

# Non-linear vibrations of tensegrity structures

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## List of Contents

|   |           |
|---|-----------|
| <b>Abstract.....</b>  | <b>v</b>  |
| <b>Chapter 1: Introduction .....</b>  | <b>1</b>  |
| 1.1. Background.....  | 1         |
| 1.2. Motivation.....  | 2         |
| 1.3. Objectives .....   | 2         |
| 1.4. Overview.....  | 3         |
| <b>Chapter 2: Tensegrity.....</b>   | <b>4</b>  |
| 2.1. Definition .....   | 4         |
| 2.2. Application.....   | 4         |
| 2.3. Form-finding .....   | 4         |
| 2.4. Equilibrium matrix.....  | 5         |
| 2.5. Tangent stiffness matrix.....  | 6         |
| 2.6. Mass matrix.....   | 7         |
| <b>Chapter3: Direct time integration method .....</b>                           | <b>11</b> |
| 3.1. Definition .....   | 11        |
| 3.2. Explicit methods .....   | 13        |
| 3.2.1. Central difference method.....   | 13        |
| 3.3. Implicit methods .....   | 16        |
| 3.3.1. Newmark family of methods .....  | 16        |
| 3.3.1.1. Newmark's method for linear problems .....                             | 17        |
| 3.3.1.2. Newmark's method for non-linear problem .....                          | 19        |
| 3.3.1.3. Stability of Newmark's method .....                                    | 22        |
| 3.4. Selection of direct integration method.....                                | 22        |
| <b>Chapter4: Problems of study .....</b>  | <b>24</b> |
| 4.1. Problem with one degree of freedom.....                                    | 24        |
| 4.1.1. Linear mass-spring problem .....   | 24        |
| 4.1.1.1. Linear mass-spring system to solve with central difference method..... | 24        |
| 4.1.1.2. Linear mass-spring system to solve with Newmark's method .....         | 24        |
| 4.2. Problem with two degrees of freedom.....                                   | 25        |

|  |           |
|--|-----------|
| 4.2.1. Elastic pendulum.....                                     | 25        |
| 4.2.1.1. Elastic spring pendulum.....                            | 25        |
| 4.2.1.2. Elastic bar pendulum.....                               | 26        |
| 4.2.2. Problem of bar- spring .....                              | 27        |
| 4.2.2.1. Problem of low stiff bar- spring.....                   | 27        |
| 4.2.2.2. Problem of stiff bar- spring.....                       | 29        |
| 4.2.3. Problem of cable-bar.....                                 | 29        |
| 4.2.3.1. Problem of cable-bar with slackening.....               | 29        |
| 4.2.3.2. Problem of cable-bar with slackening and snapping ..... | 30        |
| 4.3. Problem with multi degrees of freedom.....                  | 31        |
| 4.3.1. Problem of x-frame tensegrity .....                       | 31        |
| 4.3.1.1. Problem of x-frame tensegrity without snapping.....     | 31        |
| 4.3.1.2. Problem of x-frame tensegrity with snapping.....        | 32        |
| <b>Chapter 5: Results.....</b>                                   | <b>33</b> |
| 5.1. Results of one degree of freedom problems .....             | 33        |
| 5.1.1. Results of first spring-mass problem .....                | 33        |
| 5.1.1.1. Solution with central difference method.....            | 33        |
| 5.1.1.2. Solution with exact method .....                        | 33        |
| 5.1.2. Results of second spring-mass problem.....                | 35        |
| 5.1.2.1. Solution with Newmark's method .....                    | 35        |
| 5.1.2.2. Solution with exact method .....                        | 36        |
| 5.2. Results of two degrees of freedom problems.....             | 37        |
| 5.2.1. Results of pendulum problems.....                         | 37        |
| 5.2.1.1. Results of spring pendulum problem .....                | 37        |
| Solution with Newmark's method .....                             | 37        |
| Verification of results .....                                    | 40        |
| Solution with central difference method.....                     | 42        |
| 5.2.1.2. Results of bar pendulum problem .....                   | 45        |
| Solution with Newmark's method .....                             | 45        |
| Verification of results .....                                    | 46        |

|  |           |
|--|-----------|
| Solution with central difference method.....                     | 48        |
| 5.2.2. Results of bar-spring problems .....                      | 48        |
| 5.2.2.1. Results of low stiff bar-spring.....                    | 48        |
| Solution with Newmark's method .....                             | 48        |
| Solution with central difference method.....                     | 50        |
| 5.2.2.2. Results of stiff bar-spring (cable) .....               | 52        |
| Solution with Newmark's method .....                             | 52        |
| Verification of results .....                                    | 55        |
| 5.2.3. Problem of cable-bar.....                                 | 57        |
| 5.2.3.1. Problem of cable-bar with slackening.....               | 57        |
| Solution with Newmark's method .....                             | 57        |
| Solution with central difference method.....                     | 59        |
| 5.2.3.2. Problem of cable-bar with slackening and snapping ..... | 59        |
| Solution with Newmark's method .....                             | 59        |
| 5.3. Results of multi degrees of freedom problems .....          | 61        |
| 5.3.1. Results of X-frame tensegrity .....                       | 61        |
| 5.3.1.1. X-frame without slackening.....                         | 61        |
| Solution with central difference method.....                     | 61        |
| Solution with Newmark's method .....                             | 63        |
| 5.3.1.2. X-frame with slackening.....                            | 64        |
| Solution with central difference method.....                     | 64        |
| Solution with Newmark's method .....                             | 67        |
| <b>Chapter 6: Discussion .....</b>                               | <b>68</b> |
| <b>Chapter 7: Conclusion.....</b>                                | <b>70</b> |
| <b>References.....</b>   | <b>71</b> |

# Abstract

A study has been done on different methods to solve the linear and non-linear problems with single or multi degrees of freedom structures. To do that direct time integration methods are used to solve the dynamic equilibrium equations. It has been tried to perform general methods to apply in most structures. In this thesis structures are made of cables and bars but the solution methods which is presented can be applied for any structure, and to utilize those implicit and explicit methods for all structures it is needed to know the tangent stiffness matrix and mass matrix for the structure. Then, it would be possible to analyze the dynamic response of structures under general loads and by general it can be understood that by application of arbitrary forces on different nodes of structure, the method generates result based on applied forces. It is crucial to get the right tangent stiffness matrix and mass matrix to know the dynamic equation in each node. Hence, the method will then work correctly to solve the dynamic problems. More, a parametric study has been done to see the effects of times step, stiffness of elements, length of elements, and other mechanical properties of elements, and this parametric study enables one to produce new results by changing every parameter. Also, continuation of the study on x-frame tensegrity has been done by solving them to check out dynamic response of structure with proposed methods of this thesis. Moreover, a method is presented to use the codes of solver methods of current thesis to apply them for other structures. Hence, as a future work, one can combine the codes of structures and solver codes of this thesis for dynamic response of structure. In fact the main effort of this thesis is on presenting different methods to solve various structures.

# Chapter 1: Introduction

## 1.1. Background

Dynamic response of structures is interesting for structural engineers and has high importance among earthquake engineers. Actually, the present study is continuation of response of structure in dynamics. First, a structure is presented, and that would be any kind of structure, one can indicate a structure like tensegrity that is the main structure in this thesis, but that can be any kind of structure to consider it for earthquake response. Since the subject of this project is non-linear vibration of tensegrity structures, the word of earthquake suits the meaning of whole project for a structure like tensegrity. Tensegrities are structures made of cables and bars which have specific usages like creating sculptures, domes and so on. In addition, tensegrity structures has low stiffness but studying this kind of structures enables one to calculate more complex structures in future because the procedure of calculations are the same just the elements of structures can change like changing bar to beam and so on. Apart from low stiffness of tensegrity structures, there are some topics like slackening of cables in structure which can be important and that must be taken into account.

The reason for choosing a tensegrity structure is the experiences in this topic at the Department of Mechanics of Royal institute of Technology. As a result, a study on different methods of solving non-linear problems has been done to meet the expectations of this project.

To solve the dynamics of tensegrity structures, direct time integration methods were used. Implicit and explicit method like Newmark's method and central difference method are performed as most well know methods of direct time integration methods to solve tensegrity structures. Direct time integration methods have been used widely to capture dynamic response of structures. Most recently, Irfan Baig and Grätsch (2009) took a survey on recommendations for practical use of numerical methods for linear and non-linear dynamics but most of their research was on single degree of freedom systems. So, this thesis is a continuation their research on multi degrees of freedom systems to solve with numerical methods.

Moreover, for tensegrity structures, to solve with direct time integration methods, it is important whether they are prestressed or not. For structure to be stable they should be prestressed and in both cases they can be analyzed with direct time integration methods. However, for obtaining stiffness of prestressed frameworks, Guest (2006) has presented a unifying approach to get the stiffness matrix of tensegrity structure. Furthermore, to prestress a tensegrity structure, Pellegrino and Calladine (1985) introduced a matrix analysis of

indeterminate frameworks, then Pellegrino (1993) presented SVD<sup>1</sup> method to analyze equilibrium matrix of tensegrity structure. Most importantly, G radin and Rixen (1994) published a book as Mechanical Vibration which is the main reference of this project as it includes algorithms which are used to solve for the dynamic response of structures. Also, Murakami (2000) has done a research on static and dynamic analysis of tensegrity structures with linearized Lagrangian equations. Furthermore, Tran and Lee (2010) introduced a method for initial self-stress design of tensegrity structure. Moreover, Oliveto and Siavaselman (2011) performed dynamic analysis of tensegrity structure in the small displacement regime.

In fact, there were many efforts to understand details of tensegrity structures in static and dynamic cases. Most efforts have been on prestressing, form-finding, statical and dynamical analysis of tensegrity structures. In this thesis, it is tried to find an applicable method to solve the tensegrity structures in the dynamic cases. So, the main effort of this thesis is finding a general method to solve the tensegrity structures to find dynamic responses with given external load.

As a matter of fact, by taking previous efforts into account, it can be assumed that a tensegrity structure that could be or not to be prestressed, must be solved for dynamic cases. Then, direct time integration methods will be presented to solve tensegrity structures in dynamic cases.

## 1.2. Motivation

Capturing the response of tensegrity structures with different modules is the main motivation of this project but as it has been pointed out before if one is successful to analyze the dynamic behavior of tensegrity structure, it will be possible to perform the solution methods to more complex and complicated structures. In that case that is really crucial how to record the tangent stiffness matrix and mass matrix of complex structures.

## 1.3. Objectives

The objective of this thesis is to develop a numerical procedure for the dynamic analysis of tensegrity structures which are made of cables and trusses which are subjected to random external forces. The equations of motions of structure are derived from standard method of structural dynamics. Moreover, the study has been done to see the effects of time step, stiffness of bars, stiffness of cables, slackening of cables, and initial conditions of structure. Since, that is a parametric study; the effect of each parameter can be captured in the final result of the dynamic analysis of structure.

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<sup>1</sup> Singular Value Decomposition

## 1.4. Overview

The solution of tensegrity structures is achieved by usage of implicit or explicit methods for tensegrity structures in dynamic problems. The implicit and explicit methods which have been used in this thesis are written and developed to calculate dynamic responses.

In the second chapter, different points in matter of tensegrity are presented. In the third chapter, direct time integration methods have been introduced. Implicit and explicit methods with details of them are presented in third the chapter. More, in the fourth chapter, the problems of the study are presented. Actually, problems are defined to find out the differences between methods to solve the problems in single or multi degrees of freedom structures. Then, results are presented in fifth chapter. Finally, last two chapters are devoted to discussion and conclusions to find the final outcomes of this project.

## Chapter 2: Tensegrity

### 2.1. Definition

There are many studies about concept of tensegrity. Tensegrity structures are spatial structural systems composed of struts and cables with pin-jointed connections, and their stability comes from the self-stress state in their tensioned and compressed members. Tensegrity structures can be used in modern structural engineering. To find a widely acceptable definition for tensegrity, the definition of Motro is more respectable which says: “A tensegrity is a system in stable self-equilibrated state comprising a discontinuous set of compressed components inside a continuum of tensioned components”. [1]

One of the main requirements for a structure to be categorized as a tensegrity structure is that its initial prestressed configuration must be in stable equilibrium in absence of external forces. [2]

### 2.2. Application

These kinds of structures denoted tensegrity are interesting among architects and engineers. Tensegrity structures in most cases are very light and that is an important property to be used in space. Specially, if the structure inherited the deployability, that would be interesting to be used in space structures.

Furthermore, stiffness of tensegrity structures is not great, and because of not being stiff, they have special usages. More, structural damping is low for this kind of structure, and that would lead to increase sensitivity with dynamic loading.

The stability of tensegrity structures should be supported by self-stress of tensions and compressions in elements of structure. Moreover, non-linear behavior which comes from geometry non-linearity is expected when dynamic loading performed on the structure.

### 2.3. Form-finding

Form-finding is an expression which mostly used when subject is the tensegrity, and it is a procedure of finding self-stress state in whole structure of tensegrity and how to join different elements of structure together.

Also, the procedures of defining forces in each element of structure are called designing of self-stress for initial condition. As it has mentioned before, in tensegrity, cables are used for tensions and bars to take compression forces. [3]

Every structure made of a module or it can be multimodular. The modules of tensegrity can be x-module, T3 module or other types.

## 2.4. Equilibrium matrix

Moreover, to solve the initial self-stress of tensegrity structures, the equilibrium matrix  $\mathbf{A}$  must be defined. That can be a square matrix but in most cases it is a rectangular matrix and that can be obtained from the equilibrium equation:

$$\mathbf{A}\boldsymbol{\sigma} = \mathbf{l}$$

Where  $\boldsymbol{\sigma}$  is generalized stresses and  $\mathbf{l}$  is the vector of generalized loads. In general, the coefficients of the equilibrium matrix are functions of the geometrical configuration of the assembly. The dimension of this matrix is very important by knowing the number of rows and columns,  $n_r$  and  $n_c$  can be defined. More, the size of  $\mathbf{l}$  is  $n_r$  but the size of  $\boldsymbol{\sigma}$  is  $n_c$ .

By defining the rank  $r$  of the equilibrium matrix, the key parameters which show behavior of structure can be written as:

$$m = n_r - r$$

$$s = n_c - r$$

By deriving the parameters of  $m$  and  $s$  other properties of structure can be found in Table 2-1.

The values of  $m$  and  $s$  do not have high importance but it is essential to know that those parameters are equal to zero or not.

Then, to solve the formulation of initial self-equilibrium equations that must be noted that external and self-weight of the structure can be neglected. So, to solve the system, singular value decomposition (SVD) of equilibrium matrix is used. By using SVD, the equilibrium matrix will be changed to what can be seen below:

$$\mathbf{A} = \mathbf{U}\mathbf{V}\mathbf{W}^T$$

Where  $\mathbf{U}$  is an orthogonal matrix with dimension of  $n_r \times n_r$ . Likewise,  $\mathbf{W}$  is an orthogonal matrix with dimension of  $n_c \times n_c$ , and  $\mathbf{V}$  is a matrix with dimension of  $n_r \times n_c$ .

More, to choose the rank of the equilibrium matrix it must be noted that:

$$r < \min(n_r, n_c)$$

| <b>Type</b> | <b>Static properties</b> | <b>Kinematic properties</b> | <b>Statically determinate?</b> | <b>Kinematically determinate?</b> |
|-------------|--------------------------|-----------------------------|--------------------------------|-----------------------------------|
| <b>I</b>    | $s=0$                    | $m=0$                       | Statically determinate         | Kinematically determinate         |
| <b>II</b>   | $s=0$                    | $m>0$                       | Statically determinate         | Kinematically indeterminate       |
| <b>III</b>  | $s>0$                    | $m=0$                       | Statically indeterminate       | Kinematically indeterminate       |
| <b>IV</b>   | $s>0$                    | $m>0$                       | Statically indeterminate       | Kinematically indeterminate       |

**Table 2-1: Classifications of structural assemblies [4]**

Also, the  $\mathbf{V}$  matrix has  $r$  non-zero elements on the leading diagonal, and the rest of the elements of matrix are zero. Hence, the  $\mathbf{V}$  matrix contains up to  $\min(n_r, n_c)$  singular values. [5, 6]

To analyze static and dynamic responses of pre-stressed tensegrity structures, it is extremely important to compute infinitesimal mechanisms of tensegrity modules and to interpret global deformation modes of tensegrity structures in terms of infinitesimal mechanisms modes. [7]

## 2.5. Tangent stiffness matrix

After finding the equilibrium matrix, tangent stiffness of whole structure can be specified as:

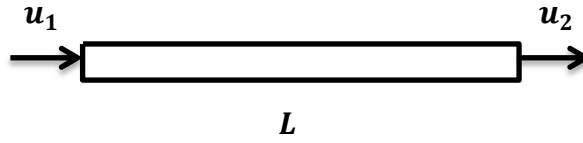
$$\mathbf{K} = \mathbf{A}\hat{\mathbf{G}}\mathbf{A}^T + \mathbf{S}$$

Where  $\mathbf{A}$  is the equilibrium matrix for the whole structure,  $\hat{\mathbf{G}}$  is a diagonal matrix of modified axial stiffness, and  $\mathbf{S}$  is the stress matrix for the entire structure. The tangent stiffness matrix consist of material stiffness and geometric stiffness which in first one, the structure is maintained in its initial geometry, and second one is about the time when stiffness changed because of reorientation of stressed members. Also, due to increase in axial forces.

It is crucial to know that when external load and self-weight are ignored, the tensegrity does not need any fixed node, and geometry of the tensegrity structure can be defined by position of nodes, and the whole system can be assumed as free-standing, forming a rigid body in space.[8]

## 2.6. Mass matrix

For a linear bar element shown below



Where the length is  $L$ , cross sectional area  $A$ , and the density  $\rho$ . A linear displacement function is assumed in the  $X$  direction as:

$$u = a_1 + a_2x$$

The number of coefficients in the displacement function  $a_i$  is equal to the number of degrees of freedom associated with the element. Moreover, the displacement function in terms of shape function can be defined as below:

$$u = [N_1 \ N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad N_1 = 1 - \frac{x}{L} \quad N_2 = \frac{x}{L}$$

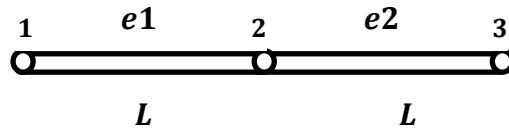
The typical method to derive the consistent mass matrix is the principle of virtual work. By substitution of shape functions in the above equation, mass matrix equation gives:

$$\mathbf{m} = \int_{V_e} \rho \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} dV = A\rho \int_0^L \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} dx = A\rho \int_0^L \begin{bmatrix} (1 - \frac{x}{L})^2 & (1 - \frac{x}{L})\frac{x}{L} \\ (1 - \frac{x}{L})\frac{x}{L} & \frac{x^2}{L^2} \end{bmatrix} dx$$

Then, by evaluating the above integral, consistent mass matrix is computed as follow:

$$\mathbf{m} = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

By assuming that the length of bar is  $2L$  and discretizing the bar into two elements with length  $L$  for each element, lumped mass matrix is defined as below:



$$\mathbf{m}_1 = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{m}_2 = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then the global lumped mass matrix is defined as:

$$\mathbf{M} = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In general, the result of consistent mass matrix is closer to available analytical and experimental results than those which use the lumped mass matrix. Surprisingly, for bar elements, the results of lumped mass matrix are as good as or even better than the result of those are used consistent mass matrix. However, from the mathematical point of view, consistent mass matrix approach yield an upper bound on the frequencies but for lumped mass matrix approach has no mathematical proof of boundless.

As a general rule, the construction of the structure mass matrix  $\mathbf{M}$  largely parallels that of tangent stiffness matrix  $\mathbf{K}$ . Mass matrices for individual elements are formed in local coordinates then transformed to global, and merged into the tangent mass matrix following exactly the same techniques used for  $\mathbf{K}$ .

An important difference with the stiffness matrix is the possibility of using a diagonal mass matrix based on direct lumping. More, a lumped mass matrix entails significant computational advantages for calculations of  $\mathbf{M}^{-1}$  [20].

Mass matrix must satisfy special conditions and it can be used for verification and debugging. So, a mass matrix should have properties as follow:

- ✓ Matrix symmetry, which means  $(\mathbf{M}^e)^T = (\mathbf{M}^e)$ .
- ✓ Physical symmetry must be reflected in mass matrix.
- ✓ Conservation. At a minimum, total element mass must be preserved. This is easily verified by applying a uniform translational velocity and checking that linear momentum is conserved.
- ✓ Positivity, which shows  $\mathbf{M}^e$  must be nonnegative. Also, for any nonzero velocity field defined by the node values  $\dot{u}_e \neq 0$ ,  $(\dot{u}_e)^T \mathbf{M}^e \dot{u}_e \geq 0$ .

Also, like the tangent stiffness matrix, mass matrices are developed in a local or element frame, and they must be globalized before assembling to tangent mass matrix. The transformation applied as follow:

$$\mathbf{M}^e = (\mathbf{T}^e)^T \overline{\mathbf{M}}^e \mathbf{T}^e$$

Where  $\overline{\mathbf{M}}^e$  is element's mass matrix, referred to the local frame, and  $\mathbf{T}^e$  is the local to global displacement transformation matrix, and it is derived from transformation which is used for stiffness globalization. Moreover, for stiffness matrices,  $\mathbf{T}^e$  is often rectangular if the local

stiffness has lower dimensionality. Globalization to 2D and 3D involves application of  $2 \times 4$  and  $2 \times 6$  transformation matrices.

It is important to know that unlike stiffness, translational masses never vanish. So, translational mass cannot be zero. Conclusion is that all translational masses must be retained in the local mass matrix.

For a two node bar element which has two degrees of freedom each node, the local consistent and lumped mass matrices are shown below:

$$\overline{\mathbf{M}}^{\mathbf{e}}_{\mathbf{c}} = \frac{\rho AL}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \quad \overline{\mathbf{M}}^{\mathbf{e}}_{\mathbf{L}} = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then by globalizing the local consistent and mass matrices by transformation matrix ( $\mathbf{M}^{\mathbf{e}} = (\mathbf{T}^{\mathbf{e}})^{\mathbf{T}} \overline{\mathbf{M}}^{\mathbf{e}} \mathbf{T}^{\mathbf{e}}$ ) the results are  $\mathbf{M}^{\mathbf{e}}_{\mathbf{c}} = \overline{\mathbf{M}}^{\mathbf{e}}_{\mathbf{c}}$  and  $\mathbf{M}^{\mathbf{e}}_{\mathbf{L}} = \overline{\mathbf{M}}^{\mathbf{e}}_{\mathbf{L}}$ . Hence, the mass matrices repeat.

$$\mathbf{M}^{\mathbf{e}}_{\mathbf{L}} = (\mathbf{T}^{\mathbf{e}})^{\mathbf{T}} \overline{\mathbf{M}}^{\mathbf{e}}_{\mathbf{L}} \mathbf{T}^{\mathbf{e}} = \frac{\rho AL}{2} (\mathbf{T}^{\mathbf{e}})^{\mathbf{T}} \mathbf{I}_4 \mathbf{T}^{\mathbf{e}} = \frac{\rho AL}{2} (\mathbf{T}^{\mathbf{e}})^{\mathbf{T}} \mathbf{T}^{\mathbf{e}} = \frac{\rho AL}{2} \mathbf{I}_4 = \overline{\mathbf{M}}^{\mathbf{e}}_{\mathbf{L}}$$

And for consistent matrix is proved as can be seen below:

$$\overline{\mathbf{M}}^{\mathbf{e}}_{\mathbf{c}} = \begin{bmatrix} \tilde{\mathbf{M}} & 0 \\ 0 & \tilde{\mathbf{M}} \end{bmatrix}, \mathbf{T}^{\mathbf{e}} = \begin{bmatrix} c\mathbf{I}_2 & s\mathbf{I}_2 \\ -s\mathbf{I}_2 & c\mathbf{I}_2 \end{bmatrix}, \text{ and } \tilde{\mathbf{M}} = \frac{\rho AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, c = \cos \varphi, s = \sin \varphi$$

$$\mathbf{M}^{\mathbf{e}}_{\mathbf{c}} = \begin{bmatrix} c\mathbf{I}_2 & -s\mathbf{I}_2 \\ s\mathbf{I}_2 & c\mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{M}} & 0 \\ 0 & \tilde{\mathbf{M}} \end{bmatrix} \begin{bmatrix} c\mathbf{I}_2 & s\mathbf{I}_2 \\ -s\mathbf{I}_2 & c\mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} (c^2 + s^2)\tilde{\mathbf{M}} & (cs - cs)\tilde{\mathbf{M}} \\ (cs - cs)\tilde{\mathbf{M}} & (c^2 + s^2)\tilde{\mathbf{M}} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{M}} & 0 \\ 0 & \tilde{\mathbf{M}} \end{bmatrix} = \overline{\mathbf{M}}^{\mathbf{e}}_{\mathbf{c}}$$

The same conclusion holds for 3D. So, the contents and order of  $\tilde{\mathbf{M}}$  are irrelevant to the result. Hence, the following generalization follows [20].

