braided Lie algebras. So we end by examining the representation theory of braided Lie algebras and braided commutator Lie algebras. In particular, we find analogues of the adjoint representation, the tensor product representation, and the contragredient representation.

Contents

1 Introduction

During the past 20 years, physics has seen the need to introduce a number of seemingly unrelated structures to describe the symmetries which have taken a leading role in most modern physical theories. Classically, these structures took the form of Lie algebras and groups. Nowadays we often must look at generalizations or variations of these such as Lie superalgebras and color Lie algebras. Because of this, it is becoming harder to choose and distinguish among these structures. Hence, it is necessary to organize these structures in manner that suggests we can interpret the old structures as different examples of something new.

As a basis for our motivation we should choose– being wary of the multitude of "generalizations" extant– a list of these "generalized Lie algebras" that we wish to have as examples of our generalization and work from there. To motivate our choice of examples, we summarize an admittedly incomplete overview of various "generalized Lie algebras." Later we review the definitions of those generalizations which we wish include as examples in more depth.

We assume the reader is familiar with the basic notions of a Lie algebra and its representation theory. The notion of a "Lie superalgebra" was found interesting on physical terms as a structure arising in the supersymmtery regime which continues to be popular today. This structure can be seen to consist of commuting and anticommuting parts and, hence, allows one to unify quantities obeying boson and fermion statistics into a single mathematical structure.

In 1977 Rittenberg and Wyler [20] introduced what are now known as color Lie algebras, ϵ-Lie algebras, Γ-Lie algebras, or anyonic Lie algebras [18]. We shall adopt the nomenclature "color Lie algebra" herein. These are generalizations of Lie superalgebras from grading over $\mathbb{Z}_2$ to grading over an arbitrary abelian group usually denoted by Γ. The name ϵ-Lie algebra is also used since the structure is not only dependent on Γ but an antisymmetric bicharacter which is usually denoted by ϵ.

In the early 90's, Majid [15][16] developed a "braided Lie algebra" with the motivation of finding an algebra that had as its universal enveloping algebra a braided bialgebra $U(\mathcal{L})$.

In [21] we have the definition of a m-Lie algebra, which we shall call a commutator Lie algebra, some examples, and some distinctions among these and Majid's braided Lie algebras.

1.1 Group Graded Vector Spaces and Lie Algebras

1.1.1 Group Graded Vector Spaces and Supervector Spaces

Let us summarize what we mean by Lie superalgebras and color Lie algebras. We'll need a few definitions first.

Definition 1.1 *Let Γ be a finite abelian group. An* **antisymmetric Γ-bicharacter** χ *is a map*

$$\chi\colon\ \Gamma \times \Gamma \to \mathbb{S}^1 \subseteq \mathbb{C}, \tag{1}$$

where $\mathbb{S}^1$ denotes the unit circle in $\mathbb{C}$, that is a character in each argument, i.e.

$$\chi(\gamma,\delta\delta') = \chi(\gamma,\delta)\,\chi(\gamma,\delta')\,, \qquad \chi(\gamma\gamma',\delta) = \chi(\gamma,\delta)\,\chi(\gamma',\delta) \tag{2}$$

for all $\gamma,\gamma',\delta,\delta'\in\Gamma$, and that satisfies

$$\chi(\gamma,\delta)\,\chi(\delta,\gamma) = 1 \tag{3}$$

for all $\gamma,\delta\in\Gamma$. (We write the group operation multiplicatively.)

Below, all vector spaces are over $\mathbb{C}$.

Definition 1.2 *A* **Γ-graded vector space** *is a vector space X that can be written as a direct sum*

$$X = \bigoplus_{\gamma\in\Gamma} X_\gamma \tag{4}$$

of vector subspaces X_γ. The subspaces X_γ are called **homogeneous subspaces***, and their elements are called the* **homogeneous elements** *of X of* **grade** *γ. For a homogeneous element x one writes its grade as $|x|$ or $\gamma(x)$.*

Below, "graded" means "Γ-graded", unless stated otherwise.

Example 1.1 When $\Gamma = \mathbb{Z}_2$ the graded vector space is called a **supervector space**. In this case, besides the trivial bicharacter χ_0 defined by $\chi_0(\gamma,\delta) := 1$ for all $\gamma,\delta\in\Gamma$, there is only one other antisymmetric bicharacter, given by (writing $\mathbb{Z}_2 = \{0,1\}$)

$$\chi(0,0) = \chi(0,1) = \chi(1,0) = 1 \tag{5}$$

and

$$\chi(1,1) = -1\,, \tag{6}$$

i.e.

$$\chi(|x|,|y|) = (-1)^{|x||y|}. \tag{7}$$

□

Denote by $\imath^X_\gamma : X_\gamma \to X$, $r^X_\gamma : X \to X_\gamma$, and $p^X_\gamma : X \to X$ defined by

$$p^X_\gamma := \imath^X_\gamma \circ r^X_\gamma \tag{8}$$

the embedding, restriction, and idempotent (projector) maps corresponding to the vector subspaces $X_\gamma \subseteq X$. Note that

$$r^X_\gamma \circ \imath^X_\gamma = id_{X_\gamma}\,, \tag{9}$$

and that the idempotents are orthogonal in the sense that

$$p^X_\gamma \circ p^X_\delta = \begin{cases} p^X_\gamma & \text{if } \gamma=\delta \\ 0 & \text{if } \gamma\neq\delta \end{cases} \tag{10}$$

Definition 1.3 *A* **graded map** *between two graded vector spaces X, Y is a linear map $f\colon\ X \to Y$ which is compatible with the grading in the sense that there exists a $\gamma_f \in \Gamma$ such that*

$$f \circ p^X_\gamma = p^Y_{\gamma_f \gamma} \circ f \tag{11}$$

for all $\gamma \in \Gamma$.

(In words, f shifts the grading by a constant 'amount'.)

1.1.2 Color Lie Algebras and Lie Superalgebras

Now, a Lie superalgebra is an algebra on supervector spaces such that, for elements of this algebra,

$$\begin{gathered} [x, y] = (-1)^{1+|x||y|}[y, x] \\ \circlearrowleft_{x,y,z} (-1)^{|x||z|}[x, [y, z]] = 0 \\ [L_\varepsilon, L_{\varepsilon'}] \subseteq L_{\varepsilon+\varepsilon'} \text{ for } \varepsilon, \varepsilon' \in \mathbb{Z}_2 \end{gathered} \tag{12}$$

where we use the symbol $\circlearrowleft_{x,y,z}$ to mean "sum over all the cyclic permutations of x, y, and z". The two homogeneous subspaces of the Lie superalgebra are called the *bosonic* and *fermionic* parts

$$L = L_0 \oplus L_1 \tag{13}$$

where L_0 is said to be bosonic and L_1 fermionic.

A color Lie algebra is simply a generalization of this algebra from a grading over $\mathbb{Z}_2$ to some finite abelian group Γ and from the bicharacter[1] $(-1)^{|x||y|}$ to some general antisymmetric bicharacter χ. (Hence, it is sometimes called a Γ Lie algebra.) A color Lie algebra decomposes like

$$L = \bigoplus_{\gamma \in \Gamma} L_\gamma \tag{14}$$

and the homogeneous elements obey

$$\begin{gathered} [x, y] = -\chi(|x|, |y|)[y, x] \\ \circlearrowleft_{x,y,z} \chi(|x|, |z|)[x, [y, z]] = 0 \\ [L_\gamma, L_{\gamma'}] \subseteq L_{\gamma\gamma'} \text{ for } \gamma, \gamma' \in \Gamma \end{gathered} \tag{15}$$

where this time the group multiplication is just denoted by juxtaposition.

[1] That $(-1)^{|x||y|}$ is indeed an antisymmetric bicharacter is easily seen by direct calculation.

2 Category Theory and the Graphical Calculus

The purpose of this section is twofold. The first is to introduce some concepts, definitions, examples, and theorems from category theory in a manner that suggests its competence for describing and relating various apparently distinct mathematical branches as a unified body of relatively few concepts.[2] Most mathematical objects are made of some *stuff* with some additional *structure* that obeys certain *properties*.[3] We formulate our definitions in a manner that makes this explicit.

The second purpose of this section is to motivate and introduce the graphical calculus. Thus, we shall often state things in as much as three different ways:

1. in categorical notation such as inclusions and equations
2. in commutative diagrams, and
3. in the graphical calculus.

This should give the reader a number of ways to compare notations and convince oneself that these notations are unambiguous, sensible, and insightful. Then, in following sections, things will be stated primarily in terms of the graphical calculus.

2.1 Basic Theory

So let's start at the beginning.

Definition 2.1 *A* **category** $\mathcal{C}$ *consists of the following* stuff*:*

1. *a class, denoted* $\mathrm{Ob}(\mathcal{C})$*, whose elements are called* **objects***, and*
2. *a collection of sets, denoted* $\mathrm{Mor}(\mathcal{C})$*, one for every (ordered) pair of objects. Elements of* $\mathrm{Mor}(\mathcal{C})$ *will be denoted* $\mathrm{Hom}(A, B)$ *for objects* $A, B \in \mathrm{Ob}(\mathcal{C})$*. Elements of* $\mathrm{Hom}(A, B)$ *are called* **morphisms***. So* $\mathrm{Mor}(\mathcal{C})$ *is the family of sets of morphisms in* $\mathcal{C}$*. For a morphism* $f \in \mathrm{Hom}(A, B)$ *we may write* $f : A \to B$*.*

The category $\mathcal{C}$ *comes equipped with the following* structure*:*

1. *for every object* $A \in \mathrm{Ob}(\mathcal{C})$*, an* **identity morphism** *denoted* $id_A \in \mathrm{Hom}(A, A)$*, and*

[2] Another motivation for the introduction of the concepts below is this: In mathematics, often one finds theorems phrased, "For all topological spaces satisfying..." or "For all vector spaces endowed with...". The language of categories does away with the need for such universal quantifiers in statements that pertain to classes of objects such as topological spaces or vector spaces, for example.

[3] The terms "stuff", "structure", and "properties" are in fact formal notions in mathematics. See [3]. Those familiar with logic can translate stuff = types, structure = predicates, and properties = axioms.

2. *for every pair of morphisms* $f \in \mathrm{Hom}(A,B)$ *and* $g \in \mathrm{Hom}(B,C)$, *a* **composite morphism** *in* $\mathrm{Hom}(A,C)$ *denoted* $g \circ f$.

$$A \xrightarrow{f} B \xrightarrow{g} C, \qquad A \xrightarrow{g \circ f} C$$

Also, $\mathcal{C}$ *obeys the following* properties*:*

1. *The "left and right unit laws" hold:* $\forall(f \in \mathrm{Hom}(A,B))$,

$$id_A \circ f = f = f \circ id_B.$$

2. *The "associative law" holds:* $\forall(f \in \mathrm{Hom}(A,B)), \forall(g \in \mathrm{Hom}(B,C)), \forall(h \in \mathrm{Hom}(C,D))$,

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Some examples can be found in the table below.

Objects	**Morphisms**	**Notation**
sets	functions	$\mathcal{S}et$
topological spaces	continuous mappings	$\mathcal{T}op$
groups	group homomorphisms	$\mathcal{G}rp$
vector spaces over a field $\Bbbk$	$\Bbbk$-linear mappings	$\mathcal{V}ect_{\Bbbk}$
vector spaces graded over a finite abelian group Γ	Γ-graded linear mappings	$\mathcal{V}ect_{\Gamma}$

Table 1: Examples of Categories

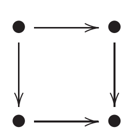
Definition 2.2 *Let* $f \in \mathrm{Hom}(A,B) \in \mathrm{Mor}(\mathcal{C})$ *for some category* $\mathcal{C}$. *We say that* f *is a* **monomorphism** *if* $\forall(C \in \mathrm{Ob}(\mathcal{C})), \forall(g,h \in \mathrm{Hom}(C,A))$,

$$f \circ g = f \circ h \quad \Rightarrow \quad g = h. \tag{16}$$

We can see this statement in at least two ways graphically. First we use the more traditional commutative diagram from algebra.

