A Simulation Approach to High-Frequency Plasma Waves

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Licentiate Thesis

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Abstract

Electrostatic waves in the form of Broadband Electrostatic Noise (BEN) have been observed in the Earth’s auroral region associated with high geomagnetic activity. This broad frequency spectrum consists of three electrostatic modes, namely electron plasma, electron acoustic and beam-driven modes. These modes are excited in a plasma containing three electron components: hot, cool and beam electrons.

A 1D Particle-in-Cell (PIC) simulation was developed to investigate the characteristics of the electrostatic waves found in such a plasma. Dispersion, phase space and spatial electric field diagrams were constructed from the output of the PIC simulation which were used to describe the wave dispersion and spatial field structures found in a plasma. The PIC code used a three electron component plasma with Maxwellian distributions to describe the electron velocity distributions. Beam-driven waves were found to dominate the frequency spectrum while electron plasma and electron acoustic waves are damped for a high beam velocity. Furthermore, for a high beam velocity, solitary waves are generated by electron holes (positive potentials), giving rise to a bipolar spatial electric field structure moving in the direction of the beam. Increasing the beam temperature allows the beam electrons to mix more freely with the hot and cool electrons, which leads to electron plasma and electron acoustic waves being enhanced while beam-driven waves are damped. Decreasing the beam density and velocity leads to damping of beam-driven waves, while electron plasma and electron acoustic waves are enhanced.

Measurements in Saturn’s magnetosphere have shown the co-existence of two electron (hot and cool) components. The electron velocities are best described by a $\kappa$-distribution (instead of a Maxwellian) which has a high-energy tail. Using an adapted PIC simulation, the study of electron plasma and electron acoustic waves was extended by using a $\kappa$-distribution to describe the electron velocities with low $\kappa$ indices. Electron acoustic waves are damped over most wave number ranges. Electron plasma waves are weakly damped at low wave numbers and damped for all other wave numbers.
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List of Papers

This thesis includes the following papers authored by Etienne J. Koen, listed in chronological order:


These papers are referred to below as Paper I, Paper II and Paper III respectively.
1 Introduction

The Sun emits a highly conductive plasma into the interplanetary space at supersonic speeds of about 500 km/s as a result of the supersonic expansion of the solar corona. This plasma is called the solar wind and consists mainly of electrons and protons. Since the plasma is highly conductive, the solar magnetic field is frozen into the plasma and drawn outward by the expanding solar wind. When the solar wind hits the Earth’s dipolar magnetic field, it is slowed down and deflected around it. This region where the solar wind is slowed down from supersonic to subsonic generates a bow shock wave. The region of thermalised subsonic plasma behind the bow shock is called the magnetosheath [Kivelson and Russell, 1995].

The boundary separating the interplanetary magnetic field from the terrestrial magnetic field is called the magnetopause and the cavity produced by the terrestrial field is referred to as the magnetosphere. The plasma within the magnetosphere consists mainly of electrons and protons. The sources of these particles are the solar wind and the terrestrial ionosphere. The near-Earth space environment is illustrated in Figure 1.1.

Figure 1.1: The solar-terrestrial environment [Kaler, 1992].

1.1 Magnetosphere and Ionosphere

A fraction of the Earth’s neutral atmosphere is ionised by solar ultraviolet radiation. At altitudes above 80 km collisions are too infrequent to result in rapid recombination and a permanent ionised population is formed called the ionosphere. The ionosphere extends to rather high altitudes (∼ 600 km) and, at low- and mid-latitudes, gradually merges into the plasmasphere. The plasmasphere contains a cool but dense plasma of ionospheric origin, which
co-rotates with the Earth. In the equatorial plane, the plasmasphere extends out to about $4R_E$, where the density drops down sharply to about $1 \text{cm}^{-3}$. This boundary is called the plasmapause.

In the inner part of the magnetosphere particles are subjected to a number of forces giving rise to trajectories composed of three components (Figure 1.2):

- gyration, caused by the velocity components in the plane which is perpendicular to the magnetic field,
- magnetic-mirror trapping, caused by the variation of the magnetic field strength along a field line,
- drift motion due to the gradient and curvature of the magnetic field.

![Figure 1.2: Trajectories of charged particles in the inner magnetosphere [Spjeldvik and Rothwell, 1985].](image)

### 1.2 Concepts from Plasma Physics

A plasma is a *quasineutral* gas of charged and neutral particles which exhibits *collective behavior*. The term *quasineutral* implies that within the same volume element there are roughly the same number of positive and negative charges whereas *collective behavior* means that motions not only depend on local conditions but on the state of the plasma in remote regions as well [Chen, 1984; Baumjohann and Treumann, 1997].