The repetition rule can be expected to fail in the following scenarios:

- The element has non-translational degrees of freedoms.
- The mass blocks are different in content or size. This occurs if different models are used in different directions. Examples are furnished by beam-column, element with curved sides or faces, and shell elements.
- Nodes are referred to different coordinate frames in the global system. This can happen if certain nodes are referred to special frames to facilitate the application of boundary conditions.

## Chapter3: Direct time integration method

### 3.1. Definition

To solve the structural dynamic equations of motion for arbitrary excitation, two methods are useful; modal superposition techniques and direct time-integration methods. The modal superposition technique is based on results from linear modal analysis, and consists of expressing the dynamic response in an eigenmode series expansion, and that is an effective method since the fundamental modes predominate in the response but for a high number of modes, this technique should be replaced by direct integration method.

The most general approach for the solution of the dynamic response of structural systems is the direct numerical integration of the dynamic equilibrium. Direct time integration techniques are not limited to the linear case as they can be easily extended to non-linear systems and this method makes it possible to use for high frequency components. In using direct time integration, the attempt is to satisfy dynamic equilibrium at discrete points in time, and to do that, most methods use equal time steps for each time interval. All methods are classified as implicit or explicit direct integration method.

In general, direct multistep integration methods for first order systems of the form  $\dot{\mathbf{u}} = \mathbf{A}\mathbf{u}$  can be stated as:

$$\mathbf{u}_{n+1} = \sum_{j=1}^m \alpha_j \mathbf{u}_{n+1-j} - h \sum_{j=0}^m \beta_j \dot{\mathbf{u}}_{n+1-j}$$

Where  $h = t_{n+1} - t_n$  is the computational time step,  $\mathbf{u}_n = \mathbf{u}(t^n)$  and

$$\mathbf{u}_{n+1} = \begin{bmatrix} \mathbf{q}_{n+1} \\ \dot{\mathbf{q}}_{n+1} \end{bmatrix}$$

is the state vector which is calculated at time  $t_{n+1}$  from the state vector at the  $m$  preceding times, from their derivatives and from the derivatives of  $\mathbf{u}_{n+1}$  itself.

- $\beta \neq 0$  leads to an implicit scheme which is a scheme where the evaluation  $\mathbf{u}_{n+1}$  requires the solution of system of equations. Actually, the state vector at time  $t_{n+1}$  is a function of its own time derivative. So, for the non-linear case the method will be iterative. Actually, an implicit method attempt to satisfy the differential equation at time  $t$  from the solution at time  $t - \Delta t$ .

This method requires the solution of a set of linear equation at each time step, and larger time steps can be used.

- $\beta = 0$  corresponds to an explicit scheme which is a scheme where the evaluation of  $\mathbf{u}_{n+1}$  does not require the solution of any system of equations and instead can be concluded from the results of previous time steps. In fact, explicit methods use the differential equation at time  $t$  to predict solution at time  $t + \Delta t$ , and very small time steps are needed. An explicit method is conditionally stable with respect to size of time step.

In fact, explicit methods do not involve the solution of linear equations at each step, and these methods use differential equation at time  $t$  to predict solution at time  $t + \Delta t$ . In reality for real structures with stiff elements, a very small time step is needed to reach a stable solution. Therefore, all explicit methods are conditionally stable and the stability depends on the chosen time step.

On the other hand, implicit methods struggle to satisfy the differential equation at time  $t$  when at time  $t - \Delta t$  the solution is found. Hence, at each time step, implicit methods require solution of a set of linear equations, and larger time steps can be used. Moreover, these methods can be conditionally or unconditionally stable.

There are many accurate, higher orders, multi-step methods for numerical solution of differential equations, which are assuming that higher derivatives are smooth but the second derivative of displacement, which is acceleration, is not a smooth function, and that cause non-linearity of solutions for structures. So, the conclusion is that single-step, implicit, unconditional stable methods can be used to solve practical structures.

For implicit method, an interesting and useful method is the Newmark method that can be called Newmark family of methods, and for the explicit one, an example is Central difference method.

In order to obtain an effective solution of a dynamic response, it is important to choose an appropriate time integration scheme. This choice depends on the finite element idealization, which in turn depends on the actual physical problem to be analyzed. It follows therefore that the selection of an appropriate finite element idealization of a problem and the choice of an effective integration scheme for the response solution are closely related and must be considered together. The finite element model and time integration schemes are chosen differently depending on whether a structural dynamic or a wave propagation problem is solved.

For structural dynamics, the basic consideration in the selection of an appropriate finite element model of a structural dynamics problem is that only the lowest modes of a physical system are being excited by the load vector. There is no need to represent the higher

frequencies of the actual physical system accurately in the finite element system because the dynamic response contribution in these frequencies is negligible.

The observation that the use of higher order elements can be effective with implicit time integration in the analysis of structural dynamics problems is consistent with the fact that higher order elements have generally been found to be efficient in static analysis, and structural dynamics problems can be thought of as static problems including inertia effects. If the finite element idealization consists of many elements it can be more efficient to use explicit time integration with a lumped mass matrix in which case no effective matrix is assembled and triangularized but a much smaller time step must generally be used in the solution. [9]

An important observation was that the cost of a direct integration analysis is directly proportional to the number of time steps required for solution. [10]

## 3.2. Explicit methods

### 3.2.1. Central difference method

This method is an explicit method, and this method is Newark's method with  $\gamma = 1/2$  and  $\beta = 0$ . So, two main equations of Newark's method change to following equations by assumption of  $h_{n+1} = t_{n+1} - t_n$ :

$$\dot{\mathbf{u}}_{n+1} = \dot{\mathbf{u}}_n + \frac{h_{n+1}}{2} (\ddot{\mathbf{u}}_n + \ddot{\mathbf{u}}_{n+1})$$

$$\mathbf{u}_{n+1} = \mathbf{u}_n + h_{n+1} \dot{\mathbf{u}}_n + \frac{h_{n+1}^2}{2} \ddot{\mathbf{u}}_n$$

To write an algorithm for Central difference method, half-time interval is calculated as:

$$h_{n+\frac{1}{2}} = \frac{h_n + h_{n+1}}{2}$$

Consequently, for a constant time step  $h$ , acceleration is calculated as:

$$\ddot{\mathbf{u}}_n = \frac{\mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1}}{h^2}$$

Also, velocity at half-time interval is showed below:

$$\dot{\mathbf{u}}_{n+\frac{1}{2}} = \dot{\mathbf{u}}\left(t_{n+\frac{1}{2}}\right) = \frac{1}{h_{n+\frac{1}{2}}}(\mathbf{u}_{n+1} - \mathbf{u}_n)$$

So, acceleration based on half-time is written as:

$$\ddot{\mathbf{u}}_n = \frac{1}{h_{n+\frac{1}{2}}}(\dot{\mathbf{u}}_{n+\frac{1}{2}} - \dot{\mathbf{u}}_{n-\frac{1}{2}})$$

The stability domain is such  $\omega_{cr}h \leq 2$  where the highest frequency in the model is  $\omega_{cr}$ . This stability condition named as Courant condition, and  $\omega_{cr}$  is taken as the highest of the eigenfrequencies of the individual elements since it corresponds to an upper bound to the eigenfrequencies of the complete model.

Central difference method is generally much less computationally expensive and becomes almost trivial when the so-called lumped mass finite element formulation is used. The explicit methods are very effective at handling various non-linearities. Moreover, in implicit methods, non-linearity generally causes some residual imbalance between internal and external forces which is iterated to equilibrium because of using iterations in Newton-Raphson procedures.

There are some advantages for explicit methods like simplicity and directness but disadvantages for stability and accuracy required very small time step which in implicit the time step can be large in comparison with explicit methods and they are stable. After all, here there are some advantages and disadvantages for explicit methods:

Advantages for explicit methods [21]:

- ❖ Robustness and efficient
- ❖ Useful for more advanced materials
- ❖ Directness of reaching to answer

Disadvantages:

- ❖ Small time step required for calculation
- ❖ Expensive about computational speed
- ❖ Solution can diverge

In the picture below, flowchart of the central difference method can be seen:

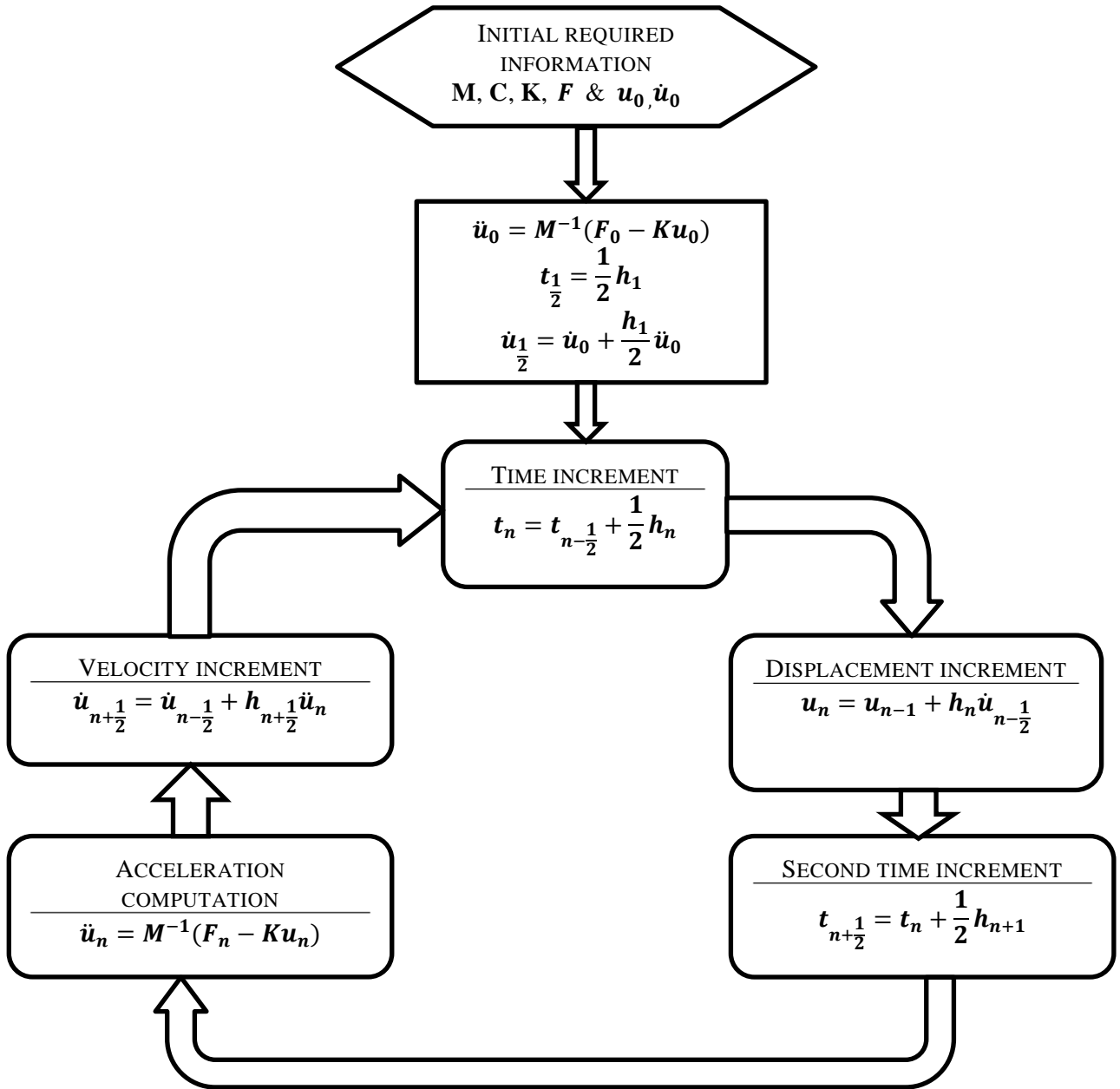


Figure 3-1: Flowchart for central difference method, [9].

For single degree of freedom systems the central difference method is most accurate, and the linear acceleration method is more accurate than the average acceleration. If only single degree of freedom systems are to be integrated there is no need to use an approximated method. It appears that the modified average acceleration method with addition of minimum proportional damping is a general procedure that can be used for dynamic analyses of all structural systems. For earthquake analysis of linear structures, it should be noted that the direct integration of the dynamic equilibrium equations is normally not numerically efficient in comparison with superposition method using LDR vectors. [22]

The advantage of using Newmark's method over the central difference method are that Newmark's method can be made unconditionally stable if  $\beta = 1/4$  and  $\gamma = 1/2$ . Also, larger time steps can be used, and the results will be better and more accurate.

From the computational point of view, central difference method can be computationally more efficient but to get more accuracy it is recommended to use accurate explicit methods like Runge-Kutta method which is approximating the slope of the secant of the solution, from one time step to the next one. However the Runge-Kutta method can be four times more expensive than central difference method for a given time step.

However, for multi degrees of freedom systems the central difference method will not be the method of choice. Besides, for multi degrees of freedom systems, the choice of a time integration scheme depends on the nature of the problem. In cases which most part of the solution lies on lower modes, the Newmark method is generally preferred, because larger time step can be used. Unlikely, for problems like crash analysis which the solution lies on higher modes, explicit methods are useful. [23]

### 3.3. Implicit methods

#### 3.3.1. Newmark family of methods

The Newmark method is a single-step direct integration method. In 1959 Newmark presented a family of single-step direct integration method to solve structural dynamic problems. It was originally introduced for solution of second order differential equation arising from dynamic equilibrium in structural dynamics. In past 53 years, the method has been applied to dynamic problems to solve many practical engineering structures, and it has been modified and

improved by many researchers. However, the versatility of Newmark's method is evidenced by its adaptation in many commercial available computer programs.

Newmark method is by far the most popular tool for direct integration of deterministic dynamical systems. [11]

To illustrate the use of Newark's method consider the solution of the linear dynamic equilibrium equations as:

$$\mathbf{M}\ddot{\mathbf{u}}_t + \mathbf{C}\dot{\mathbf{u}}_t + \mathbf{K}\mathbf{u}_t = \mathbf{F}_t$$

For linear system  $[\mathbf{M}]$ ,  $[\mathbf{C}]$  and  $[\mathbf{K}]$  are independent of time and therefore remain unchanged during the integration procedure.

The direct use of Taylor's series provides a rigorous approach like this:

$$\mathbf{u}_t = \mathbf{u}_{t-\Delta t} + \Delta t \dot{\mathbf{u}}_{t-\Delta t} + \frac{\Delta t^2}{2} \ddot{\mathbf{u}}_{t-\Delta t} + \dots$$

$$\dot{\mathbf{u}}_t = \dot{\mathbf{u}}_{t-\Delta t} + \Delta t \ddot{\mathbf{u}}_{t-\Delta t} + \frac{\Delta t^2}{2} \dddot{\mathbf{u}}_{t-\Delta t} + \dots$$

Newmark truncated equations like this:

$$\mathbf{u}_t = \mathbf{u}_{t-\Delta t} + \Delta t \dot{\mathbf{u}}_{t-\Delta t} + \frac{\Delta t^2}{2} \ddot{\mathbf{u}}_{t-\Delta t} + \beta \Delta t^3 \ddot{\mathbf{u}}$$

$$\dot{\mathbf{u}}_t = \dot{\mathbf{u}}_{t-\Delta t} + \Delta t \ddot{\mathbf{u}}_{t-\Delta t} + \gamma \Delta t^2 \ddot{\mathbf{u}}$$

By assumption of linear acceleration within the time step  $\ddot{\mathbf{u}}$  is:

$$\ddot{\mathbf{u}} = \frac{(\ddot{\mathbf{u}}_t - \ddot{\mathbf{u}}_{t-\Delta t})}{\Delta t}$$

Finally, standard form of Newmark's equation becomes:

$$\mathbf{u}_t = \mathbf{u}_{t-\Delta t} + \Delta t \dot{\mathbf{u}}_{t-\Delta t} + \left(\frac{1}{2} - \beta\right) \Delta t^2 \ddot{\mathbf{u}}_{t-\Delta t} + \beta \Delta t^2 \ddot{\mathbf{u}}_t$$

$$\dot{\mathbf{u}}_t = \dot{\mathbf{u}}_{t-\Delta t} + (1 - \gamma) \Delta t \ddot{\mathbf{u}}_{t-\Delta t} + \gamma \Delta t \ddot{\mathbf{u}}_t$$

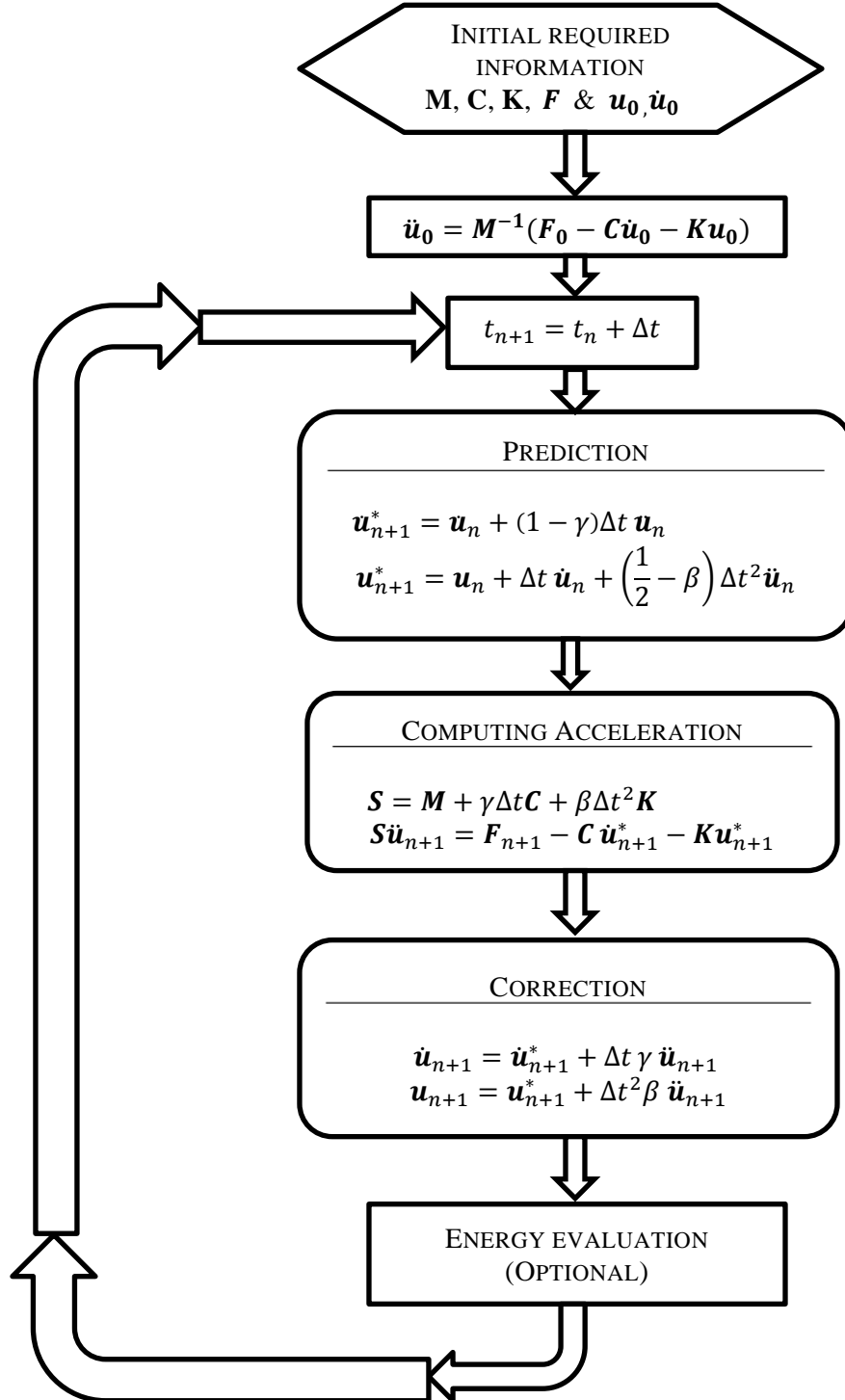
So, for each DOF (Degree Of Freedom) of structural systems, the Newmark's method is required iteratively. [9, 12, 13]

### 3.3.1.1. Newmark's method for linear problems

For linear systems, iteration matrix is associated as:

$$[M + \gamma \Delta t C + \beta h^2 K]$$

In the flowchart below the Newmark method is introduced for linear systems:



**Figure 3-2: Flowchart of Newmark's method for linear cases [9].**

The traditional Newmark family might exhibit deficiencies in the energy conservation and hence numerical stability when it is applied to non-linear transient problems. [14]

### 3.3.1.2. Newmark's method for non-linear problem

For the solution of non-linear equation of motion, direct numerical integration methods are generally recommended. The basic Newmark constant acceleration method can be extended to non-linear dynamic analysis which requires that iteration should be performed at each time step in order to satisfy equilibrium. Furthermore, the incremental stiffness matrix should be performed and triangularized at each iteration or selective points in time. To minimize the computational requirements, various kind of numerical tricks, including element by element methods, have been developed. Moreover, in some cases, the triangularization of the effective incremental stiffness matrix can be avoided by introduction of iterative solution method. In fact, many non-linear problems admit solutions by linear techniques by enlarging the domain of the problem. [17] In this section, the solution of the non-linear dynamic equations will be discussed without details of actual nature of non-linearities that can be geometric, material or other. So, the equation of motion in non-linear case can be defined as follow by assumption that  $\mathbf{u}_0$  and  $\dot{\mathbf{u}}_0$  are given:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{f}(\mathbf{u}, \dot{\mathbf{u}}) = \mathbf{g}(\mathbf{u}, t)$$

The assumption is that the inertia coefficients do not depend on the configuration, and actually it means that the reference state is fixed and the motion is in Cartesian coordinates. Then, to start with implicit method in non-linear cases, the residual vector should be identified as  $\mathbf{r}$  and can be written as follow:

$$\mathbf{r}(\mathbf{u}) = \mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{f}(\mathbf{u}, \dot{\mathbf{u}}) - \mathbf{g}(\mathbf{u}, t) = \mathbf{0}$$

Where equations of Newmark's method can be rewritten as follow:

$$\ddot{\mathbf{u}}_{n+1} = \frac{1}{\beta h^2} (\mathbf{u}_{n+1} - \mathbf{u}_{n+1}^*)$$

$$\dot{\mathbf{u}}_{n+1} = \dot{\mathbf{u}}_{n+1}^* + \frac{\gamma}{\beta h} \ddot{\mathbf{u}}_{n+1} (\mathbf{u}_{n+1} - \mathbf{u}_{n+1}^*)$$

Then predictors are defined below:

$$\ddot{\mathbf{u}}_{n+1} = 0$$

$$\dot{\mathbf{u}}_{n+1}^* = \dot{\mathbf{u}}_n + (1 - \gamma)h\ddot{\mathbf{u}}_n$$

$$\mathbf{u}_{n+1}^* = \mathbf{u}_n + h\dot{\mathbf{u}}_n + \left(\frac{1}{2} - \beta\right)h^2\ddot{\mathbf{u}}_n$$

Residual vector is represented in terms of  $\mathbf{u}_{n+1}$  as  $\mathbf{r}(\mathbf{u}_{n+1}) = 0$ .