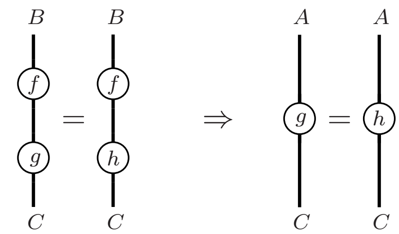
$$\begin{array}{ccc} C & \xrightarrow{g} & A \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array} \quad \text{commutes} \quad \Rightarrow \quad g = h. \tag{17}$$

Here we saw the objects as points and the morphisms as arrows.



$\bullet$ = objects
$\rightarrow$= morphisms (18)

Another notation,[4] which we shall call the *graphical calculus*[5], reverses these assignments and we draw *morphisms* as "dots" and *objects* as arrows. Thus, (17) becomes



(19)

were we have made the convention that the diagrams are to be read "from the bottom up". As we shall see, this second notation becomes very flexible and intuitive in the context of category theory.[6] Hence, from this point forward, much of what is formulated will be done, when possible, in this graphical calculus.

Returning to the development of category theory, we give some examples of monomorphisms below in Table 2:

category	**monomorphisms**
$\mathcal{S}et$	injective functions
$\mathcal{G}rp$	injective group homomorphisms
$\mathcal{T}op$	injective continuous mappings
$\mathcal{V}ect_{\Bbbk}$	$\Bbbk$-linear embeddings
$\mathcal{V}ect_{\Gamma}$	Γ-graded embeddings

Table 2: Examples of Monomorphisms

So a monomorphism is a sort of injective morphism.

Definition 2.3 *An* **epimorphism** *in a category* $\mathcal{C}$ *is a morphism* $f : B \to A$ *such that* $\forall(C \in \mathrm{Ob}(\mathcal{C}))$, $\forall(g, h \in \mathrm{Hom}(A, C))$*:*

$$(g \circ f = h \circ f) \Rightarrow (g = h). \tag{20}$$

We can again draw a commutative diagram for eq.(20):

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[h]{} & C \end{array} \quad \text{commutes} \quad \Rightarrow \quad g = h. \tag{21}$$

Some examples of epimorphisms are given below in Table 3. Thus, an epimorphism is a

[4]Apparently this notation is due to Roger Penrose, being used first to relieve the mathematics of general relativity of its "index-ridden equations". cf.[2] However, the first real nontrivial application was to André Joyal's and Ross Street's notion of a *braided* category in 1986 cf.[4]

[5]after [9]

[6]This notation also appears similar to that of Feynman diagrams and can, in fact, be used to see the processes described by Feynman diagrams as morphisms in the category $\mathcal{H}ilb$ of Hilbert spaces and bounded linear operators. Indeed, Feynman diagrams are just "a notation for intertwining operators between positive-energy representations of the Poincaré group." [2]

category	**epimorphisms**
$\mathcal{S}et$	surjective functions
$\mathcal{G}rp$	surjective group homomorphisms
$\mathcal{T}op$	surjective continuous mappings
$\mathcal{V}ect_{\Bbbk}$	$\Bbbk$-linear restrictions
$\mathcal{V}ect_{\Gamma}$	Γ-graded restrictions

Table 3: Examples of Epimorphisms

sort of surjective morphism.

Definition 2.4 *A* **subobject** *of an object* $A \in \mathrm{Ob}(\mathcal{C})$ *is an object* $A' \in \mathrm{Ob}(\mathcal{C})$ *along with a monomorphism* $\phi : A' \to A$*.*

category	**subobjects**
$\mathcal{S}et$	subsets
$\mathcal{G}rp$	subgroups
$\mathcal{T}op$	subspaces
$\mathcal{V}ect_{\Bbbk}$	vector subspaces
$\mathcal{V}ect_{\Gamma}$	Γ-graded vector subspaces

Table 4: Examples of Subobjects

From these examples, it becomes apparent that categories provide a language for establishing an underlying unity among apparently different mathematical objects: these "branches" of mathematics have a certain amount of postulated stuff existing as well as some structure and properties. What we need now is a way to relate these categories.

Definition 2.5 *A* **functor** F *between two categories* $\mathcal{C}$ *and* $\mathcal{D}$*, denoted* $F : \mathcal{C} \to \mathcal{D}$*, consists of:*

1. *a function* $F_{\mathrm{Ob}} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ *and*

2. *for every pair of objects* $A, B \in \mathrm{Ob}(\mathcal{C})$*, a function*

$$F_{\mathrm{Mor}} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(F_{\mathrm{Ob}}(A), F_{\mathrm{Ob}}(B))$$

such that:

1. F_{Mor} *preserves identities: for any object* $A \in \mathrm{Ob}(\mathcal{C})$*,*

$$F_{\mathrm{Mor}}(id_A) = id_{F_A}$$

2. *F_{Mor} preserves composition: for any objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$ and any morphisms $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$ in $\mathcal{C}$,*

$$F_{\mathrm{Mor}}(f \circ g) = F_{\mathrm{Mor}}(f) \circ F_{\mathrm{Mor}}(g).$$

The standard example is the fundamental group for topological spaces.[7] The functor π_1 gives a group for every object (topological space) in $\mathcal{T}op$ and a group homomorphism for every continuous mapping, i.e. it is a functor $\pi_1 : \mathcal{T}op \to \mathcal{G}rp$.

We will also want to relate functors like

$$\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D} \quad (\Downarrow \alpha) \tag{22}$$

Thus we have the following:

Definition 2.6 *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A* **natural transformation** *α between two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$, denoted $\alpha : F \Rightarrow G$, consists of:*

- *a function $\alpha : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$ given by, $\forall(A \in \mathrm{Ob}(\mathcal{C}))$,*

$$\alpha(A) = \alpha_A, \tag{23}$$

where

$$\alpha_A : F_{\mathrm{Ob}}(A) \to G_{\mathrm{Ob}}(A) \tag{24}$$

such that:

- $\forall(A \in \mathrm{Ob}(\mathcal{C})), \forall(B \in \mathrm{Ob}(\mathcal{C})), \forall(f \in \mathrm{Hom}(A, B) \in \mathrm{Mor}(\mathcal{C}),$

$$G_{\mathrm{Mor}}(f) \circ \alpha_A = \alpha_B \circ F_{\mathrm{Mor}}(f). \tag{25}$$

It is illuminating to see eq.25 as a commutative diagram:

$$\begin{array}{ccc} F_{\mathrm{Ob}}(A) & \xrightarrow{F_{\mathrm{Mor}}(f)} & F_{\mathrm{Ob}}(B) \\ \Big\downarrow{\scriptstyle \alpha_A} & & \Big\downarrow{\scriptstyle \alpha_B} \\ G_{\mathrm{Ob}}(A) & \xrightarrow{G_{\mathrm{Mor}}(f)} & G_{\mathrm{Ob}}(B) \end{array} \tag{26}$$

One can also define a composition of natural transformations and an identity natural transformation in the obvious way. It immediately follows that the left and right unit laws and associativity hold for these definitions.

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[7] Many of the invariants of algebraic topology are in fact functors and this was the motivation for Eilenberg and Mac Lane to formulate the definition of a functor in 1945 (as well as the definition of a category). See [2].

Definition 2.7 *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A* **natural isomorphism** *α between two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$, denoted $\alpha : F \Rightarrow G$, is a natural transformation that has an* **inverse***, that is, a natural transformation $\beta : G \Rightarrow F$ such that $\alpha \circ \beta = \mathbf{1}_G$ and $\beta \circ \alpha = \mathbf{1}_F$.*

It can be shown that a natural transformation $\alpha : F \Rightarrow G$ is a natural isomorphism iff for every object $A \in \mathrm{Ob}(\mathcal{C})$, the morphism α_A is invertible in the obvious sense of the word.

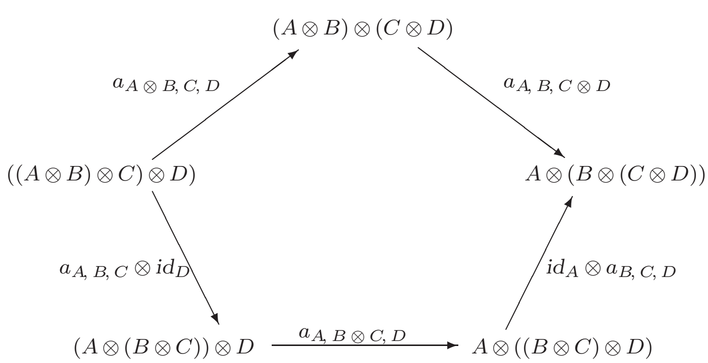
2.2 Braided Monoidal Categories

Definition 2.8 *A* **monoidal category***, or* **tensor category***, consists of:*

1. *a category $\mathcal{C}$*
2. *a functor called the* **tensor product** *$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, where we write $\otimes_{\mathrm{Ob}}(A, B) = A \otimes B$ for objects $A, B \in \mathrm{Ob}(\mathcal{C})$ and $\otimes_{\mathrm{Mor}}(f, g) = f \otimes g$ for morphisms f and g in $\mathrm{Mor}(\mathcal{C})$ and the ambiguity of the notation is abnegated by the context*
3. *an object called the* **identity object** *denoted by $\mathbf{1} \in \mathrm{Ob}(\mathcal{C})$*
4. *a natural isomorphism called the* **associator***:*
$$a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C) \tag{27}$$
5. *a natural isomorphism called the* **left unit law***:*
$$\ell_A : \mathbf{1} \otimes A \to A \tag{28}$$
6. *a natural isomorphism called the* **right unit law***:*
$$r_A : A \otimes \mathbf{1} \to A \tag{29}$$

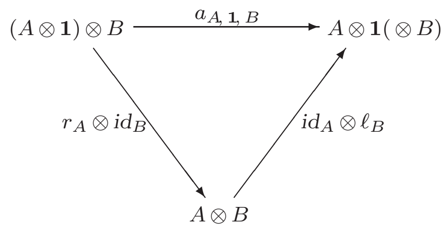
such that the following diagrams commute for all objects $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$:

1. *the* **pentagon equation** *for the associator:*



(30)

2. the **triangle equation** *for the left and right unit laws:*



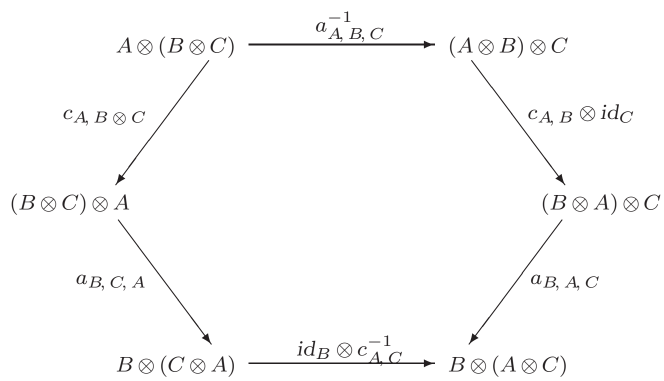
(31)