To let the plasma appear electrically neutral, the electric Coulomb potential field of an isolated charge

$$\phi_C(r) = \frac{q}{4\pi\epsilon_0 r}$$

with $\epsilon_0$ being the free space permittivity, is shielded by other charges in the plasma and assumes the Debye potential form

$$\phi_D(r) = \frac{q}{4\pi\epsilon_0 r} \exp\left(-\frac{r}{\lambda_D}\right).$$

$\lambda_D$,
The characteristic length scale, $\lambda_D$, is called the Debye Length and is the distance over which a balance is obtained between the thermal particle energy, which tends to perturb charge neutrality, and the electrostatic potential energy resulting from charge separation, which tends to restore charge neutrality.

The Debye length for a species $\alpha$ is a function of the temperature, $T_\alpha$, and the density, $n_\alpha$

$$\lambda_{D\alpha} = \left(\frac{\varepsilon_0 k_B T_\alpha}{n_\alpha e^2}\right)^{1/2} \quad (1.3)$$

where $k_B$ is the Boltzmann constant and $e$ the electron charge.

The criterion for a plasma to be quasineutral requires $\lambda_D$ to be small compared to the physical dimension of the system, $L$,

$$\lambda_D \ll L \quad (1.4)$$
in order for collective shielding to occur.

If the quasineutrality of the plasma is disturbed by some external force then the lighter species, being more mobile, are accelerated in an attempt to restore charge neutrality. Due to their inertia they will move back and forth around the equilibrium position, resulting in fast collective oscillations around the more massive species. The plasma frequency $\omega_{p\alpha}$ is dependent on the mass $m_\alpha$ and density $n_\alpha$ and can be written as

$$\omega_{p\alpha} = \left(\frac{n_\alpha e^2}{m_\alpha \varepsilon_0}\right)^{1/2}. \quad (1.5)$$

The thermal velocity of species $\alpha$ is defined as

$$v_{th\alpha} = \left(\frac{2k_B T_\alpha}{m_\alpha}\right)^{1/2} \quad (1.6)$$

which is related to the plasma frequency and Debye length

$$\omega_{p\alpha} = \frac{v_{th\alpha}}{\lambda_{D\alpha}} \quad (1.7)$$

When the particle is in the presence of a background magnetic field, the velocity component perpendicular to the magnetic field of the particle will cause the particle to follow a cyclotron motion along the magnetic field. This cyclotron frequency, $\omega_c$, of the particle is dependent on the back magnetic field and the mass of the particle

$$\omega_c = \frac{|q|B}{m}. \quad (1.8)$$

The radius of the orbit is the Larmor radius

$$r_L \equiv \frac{v_\perp}{\omega_c} = \frac{mv_\perp}{|q|B} \quad (1.9)$$

where $v_\perp$ is defined as the particle’s velocity perpendicular to the magnetic field.
1.3 The Maxwellian and $\kappa$-distributions

For a plasma in thermal equilibrium the most probable distribution to describe the velocities of the particles is a Maxwellian given by

$$f(v) \propto \exp \left( -\frac{v^2 - u_s^2}{v_{th}^2} \right)$$

with $u_s$ as the most probable speed. The width of the distribution is determined by the thermal velocity, $v_{th}$.

The velocity spectrum for a particle distribution at low energies is generally well approximated by the Maxwellian distribution. However, the tail of the distribution (high velocities) in some instances may follow a power-law decrease rather than the exponential decrease of the Maxwellian distribution. The $\kappa$ distribution was formulated to represent this situation [Summers and Thorne, 1991]

$$f(v) = \frac{\Gamma(\kappa + 1)}{(\pi \kappa \theta^2)^{3/2}} \frac{1 + \frac{v^2}{\kappa \theta^2}}{\Gamma(\kappa - 1/2)}^{-(\kappa+1)}$$

where $\theta = [(2\kappa - 3)/\kappa]^{1/2}(k_B T/m)$ is the most probable speed, with $T$ being the temperature of the equivalent Maxwellian with the same average kinetic energy. As noted, $\theta$ is related to the thermal velocity $\theta = [(2\kappa - 3)/\kappa]^{1/2}v_{th}$ [Summers and Thorne, 1991; Hellberg et al., 2009]. The distribution is only defined for $\kappa > 3/2$. The parameter $\kappa$ provides a measure of the slope of the energy spectrum of the suprathermal particles forming the tail of the distribution. The function $\Gamma(z)$ is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$  

At high energies the $\kappa$-distribution falls off more slowly than does a Maxwellian, like a power law in kinetic energy with a spectral index, $\kappa$. In the limit $\kappa \to \infty$, it becomes a Maxwellian with temperature $\theta = v_{th}$ (Figure 1.3).