The iteration matrix is written by:

$$\mathbf{S}(\mathbf{u}) = \mathbf{K}^t + \frac{\gamma}{\beta h} \mathbf{C}^t + \frac{1}{\beta h^2} \mathbf{M}$$

$\mathbf{K}^t$  is the tangent stiffness matrix,  $\mathbf{C}^t$  is tangent damping matrix, and  $\mathbf{M}$  is mass matrix. The non-linear system is solved by Newton-Raphson iteration method. Correction in time step  $k$  of time step  $n+1$  is calculated by solving the linearized equation

$$\mathbf{S}\Delta\mathbf{u}^k = -\mathbf{r}(\mathbf{u}_{n+1}^k)$$

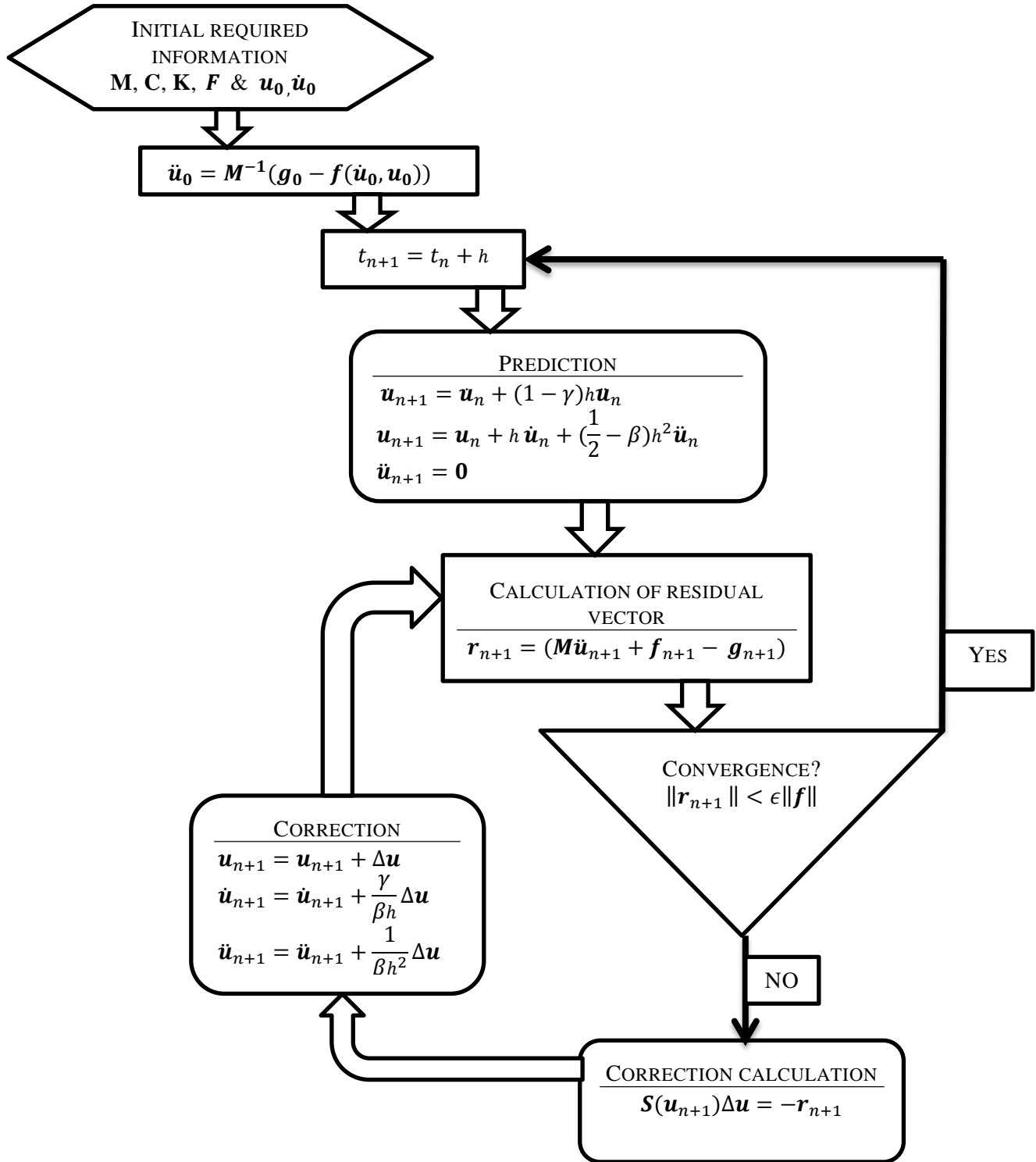
Also the corrections for acceleration and velocity is calculated as follow

$$\Delta\ddot{\mathbf{u}}^k = \frac{1}{\beta h^2} \Delta\mathbf{u}^k$$

$$\Delta\dot{\mathbf{u}}^k = \frac{\gamma}{\beta h} \Delta\mathbf{u}^k$$

The Newmark constant acceleration method, with the addition of very small amount of stiffness proportional damping, is recommended for dynamic analysis of non-linear structural systems. It should be noted that in all direct integration methods, it is important to make sure that the stiffness proportional damping does not eliminate important high-frequency response. However, when problem is a non-linear one, one cannot prove that any one method will always converge, and the amount of error in conservation of energy for every solution must be checked. More, that is a predictor-corrector method which the method predict then if it did not converge, the method would correct the prediction. [9]

In Figure 3-3, implicit integration of the response for a non-linear system can be seen.



**Figure 3-3: Flowchart of Newmark's method for non-linear cases [9]**

To explain the above algorithm, it should be noted that for each time step, iteration start from a prediction of displacement, velocity and acceleration which is zero. The reason to choose prediction of acceleration as zero is fairly stable iteration procedure, and to reach

convergence the Newton-Raphson is used. Moreover, for each time step, if the prediction was not correct, Newton-Raphson iteration would occur to reach the convergence. So, in the non-linear case, the time step drives the accuracy of integration, and plays the main role in stability of iteration process. Hence, the size of time step is a crucial issue.

### 3.3.1.3. Stability of Newmark's method

For zero damping this method is conditionally stable if:

$$\gamma \geq \frac{1}{2}, \beta \leq \frac{1}{2} \text{ and } \Delta t \leq \frac{1}{\omega_{max} \sqrt{\frac{\gamma}{2} - \beta}}$$

And  $\omega_{max}$  is the maximum frequency in structure of system, and this method is unconditionally stable if:

$$2\beta \geq \gamma \geq \frac{1}{2}$$

When  $\gamma$  is larger than  $\frac{1}{2}$  errors are introduced. These errors are associated with numerical damping and period elongation.

Also, for big structures with many degrees of freedom, the time step can be written as below:

$$\frac{\Delta t}{T_{min}} \leq \frac{1}{2\pi \sqrt{\frac{\gamma}{2} - \beta}}$$

In real structures, computer models normally contain a large number of periods that are smaller than the integration time step. So, it is better to select a numerical integration method that is unconditional for all time steps.  $T_{min}$  is the minimum time period of structure. [22]

The accuracy of the numerical response can be estimated by evaluating the variation of the energies. [19]

## 3.4. Selection of direct integration method

There are many different direct numerical integration methods, and the difference is using different integration parameters. Some of them can be seen in Table 3-1:

| <i>Method</i>               | $\gamma$ | $\beta$ | <i>Amplitude error</i> | $\Delta t/T_{min}$ | <i>Accuracy</i>   |
|-----------------------------|----------|---------|------------------------|--------------------|---|
| <i>Central difference</i>   | 1/2      | 0       | 0                      | 0.3183             | <i>Excellent for small <math>\Delta t</math><br/>Unstable for large <math>\Delta t</math></i> |
| <i>Linear acceleration</i>  | 1/2      | 1/6     | 0                      | 0.5513             | <i>Very good for small <math>\Delta t</math><br/>Unstable for large <math>\Delta t</math></i> |
| <i>Average acceleration</i> | 1/2      | 1/4     | 0                      | $\infty$           | <i>Good for small <math>\Delta t</math><br/>No energy dissipation</i>                         |
| <i>Fox &amp; Godwin</i>     | 1/2      | 1/12    | 0                      | -----              | -----   |

**Table 3-1: Summary of direct time integration methods**

The Fox & Godwin algorithm leads to a periodicity error of the third order, but is conditionally stable, and the average constant acceleration algorithm is the best unconditionally stable method.

In Newmark family of methods, the average acceleration method is the most robust method since it is unconditionally stable, and it can be used for step by step dynamic analysis of large complex structural systems which has many degrees of freedom and large number of high frequencies and short periods. Also, the average acceleration method can be used to solve linear and non-linear problems. More, calculation of tangent stiffness matrix and forces from displacements at each time step can be quite complicated for non-linear multi degrees of freedom systems. However, there is a problem in performing average acceleration method, and the problem is short periods which are smaller than the time step, oscillate indefinitely after they are excited, and the oscillation in higher modes can be reduced by addition of stiffness proportional damping.[15]

The choice of method for time-history analysis is strongly problem-dependent. The efficiency of a given method depends on whether the problem is of a wave propagation or a structural dynamics type, the time span for which analysis is required, whether response is linear or non-linear, and the topology of the finite element mesh. General speaking, direct integration methods are preferably for non-linear problems. [13]

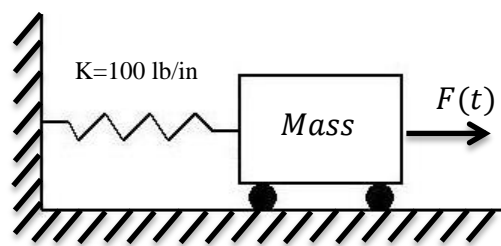
## Chapter4: Problems of study

### 4.1. Problem with one degree of freedom

#### 4.1.1. Linear mass-spring problem

##### 4.1.1.1. Linear mass-spring system to solve with central difference method

This problem is a single degree of freedom, spring- mass system. To start, in the figure below a spring mass system is shown with the one dimensional spring mass.



**Figure 4-1: SDOF problem of spring-mass system to solve with central difference method**

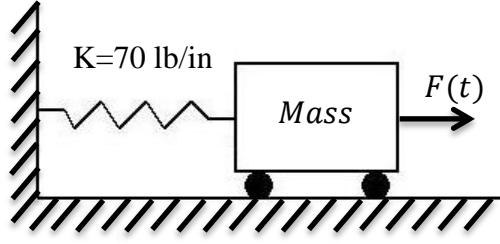
$$\text{for } 0 \leq t \leq 0.2s \Rightarrow F(t) = 2000 - 10000t \text{ [lb]}$$

To solve this problem, first the Central difference method is used. Note that  $M = 31.83 \text{ lb} \cdot \text{s}^2/\text{in}$  and after the time of  $0.2s$  the  $F(t)$  becomes zero. As it was mentioned in the part of introducing central difference method in 3.2.1,  $u_0$  and  $\dot{u}_0$  must be known, and both of them are zero in the current example. So, by implementing the central difference method for the current single degree of freedom problem, it can be solved to get displacement, velocity, and acceleration of the system.

To see the result, a graph has plotted in 5.1.1.1 and 5.1.1.2 in  $5s$  to check out how the displacement, velocity, and acceleration of the system are changed.

##### 4.1.1.2. Linear mass-spring system to solve with Newmark's method

This problem is the same as previous problem but the parameters of mass, stiffness and force has changed as you see in Figure 4-2:



**Figure 4-2: SDOF problem of spring-mass system to solve with Newmark's method**

$$\text{for } 0 \leq t \leq 0.25s \Rightarrow F(t) = 100 - 200t \quad [\text{lb}]$$

$$\text{for } 0.25s < t \leq 2s \Rightarrow F(t) = 57.142 - 28.57t \quad [\text{lb}]$$

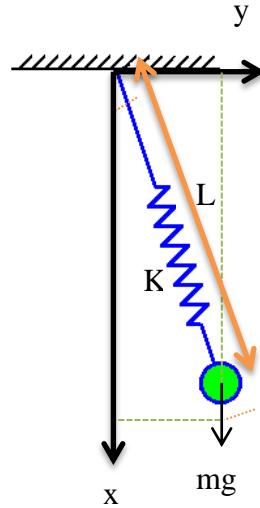
After 2 seconds the  $F(t)$  is zero. Like, the previous problem  $u_0$  and  $\dot{u}_0$  are zero, and  $M = 1.77 \text{ lb} \cdot \text{s}^2/\text{in}$ .

## 4.2. Problem with two degrees of freedom

### 4.2.1. Elastic pendulum

#### 4.2.1.1. Elastic spring pendulum

In this section a spring pendulum will be discussed which is fixed on one of its nodes. The problem is defined as it can be seen in the figure below.



**Figure 4-3: Elastic spring pendulum system**

It has two degrees of freedom with an absolute coordinates  $(x, y)$ . Moreover,  $l_0$  is the initial length of the pendulum, and  $l = \sqrt{x^2 + y^2}$ . By writing the equations of motion, two equations are obtained as follow:

$$m\ddot{x} + k(l - l_0)\frac{x}{l} - mg = 0$$

$$m\ddot{y} + k(l - l_0)\frac{y}{l} = 0$$

These equations are non-linear equations because of the length expression. Also, tangent stiffness matrix is given as below:

$$K^t = \begin{bmatrix} k(1 - \frac{l_0}{l} + \frac{l_0 x^2}{l^3}) & k \frac{l_0 xy}{l^3} \\ k \frac{l_0 xy}{l^3} & k(1 - \frac{l_0}{l} + \frac{l_0 y^2}{l^3}) \end{bmatrix}$$

Additionally, some other parameters are given like  $m = 1\text{kg}$ ,  $k = 30\text{ N/m}$ ,  $g = 10\text{m/s}^2$ ,  $l_0 = 1\text{ m}$ ,  $h = 3 \times 10^{-2}\text{ s}$ ,  $\|\mathbf{r}\| = mg \times 10^{-5}$ ,  $x_0 = 0$ ,  $y_0 = 1.5\text{ m}$ , and  $\dot{x}_0 = \dot{y}_0 = 0$ .

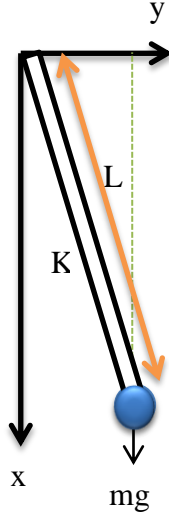
By writing down the equations of motion in matrix form, stiffness matrix can be found as follow:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} k \frac{(l - l_0)}{l} & 0 \\ 0 & k \frac{(l - l_0)}{l} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} mg \\ 0 \end{bmatrix} = 0$$

This problem is solved with two methods of central difference method and Newmark's method and they can be seen in the chapter of results.

#### 4.2.1.2. Elastic bar pendulum

In this section, to show that the codes of implicit and explicit methods are versatile codes, the previous problem is redesigned by changing a stiff bar instead of loose spring. In point of fact, a stiff bar is used to check out that the code works for linear problems or not. This problem is solved with both central difference and Newmark's method in 5.2.1.2. In figure below, elastic bar pendulum is shown.



**Figure 4-4: The elastic bar pendulum system**

The only difference of current problem with previous one is that stiffness of spring ( $k$ ) changed with stiffness of bar which is  $EA/l$ . So, the dynamic equations can be seen below:

$$m\ddot{x} + \frac{EA}{l}(l - l_0)\frac{x}{l} - mg = 0$$

$$m\ddot{y} + \frac{EA}{l}(l - l_0)\frac{y}{l} = 0$$

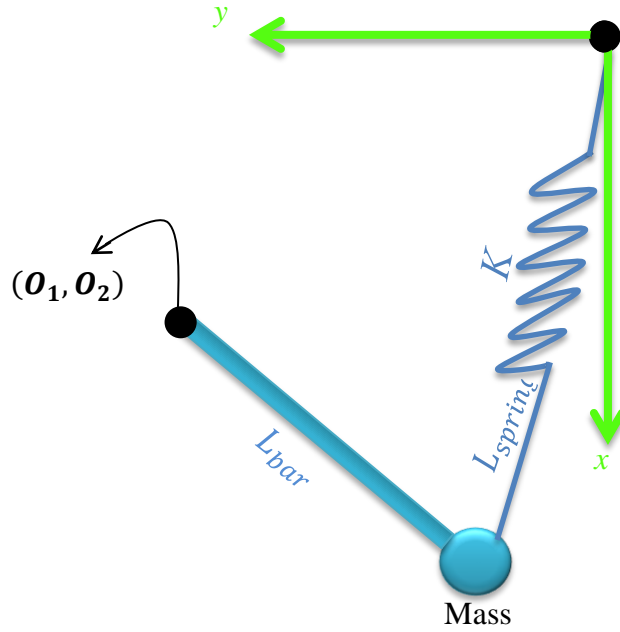
Moreover, the initial conditions are defined as follow:  $x_0 = 0$ ,  $y_0 = 1m$ , and  $\dot{x}_0 = \dot{y}_0 = 0$ . All the rest of parameters are the same as parameters in problem 4.2.1.1 just  $k$  is changed to  $EA/l$  except  $h$  which is time step. By choosing  $E=204$  GPa and  $A=19.6 \times 10^{-6}$  m<sup>2</sup>, the problem is solved. In this case  $EA/L$  is much higher than stiffness of spring that was  $k=30$  N/m in previous problem. To understand the differences, in current problem the spring stiffness of bar is about 4 MN/m but in last problem the stiffness of spring was just 30 N/m. Hence, the differences are so crystal clear and linear behavior is expected.

## 4.2.2. Problem of bar- spring

### 4.2.2.1. Problem of low stiff bar- spring

There is a bar and spring in this problem. Initial length of bar and spring are given. To solve the current problem, equations of motions should be calculated. Then by writing the equations of motions in matrix form, the stiffness matrix can be found. There are two methods to solve this problem, implicit and explicit method. However, explicit methods are

faster than implicit method. However, implicit methods are more accurate than explicit methods. The geometry of the problem is defined in the figure below.



**Figure 4-5: The elastic spring-bar system**

Dynamic equations of system can be written as below.

$$m\ddot{x} = mg - k(L_s - L0_s) \frac{x}{L_s} - \frac{EA}{L_b} (L_b - L0_b) \frac{(x - O_1)}{L_b}$$

$$m\ddot{y} = \frac{EA}{L_b} (L_b - L0_b) \frac{(O_2 - y)}{L_b} - k(L_s - L0_s) \frac{y}{L_s}$$

Above equations can be written in matrix form as follow:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} k \frac{(L_s - L0_s)}{L_s} + \frac{EA (L_b - L0_b)}{L_b} & 0 \\ 0 & k \frac{(L_s - L0_s)}{L_s} + \frac{EA (L_b - L0_b)}{L_b} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{EA (L_b - L0_b)}{L_b} \frac{O_1}{L_b} + mg \\ \frac{EA (L_b - L0_b)}{L_b} \frac{O_2}{L_b} \end{bmatrix} = 0$$

Stiffness matrix can be obtained from above equation:

$$K = \begin{bmatrix} k \frac{(L_s - L0_s)}{L_s} + \frac{EA (L_b - L0_b)}{L_b} & 0 \\ 0 & k \frac{(L_s - L0_s)}{L_s} + \frac{EA (L_b - L0_b)}{L_b} \end{bmatrix}$$

Also  $L_b = \sqrt{(x - O_1)^2 + (y - O_2)^2}$ , and  $L_s = \sqrt{x^2 + y^2}$ . Furthermore, initial displacement and velocity of current two degrees of freedom are given. Initial conditions can be defined as  $m = 1\text{kg}$ ,  $k = 30\text{ N/m}$ ,  $g = 10\text{m/s}^2$ ,  $l_0\text{spring} = 1\text{m}$ ,  $l_0\text{bar} = 1\text{m}$ ,  $h = 5 \times 10^{-3}\text{s}$ ,

$\|r\| = mg \times 10^{-5}$  ,  $x_0 = 1.5$  m ,  $y_0 = 1 - \sqrt{3}/2$  m ,  $E \times A = 204.10^9 \times 196.10^{-7}$  N and  $\dot{x}_0 = \dot{y}_0 = 0$  . Results are available in 5.2.2.1.

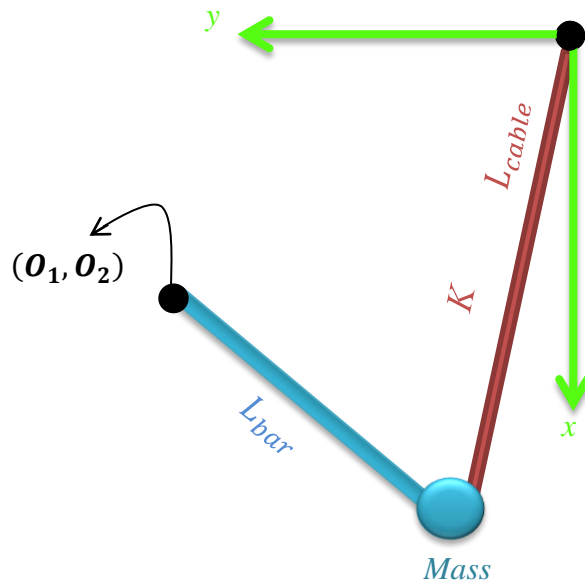
#### 4.2.2.2. Problem of stiff bar- spring

In this part, problem of 4.2.2.1 is solved but two differences applied. First, stiffness of bar increased dramatically ( $EA_{bar} = 10^5$  N). Secondly, stiffness of spring increased and instead of spring stiffness,  $EA/L$  has used which is stiffness of cable ( $EA_{cable} = 10^4$  N). In the current problem, mass is 100kg. In fact, stiff spring is assumed to behave like a cable. Equations of motion are available in 4.2.3.1. It must be noted that just stiffness of spring ( $k$ ) has changed with stiffness of cable ( $EA/L$ ). In fact one can call this system a stiff bar-cable. However, in comparison with problem of 4.2.2.1, three items have changed as follow: stiffness of bar changed, stiffness of spring has changed, and instead of  $k$  for stiffness of spring,  $EA/L$  is used. In fact both elements are bar elements but because of low stiffness of an element in comparison with another one, we called them bar-spring system.

#### 4.2.3. Problem of cable-bar

##### 4.2.3.1. Problem of cable-bar with slackening

This problem is like problems of spring-bar but instead of using a spring, a cable has used. The  $EA$  of cable has been chosen to be one tenth of the bar's  $EA$ , and the mass is 115kg. Then, the geometry of this problem is presented below:



**Figure 4-6: The elastic cable-bar system**

Dynamic equations can be written like below equations.

$$m\ddot{x} = mg - \frac{EA}{L_{cable}}(L_{cable} - L0_{cable})\frac{x}{L_{cable}} - \frac{EA}{L_b}(L_b - L0_b)\frac{(x - O_1)}{L_b}$$

$$m\ddot{y} = \frac{EA}{L_b}(L_b - L0_b)\frac{(O_2 - y)}{L_b} - \frac{EA}{L_{cable}}(L_{cable} - L0_{cable})\frac{y}{L_{cable}}$$

It is important to know that stiffness of spring ( $k$ ) in problem 4.2.2.1, has changed to  $EA/L_{cable}$ . There is a difference between how spring and cable are performing in problems. As the natural behavior of cable, a slackening condition must be added to the problem, and that can be seen below:

```
if Lcable<L0cable
    EAcable=0;
else
    EAcable=real EA of cable;
end
```

Additionally, the initial conditions are:  $m=115\text{kg}$ ,  $x_0 = 1.5 \text{ m}$ ,  $y_0 = 1 - \sqrt{3}/2 \text{ m}$ ,

$EA_{bar} = 10^5 \text{ N}$ ,  $EA_{cable} = 10^4 \text{ N}$ , and  $\dot{x}_0 = \dot{y}_0 = 0$ . Results are available in 5.2.3.1.

#### 4.2.3.2. Problem of cable-bar with slackening and snapping

This problem is like the problem of 4.2.3.1 but the cable snaps after specific elongation. Then, after snapping, stiffness of cable becomes zero. Also, initial conditions are defined as:  $m=115\text{kg}$ ,  $x_0 = 1 \text{ m}$ ,  $y_0 = 0 \text{ m}$ ,  $EA_{bar} = 10^5 \text{ N}$ ,  $EA_{cable} = 10^4 \text{ N}$ , and  $\dot{x}_0 = 3 \text{ m/s}$ ,  $\dot{y}_0 = 0$ . More, it can be defined that length of cable changes to zero after slackening (braking or tearing down). These conditions are added to the code of last problem.

```
if Ls<L0s
    EAcable=0;

else if Ls<Lsnapping
    EAcable=10000;

else if Ls>=Lsnapping
    EAcable=0;
    Ls=0;
end
end
```

To solve the problem it is needed to put above conditions after Newton-Raphson iteration for each time step .The effect of snapping and slackening of the cable can be seen in 5.2.3.2.