Definition 2.9 *A* **braided monoidal category** *consists of:*

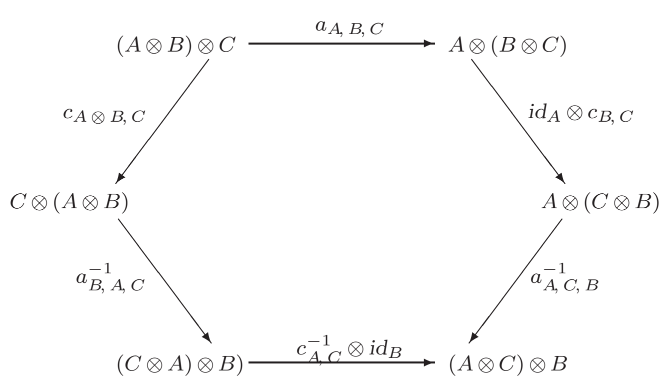
1. *a monoidal category* $\mathcal{C}$
2. *a natural isomorphism called the* **braiding***:*

$$c_{A,B} : A \otimes B \Rightarrow B \otimes A \tag{32}$$

such that the following diagrams, called the **hexagon equations** *for the braiding, commute for all objects* $A, B, C \in \mathrm{Ob}(\mathcal{C})$*:*



(33)



(34)

Definition 2.10 *A monoidal category* $\mathcal{C}$ *is said to be* **symmetric** *if the braiding is such that* $\forall (A, B \in \mathrm{Ob}(\mathcal{C})),\ c_{A,B} = c^{-1}_{B,A}$*. We call a monoidal category* $\mathcal{C}$ **strict** *if the associator*

$a_{A,B,C}$, the left unit law ℓ_A, and the right unit law r_A are all identity morphisms. In such cases, we may write, for $A, B, C \in \mathrm{Ob}(\mathcal{C})$,

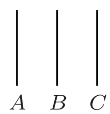
$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \tag{35}$$

$$\mathbf{1} \otimes A = A \tag{36}$$

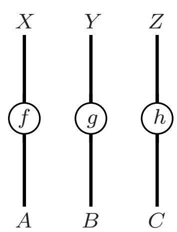
$$A \otimes \mathbf{1} = A. \tag{37}$$

We should note here that Mac Lane has proved that every monoidal (resp. braided and symmetric) category is equivalent to a strict monoidal (resp. braided and symmetric) category, in a sense which can be made more precise than we shall state here. See [1]. Thus, in essence, all we really need to work with are strict monoidal categories. This simplifies things considerably! And this is where the utility of our graphical calculus notation kicks in.

In a strict monoidal category, since we are no longer concerned with the order in which we tensor objects, we can represent tensored objects horizontally with no additional parentheses:

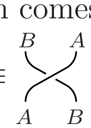
$A \otimes B \otimes C \equiv$  (38)

Similarly, if we have morphisms $f \in \mathrm{Hom}(A, X), g \in \mathrm{Hom}(B, Y)$, and $f \in \mathrm{Hom}(C, Z)$, we may write $f \otimes g \otimes h \in \mathrm{Hom}(A \otimes B \otimes C, X \otimes Y \otimes Z)$ as

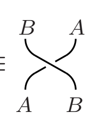
$f \otimes g \otimes h \equiv$  (39)

without worrying about f, g, and h sliding up or down our wires a bit.

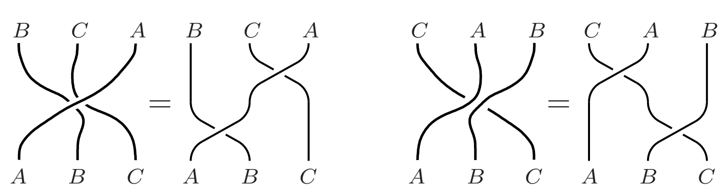
Up to now, this graphical calculus may seem like a mere curiosity. Strict *braided* monoidal categories are where this notation comes alive. If we denote braidings by

$c_{A,B} \equiv$  (40)

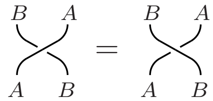
and inverse braidings by

$c^{-1}_{B,A} \equiv$  (41)

we can write the hexagon equations (eqs.33 and 34) for a *strict* braided monoidal category as

 (42)

For a *symmetric* braided monoidal category, the requirement that $c_{A,B} = c^{-1}_{B,A}$ becomes



(43)

so that the braiding is trivial.

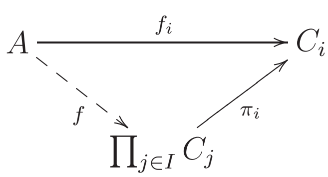
2.3 Additional Structure

The categories in which we will be interested, namely $\mathcal{V}ect$ and $\mathcal{V}ect_\Gamma$ are *abelian* categories. In particular, they have the concept of "addition" of morphisms. Before we can state exactly what an abelian category is, we will need a few more concepts.

Definition 2.11 *An* **initial object** *in a category* $\mathcal{C}$ *is an object* A *in* $\mathcal{C}$ *such that, for every object* $X \in \mathrm{Ob}(\mathcal{C})$*, there is exactly one morphism* $A \to X$*. A* **terminal object** *in a category* $\mathcal{C}$ *is an object* $B \in \mathrm{Ob}(\mathcal{C})$ *such that, for every object* $X \in \mathrm{Ob}(\mathcal{C})$*, there is exactly one morphism* $X \to B$*. A* **zero object** *in a category* $\mathcal{C}$ *is an object* 0 *that is both an initial object and a terminal object.*

All initial objects (respectively, terminal objects, and zero objects), if they exist, are isomorphic in $\mathcal{C}$.

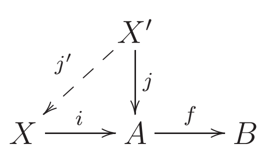
Definition 2.12 *Let* $\{C_i\}_{i\in I}$ *be a set of objects in a category* $\mathcal{C}$*. A* **direct product** *of the collection* $\{C_i\}_{i\in I}$ *is an object* $\prod_{i\in I} C_i$ *of* $\mathcal{C}$*, with morphisms* $\pi_i : \prod_{j\in I} C_j \to C_i$ *for each* $i \in I$*, such that for every object* $A \in \mathrm{Ob}(\mathcal{C})$*, and any collection of morphisms* $f_i \in \mathrm{Hom}(A, C_i)$ *for every* $i \in I$*, there exists a unique morphism* $f : A \to \prod_{i\in I} C_i$ *making the following diagram commute for all* $i \in I$*:*



(44)

Definition 2.13 *Given a morphism* $f \in \mathrm{Hom}(A, B)$ *in* $\mathcal{C}$*, a* **kernel** *of* f *is a morphism* $i \in \mathrm{Hom}(X, A)$ *such that:*

- $f \circ i = 0$.
- *For any other morphism* $j \in \mathrm{Hom}(X', A)$ *such that* $f \circ j = 0$*, there exists a unique morphism* $j' \in \mathrm{Hom}(X', X)$ *such that the diagram*



(45)

commutes.

Likewise, a **cokernel** *of f is a morphism $p \in \mathrm{Hom}(B, Y)$ such that:*

- $p \circ f = 0$.
- *For any other morphism $j \in \mathrm{Hom}(B, Y')$ such that $j \circ f = 0$, there exists a unique morphism $j' \in \mathrm{Hom}(Y, Y')$ such that the diagram*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{p} & Y \\ & & {\scriptstyle j}\downarrow & \swarrow {\scriptstyle j'} & \\ & & Y' & & \end{array} \tag{46}$$

  commutes.

The kernel and cokernel of a morphism f in $\mathcal{C}$ will be denoted $\ker(f)$ and $\mathrm{cok}(f)$, respectively.

Definition 2.14 *A category $\mathcal{C}$ is said to be* **abelian** *if it satisfies:*

1. *For any two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, the set of morphisms $\mathrm{Hom}(A, B)$ admits an abelian group structure, with group operation denoted by $+$, satisfying the following "naturality" requirement: given any diagram of morphisms*

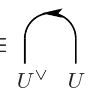
$$A \xrightarrow{f} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C \xrightarrow{h} D \tag{47}$$

   we have $(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$ and $h \circ (g_1 + g_2) = h \circ g_1 + h \circ g_2$. That is, composition of morphisms must distribute over addition in $\mathrm{Hom}(\cdot\,,\cdot)$. The identity element in the group $\mathrm{Hom}(\cdot\,,\cdot)$ will be denoted by 0.
2. *$\mathcal{C}$ has a zero object.*
3. *For any two objects A, B in $\mathcal{C}$, the categorical direct product $A \times B$ exists in $\mathcal{C}$.*
4. *Every morphism in $\mathcal{C}$ has a kernel and a cokernel.*
5. $\ker(\mathrm{cok}(f)) = f$ *for every monomorphism f in $\mathcal{C}$.*
6. $\mathrm{cok}(\ker(f)) = f$ *for every epimorphism f in $\mathcal{C}$.*

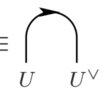
We also would like in some cases to have the notion of a *dual object*. In braided categories, it is natural to require that $\mathcal{C}$ is *sovereign*[8]:

[8]See [9].

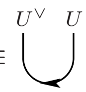
Definition 2.15 *A braided category $\mathcal{C}$ is said to be* **sovereign** *if for every object $U \in \mathrm{Ob}(\mathcal{C})$, there is an object $U^\vee \in \mathrm{Ob}(\mathcal{C})$ called the left and right* **dual object** *of U and there are left and right* **evaluation morphisms**

$d_U \in \mathrm{Hom}(U^\vee \otimes U, \mathbf{1})$ *denoted* $d_U \equiv$ 

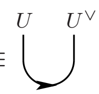
and

$\widetilde{d}_U \in \mathrm{Hom}(U \otimes U^\vee, \mathbf{1})$ *denoted* $\widetilde{d}_U \equiv$ 

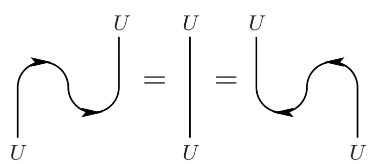
as well as left and right **coevaluation morphisms**

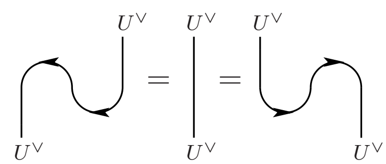
$b_U \in \mathrm{Hom}(\mathbf{1}, U^\vee \otimes U)$ *denoted* $b_U \equiv$ 

and

$\widetilde{b}_U \in \mathrm{Hom}(\mathbf{1}, U \otimes U^\vee)$ *denoted* $\widetilde{b}_U \equiv$ 

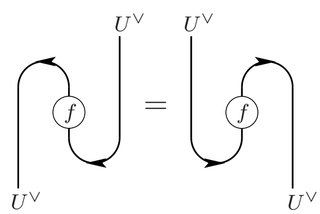
which satisfy



and


(48)

as well as



(49)

for every morphism $f \in \mathrm{Hom}(U, U)$.

We have now developed a sufficient amount of category theory to discuss Lie algebras. From now on we will suppress the object labeling in the graphical equations when it is redundant or implicit in the context.

3 Braided Commutator Algebras, Braided Lie Algebras, and Braided Commutator Lie Algebras

In this section we will define first a braided braided commutator algebra and then a braided Lie algebra. Next, the definition of a braided commutator Lie algebra is given and finally, a theorem which gives a sufficient condition that a braided commutator algebra is a braided commutator Lie algebra.

We start by stating the definition for an associative algebra, since it is in terms of these which we define braided commutator algebras. However, we do so in the context of category theory and the language of the graphical calculus. In the case $\mathcal{C} = \mathcal{V}ect$, this reduces to the familiar definition of an algebra. For details, see [9].

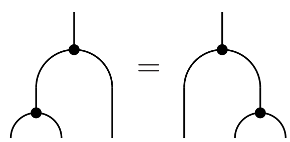
Definition 3.1 *A* **unital associative algebra (=monoid)** *A in a strict monoidal category* $\mathcal{C}$ *is:*

- *an object* $\dot{A}$
- *equipped with two morphisms:* $\mu \in \mathrm{Hom}(\dot{A} \otimes \dot{A}, \dot{A})$ *called the* **product** *and* $\eta \in \mathrm{Hom}(\mathbf{1}, \dot{A})$ *called the* **unit***, denoted as*

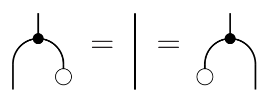
(50)

  in our graphical notation.