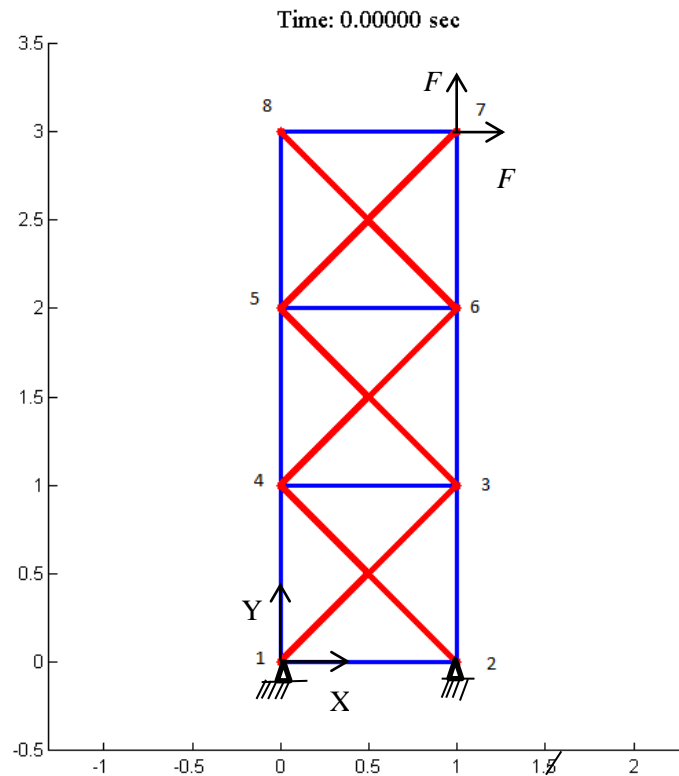
### 4.3. Problem with multi degrees of freedom

#### 4.3.1. Problem of x-frame tensegrity

##### 4.3.1.1. Problem of x-frame tensegrity without snapping

By putting three modules of x-frame tensegrity [16] on top of each other a structure can be made. To start the analysis there are some point that should be noted. First, the structure is not prestressed. So, the cable slackening is immediate. Secondly, the slackening condition which says if length of cable smaller than initial length of cable was true, the effect of that one would be calculated in the file of structure to generate the right tangent stiffness matrix and right mass matrix. Also, the effect of slackening should be in the file of structure to modify the tangent stiffness matrix in a case that there is snapping in the structure which adds another condition to the structure. In tensegrity x-frame module, it has assumed that the initial length of cable is 1m and for bar is  $\sqrt{2}$  m. More,  $EA_{bar} = 10^5$  N,  $EA_{cable} = 10^4$  N.

After all, the initial position of structure can be seen in the picture below:



**Figure 4-7: System of tensegrity made of three modules of x-frame**

This structure is solved with central difference method as the first method to solve it.

#### 4.3.1.2. Problem of x-frame tensegrity with snapping

In this section a condition must be added to problem of 4.3.1.1 for snapping. In fact the condition says that if length of cable became more than specific length, then snapping would happen. So, it affects tangent stiffness matrix of whole structure. Then, the structure can be solved by direct time integration methods. Therefore, the only difference of current problem with 4.3.1.1 is choosing snapping length for cable. Lastly, results are available in 5.3.1.2.

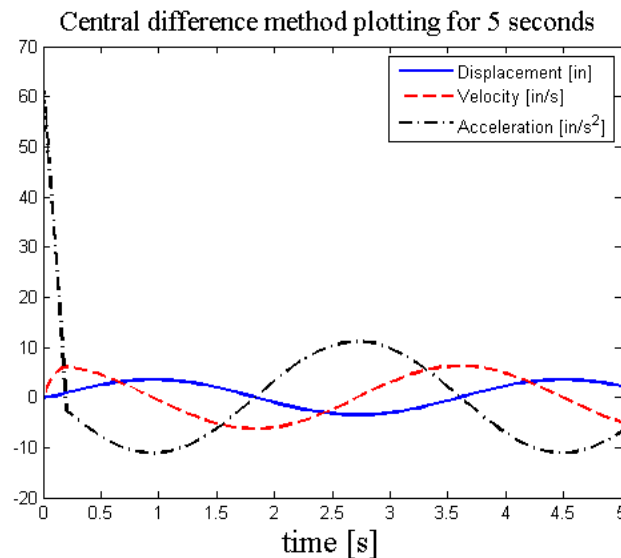
## Chapter 5: Results

### 5.1. Results of one degree of freedom problems

#### 5.1.1. Results of first spring-mass problem

##### 5.1.1.1. Solution with central difference method

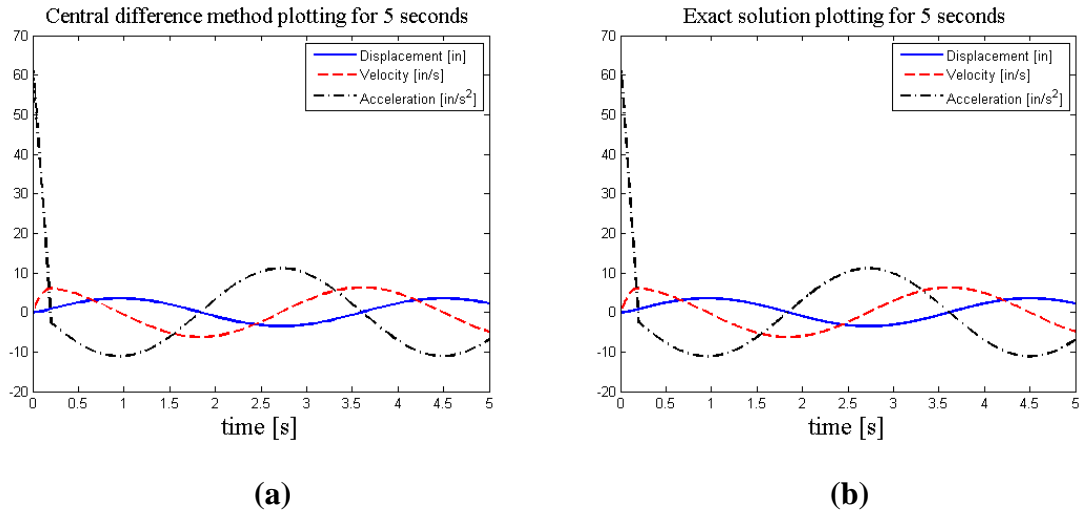
To solve a linear problem with spring-mass which was shown in 4.1.1.1, first the Central difference method is used. Note that  $M = 31.83 \text{ lb. s}^2/\text{in}$  and after the time of  $0.2\text{s}$  the  $F(t)$  is zero. As it was mentioned in the part of introducing central difference method,  $u_0$  and  $\dot{u}_0$  must be known, and both of them are zero in the current problem. So, by implementing the central difference method for the current problem that is a single degree of freedom problem, it can be solved to get displacement, velocity, and acceleration of the system. To see the result, a graph is plotted in the Figure 5-1 in 5s to check out how the displacement, velocity, and acceleration of the system are changed.



**Figure 5-1: Results of central difference method for 5 seconds**

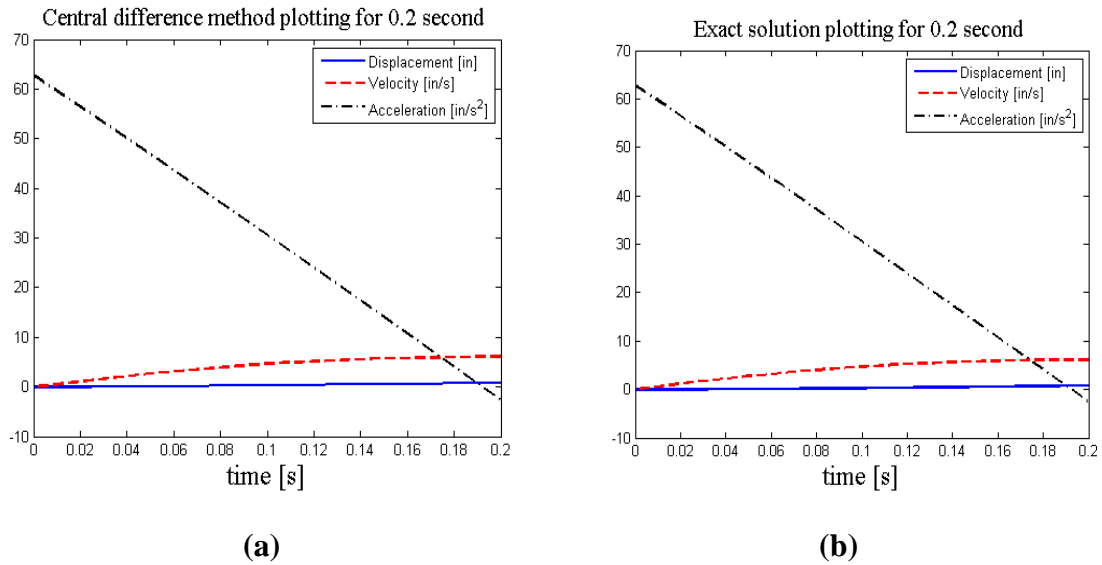
##### 5.1.1.2. Solution with exact method

To verify the results of 5.1.1.1, this problem has solved with *ODE45* in Matlab which gives a very exact solution. In the pictures below, two pictures can be seen. The left one is solution of central difference method, and the right one is the almost exact solution with *ODE45* in Matlab.



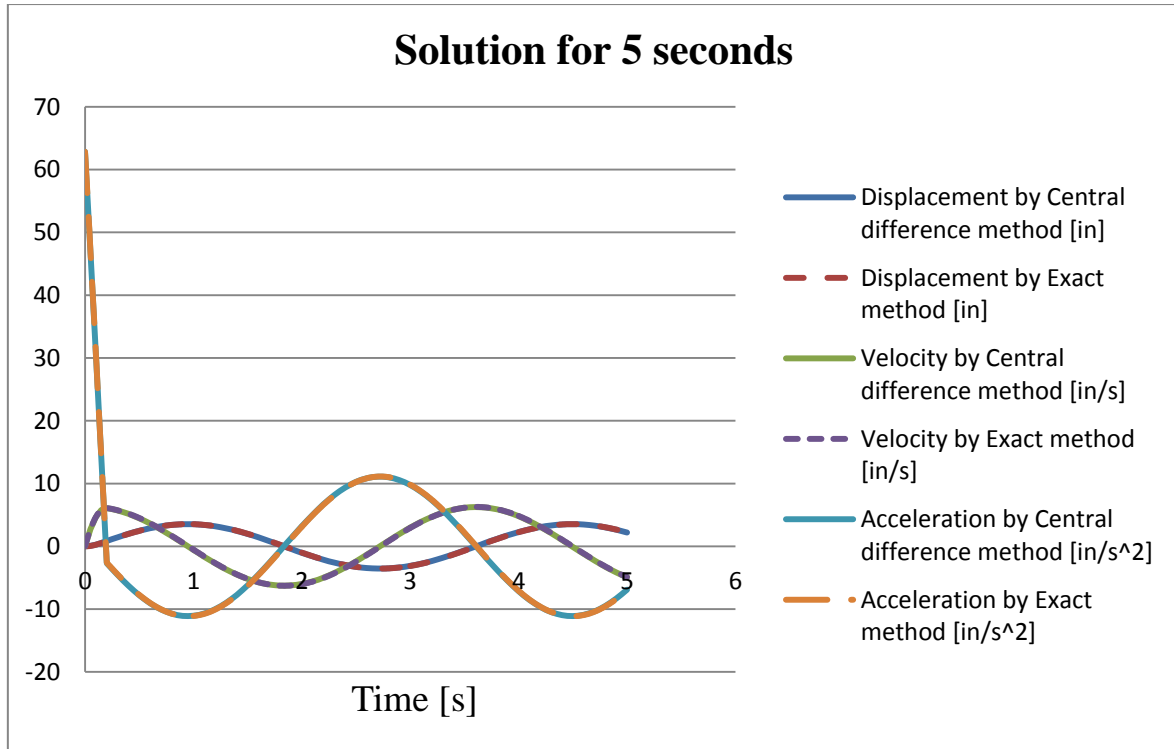
**Figure 5-2: Comparison of results with two methods of central difference and exact method for 5 seconds**

It is clear that they are very similar. Also, to have a closer look, first 0.2s of this example is magnified in pictures below, and like the last two pictures, the left one is solution with central difference method and the right one is the exact solution of current problem.



**Figure 5-3: Comparison of results with two methods of central difference and exact method for 0.2 seconds**

It should be included that the time step for central difference method in above pictures was 0.05s but in exact solutions it was 0.005s. By putting two graphs of different solutions together below graph will be obtained.



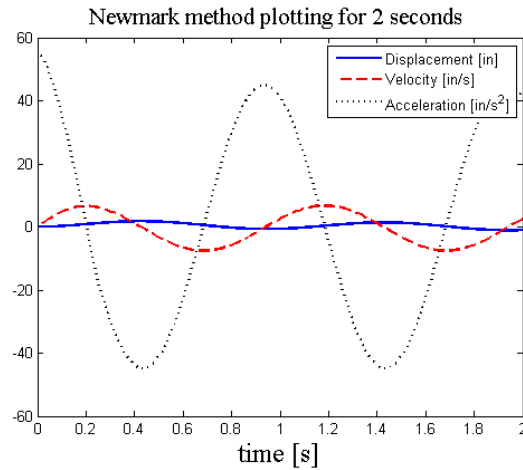
**Figure 5-4: Comparison of results with two methods of central difference and exact method for 5 seconds**

It is clear that there is not any difference between results of exact method and central difference method. So, the error of central difference method is negligible.

## 5.1.2. Results of second spring-mass problem

### 5.1.2.1. Solution with Newmark's method

In this section, problem of 4.1.1.2 is solved. After 2 seconds the  $F(t)$  is zero. Like, the problem of 4.1.1.1,  $u_0$  and  $\dot{u}_0$  are zero, and  $M = 1.77 \text{ lb.s}^2/\text{in}$ . Lastly, after solving the problem by Newmark's method, displacement, velocity, and acceleration are calculated, as can be seen in the figure below.

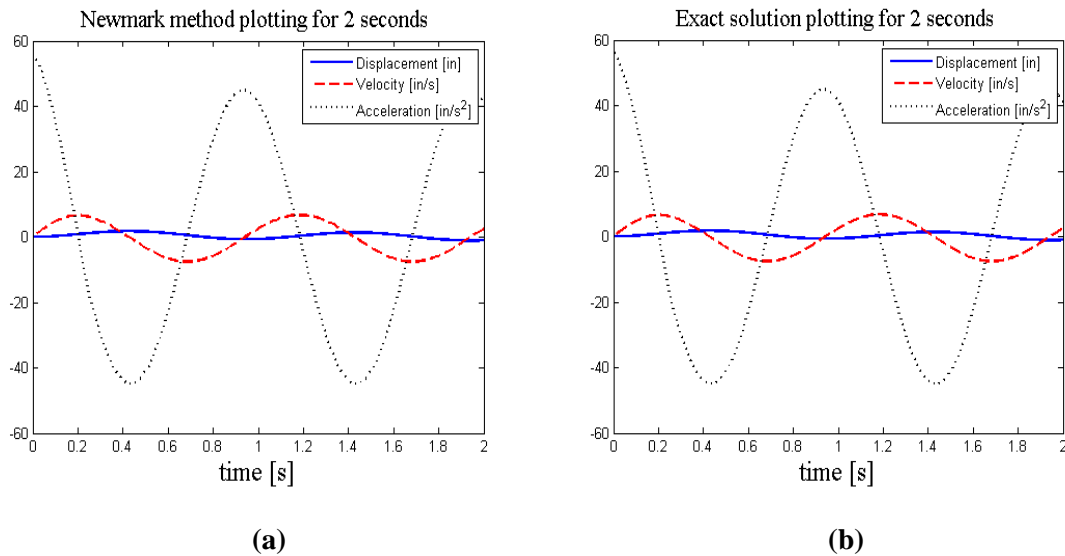


**Figure 5-5: Figure 5-1: Results of 4.1.1.2 for 2 seconds**

In those two problems that they have solved with two different methods, the parameters were different. In next section, results of a pendulum will be discussed, and it will be solved with both Newmark and central difference method. Then, it would be possible to compare the calculation time and time steps which have used to solve the problem when the same parameters chose to solve the problem.

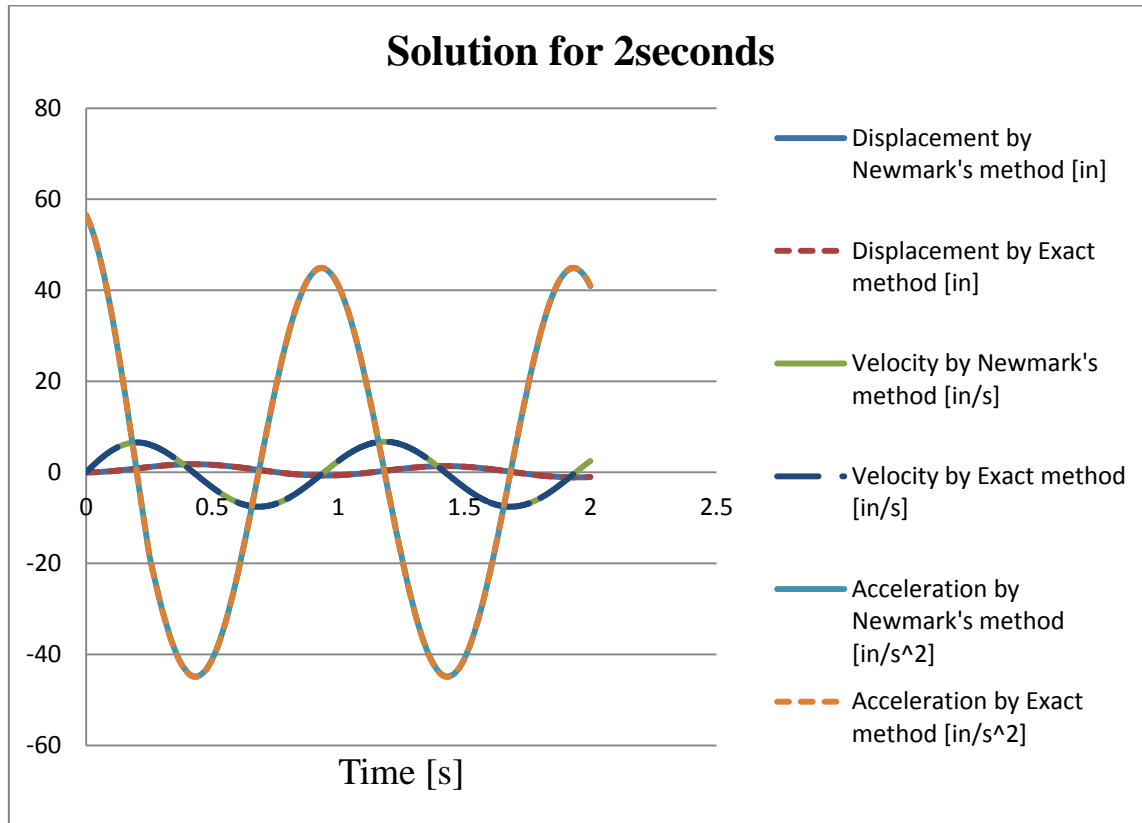
#### 5.1.2.2. Solution with exact method

To compare the results of Newmark's method in 5.1.2.1, by exact method, *ODE45* of Matlab has used to get the exact solution. In the pictures below, in left side, results of Newmark's method, and on the right side the exact solution are comparable.



**Figure 5-6: Results of 4.1.1.2 with two methods of central difference and exact method for 2 seconds**

By putting these two plot on each other, the below graph will be obtained for displacement and velocity of solutions with Newmark and exact method.



**Figure 5-7: Comparison of results for 4.1.1.2 with two methods of central difference and exact method for 2 seconds**

It is crystal clear that the error between those methods is negligible, and it must be mentioned that time step in Newmark's method was 0.01s, and in exact method was 0.0001s. More, the Matlab code of exact method which is written by *ODE45* in Matlab is available in appendix.

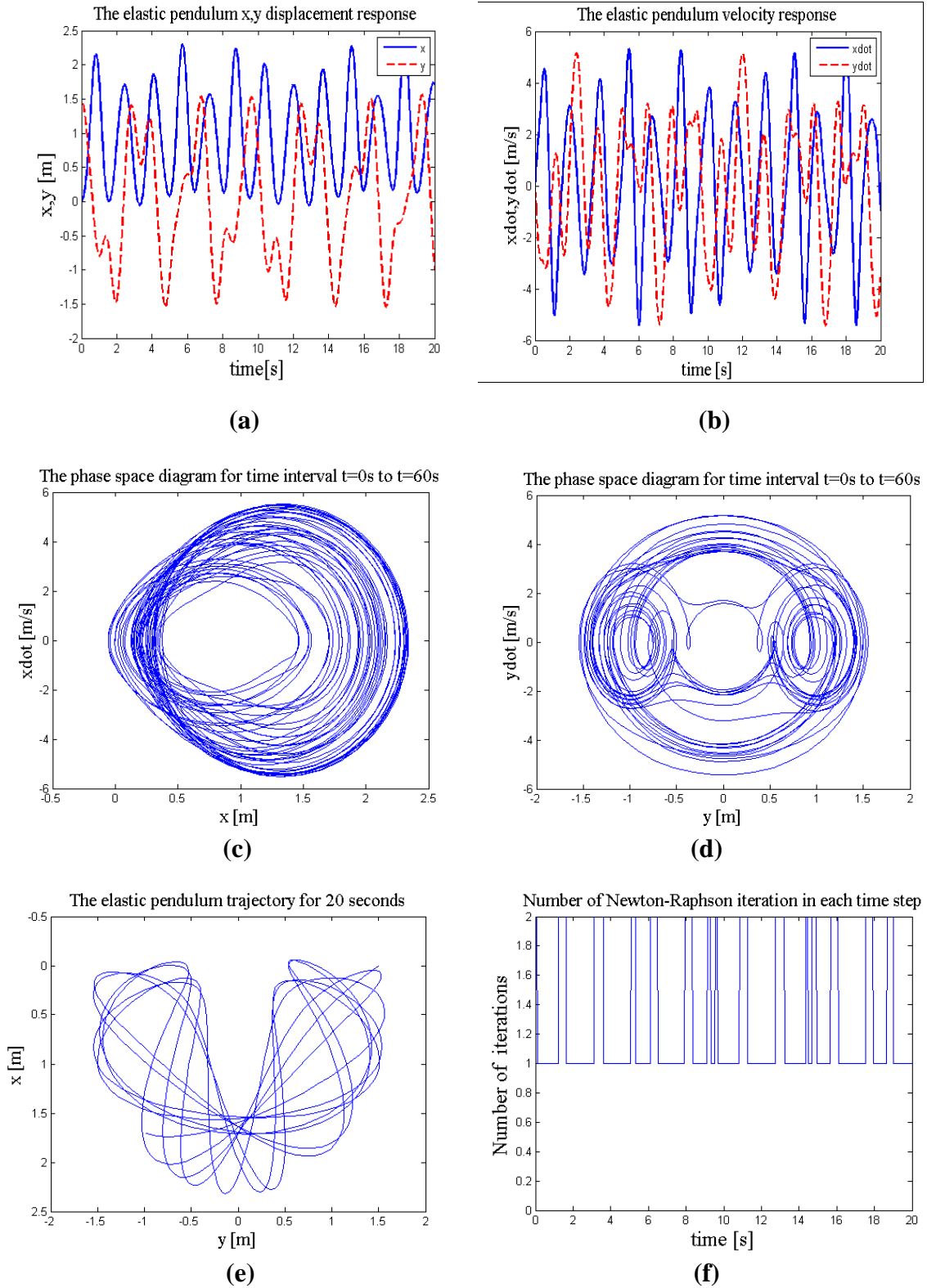
## 5.2. Results of two degrees of freedom problems

### 5.2.1. Results of pendulum problems

#### 5.2.1.1. Results of spring pendulum problem

##### Solution with Newmark's method

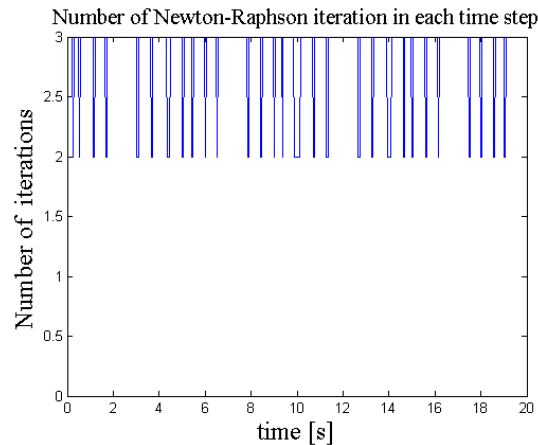
In this part, problem 4.2.1.1 is solved . Therefore, to solve this problem, first the Newmark's method is performed with  $\gamma = 1/2$  and  $\beta = 1/4$ . Plus, the matlab code of this example can be found in appendix. The results are available in figures below:



**Figure 5-8: Results of spring pendulum by Newmark's method**

So, there are some points at the time of solving such problems which will be point out. Furthermore, if stiffness matrix used instead of tangent stiffness matrix in calculation of correction  $\Delta u$ , by iteration matrix  $(\mathbf{u}_{n+1})$ , then the results will not change, but that will dramatically affect the number of Newton-Raphson calculation, especially in multi degrees of

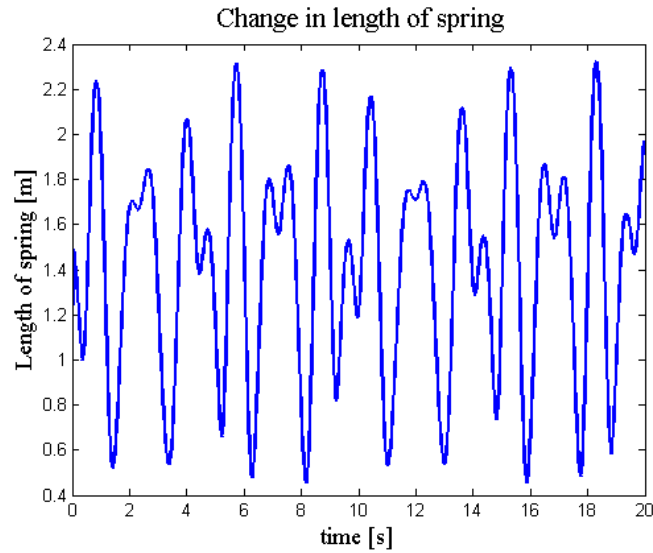
freedom problems. More, as it was mentioned in the algorithm of Newmark's method for non-linear problems, to calculate iteration matrix  $\mathbf{S}(\mathbf{u}_{n+1})$ , tangent stiffness matrix has used but if iteration matrix  $\mathbf{S}(\mathbf{u}_{n+1})$ , calculated with stiffness matrix which can be found in matrix of equation of motion, the final result would not change. Then, by using stiffness matrix to calculate iteration matrix the result would be the same but the number of Newton-Raphson iterations in each time step is changed, as consequence, and it is changed like the figure below:



**Figure 5-9: Number of Newton-Raphson iteration in each time step**

In fact, by using tangent stiffness matrix to calculate iteration matrix ( $\mathbf{u}_{n+1}$ ), number of Newton-Raphson iterations are between 1-2 but by performing stiffness matrix to calculate iteration matrix, the number of Newton-Raphson iterations changed between 2-3. It means more calculation is required to get the same results. Furthermore, it really crucial to use tangent stiffness matrix to calculate the right correction when Newmark's method is used to solve the problem, specially that can be a reason to save huge time for calculations of large structures.

In the Figure 5-10, change in length of spring is presented. The changes in length of spring show that problem is highly non-linear.

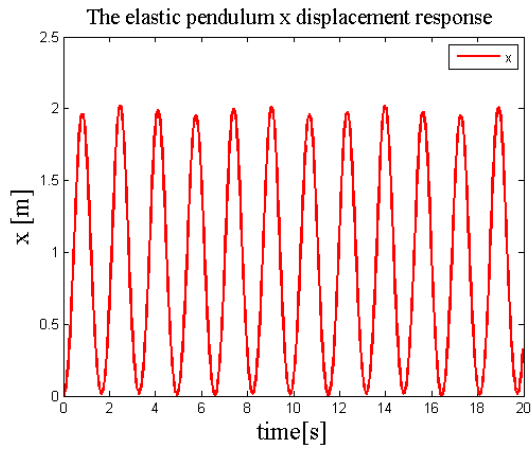


**Figure 5-10: Change in length of spring pendulum for 20 seconds with Newmark's method**

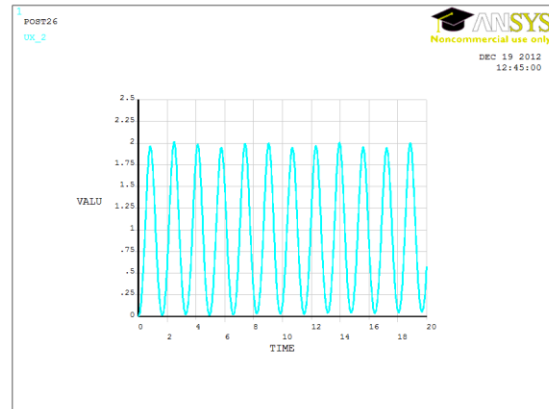
As the next step, current problem with the same parameters will be solved with explicit method of central difference method with exactly the same time step. Resultantly, the output of both methods can be compared to understand the differences.

### Verification of results

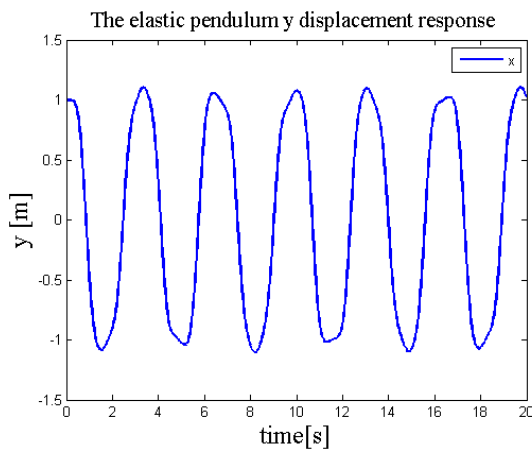
In this part, initial condition of  $y_0$  on 4.2.1.1 has changed to 1m, and resolved. It is also solved with transient dynamic analysis of ANSYS. To model the problem in ANSYS, element of COMBIN14 has used which is an element to model spring and damper together but the damping assigned as zero. Also, MASS21 element on ANSYS is performed to model the mass element at the end of spring. More, to apply the transient dynamic analysis of ANSYS, three options are available to choose for analysis consisting of full, reduced, and mode superposition method. The full method is the most accurate one among those three methods, and does not need to define master degrees of freedom, and that can capture all types of non-linearities. In this thesis, full method of transient dynamic analysis of ANSYS has been used to verify the results of written Matlab codes because the full method can obtain the most accurate result among three available methods in ANSYS. Moreover, to see more details of transient dynamic analysis of ANSYS, manual of this section must be followed. On the left hand side of Figure 5-11 results of Newmark's method with written codes on Matlab and on the right hand side results of ANSYS are shown.



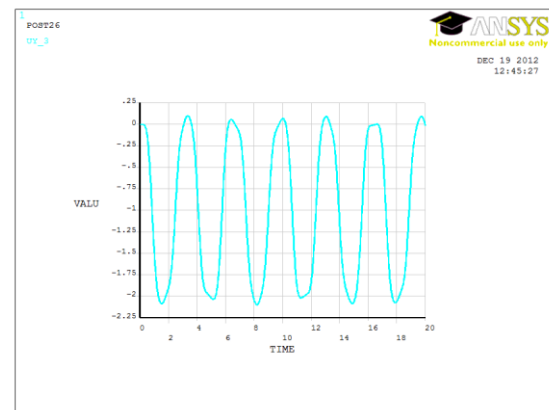
(a)



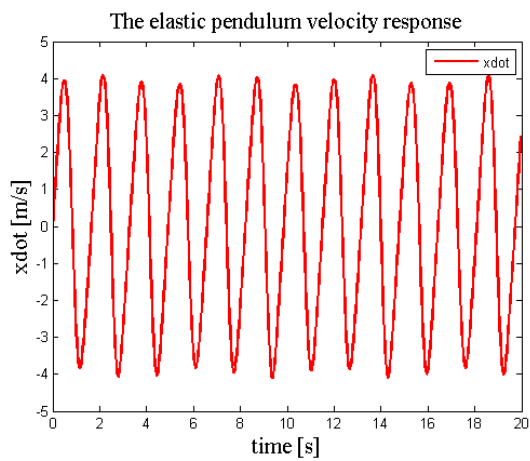
(b)



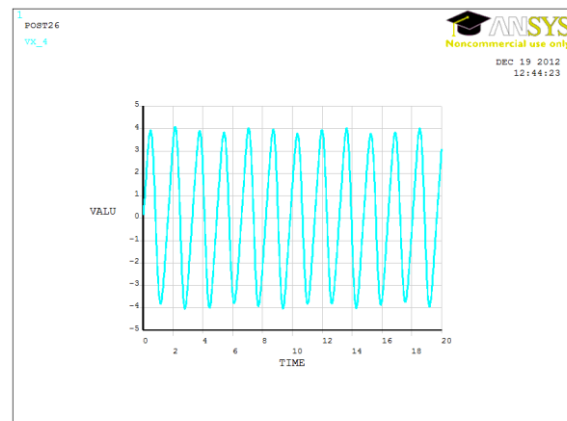
(c)



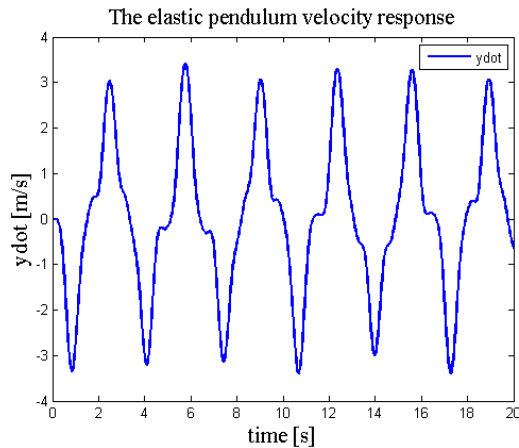
(d)



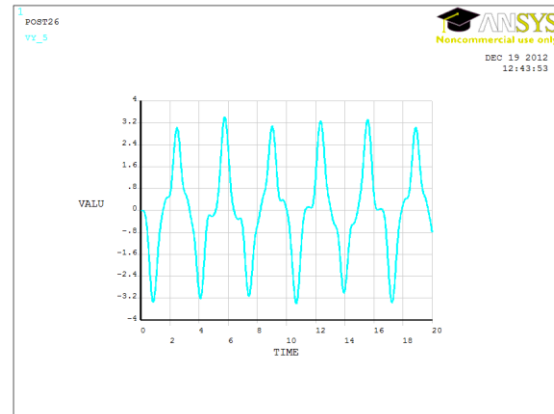
(e)



(f)



(g)



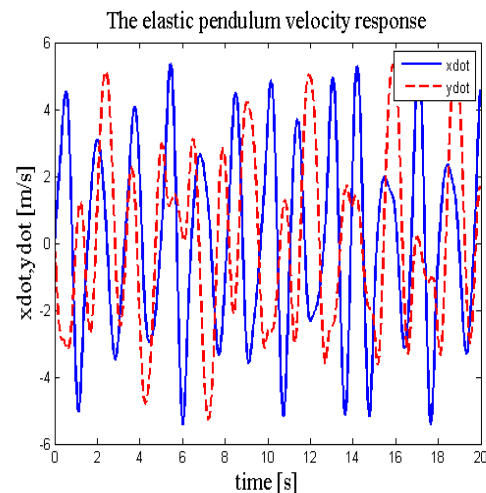
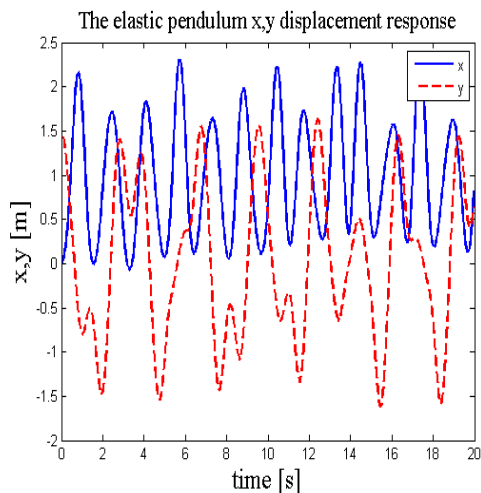
(h)

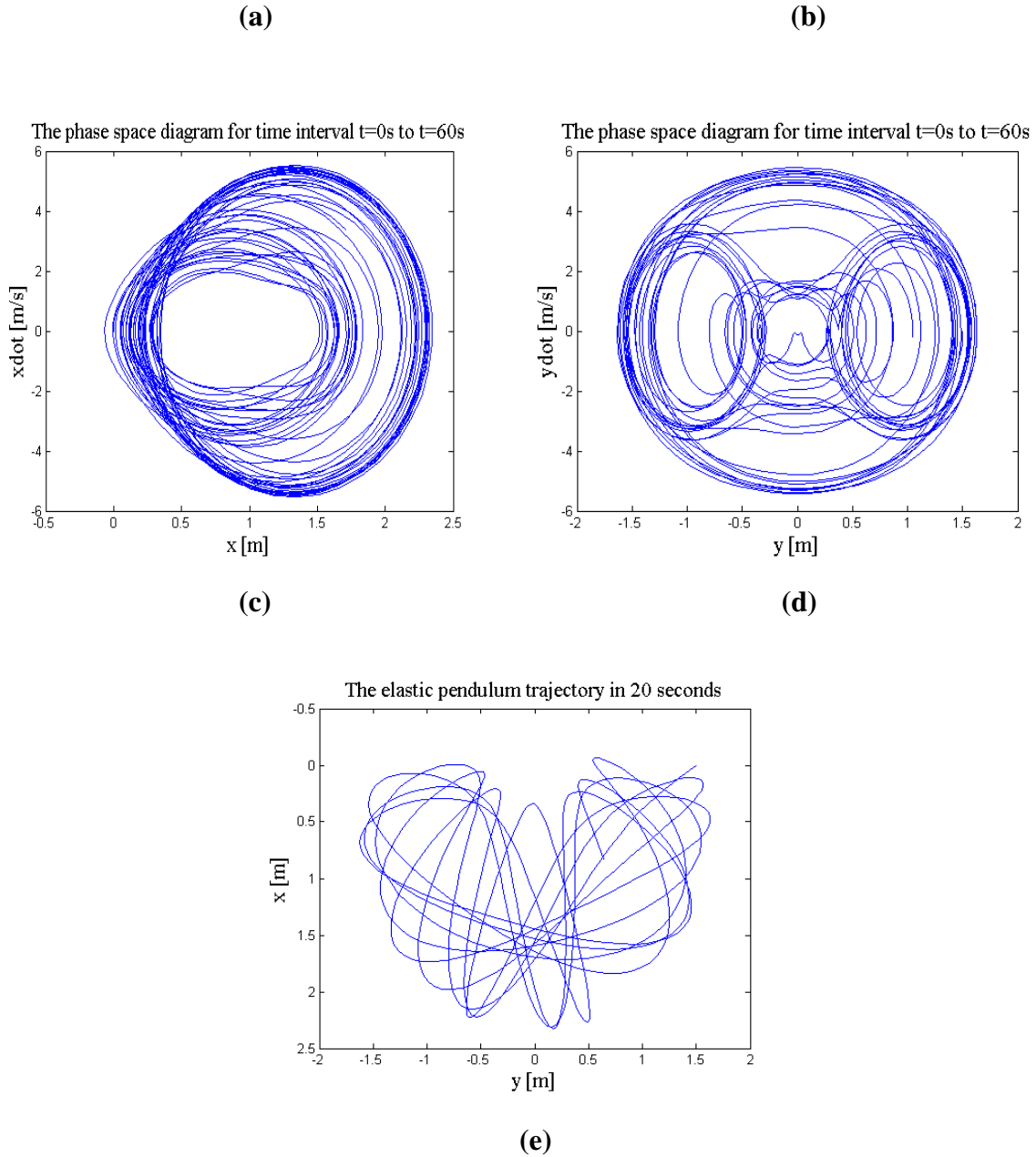
**Figure 5-11: Verification of Newmark's method with transient dynamic analysis of ANSYS for a spring pendulum**

That must be noted that in Figure 5-11 in section of (c) and (d) in y direction the results of ANSYS on section (d) has started from 0 to -2m but on part (c), the range started from 1m to -1m. In fact they are the same figures but ANSYS has used relative coordinate on part (d) of Figure 5-11, and one can makes sure that both pictures of (c) and (d) are the same as each other just (c) is plotted with real assigned numbers but (d) is plotted with relative coordinate system.

## Solution with central difference method

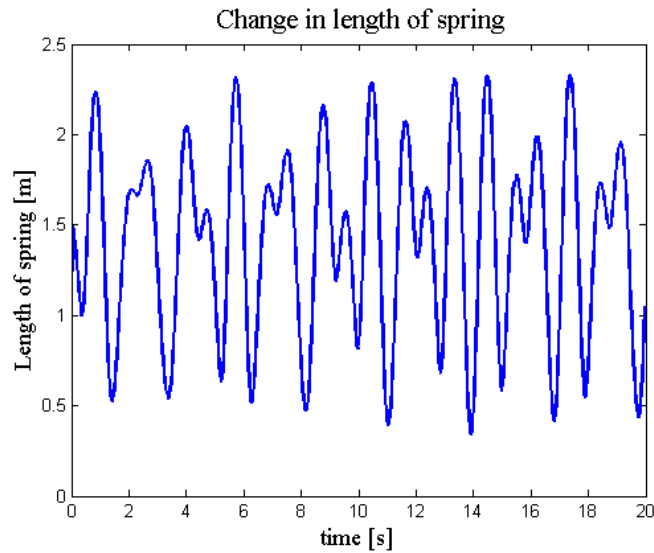
In the current section, the problem of 4.2.1.1 is solved with central difference method. Again, the Matlab codes are available in the section of appendix. After, by solving that with central difference method, below figures are obtained:





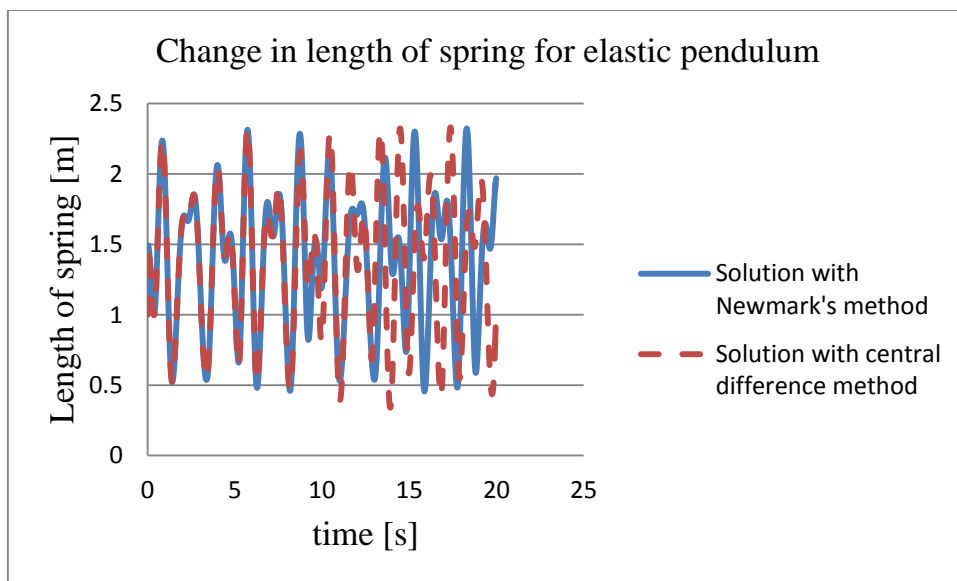
**Figure 5-12: Results of spring pendulum by central difference method**

Thus, it can be seen that the results are somehow the same, and the time step in both methods were 0.03s. In central difference method, Newton-Raphson iteration is not used. When, Newmark's method is used to solve the problem the elapsed time was 0.052596s but in case of using central difference method the time was 0.011891s. So, by performing the same time step, central difference method is more than four times faster than Newmark's method in this special case. However, the trajectory is not exactly the same, and the accuracy of central difference method can increase by choosing smaller time steps which spend more time to calculate the displacements, velocities, and accelerations. Also, the change of length versus time is plotted in Figure 5-13.



**Figure 5-13: Change in length of spring for 20 seconds with central difference method**

In the picture below, the difference in change of spring's length versus time has been compared between solutions of Newmark's method, and central difference method. It can be seen that in the first 10 seconds they are somehow the same but after 10 seconds they started to deviate from each other.



**Figure 5-14: Comparison of changes in length of spring pendulum with Newmark and central difference method**

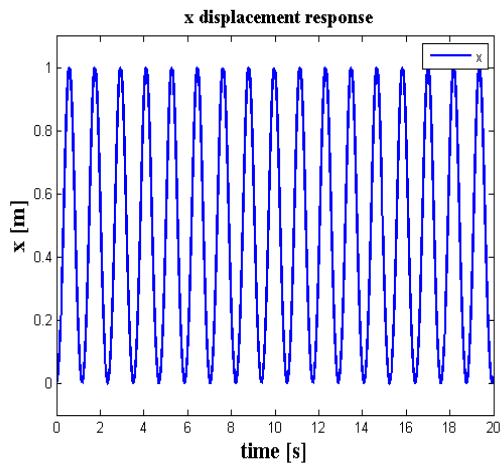
### 5.2.1.2. Results of bar pendulum problem

#### Solution with Newmark's method

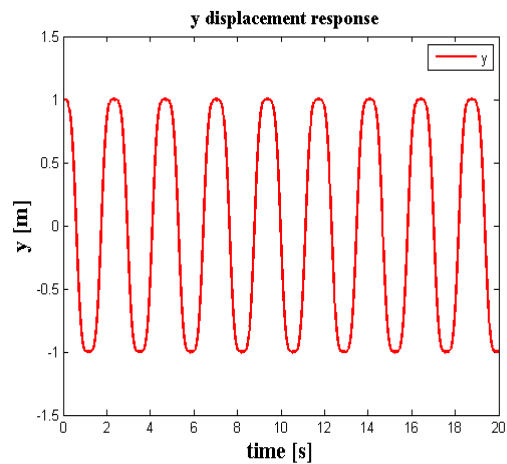
In current part, the solution for problem of 4.2.1.2 is accomplished. It is so important to choose a right time step to keep the stability of method. As it has mentioned in 3.3.1.3 of this report, it is written how to choose magnitude of time step:

$$\gamma \geq \frac{1}{2}, \beta \leq \frac{1}{2} \text{ and } \Delta t \leq \frac{1}{\omega_{max} \sqrt{\frac{\gamma}{2} - \beta}}$$

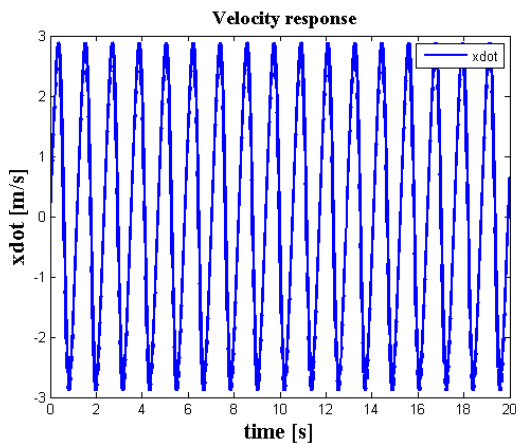
Then, by calculating the time step based on above parameters, the time step should be used to solve current problem, and that must be smaller than 0.001s. Hence, to keep the stability of method, the time step must be smaller than 0.001s. So, the time step of 0.0008s is used to solve the current problem. After all, below results are obtained:



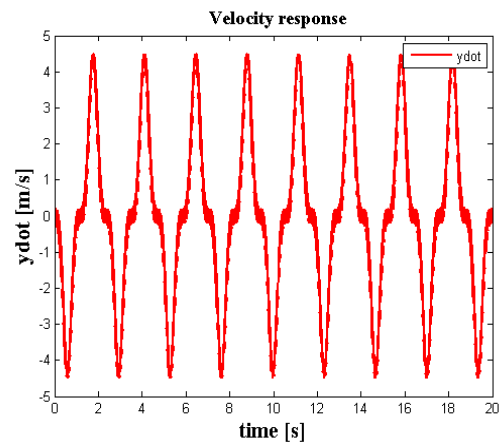
(a)



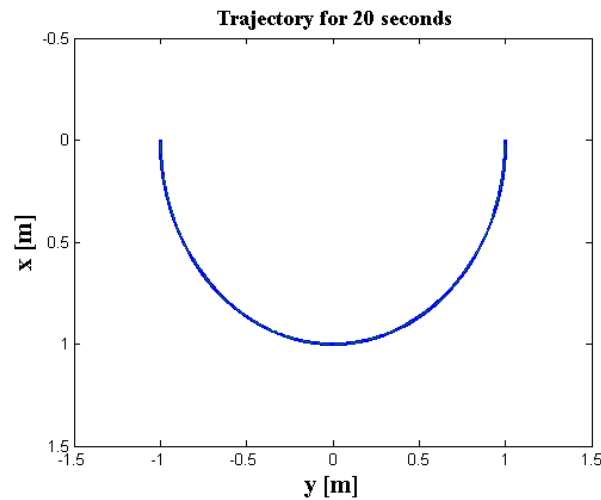
(b)



(c)



(d)



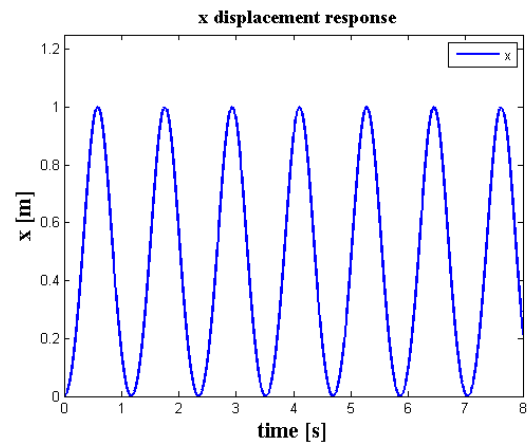
(e)

**Figure 5-15: Results of bar pendulum with Newmark's method**

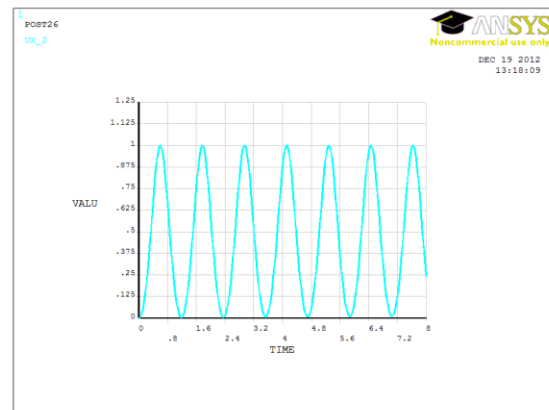
To solve a problem with this method, number of Newton-Raphson iterations for convergence plays the main role in time of calculation. Actually, the role of fast convergence is more important than having low time step, but they are two different separated concepts. However, fast convergence is needed to solve the problems. In current problem, length of bar is changing between 0.9999 and 1.0001. To see exactly what is happening during the time of solution, the animation is provided and it can be found in the appendix. That is actually data visualization, and by changing parameters the animation would change.