- *These morphisms satisfy associativity:*


(51)

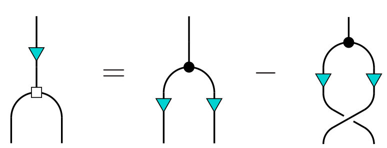
  and the left and right unit laws:


(52)

To get an associative algebra, we just drop the unit requirements. We shall often denote an algebra, unital or not, by its object and product, i.e. $A := (\dot{A}, \mu)$.

Definition 3.2 *Let* $A := (\dot{A}, \mu)$ *be an associative algebra in a braided monoidal abelian category* $\mathcal{C}$*. If, for an object* $\dot{L} \in \mathrm{Ob}(\mathcal{C})$*, there exists a monomorphism* $\phi \in \mathrm{Hom}(\dot{L}, \dot{A})$ *and a morphism* $\lambda \in \mathrm{Hom}(\dot{L} \otimes \dot{L}, \dot{L})$ *in* $\mathcal{C}$ *such that, denoting*

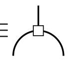
we have


(53)

then $L := (\dot{L}, \lambda)$ *is said to be a* **braided commutator algebra** *in* $\mathcal{C}$ *induced by multiplication of A through* ϕ*. We call* λ *the* **commutator** *and the associative algebra A is called an* **algebra associated to** *L.*

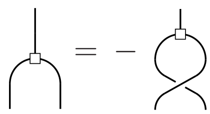
Definition 3.3 *A* **braided Lie algebra** *in a braided monoidal abelian category is:*

- *an object L*
- *equipped with a morphism $\lambda \in \mathrm{Hom}(L \otimes L, L)$ called the* **Lie bracket** *and denoted as*

$\lambda \equiv$ 

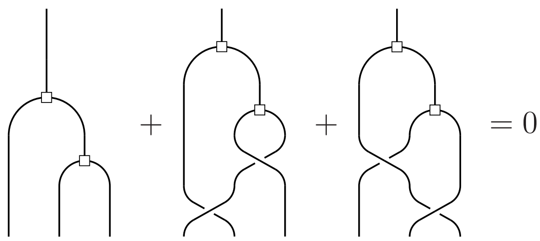
such that

- λ *has* **braided antisymmetry***:*



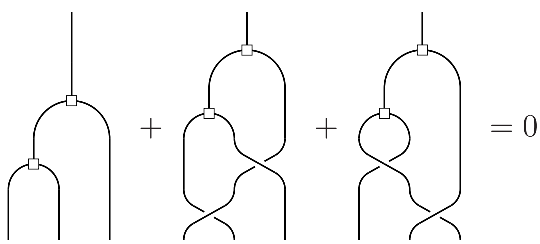
(54)

- λ *obeys the* **left braided primitive Jacobi identity***:*



(55)

- *and the* **right braided primitive Jacobi identity***:*

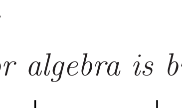


(56)

Definition 3.4 *A* **braided commutator Lie algebra** *is a braided commutator algebra that is also a braided Lie algebra.*

The following theorem gives a sufficient condition that a braided commutator algebra is a braided commutator Lie algebra.

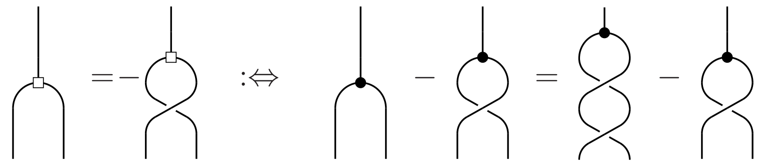
Theorem 3.1 *A braided commutator algebra is braided antisymmetric iff*



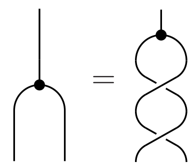
(57)

i.e. if μ symmetrizes the braiding. Moreover, an antisymmetric braided commutator algebra obeys both the left and right braided primitive Jacobi identities.

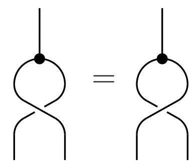
Proof. For the first part of the proof we have:



(58)

$\Leftrightarrow$ 

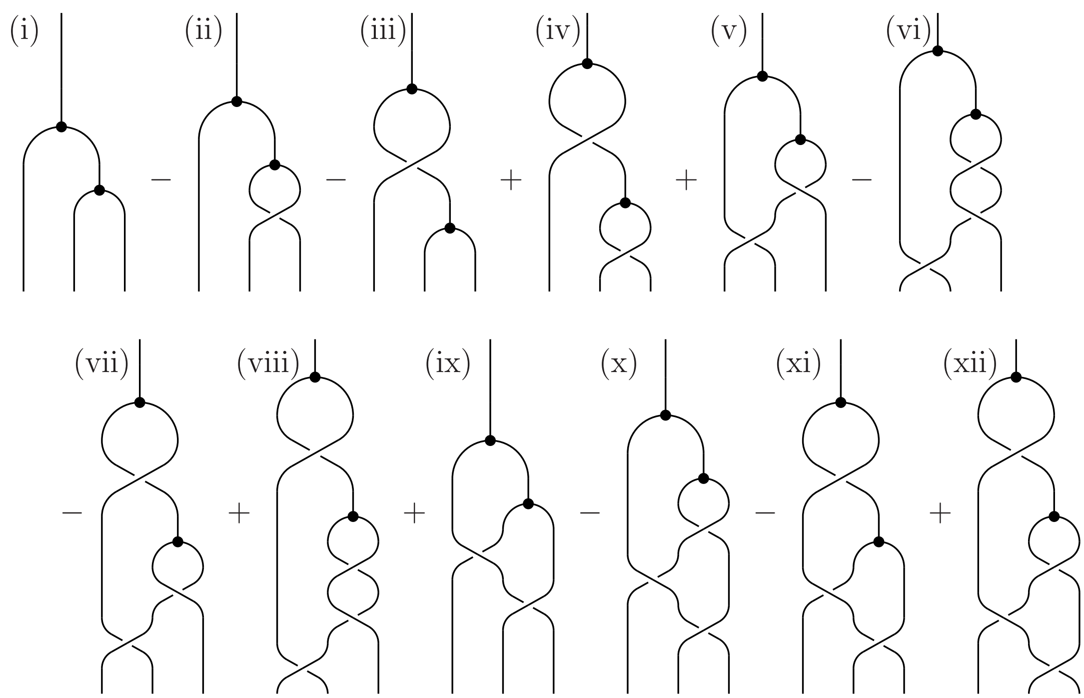
(59)

$\Leftrightarrow$ 

(60)

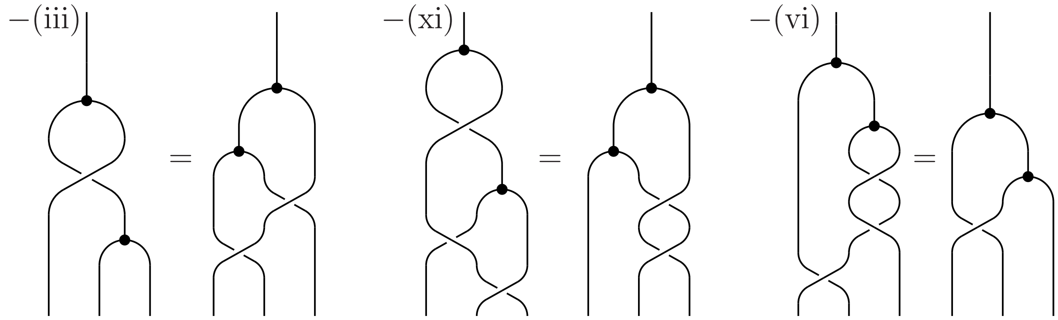
In the first equivalence we used the defining property (Eq. 53) of a braided commutator Lie algebra with $\phi = \mathrm{id}_{\dot{L}}$. In the last equivalence we have applied $c^{-1}_{\dot{L},\dot{L}}$ to both sides.

To show that braided antisymmetry implies the left braided primitive Jacobi identity is obeyed we start by expanding (55) by (53) again with $\phi = \mathrm{id}_{\dot{L}}$. Numbering the terms, we have for the LHS:



(61)

Simplifying double braidings and writing terms in a manner which suggests the use of associativity of μ, we have:



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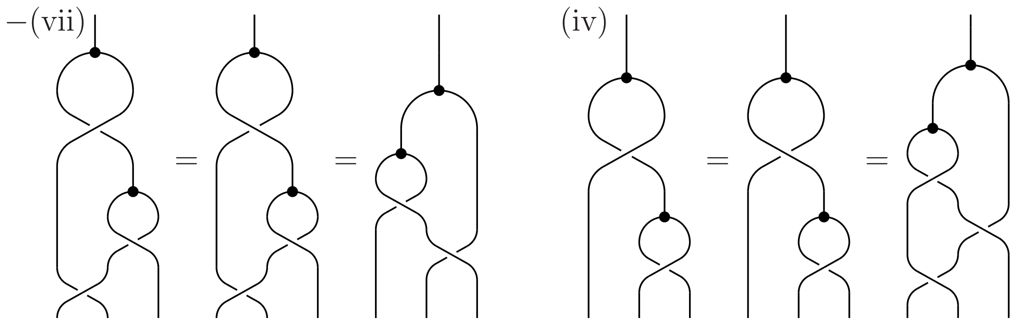
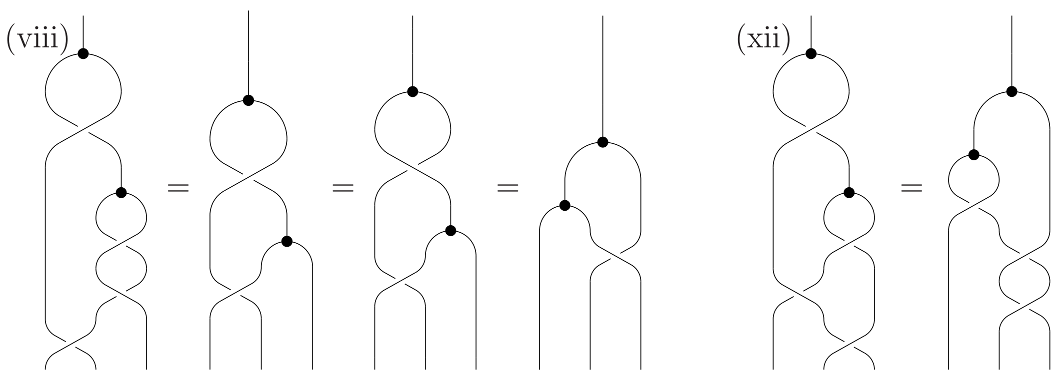


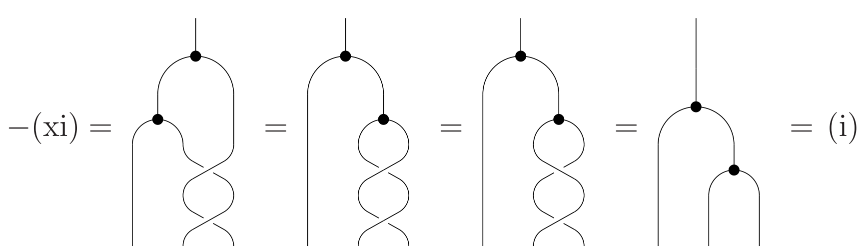
(63)



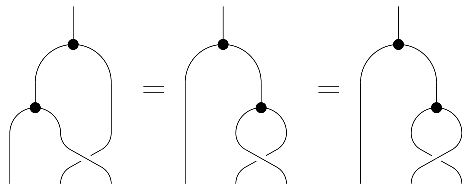
(64)

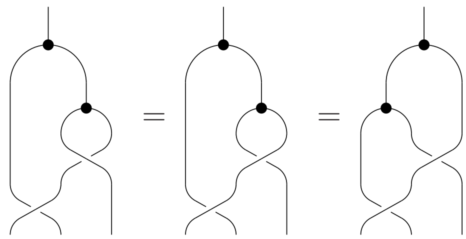
Now we are in a position to make explicit use of associativity of μ and braided anti-symmetry. We shall show that

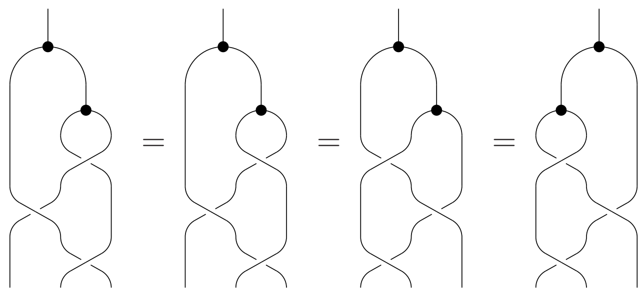
$$\begin{array}{ll} \text{(i)}=-\text{(xi)} & \text{(iv)}=-\text{(x)} \\ -\text{(ii)}=\text{(viii)} & -\text{(vi)}=\text{(xii)} \\ -\text{(iii)}=\text{(v)} & -\text{(vii)}=\text{(ix)}. \end{array}$$

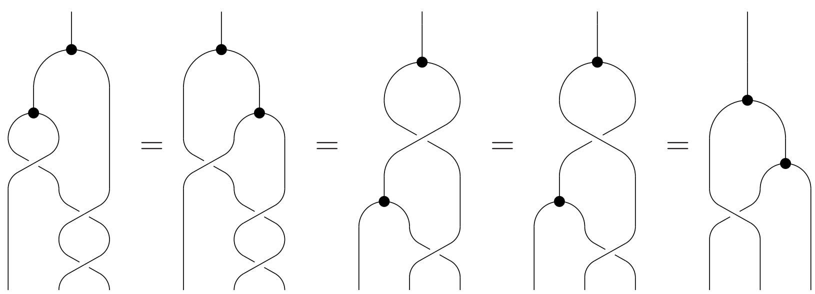


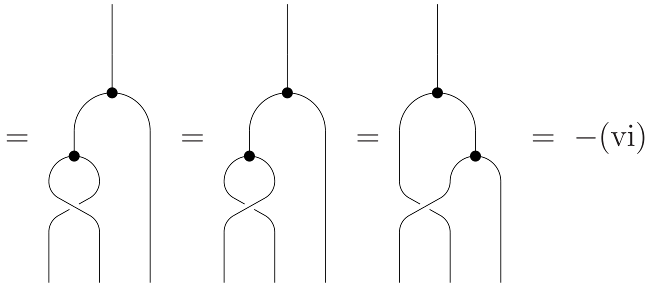
(65)

(viii) =  = −(ii) (66)

(v) =  = −(iii) (67)

−(x) =  = (iv) (68)

(xii) =  =



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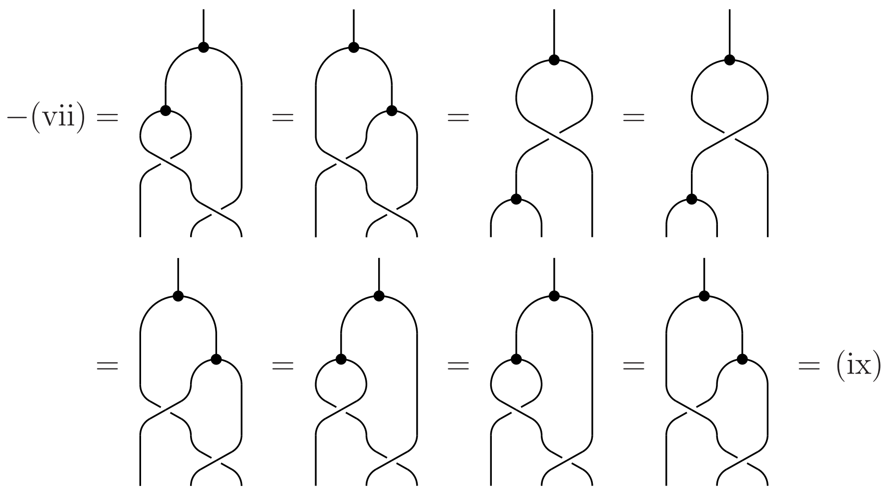
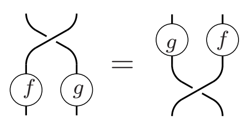


(70)

The proof that braided antisymmetry implies the right braided Jacobi identity holds proceeds exactly along the same lines as that above. □

The more general case that $\phi \neq \mathrm{id}_{\dot{L}}$ is also true. This is easy to see since, using the functoriality of c, we can pull ϕ through any braidings, i.e.



(71)

So if we have a braided commutator algebra for which μ symmetrizes the braiding, then that braided commutator algebra is also a braided Lie algebra and, hence, a braided commutator Lie algebra.

4 The Category Theory of Color Lie Algebras and Lie Superalgebras

It was mentioned earlier (see Table 1) that Γ-graded vector spaces along with Γ-graded maps form a category. In this section we will formally state this and show that this category is in fact a *braided monoidal abelian* category.

Definition 4.1 *The* **category $\mathcal{V}ect_\Gamma$ of Γ-graded vector spaces** *is the category whose objects are Γ-graded vector spaces and whose morphisms are graded maps between them.*

Theorem 4.1 *The category $\mathcal{V}ect_\Gamma$ is a braided monoidal abelian category.*

Proof. The category $\mathcal{V}ect_\Gamma$ is a subcategory[9] of the category of vector spaces. As a consequence, it is $\mathbb{C}$-linear and abelian.

Furthermore, the category $\mathcal{V}ect_\Gamma$ is monoidal: The tensor product $\otimes$ is the tensor product of vector spaces, with the obvious Γ-grading,[10] and the tensor unit $\mathbf{1}$ is the one-dimensional space $V_0 = \mathbb{C}$.

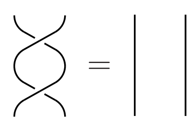
Lastly, $\mathcal{V}ect_\Gamma$ is *braided* monoidal: The category $\mathcal{V}ect_\Gamma$ inherits the "exchange braiding" π from the category of vector spaces. In terms of elements, this is a family of linear maps $\pi_{X,Y}$ such that

$$\pi_{X,Y}(x,y) = (y,x) \tag{72}$$

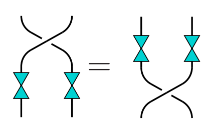
for ordered pairs of elements $x \in X$ and $y \in Y$. In terms of category theory, if we denote the embedding, restriction, idempotent,[11] and exchange braiding graphically by

(73)

π is the family of morphisms $\pi_{X,Y} \in \mathrm{Hom}(X \otimes Y, Y \otimes X)$ satisfying


(74)

and


(75)

for all objects $X, Y \in \mathrm{Ob}(\mathcal{V}ect_\Gamma)$ and all $\gamma, \delta \in \Gamma$. □

We now introduce a braiding on $\mathcal{V}ect_\Gamma$ that is different from the "trivial" exchange braiding, but uses the latter as an ingredient:

[9]Intuitively, a *subcategory* is just a category $\mathcal{S}$ which can be seen as a category $\mathcal{C}$ with some objects and morphisms removed. If, for every $A, B \in \mathrm{Ob}(\mathcal{S})$, one has that $\mathrm{Hom}_{\mathcal{S}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)$, then one says that $\mathcal{S}$ is a *full subcategory*.

[10]The tensor product is even $\Gamma \times \Gamma$-graded, but that will not play a role.

[11]Of course, these are always with respect to some homogeneous subspace indexed by $\gamma \in \Gamma$. We shall suppress labeling the graphical notations when it is clear from the context what is meant.

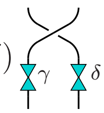
Theorem 4.2 *Choose an antisymmetric bicharacter* χ *of* Γ *and for any objects* $X, Y \in \mathrm{Ob}(\mathcal{V}ect_\Gamma)$ *define* $c_{X,Y} \in \mathrm{Hom}(X \otimes Y, Y \otimes X)$ *by*

$$c_{X,Y} := \sum_{\gamma,\,\delta\in\Gamma} \chi(\gamma,\delta)\,\pi_{X,Y} \circ (p^X_\gamma \otimes p^Y_\delta)\,. \tag{76}$$

Then $c_{X,Y}$ *is a braiding on the monoidal category* $\mathcal{V}ect_\Gamma$.

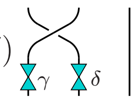
We will write Eq.(76) as

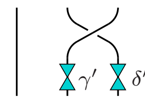
$$c := \sum_{\gamma,\,\delta\in\Gamma} \chi(\gamma,\delta) \tag{77}$$



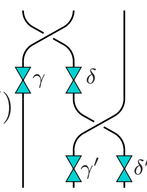
Proof. To show that $c_{X,Y}$ is indeed a braiding, we must show that it obeys the hexagon equations (eqs.42). To do this we use the properties of $\pi_{X,Y}$ and the orthogonality of the idempotents.

$$\text{R.H.S. Eq.42} = \left[\sum_{\gamma,\,\delta\,\in\,\Gamma} \chi(\gamma,\delta)\right] \circ \left[\sum_{\gamma',\,\delta'\,\in\,\Gamma} \chi(\gamma',\delta')\right]$$

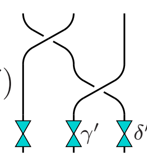




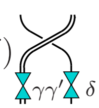
$$= \sum_{\gamma,\,\delta,\,\gamma',\,\delta',\,\in\,\Gamma} \chi(\gamma,\delta)\chi(\gamma',\delta')$$



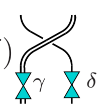
$$= \sum_{\gamma,\,\gamma',\,\delta\,\in\,\Gamma} \chi(\gamma,\delta)\chi(\gamma',\delta)$$



$$= \sum_{\gamma,\,\gamma',\,\delta\,\in\,\Gamma} \chi(\gamma\gamma',\delta)$$



$$= \sum_{\gamma,\,\delta\,\in\,\Gamma} \chi(\gamma,\delta)$$



$$= \text{L.H.S. Eq.42}$$

Here the double line denotes a tensor product of objects. The proof for the other hexagon equation proceeds analogously.□

Also note that, again by the orthogonality of idempotents,

$$c_{X,Y} \circ (p^X_\gamma \otimes p^Y_\delta) = \pi_{X,Y} \circ (p^X_\gamma \otimes p^Y_\delta)\,. \tag{78}$$

Below, by $\mathcal{V}ect_\Gamma$ we always mean the category above endowed with this braiding $\{c_{X,Y}\}$ for a chosen antisymmetric bicharacter χ. Note that strictly speaking one should use the notation $\mathcal{V}ect_{\Gamma,\chi}$ instead of $\mathcal{V}ect_\Gamma$. Also, if $\chi = \chi_0 \equiv 1$ is the trivial bicharacter, the braiding is just the exchange braiding; this uninteresting case is implicitly excluded below.

Using again orthogonality of idempotents, together with the antisymmetry property of χ, one checks that the 'square' of the braiding is given by

$$c_{Y,X} \circ c_{X,Y} = \sum_{\gamma,\,\delta\,\in\,\Gamma} \chi(\gamma,\delta)\,\chi(\delta,\gamma)\,p^X_\gamma \otimes p^Y_\delta = \sum_{\gamma,\,\delta\,\in\,\Gamma} p^X_\gamma \otimes p^Y_\delta = id_{X\otimes Y}\,. \tag{79}$$

Thus for any choice of antisymmetric bicharacter, the braiding is symmetric.

Let now $\dot{L}$ be an object of $\mathcal{V}ect_\Gamma$ and $\lambda \in \mathrm{Hom}(\dot{L}\otimes\dot{L},\dot{L})$. Write c for $c_{\dot{L},\dot{L}}$, p_γ for p^A_γ, etc.