## Verification of results

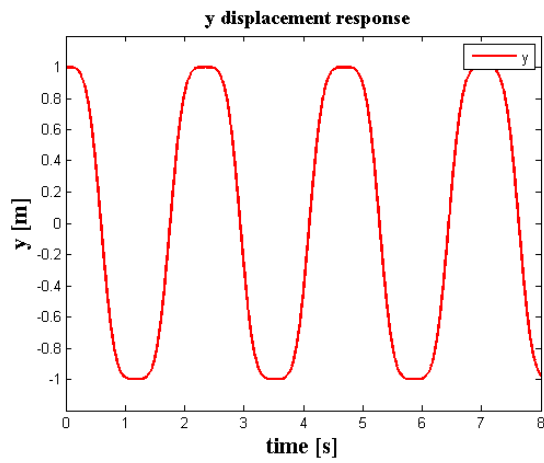
In this section results of bar pendulum with Newmark's method are verified with transient dynamic analysis of ANSYS. Again, the full method of transient dynamic analysis type of ANSYS has used to verify the results of Matlab codes for Newmark's method. To model the bar pendulum on ANSYS, element of LINK180 has used as a bar element and element of MASS21 has used as element of mass at the end of bar. Figure 5-16, it can be seen that the results are exactly the same.



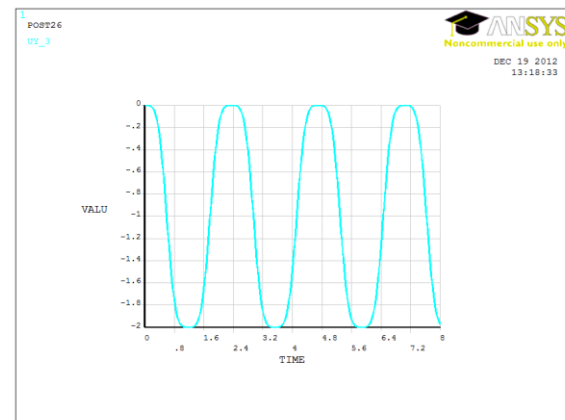
(a)



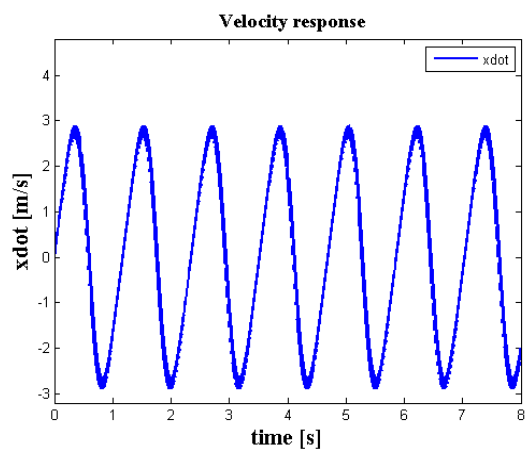
(b)



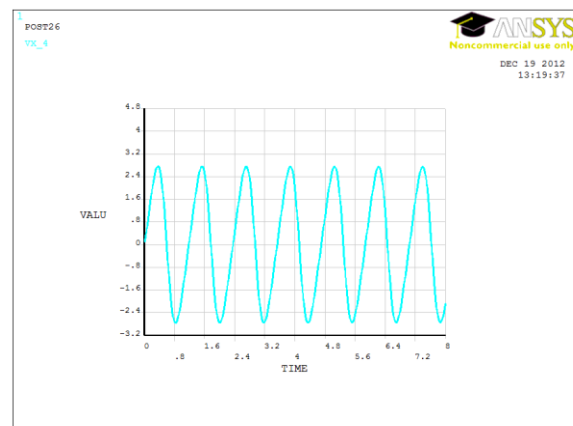
(c)



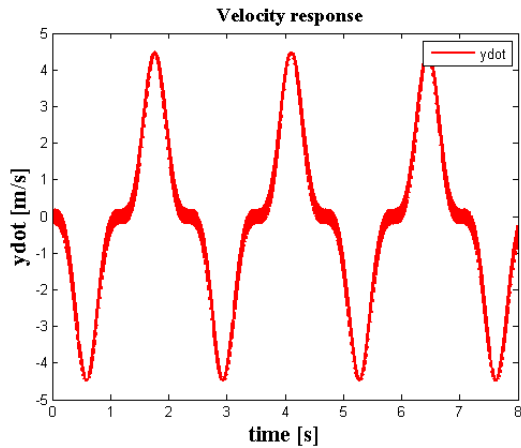
(d)



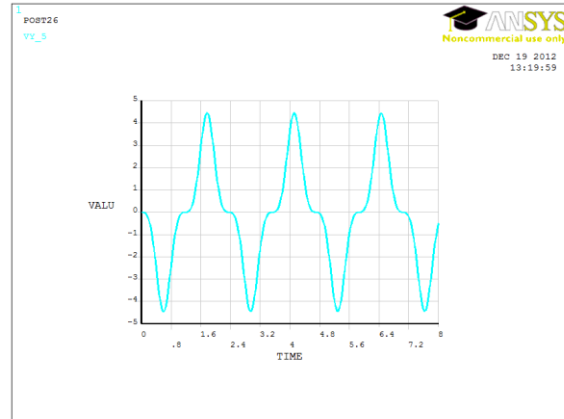
(e)



(f)



(g)



(h)

**Figure 5-16: Verification of Newmark's method with transient dynamic analysis of ANSYS for a stiff bar pendulum**

Again, that should be noted that ANSYS figures have been plotted based on relative coordinate system. So, you may find a difference of numbering system on y directions of part (c) and (d) of Figure 5-16, but they interpret the same plot in two different coordinate systems. Moreover, right side pictures of Figure 5-16 are plotted from TimeHist Postpro of ANSYS.

## Solution with central difference method

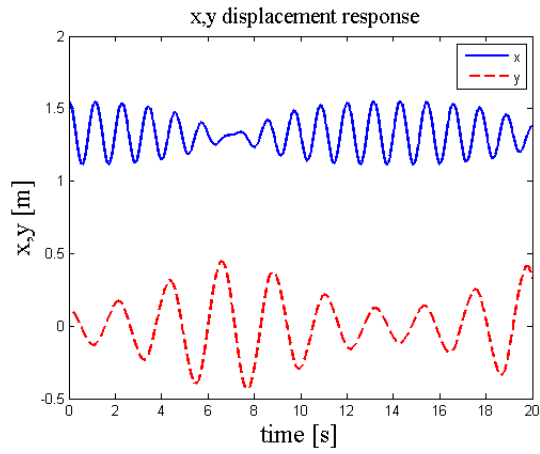
Problem of 4.2.1.2 is solved with central difference method, and the same results with results of bar pendulum with Newmark's method are reached. Actually, the results were exactly the same as previous section. It can be concluded that for linear problems, the results of Newmark's method and central difference method are the same and deviation between methods happen, when the problem is highly non-linear. Moreover, the deviation can be more evident as time goes from starting time of solution.

## 5.2.2. Results of bar-spring problems

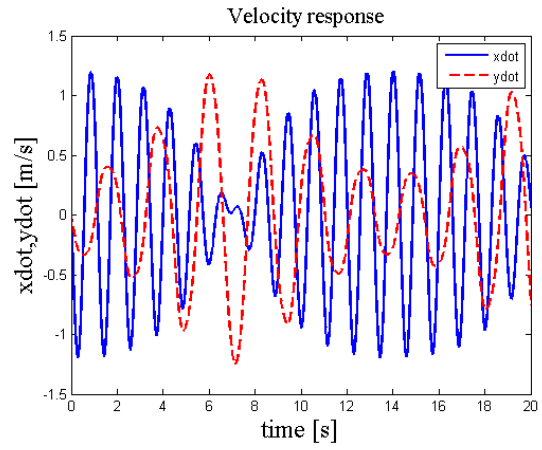
### 5.2.2.1. Results of low stiff bar-spring

#### Solution with Newmark's method

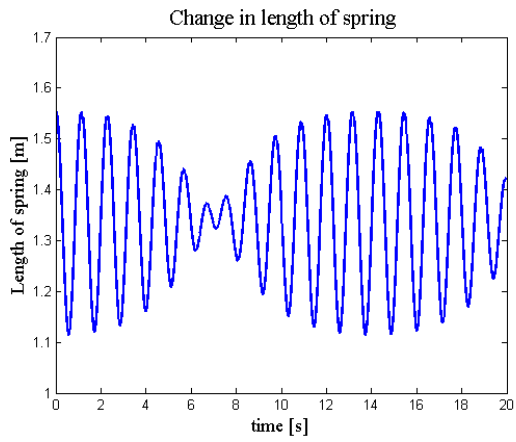
In this part, problem of 4.2.2.1 is solved. Furthermore, initial displacement and velocity of current two degrees of freedom are given. Then, just import the values to Matlab codes of Newmark's method and solve it which can be found in the appendix. Results are available in figures below:



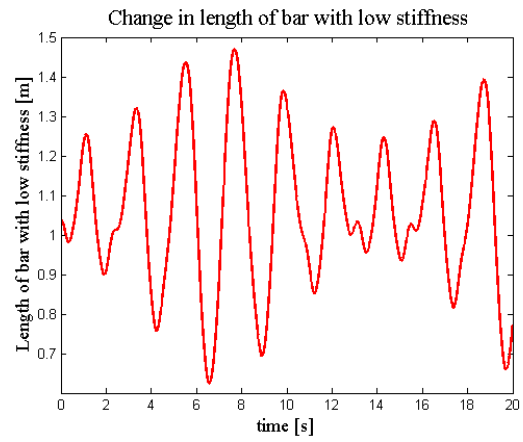
(a)



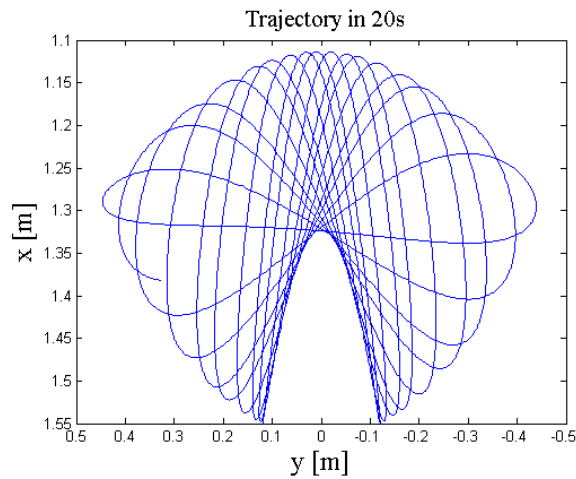
(b)



(c)



(d)



(e)

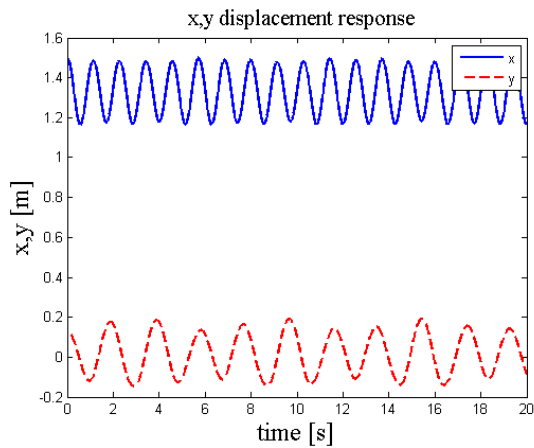
**Figure 5-17: Results of low stiff bar-spring with Newmark's method**

A considerable point to solve this problem is how to define the initial parameters like initial position of mass, initial length of bar, initial length of spring, and values for stiffness of

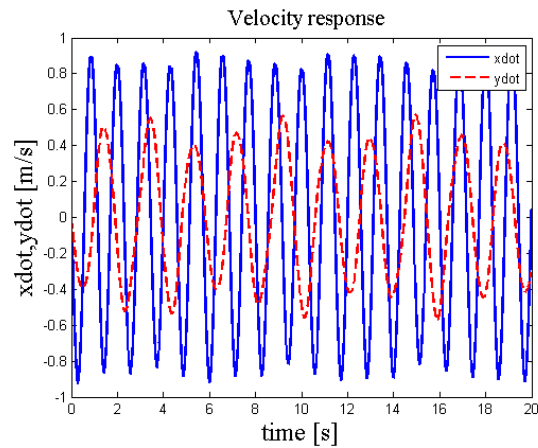
spring and elastic modulus, and area of bar. Actually, in this example, the trajectory and the whole problem are totally non-linear which came from definition of magnitude of the spring's stiffness and  $EA$  of bar element. Those have been chosen very low. In fact, the bar is changed to a very loose spring with the stiffness of spring is higher than stiffness of bar in this problem, and this cause a non-linear behavior. In next section exactly the same parameters will be used to capture the differences in non-linear solution of this problem but with central difference method which is an explicit method.

### Solution with central difference method

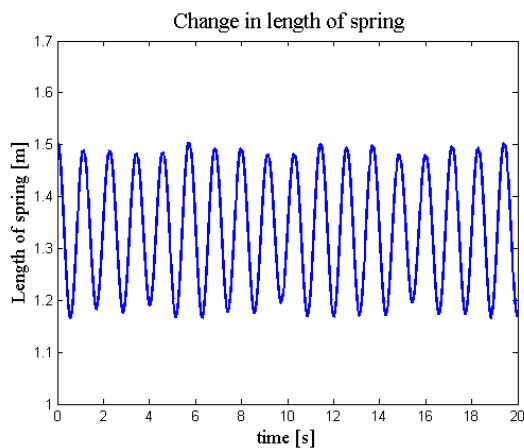
In this part, solution with central difference method, for problem of 4.2.2.1 is available. Likewise, the geometry and every single part of problem is the same as before but the method to solve the problem has changed. After all, achieved results are available in figures below:



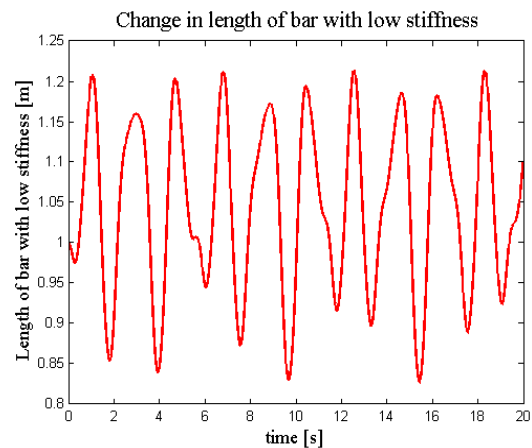
(a)



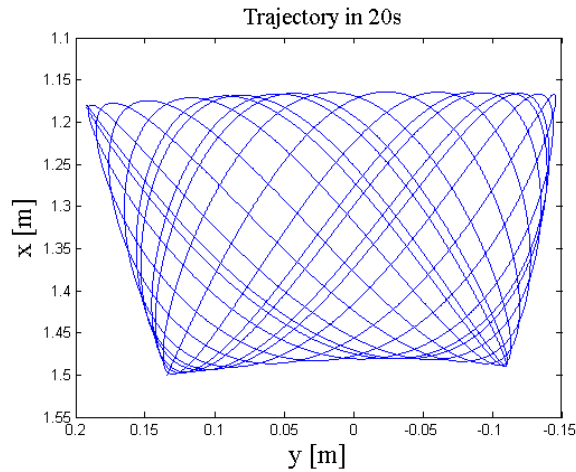
(b)



(c)



(d)



(e)

**Figure 5-18: Results of low stiff bar-spring with central difference method**

It seems that the results of Newmark's method are more accurate. However, the problem is highly non-linear, and magnitude of deviation of results is dependent on amount of non-linearity of problem. Besides, in graphs of displacement and velocity the differences are so vital. Also, in graph of trajectory, it can be seen that the range in Y direction has changed, for the Newmark's method it was between around -0.4m to 0.4m but in solution of central difference method it is between -0.15m to 0.2m.

Additionally, if the time step in central difference method decreased, the final results of problem would not change that much. So, decreasing the time step in central difference method does not help to increase the accuracy of solving the current problem. Finally, to get more accurate results, usage of Newmark's method is recommended for highly non-linear problems.

However, the sensitivity of these kinds of calculations is high for highly non-linear problems. Since, the problem is highly non-linear, in Newmark's method it is really crucial how to define the values specially stiffness of spring, elastic modulus, and area of bar because the results can end up in singularity. De facto, it is really crucial to choose logical numbers for parameters like stiffness of spring or bar. So, it is important to define the values in a way not to end up in singularity. Actually, the numbers for all parameters should be logical to get acceptable results.

In a case that a cable has been used instead of spring in this problem, everything is the same but the stiffness matrix has changed. Moreover, in the code before calculation of residual these lines must be added which can be seen below (By assuming that cable is a kind of spring which has the stiffness of k:

```

if Lcable <= L0cable
    k=0;
else
    K=the K that has defined to solve the problem
end

```

Then the code will work. Also to add snapping, based on the mechanical properties of the cable, maximum length of cable should be calculated which the cable snaps in that length. Afterward, before calculation of every residual matrix, these lines should be added which can be seen below:

```

if Lcable => LcableMAX
    k=0;
end

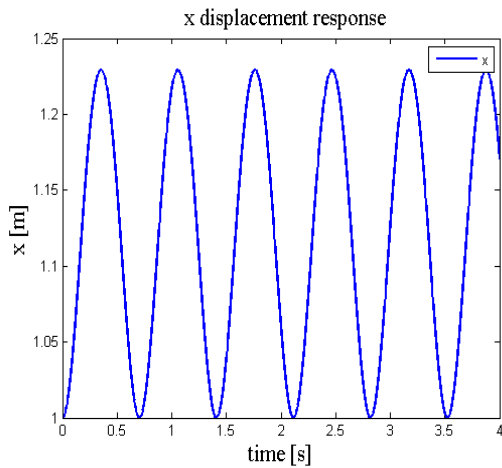
```

In the next examples, these matters will be enlightened clearly.

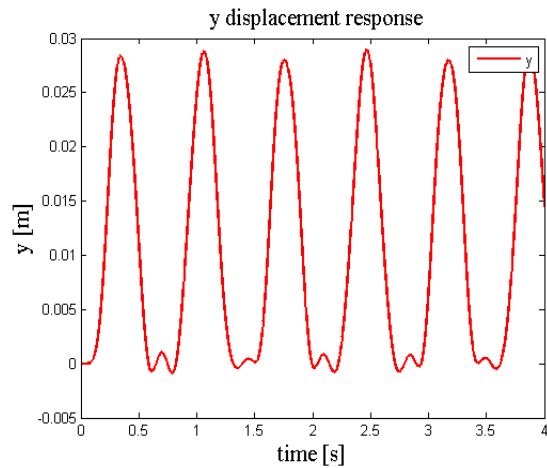
#### 5.2.2.2. Results of stiff bar-spring (cable)

##### Solution with Newmark's method

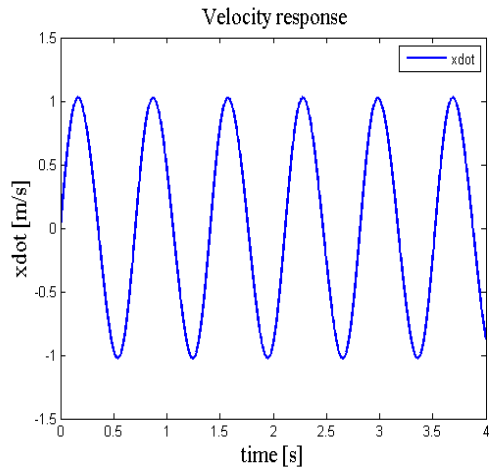
In this section, problem of 4.2.2.2 is solved. It is important to know that stiffness of spring in problem  $k$ , has changed to  $EA/L_{cable}$  but in this problem the cable is used as a stiff spring. It means the cable is working like a stiff spring without any extra condition. Additionally, in next problem, one more extra condition will be added to current problem and will be solved to check out the differences in results. The time step of 0.0002s has chosen to solve this problem. Additionally, the initial conditions are:  $x_0 = 1$  m ,  $y_0 = 0$  , and  $\dot{x}_0 = \dot{y}_0 = 0$ . So, below results are obtained:



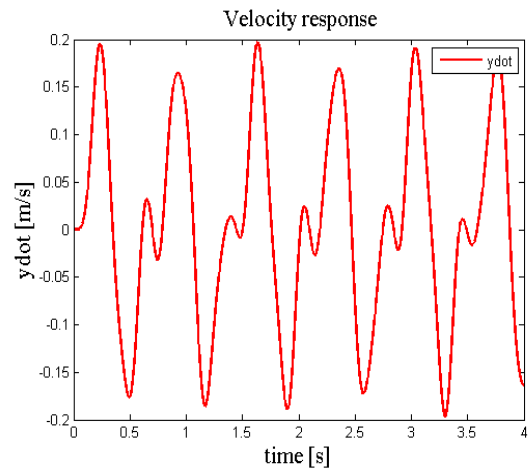
(a)



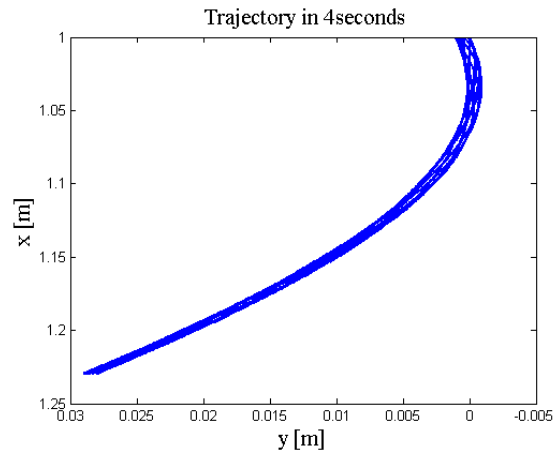
(b)



(c)



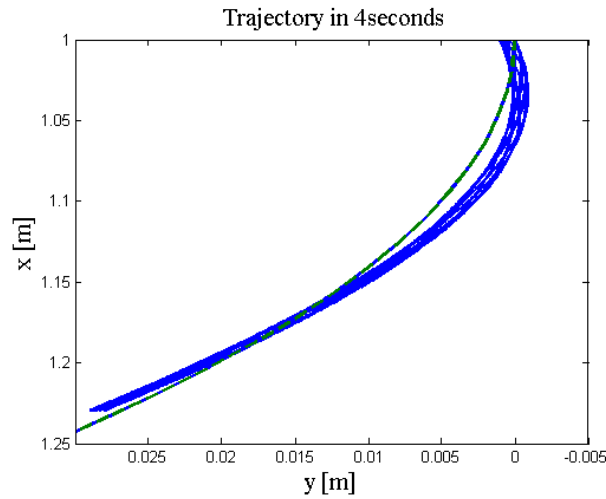
(d)



(e)

**Figure 5-19: Results of stiff bar-spring with Newmark's method**

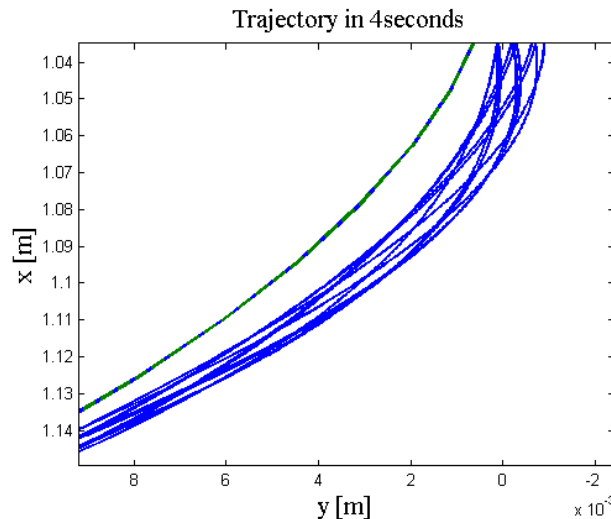
Since the bar is stiff, one can draw a circle with center of (O1,O2), and the radius of 1m which is the initial length of bar. In this way, it can be seen that how the length of bar is changed during the problem, and the result can be compared by Figure 5-20:



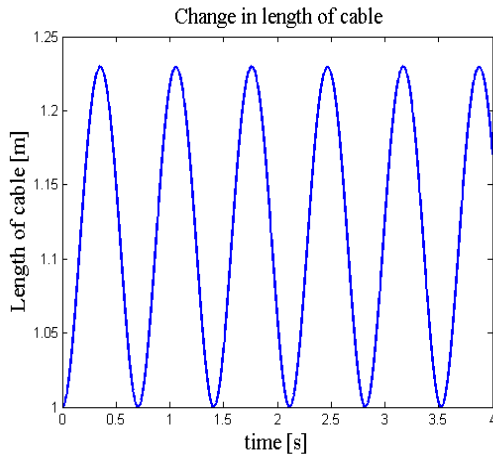
**Figure 5-20: Trajectory of stiff bar-spring in 4 seconds with Newmark's method**

The green line is the circle and the blue line is the trajectory. By capturing the length, it is found that minimum and maximum of bar's length are represented as 0.9977 and 1.0023. So, the blue line which is trajectory is very close to green line which is the trajectory of solid bar with stiffness of infinity. Then, it can be assumed that two degrees of freedoms changed to one, and it means  $x$  and  $y$  are dependent on each other and both together are making the problem to a single degree of freedom problem. Hence, the problem changes to a single degree of freedom problem.