Theorem 4.3 *If the 'bracket' λ is braided antisymmetric, i.e.*

$$\lambda \circ c = -\lambda\,, \tag{80}$$

then

$$\lambda \circ \pi = -\sum_{\gamma,\,\delta\,\in\,\Gamma} \chi(\gamma,\delta)\,\lambda\circ(p_\gamma\otimes p_\delta)\,. \tag{81}$$

This is precisely the "Γ-twisted antisymmetry" of the bracket of a *color Lie algebra.*

Proof.

$$\begin{aligned}
\lambda\circ c = -\lambda \;\;&\Rightarrow\;\; \lambda\circ c\circ c = -\lambda\circ c \\
&\Rightarrow\;\; \lambda = -\sum_{\gamma,\,\delta\,\in\,\Gamma} \chi(\gamma,\delta)\,\lambda\circ\pi\circ(p_\gamma\otimes p_\delta) \\
&\Rightarrow\;\; \lambda\circ\pi = -\sum_{\gamma,\,\delta\,\in\,\Gamma} \chi(\gamma,\delta)\,\lambda\circ\pi\circ(p_\gamma\otimes p_\delta)\circ\pi \\
&\Rightarrow\;\; \lambda\circ\pi = -\sum_{\gamma,\,\delta\,\in\,\Gamma} \chi(\gamma,\delta)\,\lambda\circ\pi\circ\pi\circ(p_\gamma\otimes p_\delta) \\
&\Rightarrow\;\; \lambda\circ\pi = -\sum_{\gamma,\,\delta\,\in\,\Gamma} \chi(\gamma,\delta)\,\lambda\circ(p_\gamma\otimes p_\delta)
\end{aligned}$$

□

Next, assume that the left braided Jacobi identity also holds (recall that this follows from braided antisymmetry if L is a braided commutator Lie algebra). When composed with $p_\gamma\otimes p_\delta\otimes p_\epsilon$, then when expressing the braiding through the exchange braiding, in the second term one gets a factor $\chi(\gamma,\delta)\,\chi(\gamma,\epsilon)^{-1}$, and in the third term a factor

$\chi(\gamma,\epsilon)^{-1}\,\chi(\delta,\epsilon)$. It follows that, again using $\circlearrowleft_{a,b,c}$ to refer to the sum over all cyclic permutations but with respect to the exchange braiding,

$$\circlearrowleft_{\gamma,\delta,\epsilon}\,\chi(\gamma,\epsilon)\,\lambda\circ(id\otimes\lambda)\circ(p_\gamma\otimes p_\delta\otimes p_\epsilon)=0\,. \tag{82}$$

This is precisely the "Γ-twisted Jacobi identity" of the bracket of a *color Lie algebra*.

Referring to $(\dot{L},\lambda)$ such that braided antisymmetry and left braided Jacobi identity are satisfied as a *braided Lie algebra in* $\mathcal{V}ect_\Gamma$, we have thus shown:

Theorem 4.4 *A (Γ-twisted) color Lie algebra is a braided Lie algebra in the category* $\mathcal{V}ect_\Gamma$*.*

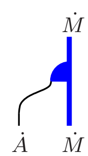
And, in particular,

Corollary 4.1 *A Lie superalgebra is a braided Lie algebra in the category of super vector spaces.*

5 Representations of Braided Commutator Algebras, Braided Lie Algebras, and Braided Commutator Lie Algebras

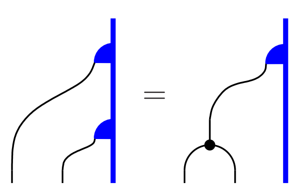
In this section we define and give examples of braided Lie algebras and braided braided commutator Lie algebras. In particular, we find generalizations of the adjoint representation, the tensor product representation, and the contragredient representation on dual objects. We begin by recalling some definitions for associative algebras and then move to braided Lie algebras and braided commutator Lie algebras.

Definition 5.1 *Suppose* $A:=(\dot{A},\mu)$ *is an associative algebra in a braided monoidal abelian category* $\mathcal{C}$ *and that there is an object* $\dot{M}\in\mathrm{Ob}(\mathcal{C})$*. If, in addition, there exists a morphism* $\rho\in\mathrm{Hom}(\dot{A}\otimes\dot{M},\dot{M})$ *denoted*

$\rho \equiv$ 

(83)

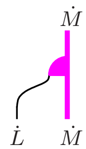
such that


(84)

then ρ *is a* **representation** *of* A *and* $\dot{M}$ *is called an* A**-module**.

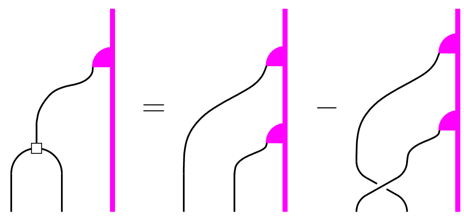
The following definition is similar to Definition 1.7 in [21].

Definition 5.2 *Let $L := (\dot{L}, \lambda)$ be a braided Lie algebra or $L := (\dot{L}, \lambda, \phi)$ be a braided commutator algebra in a braided monoidal abelian category $\mathcal{C}$. Suppose $\dot{M} \in \mathrm{Ob}(\mathcal{C})$. If there exists a morphism $\rho \in \mathrm{Hom}(\dot{L} \otimes \dot{M}, \dot{M})$ denoted*

$\rho \equiv$ 

(85)

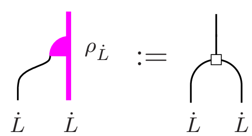
such that


(86)

then ρ is called a **representation** *of L and $M := (\dot{M}, \rho)$ is called an L-***module***.*

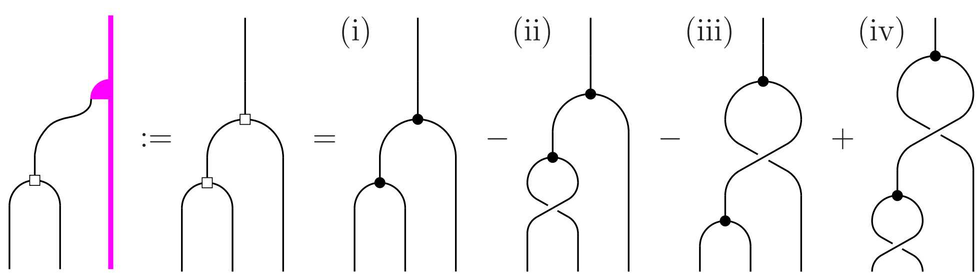
5.1 The Adjoint Representation

Let L be a braided commutator Lie algebra in a braided monoidal abelian category $\mathcal{C}$. Then $\rho_{\dot{L}} \equiv \mathrm{Ad}_{\dot{L}} := \lambda$ provides a L-module structure[12] on $\dot{L}$, i.e.



(87)

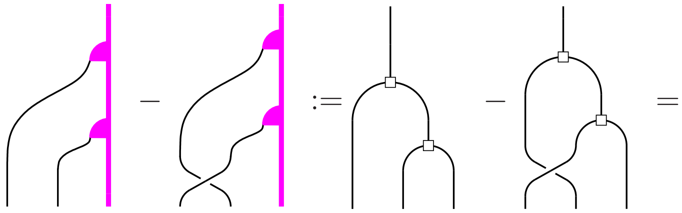
Putting eq.(87) into the LHS of eq.(86) we get



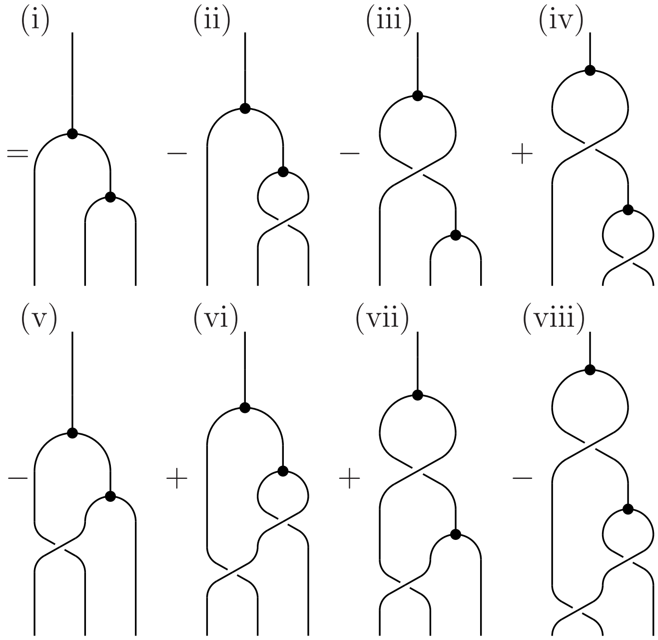
(88)

and eq.(87) into the RHS of eq.(86)

[12]The induced L-module structure is analogous. One just uses $\dot{L} \otimes \dot{U}$ in place of $\dot{L}$for the underlying object in the L-module



(89)

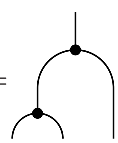
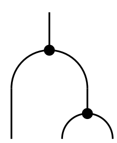


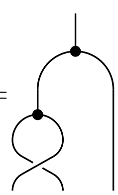
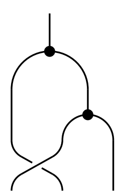
(90)

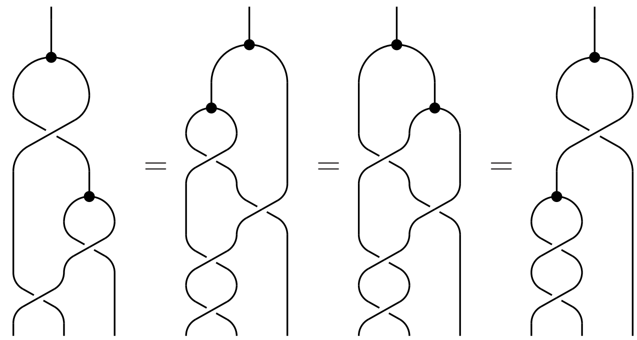
We will show that

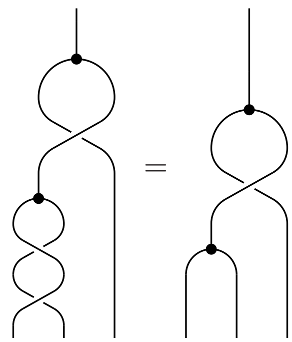
$$
\begin{array}{cc}
(88.\text{i})=(90.\text{i}) & -(88.\text{ii})=-(90.\text{v}) \\
-(90.\text{viii})=-(88.\text{iii}) & (90.\text{iv})=(88.\text{iv}) \\
(90.\text{vii})=-(90.\text{ii}) & -(90.\text{iii})=-(90.\text{vi})
\end{array}
$$

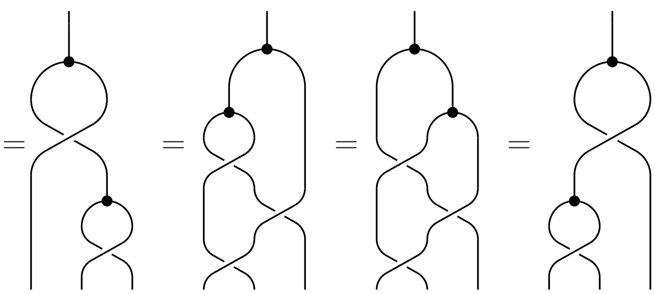
Indeed,

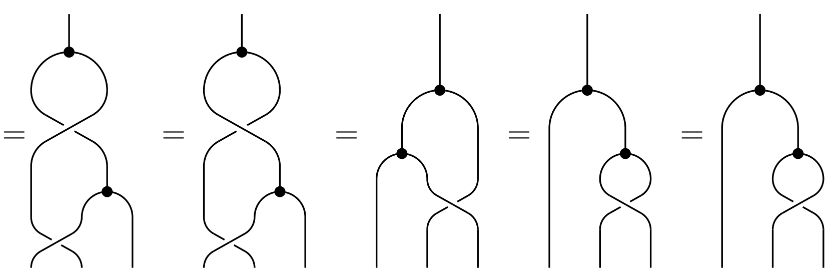
(88.i) =  =  = (90.i) (91)

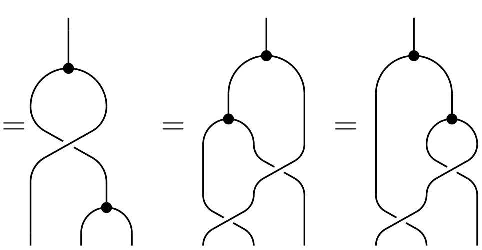
−(88.ii)=  =  = −(90.v) (92)

−(90.viii) = 

=  = −(88.iii) (93)

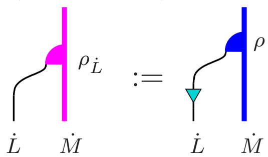
(90.iv) =  = (88.iv) (94)

(90.vii) =  = −(90.ii) (95)

−(90.iii) =  = −(90.vi) (96)

5.2 A-modules provide L-modules

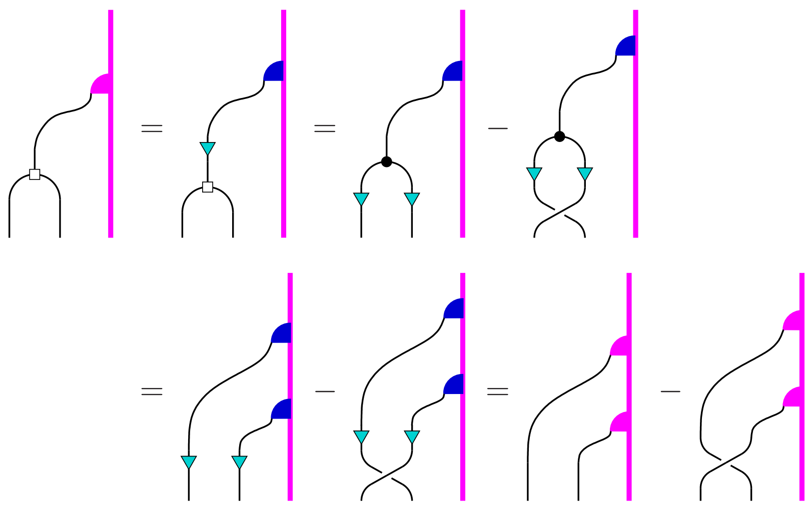
Theorem 5.1 *Suppose ρ is a representation of an associative algebra A associated to a braided commutator algebra L in some braided monoidal abelian category $\mathcal{C}$ and $M := (\dot{M}, \rho)$ is the corresponding A-module. Then, if $\phi \in \mathrm{Hom}(\dot{L}, \dot{A})$ is the monomorphism in the defining property of L, $\rho_L \in \mathrm{Hom}(\dot{L} \otimes \dot{M}, \dot{M})$ defined by*



(97)

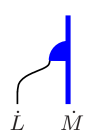
provides a module structure on L, and, hence, $\dot{M}$ is also an L-module.

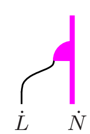
Proof. We use successively the definitions of ρ_L, λ, ρ, and once again ρ_L.


(98)

5.3 The Tensor Product Representation

If ρ is a representation for L on the module M and σ is a representation for L on the module N, i.e. we have

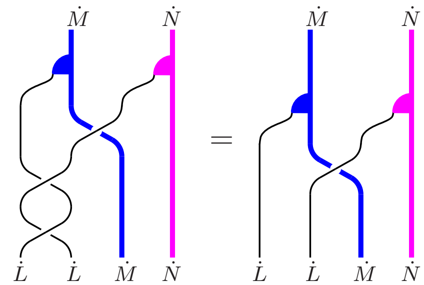
$\rho \equiv$ 

and $\sigma \equiv$ 

(99)

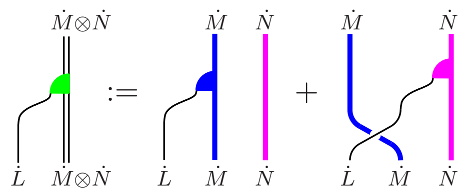
and if[13]

[13]This requirement is actually not so restrictive. In particular it holds (for all M, N) if the category is symmetric, as in e.g. $\mathcal{V}ect_\Gamma$.



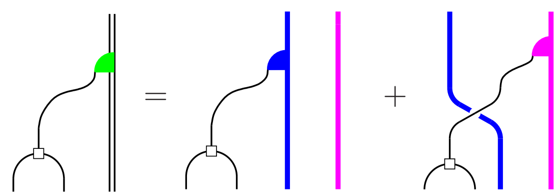
(100)

then[14] $\rho \otimes_{\dot L} \sigma$ defined by

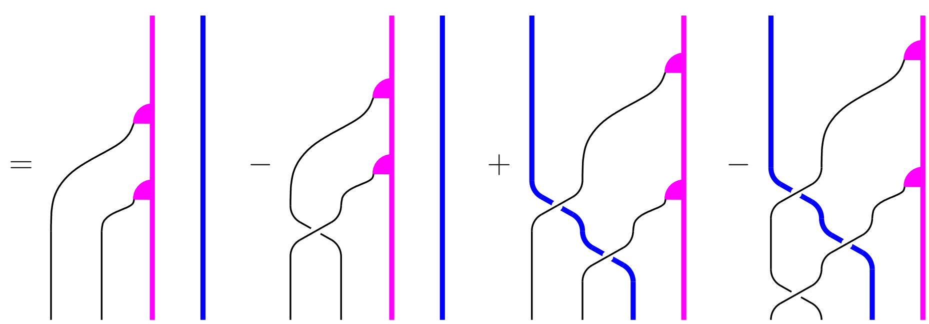


(101)

is a representation for the braided Lie algebra L on the object $\dot M \otimes \dot N$. Indeed, we have

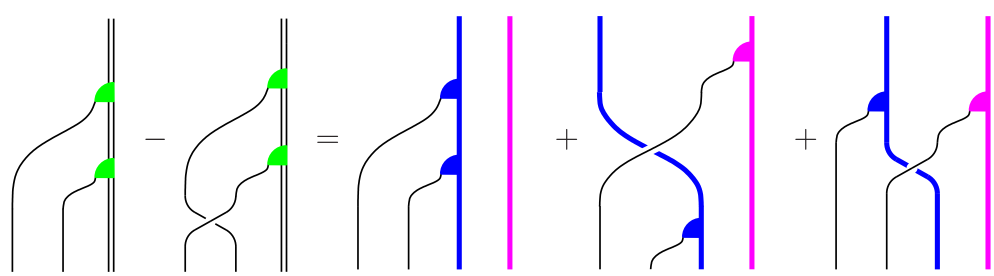


(102)

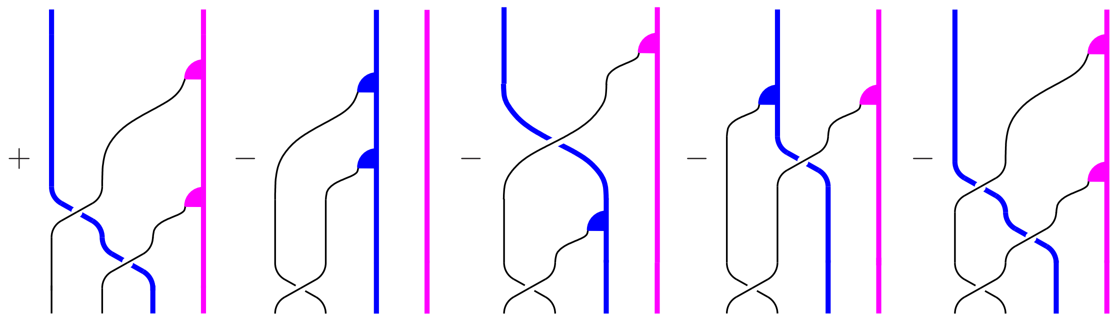


(103)

and



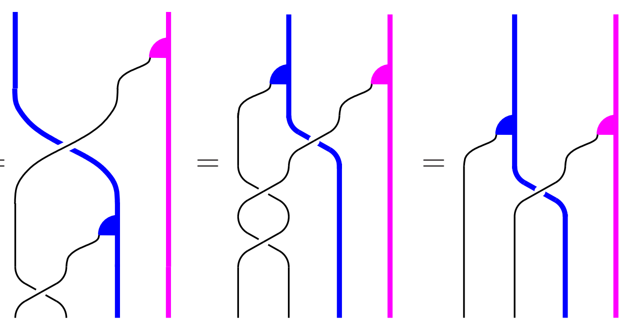
[14]Here we need to label the tensor product to specify that this is not an ordinary tensor product of morphisms.

 (104)

It follows directly that

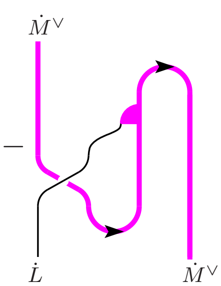
103-(i)=104-(i)	103-(iv)=104-(viii)
103-(ii)=104-(v)	104-(ii)=−104-(vii)
103-(iii)=104-(iv)	

If in addition we impose Eq.(100),[15] then we have also that 104-(iii) = −104-(vi):

-104-(vi)=  =104-(iii) (105)

5.4 The Contragredient Representation

If L is a braided commutator algebra or a braided Lie algebra in some braided monoidal abelian category that is also sovereign and if M is an L-module, then $M^\vee$ carries the structure of an L-module. In particular, if we denote the L-module by as usual, then[16]

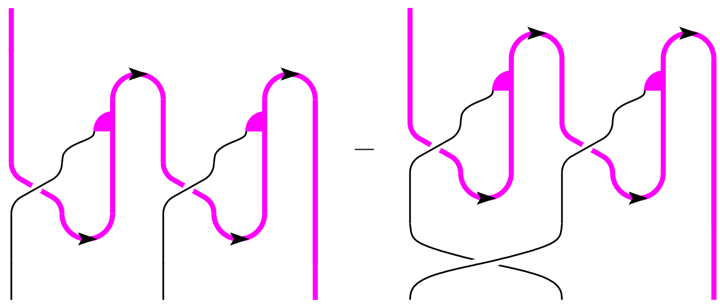
$\rho_\vee \equiv -$ 

(106)

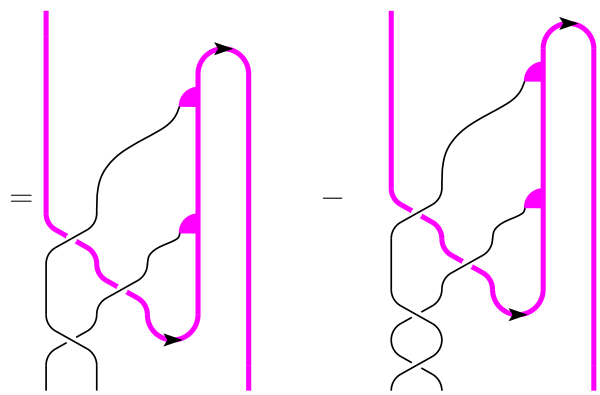
[15]Note that this is the only place that Eq.(100) is used.

[16]This is in fact just the negative of the transpose of ρ in our categories of interest. See [14]. This then reproduces the familiar result for Lie algebras in the category $\mathcal{V}ect$.