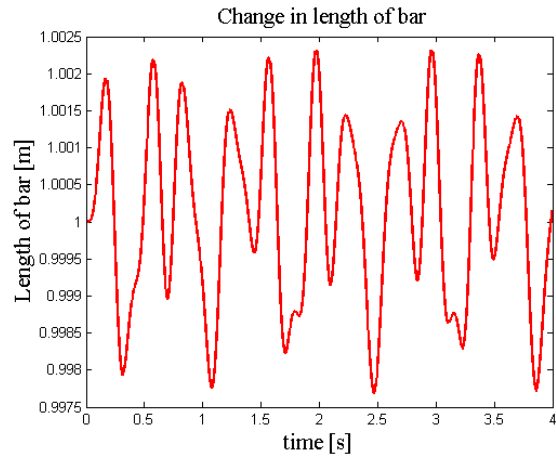
Besides, below Figure 5-21 is a closer look at the scene where max deviation of trajectory and green circle takes place.



**Figure 5-21: Trajectory of stiff bar-spring in a closer look**



(a)



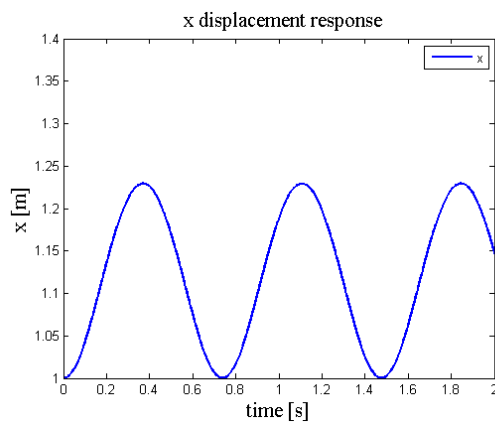
(b)

**Figure 5-22: Changes in lengths of bar and cable in first second**

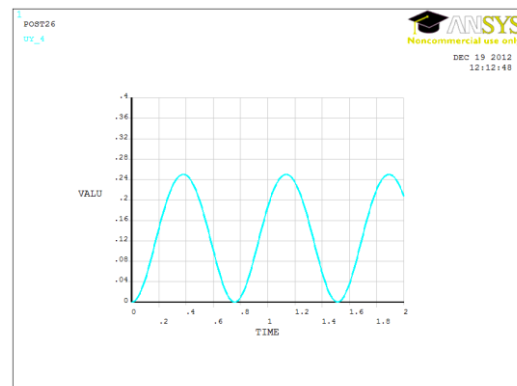
In Figure 5-22, change in length of bar and cable versus time is shown. In Figure 5-22 (b), it is evident that the length of bar does not change so much, and the elongation is so small which proves the high stiffness of bar. More, in Figure 5-22 (a) it can be seen that the length of cable changes much more than length of bar, and the reason is stiffness of cable which is one tenth of the bar's stiffness. The data visualization (animation) of current problem can be found in appendix.

## Verification of results

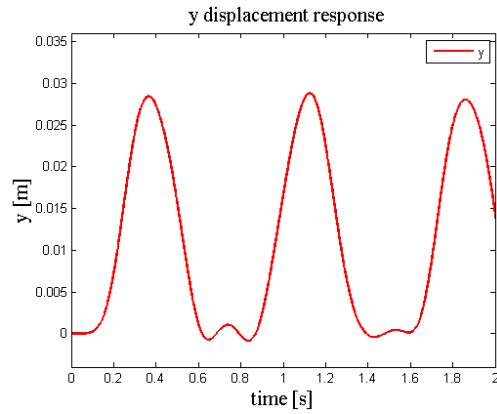
In this section results of stiff bar-cable with Newmark's method are compared with results of dynamic analysis in ANSYS. Again, the full method of transient dynamic analysis of ANSYS performed. The verification process is done for first two seconds of analysis. Results are available on below pictures. To model the system, two elements of LINK180 and just one element of MASS21 have utilized on ANSYS. Verification of results for Matlab codes with results of ANSYS is vital on pictures below.



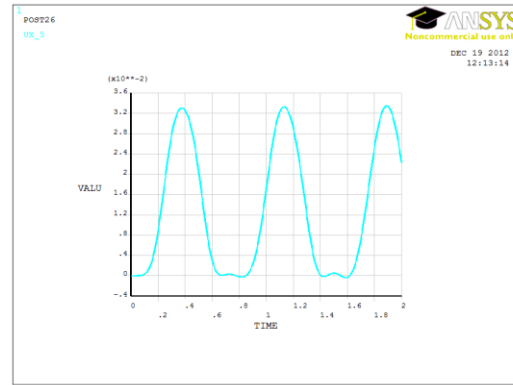
(a)



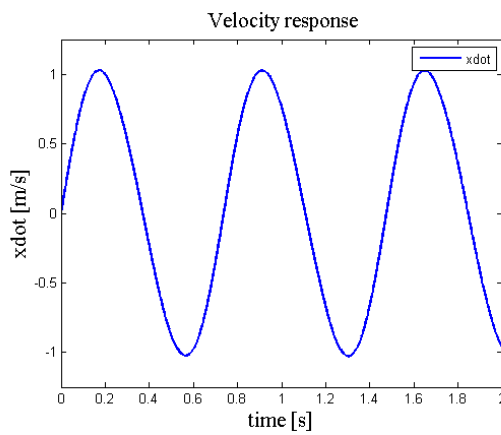
(b)



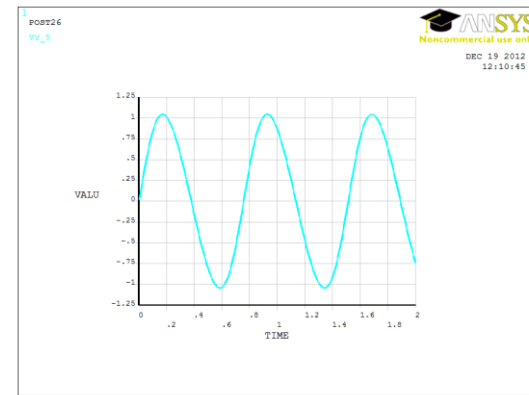
(c)



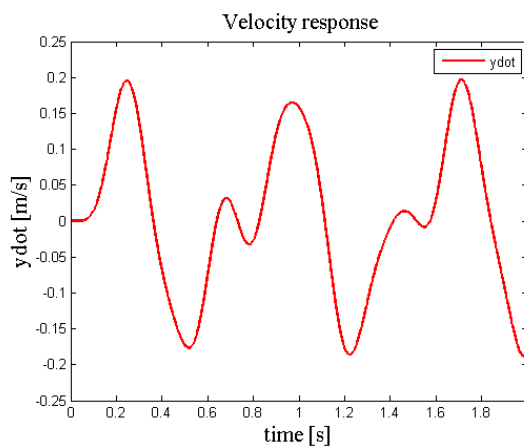
(d)



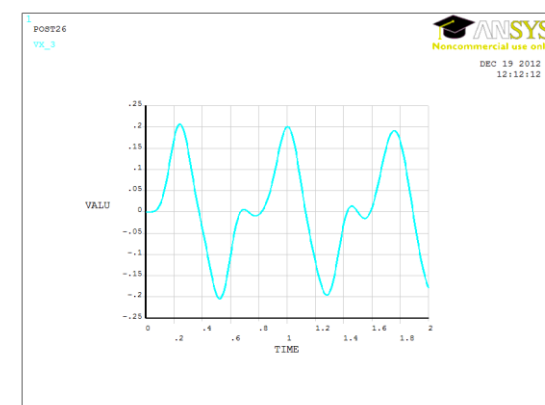
(e)



(f)



(g)



(h)

**Figure 5-23: Changes in lengths of bar and cable in first second**

Again, it must be noted that results of ANSYS are based on relative coordinate system. So, it can be understood that results of Figure 5-23, in part of (c) and (d) are the same. So, by verifying this problem, there is no need of verification for results of problems in section 4.2.3

because the base problem of them is problem 4.2.2.2 where the problem is solved with two bar elements but stiffness of a bar in system was much more than another one. So, we called them as stiff bar-spring problem but that was genuinely a bar-bar system with two different stiffness values.

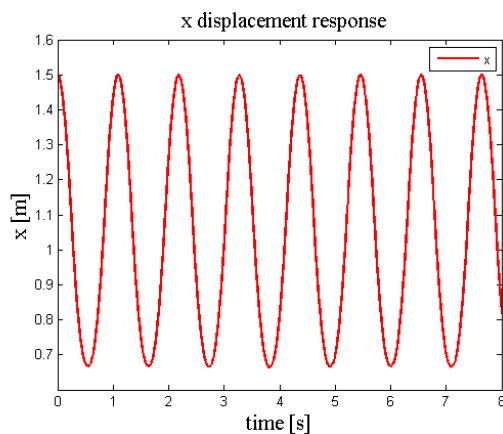
### 5.2.3. Problem of cable-bar

#### 5.2.3.1. Problem of cable-bar with slackening

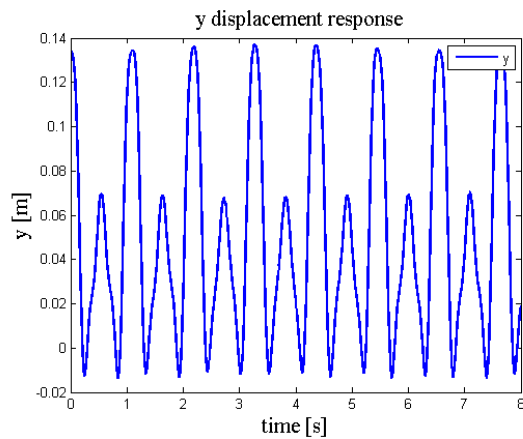
##### Solution with Newmark's method

In this part the problem of 4.2.3.1 is solved. The time step of 0.003 s has chosen to solve this problem. Since, there is a cable in the problem; a condition must be added to solve it. So, the condition can be seen as follow:

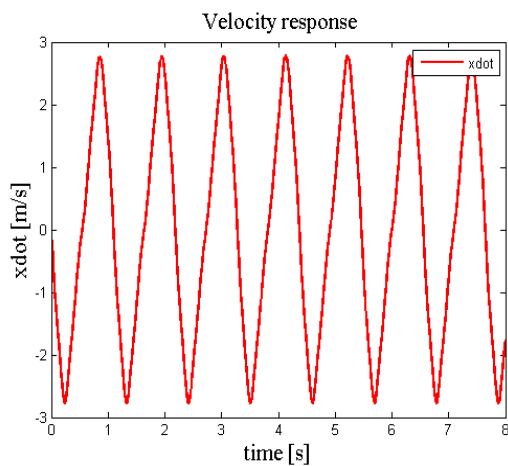
```
if Ls < L0s
    EA_cable = 0;
else
    EA_cable = real EA of cable;
end
```



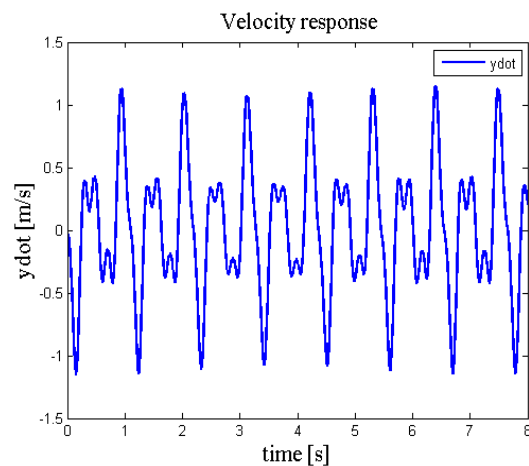
(a)



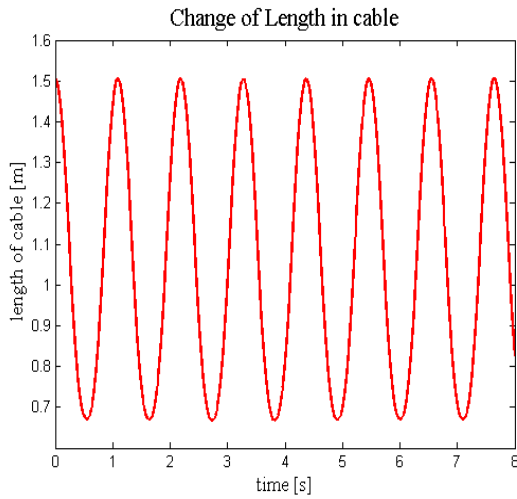
(b)



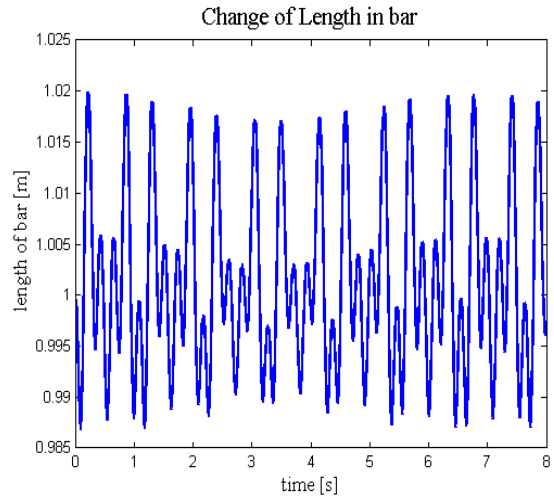
(c)



(d)



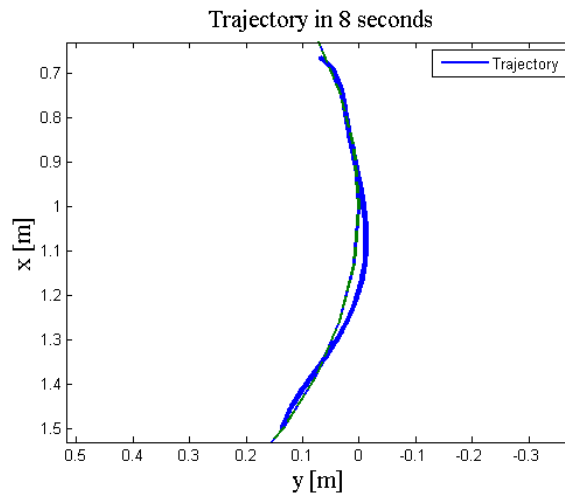
(e)



(f)

**Figure 5-24: Results of cable-bar system with slackening by Newmark's method**

Again, by making a comparison between two figures above, it is clear that the differences in length of bar is much lower than changes in length of cable. Moreover, it can be seen that length of cable never became less than 1m , and the reason is the initial condition where the mass has left. If the initial conditions were like  $x_0=1.5\text{m}$ , and  $y_0=\left(1 - \frac{\sqrt{3}}{2}\right)\text{m}$ , then that would be possible to see more deviation for changes in length of cable.



**Figure 5-25: Trajectory of cable-bar system with slackening by Newmark's method**

Like the last problem, if one draw a green circle by center of  $(O_1, O_2)$ , and radius of 1m that is initial length of bar, the trajectory in blue will be comparable with the green circle. Genuinely, both lines are not exactly on each other in Figure 5-25. So, there are some deviation between two lines of trajectory and a part of circle passes over the boundaries of trajectory, and the reason can be initial conditions of problem and a condition that is applied for definition of cable in current problem.

So, since the deviations are negligible, it can be assumed that two degrees of freedoms, changes to single degree of freedom, because the  $x$  and  $y$  are dependent of each other and can be used as parameters of circle's equation which passes through the trajectory. In fact, trajectory is a part of circle.

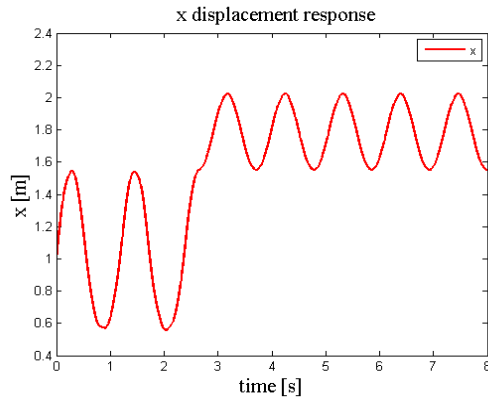
### Solution with central difference method

The problem of 4.2.3.1 has solved with central difference method and the same results like results of Newmark's method has achieved. The reason for achievement of the same results is that the problem of 4.2.3.1 is not highly non-linear. Actually, the bar's behavior in both results, was linear, and cable's behavior was non-linear. Therefore, the whole system was not highly non-linear. So, the results of both central difference method and Newmark's method were in agreement with each other.

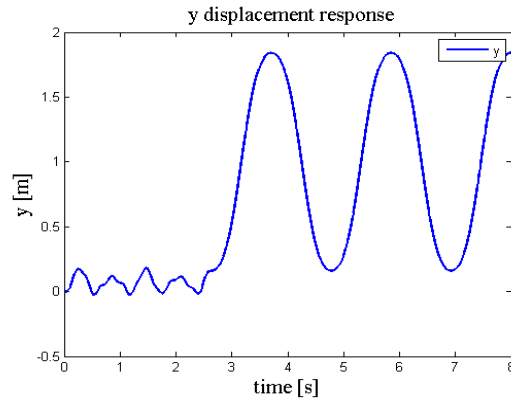
### 5.2.3.2. Problem of cable-bar with slackening and snapping

#### Solution with Newmark's method

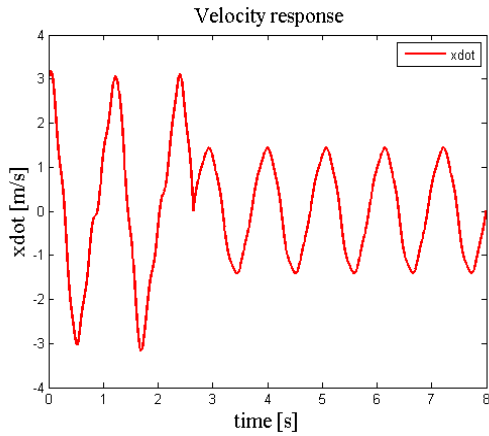
In current part, problem of 4.2.3.2 is solved and results are obtained in below figures.



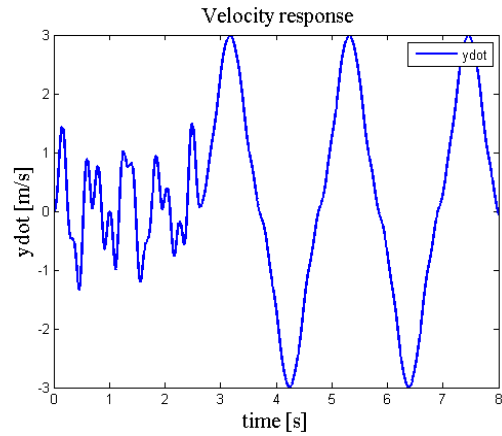
(a)



(b)



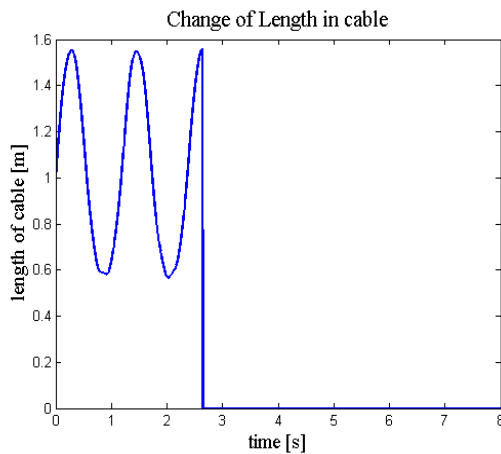
(c)



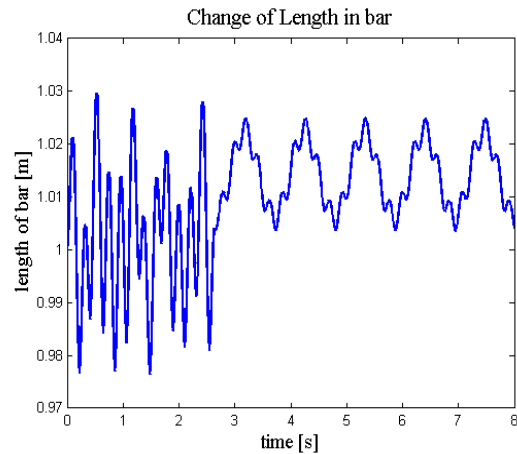
(d)

**Figure 5-26: Results of cable-bar system with slackening and snapping by Newmark's method**

In above figures, it is obvious that when slackening happened which is before 2.8 s. After slackening, behavior of system changes dramatically. De facto, after slackening, the system changes into a bar with a mass at the end. Furthermore, it is assumed that initial length of cable is 1m, and the length of slackening is 1.5591m just to check out the behavior of system at the time of slackening (rupture of cable). So, it is easy to capture the effect of slackening in every system, and it is needed to put the condition when stiffness of system is calculating in each time step.



(a)



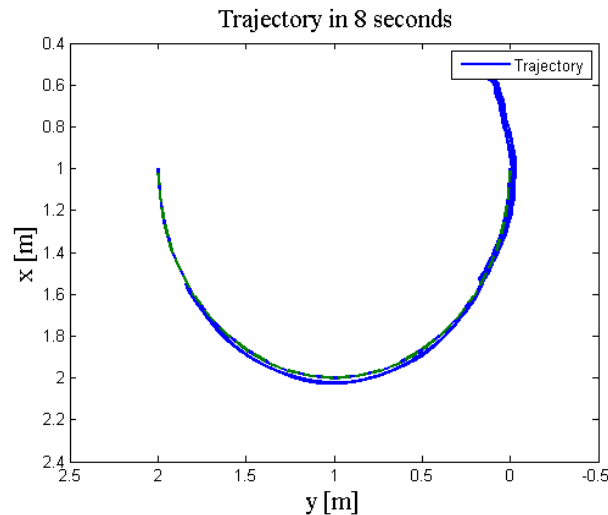
(b)

**Figure 5-27: Changes of lengths in bar and cable in cable-bar system with slackening by Newmark's method**

More, in above figures, changes in length of cable and bar are shown. That is easy to see the effect of slackening in both figures. In fact, after slackening, the length of cable became zero and changes in length of bar became less than length of bar before the time of slackening.