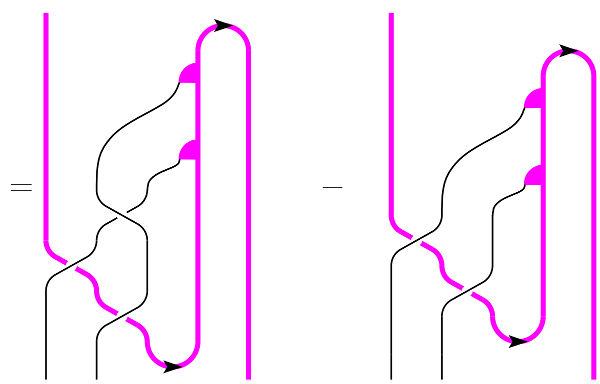
gives an L-module. Indeed, we have that



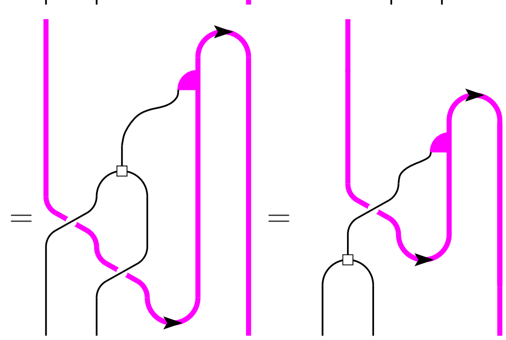
(107)



(108)



(109)



(110)

Theorem 5.2 *For* $\dim(M) \neq 0$ *and the "twist"* $\theta_{\dot{L}} = id_{\dot{L}}$*, the module* $M \otimes M^{\vee}$ *contains the trivial representation* $M_{triv} := (\mathbf{1}, \rho = 0)$ *as a submodule.*

Proof.
By "contains M_{triv} as a submodule" we mean there exists an $r \in \mathrm{Hom}(M \otimes \dot{M}^{\vee}, M_{triv})$ such that r is a morphism of $\dot{L}$-modules. More precisely, M_{triv} is a "module retract" of $M \otimes \dot{M}^{\vee}$, i.e.

$$\exists i : \mathbf{1} \to M \otimes \dot{M}^{\vee}$$

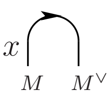
$$\exists r : M \otimes \dot{M}^{\vee} \to \mathbf{1}$$

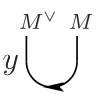
with i and r module morphisms such that

$$r \circ i = id_{\mathbf{1}}.$$

(and then $i \circ r$ is an idempotent in $\mathrm{End}(M \otimes \dot{M}^{\vee})$).

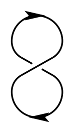
We must first show that $\mathbf{1}$ is a retract of $M \otimes \dot{M}^{\vee}$ as an object in $\mathcal{C}$. Simply take

$r := x$ 

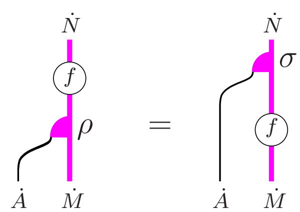
and $i := y$ 

(111)

where $x, y \in \mathrm{End}(\mathbf{1})$ and $xy = \frac{1}{\dim(M)}$. By $\dim(M)$ we mean the "categorical dimension" (cf.[16]) defined by

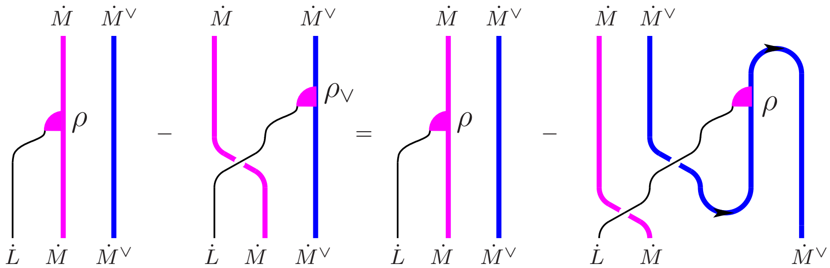
$\dim(M) :=$  (112)

It remains to be shown that r and i are indeed module morphisms. By "module morphism" we mean a morphism $f \in \mathrm{Hom}(\dot{M}, \dot{N})$ for $M := (\dot{M}, \rho)$ and $N := (\dot{N}, \sigma)$ modules of an associative algebra A, in general, such that

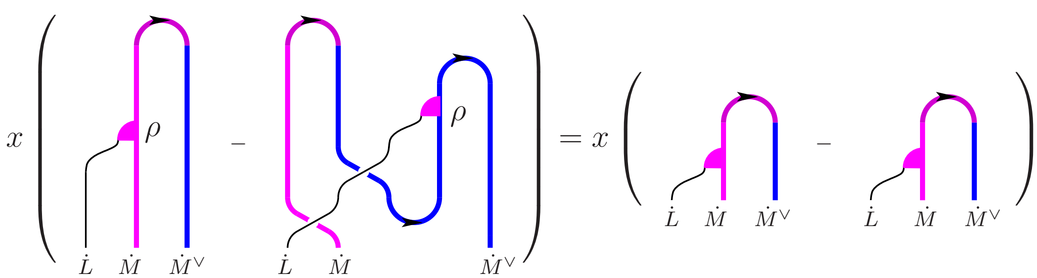


(113)

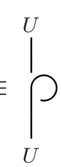
To see that r is a module morphism from $M \otimes \dot{M}^{\vee}$ to $\mathbf{1}$ notice that, since our module $M \otimes \dot{M}^{\vee}$ looks like

$\rho \otimes_{\dot{L}} \rho_{\vee} \equiv$  (114)

the left hand side of (eq.113) with $\dot{M}$ replaced by $\dot{M} \otimes \dot{M}^{\vee}$, $\dot{N}$ replaced by $\mathbf{1}$, and $f = r$ looks like

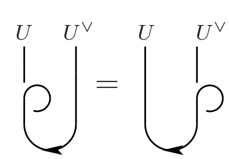
 (115)

To see that i is a module morphism is somewhat less trivial. We need the concept of a *twist* [10][14]. This is a family of isomorphisms $\{\theta_U \,|\, U \in \mathrm{Ob}(\mathcal{C})\}$ which we draw as

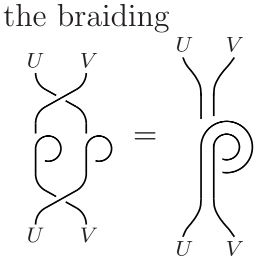
$\theta_U \equiv$  (116)

and which obey

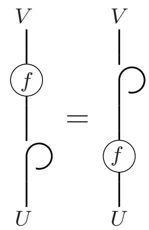
- compatibility of the twist with duality

 (117)

- compatibliity of the twist with the braiding

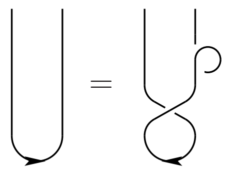
 (118)

- functoriality of the twist



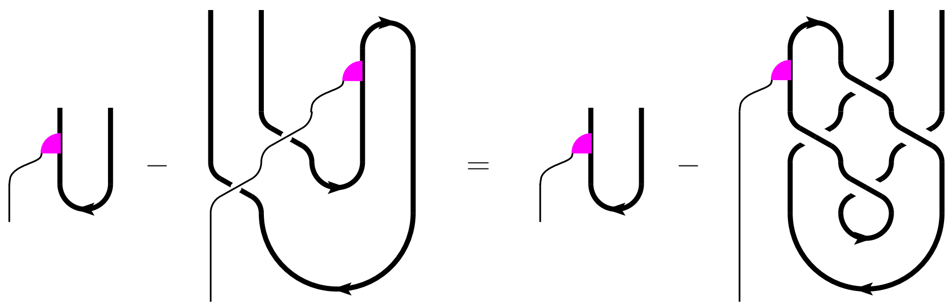
(119)

In sovereign categories we also have that (see [10])



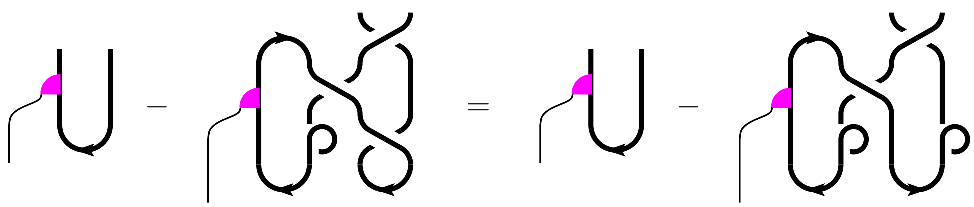
(120)

Now, the right side of (eq.113) with $f = i$ looks like



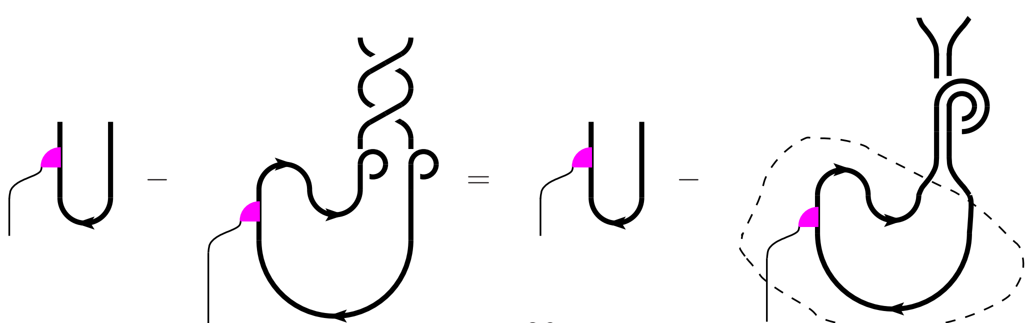
(121)

where we have employed functoriality of the braiding (eq.(71)) to get the RHS of eq.(121). Next we use eq.(120) twice:



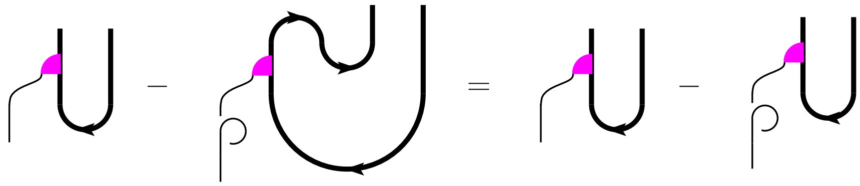
(122)

Again we use functoriality of the braiding (eq.(71)) and then compatibility of the twist with the braiding (eq.(118)):



(123)

Next we make use of functoriality of the twist where the region enclosed by the dashed line above is considered as the f in (eq.(119)):


(124)

Now we must assume that

$$\theta_{\dot{L}} = id_{\dot{L}} \tag{125}$$

to arrive at the conclusion $(\rho \otimes {}_{\dot{L}}\rho_{\vee}) \circ b_{\dot{M}} = 0$.

□

In particular, we note that color Lie algebras and Lie algebras obey eq.(125).

6 Concluding Remarks; New Directions

We have seen that some algebraic structures which appear unrelated are in fact described by the same structure in terms of categories, e.g. a braided commutator algebra, a braided Lie algebra, or a braided commutator Lie algebra. Also we have shown that some of the representation theory of the various mathematical objects can be constructed concurrently and also rather easily by considering the objects in this context.

However, there is still more that can be done and a few open ends. Firstly, we wonder if the assumptions for some of the theorems are too strong. In particular, we ask: Do only braided commutator Lie algebras possess an adjoint representation? If so, can we modify the braided primitive Jacobi identities (e.g. perhaps changing some braidings to inverse braidings) in a manner such that we can obtain an adjoint representation? Is there a more general (in the sense of relaxed assumptions) tensor product representation?

Also, we wonder what other interesting examples might exist of braided Lie algebras and braided commutator Lie algebras. We have looked only at two categories herein, namely $\mathcal{V}ect$ and $\mathcal{V}ect_{\Gamma}$.

Further investigation of these structures must surely include finding a suitable definition for a braided enveloping algebra. One would certainly want to consider those examined in [18].

One would certainly also ask if our braided Lie algebra is equivalent to Majid's braided Lie algebra.

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