However, changes in length of bar is not so much, and the reason is high stiffness of bar in comparison with stiffness of cable.

Moreover, in figure below, it is evident that the deviation of trajectory from green line which is behavior of bar with infinite stiffness, happens when the system passes from bottom of circle.



**Figure 5-28: Trajectory of cable-bar system with slackening by Newmark's method**

However, a problem with slackening and snapping can be solved with direct time integration methods. Indeed, slackening and snapping are conditions which can be added in structures, and that affects tangent stiffness of whole structure. So, they do not affect the procedure of solving problems with direct time integration methods. Besides, they affect the tangent stiffness of structure. Therefore, problems with or without slackening and snapping can be solved with direct time integration methods, and suitable method to choose for a structure among direct time integration methods depend on structure's degree of non-linearity.

### 5.3. Results of multi degrees of freedom problems

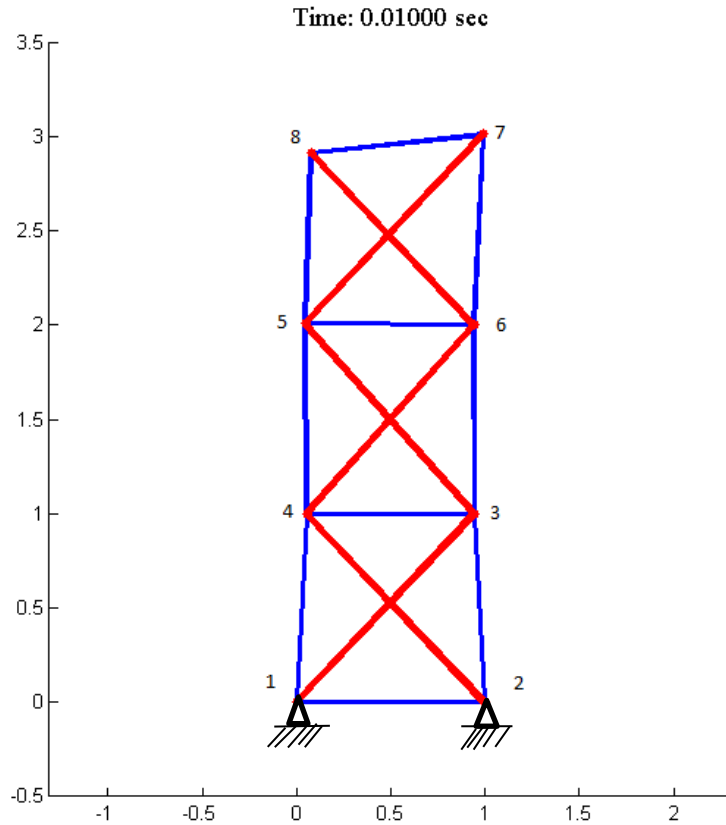
#### 5.3.1. Results of X-frame tensegrity

##### 5.3.1.1. X-frame without slackening

##### Solution with central difference method

Problem of 4.3.1.1 is solved with central difference method as the first method. Then the results of Newmark's method can be calculated and compared with the results of central difference method. The main reason of using central difference method as the first method to solve the structure is no need for convergence. So, if the method worked, it would be possible to get more accurate results with an implicit method like Newmark's method.

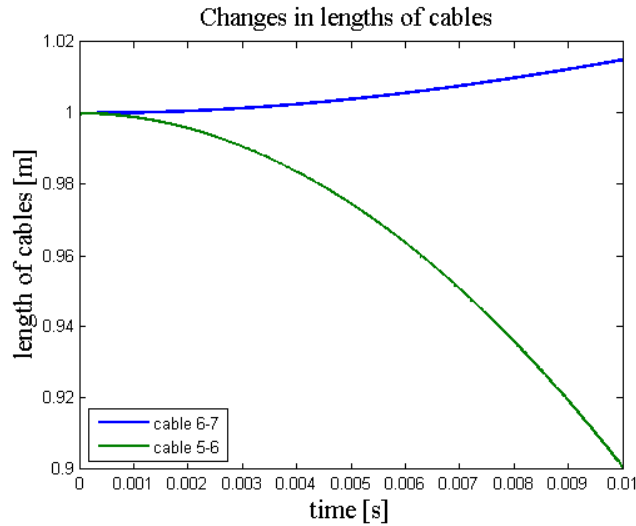
To start solving the structure, on node seven of structure two forces has applied one to right side and on to upper side. Then the final results for  $0.01s$  can be seen in picture below:



**Figure 5-29: Results of tensegrity structure by central difference method without slackening**

The time step was  $10^{-5}$ s to solve the problem. More, the time step of  $10^{-4}$ s is also possible to use and the same results is achieved but smaller time steps would change the results. Choosing the right time step depends on the higher modes of problem and smaller times steps can be utilized to capture the higher modes, and those high modes can affect the stability of solution. Anyhow, for problems with dominance of lower modes, Newmark's method is more preferred.

In figure below, changes in length of two cables between nodes of 5,6 and 6,7 are shown.



**Figure 5-30: Changes of lengths for cables which connect node 6 to 7 and node 5 to 6, with central difference method and without snapping**

So, cable 5-6 was not in tension, and cable 6-7 was in tension. It is possible to draw all differences of changes in lengths of cables but they are some cables that their changes are so similar to each other. Hence, just two cables were selected to draw their changes in specific time.

Besides, to get the higher order of accuracy and more smooth results, Newmark's method can be performed. That's also applicable for problems which are high degree of non-linearity

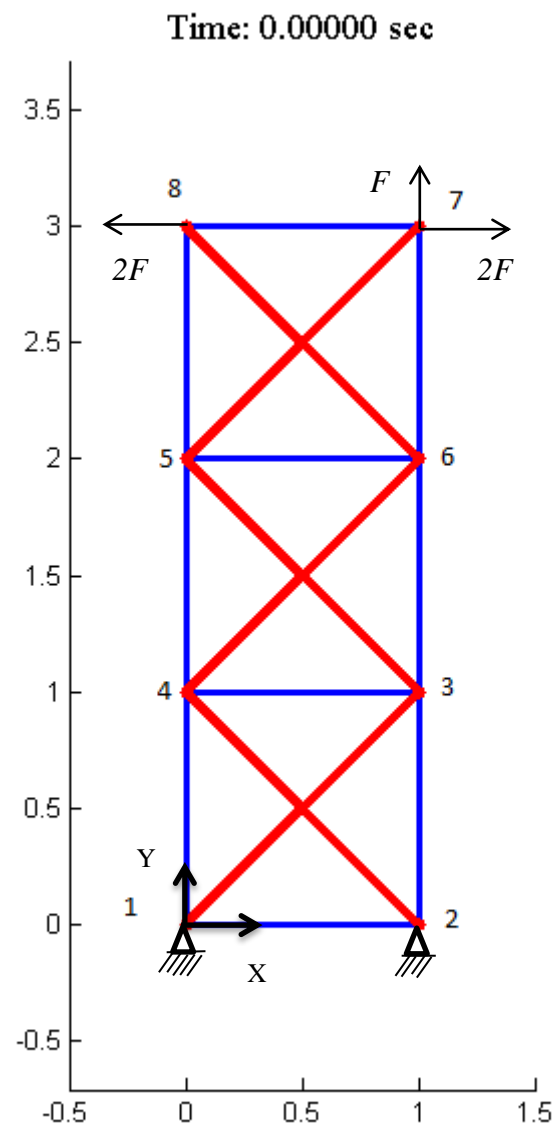
### Solution with Newmark's method

In this section problem of 4.3.1.1 is solved with Newmark's method with the same time step and conditions of solution with central difference method but the method to solve the problem changed to Newmark's method. The results of Newmark's method were exactly the same as results of central difference method. So, it can be concluded that in problem of 4.3.1.1 the results of central difference and Newmark's method are the same each other for 0.01s, and the results can deviate with each other in longer time like what happened for results of spring pendulum with central difference and Newmark's method. Actually, the results of mentioned problem were the same in first 12 seconds but they started to deviate after that time. However, even with the same time step for those two methods, central difference method was faster than Newmark's method, and that is because of convergence nature of Newmark's method. Another reason for obtaining the same result is behavior of tensegrity system which was not highly non-linear.

### 5.3.1.2. X-frame with slackening

#### Solution with central difference method

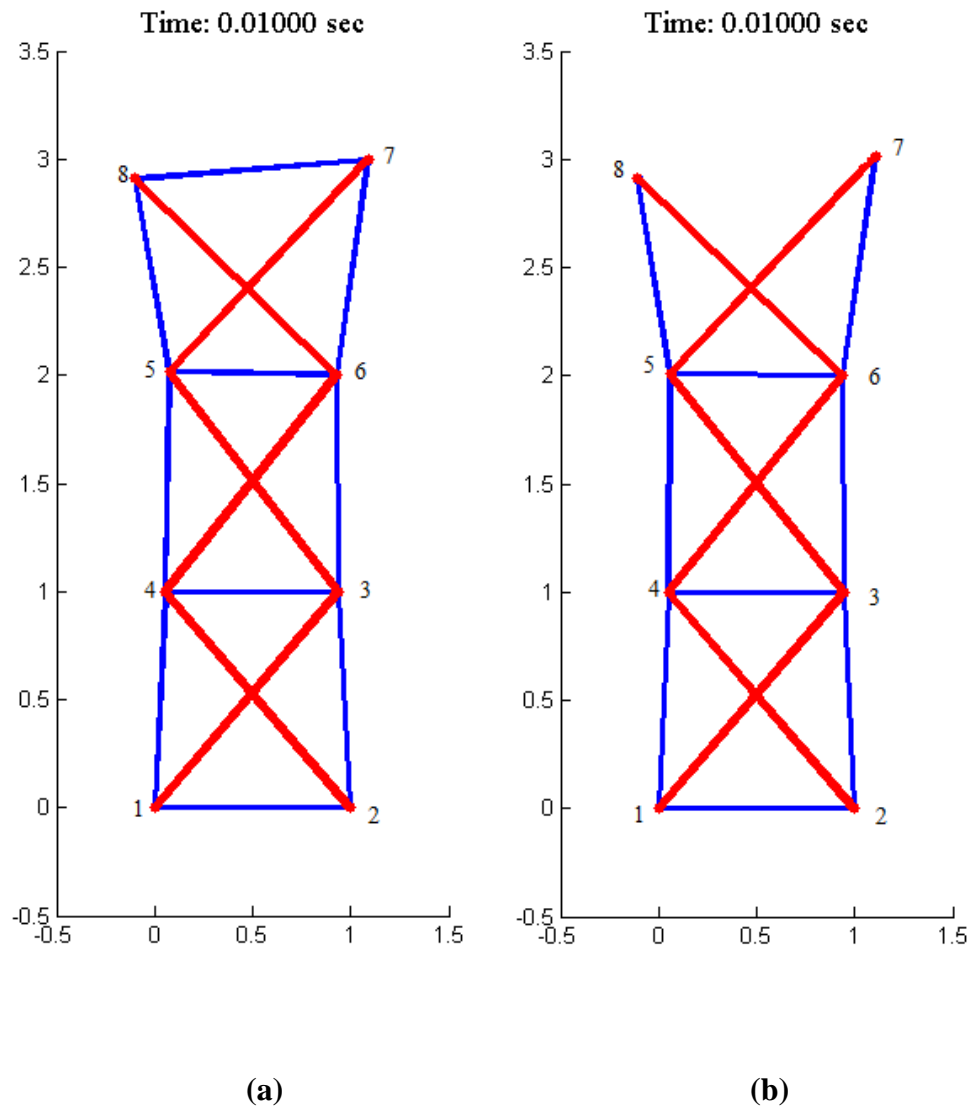
In this section problem of 4.3.1.1 is a little bit changed to see more changes in length of cables to include the condition of slackening. First, central difference method is applied on problem to solve it. In fact the problem of 4.3.1.1 is changed by changing the magnitude of forces and adding one more force on node 8. So, the problem can be seen in Figure 5-31.



**Figure 5-31: System of tensegrity made of three modules of x-frame**

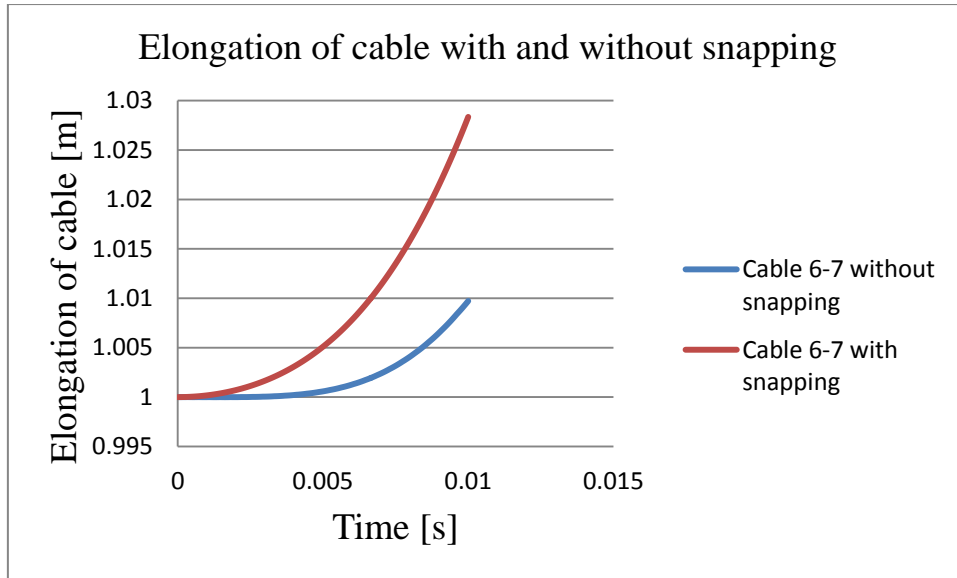
It can be assumed that if the length of cable became more than 1.1m then cable would snap. After putting that condition in structure, the problem can be solved. In figure below, on the

left side, result of problem without snapping and on the right side, result of problem with snapping is shown.



**Figure 5-32: Comparison of results with and without slackening. (a) without snapping, (b) with snapping**

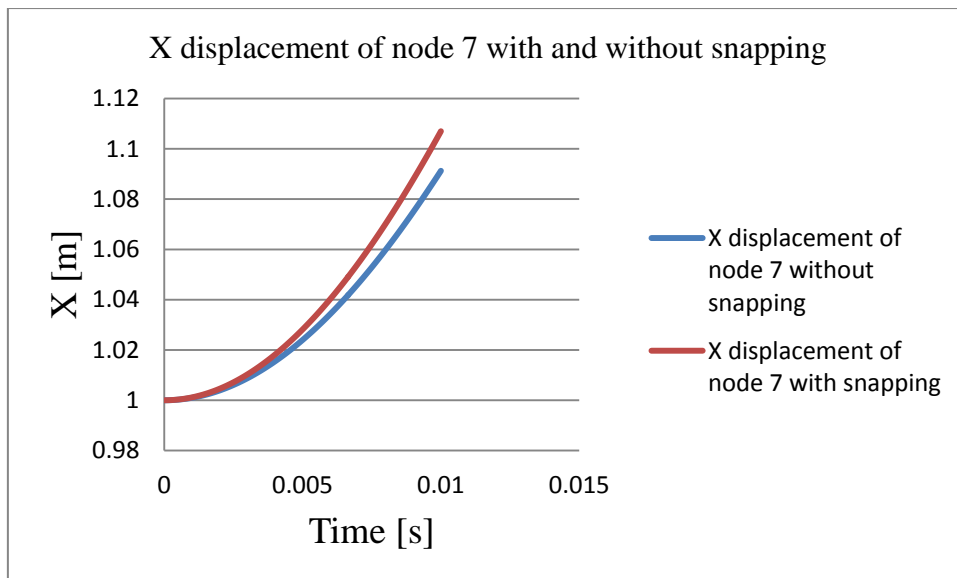
It can be seen that cable which is located between nodes 7 and 8 breaks or snaps. To see the differences with condition of snapping and without snapping, figure 5-33 shows the changes in length of cable that is located between nodes 6 and 7, with and without snapping.



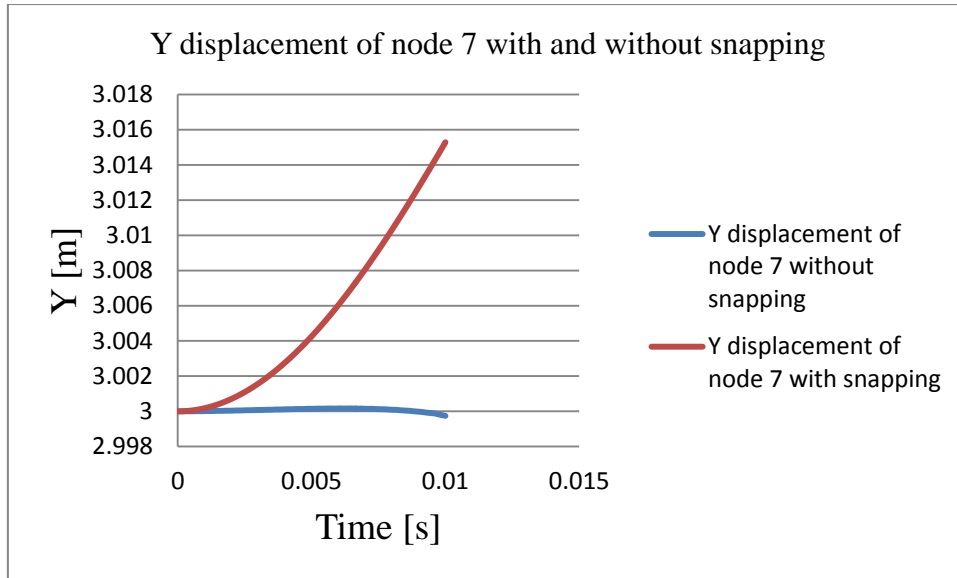
**Figure 5-33: Elongation of cable to connect nodes 6 and 7, with and without snapping**

It should be noted that the red line in Figure 5-33, shows the distance between node 6 and 7. In reality, after snapping the stiffness of cable should be zero but to understand the difference between the conditions with and without snapping, the distance between node 6 and 7 assumed as length of a cable between those nodes. However, after snapping, stiffness of slacked cable can be assumed as zero. Also, one can say that distance between two nodes can be imaginary length of snapped cable.

Also, to see the effect of snapping, Figure 5-34 is plotted to show the difference in x and y displacement of node 7 before and after application of snapping.



(a)



(b)

**Figure 5-34: Displacement of node 9 with and without snapping**

In the last three figures, the deviation of two lines in each figure is obvious, and the reason is performing two different conditions, one with and another without snapping effect.

It must be noted that to introduce snapping just add one more condition to the structure before calculation of tangent stiffness matrix in each time step. So, conditions like snapping or other conditions can be added to the problem and by taking those conditions into account, direct time integration methods are still able to solve such problems. In fact, the thing which plays the main role at the time of adding those conditions is the effect of added conditions on tangent stiffness matrix.

### Solution with Newmark's method

Results of this section are the same as results of X-frame with central difference method. It means that results of both central difference method and Newmark's method are very similar. The reason is choosing small time steps for a very short time period. Also, in that short time period, the behavior of structure was not highly non-linear. So, very similar results have been achieved.

## Chapter 6: Discussion

In this chapter, results of problems are compared, and discussion of results is performed. As it has been mentioned, the main effort of this thesis is to find a method to solve different problems in different situations. In this project, problems of study can be categorized in single and multi-degrees of freedom. More, for the case of single degree of freedom which is available in 5.1.1, two problems were solved with implicit and explicit methods. Moreover, both of them have been verified by the code ODE45 of MATLAB, and results were very similar. Furthermore, Newmark's method has an advantage over central difference method which can be unconditionally stable if  $\beta = 1/4$  and  $\gamma = 1/2$ . However, Newmark's method which is unconditionally stable for linear problems may start to show instability when used for non-linear problems, especially for long term simulations. So, smaller time steps can be used for better results. So, in non-linear problems, size of the time step plays the main role in stability of solution. In fact, the central difference method is the most useful method in single degree of freedom dynamic problems. Also, central difference method can be appropriate method for long time simulations but the time step must be specified based on the highest eigenfrequency of problem.

Moreover, for multi degrees of freedom, first pendulum problems were solved in 5.2.1, spring pendulum and bar pendulum, both solved with Newmark and central difference method. They were linear and non-linear problems, and both solved with central difference method. For pendulum problems, Newmark's method based on predictor-corrector method was applied, and it was shown that both Newmark and central difference method are applicable for linear and non-linear problems. However they were some differences in results and the differences depend on degree of non-linearity. It should be noted that Newmark's method based on prediction-correction method with Newton-Raphson iteration has very high order of accuracy, and that can be more accurate than the central difference method. In Figure 5-13 differences in changes of lengths of spring pendulum with two central difference and Newmark's method were shown. It can be concluded that as time goes on, especially after 12 seconds the results of both methods started to deviate but the range of minimum and maximum lengths were the same in both cases. It seems that Newmark's method specially with energy evaluation and taking account of momentum, is better and more accurate than central difference method for long time simulation. In fact, one way to increase the accuracy of results is energy evaluation for each time step. Also, in 5.2.2.1 for problem of low stiff bar-spring which was a highly non-linear problem with two degrees of freedom, the trajectories of two implicit and explicit methods were totally difference. However, it must be noted that because of high order accuracy in results of Newmark's method in comparison with central difference method, the results of Newmark's method is more reliable for such highly non-

linear problem. De facto, problem of 4.2.2.1 is the problem that can show the most deviation between results of Newmark and central difference method because of high order of non-linearity. Otherwise, in other linear problems or low order of non-linearity problems, results of central difference and Newmark's method would not deviate much from each other.

Besides, the exact solution for some of the solved problems may exist but a more general exact solution, may not be available in closed form. Further, it is obtained that effect of slackening or nature of cable's behavior in tensegrity structures affect tangent stiffness matrix. Then, the tangent stiffness matrix must be calculated to capture the effect of slackening or zero stiffness for each cable which its length less than unstrained length. Apart from slackening, it can be concluded that matters like snapping, just add a condition to codes of structure. Then, when tangent stiffness matrix is calculated with or without conditions like slackening, direct time integration methods would calculate displacement, velocity, and acceleration for each time step. In fact, for explicit methods like central difference methods, that would be easier to apply them to solve problems, since they do not use Newton-Raphson iterations to converge. More, in explicit methods like central difference method, the tangent stiffness matrix must be updated in each time step before calculation of acceleration in that specific time step. Besides, the situation in usage of implicit methods like Newmark's method is more complicated than explicit methods, and that's because of convergence with Newton-Raphson iteration. Furthermore, in Newmark's method with Newton-Raphson iterations, it should be noted that in each time step a tangent stiffness calculated, then if it did not converge, the tangent stiffness matrix would be recalculated and updated until the convergence.

So, many conditions like slackening can be added in structure's codes but that is apart from the procedure of direct time integration methods, and those conditions affect the tangent stiffness of structure. Then, depend on properties of expected results, suitable implicit or explicit method can be applied to solve the problem.

Further, choosing the right time step can guarantee the accuracy of solution and that can save calculation time. Also, stability of implicit and explicit methods must be noted at the time of choosing time step to keep the method stable. Otherwise, the method would be unable to capture different modes of problem.

## Chapter 7: Conclusion

Based on the problems of study in this thesis, Newmark's method with Newton-Raphson iteration is recommended as a general method that is able to solve structures with linear or highly non-linear behavior. The central difference method is useful to solve linear and non-linear problems but Newmark's method is more powerful to capture details of non-linear structures. Also, the accuracy of Newmark's method with Newton-Raphson iteration can be specified by choosing tolerance for convergence, and that is the advantage of this method which can change this method to capture high order of accuracy in results.

Also, it must be concluded that conditions like slackening should be considered. Consequently, those conditions affect the tangent stiffness matrix, and those conditions do not affect the procedure of direct time integration methods. In fact, those are two separated concepts to solve a structure.

More, it is recommended to pre solve a system, first with central difference method then solve it by Newmark's method with Newton-Raphson iterations. Then, compare the results. If the system was highly non-linear, the results of Newmark's method would deviate much from results of central difference method, and results of Newmark's method can be more accurate. If the system was linear or the order of non-linearity was not so high, the results of both methods would not deviate much from each other. Then, depending on properties of expected results, level of accuracy, and time of calculation, suitable method can be applied to solve a problem. However, if solution of the problem contains low frequencies, Newmark's method is more preferred since larger time steps may be utilized to save the calculation time.

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