![Figure 1.3: Probability distribution for the $\kappa$-distribution with varying $\kappa$.](image-url)
2 Plasma Waves

This chapter introduces some of the electrostatic waves found in a plasma. For electrostatic waves electric field oscillations are only longitudinal as opposed to electromagnetic waves where transverse oscillating electric and magnetic fields are involved.

2.1 Electron Plasma Waves

For a plasma composed of a hot electron component and a single species of positive ions with no background magnetic field, the electron plasma (Langmuir) mode will be excited. The thermal velocity $v_{th}$ of the electrons need to be taken into account. The dispersion relation is known as the Bohm-Gross dispersion relation [Chen, 1984]

\[ \omega^2 = \omega_{pe}^2 + \frac{\gamma T_e}{m_e} k^2 = \omega_{pe}^2 + \gamma k^2 v_{th}^2 \]  

(2.1)

where $\gamma = (n + 2)/2$ is the polytropic index for $n$ degrees of freedom. In one dimension, $\gamma = 3/2$ (1 degree of freedom) and (2.1) becomes

\[ \omega^2 = \omega_{pe}^2 + \frac{3}{2} k^2 v_{th}^2. \]  

(2.2)

At large wavelengths (2.2) will be weakly modified by the pressure term ($\frac{3}{2} k^2 v_{th}^2$). At wavelengths small compared to the Debye length (large $k$), the pressure term dominates and the waves become dispersionless. For such a plasma, the phase velocity of the waves at large $k$ is given by

\[ v_{ph} = \frac{\omega}{k} = \sqrt{3/2} v_{th}. \]  

(2.3)

Electrons that are moving at nearly the same speed as the phase velocity of the waves will result in a process of energy exchange between electrons and the wave. This leads to Landau damping.

2.2 Ion Acoustic Waves

In the absence of a magnetic field where the motions of the massive ions are considered, the low frequency ion acoustic mode is excited with dispersion relation

\[ \omega^2 = k^2 \left( \frac{k_B T_e}{M} \frac{1}{1 + k^2 \lambda_D^2} + \frac{\gamma_i k_B T_i}{M} \right). \]  

(2.4)

At short wavelengths ($k^2 \lambda_D^2 \gg 1$) and in the limit $T_i \to 0$, (2.4) becomes

\[ \omega^2 = k^2 \frac{n_0 e^2}{\epsilon_0 M k^2} = \frac{n_0 e^2}{\epsilon_0 M} \equiv \omega_{pi}^2 \]  

(2.5)

where $\omega_{pi}$ is the ion plasma frequency. Figure 2.1 shows the theoretical dispersion curve for (2.4) and (2.5).
2.3 Electrostatic Waves in a Beam-Driven Plasma

Consider a three-electron component plasma containing cool, hot and beam electrons, with the total density $n_{e0} = n_c + n_h + n_b$. For wave propagation in the direction of the magnetic field, the plasma can be considered to be unmagnetised yielding for the dispersion relation

$$1 + \frac{2}{k^2 \lambda_{Dc}^2} [1 + \zeta_c Z(\zeta_c)] + \frac{2}{k^2 \lambda_{Dh}^2} [1 + \zeta_h Z(\zeta_h)] + \frac{2}{k^2 \lambda_{Db}^2} [1 + \zeta_b Z(\zeta_b)] = 0,$$

(2.6)

where

$$\zeta_c = \frac{\omega}{kv_{Te}}, \quad \zeta_h = \frac{\omega}{kv_{Th}}, \quad \zeta_b = \frac{\omega - ku_d}{kv_{Tb}}$$

and

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp(-x^2)}{(x-\zeta)} dx.$$  

(2.7)

Such a plasma model supports unstable waves which are responsible for Broadband Electrostatic Noise (BEN) covering a broad frequency range which is observed in the Earth’s auroral region. The theoretical dispersion relations for the three modes associated with BEN are seen in Figure 2.2. Details on the different modes will be given in the following sections.

2.3.1 Electron Acoustic Waves

If the effect of beam electrons is neglected ($n_b \ll n_{e0}$) and $\zeta_h < 1$, $\zeta_b < 1$ and $\zeta_c > 1$, (2.6) reduces to an expression for electron acoustic waves [Gary and Tokar, 1985]:

$$\omega^2 = \omega_p^2 \frac{1 + 3 k^2 \lambda_{Dc}^2}{1 + 1/(k^2 \lambda_{Dh}^2)}.$$

(2.8)


Figure 2.2: Theoretical dispersion relations for three electrostatic modes. EPW, EAW and BE represent the electron plasma, electron acoustic and beam-driven modes, respectively.

This mode was found to be unstable when the hot to cool electron temperature was greater than $\sim 10$ and the hot electron component constituted a non-negligible fraction of the total electron density ($n_c/n_{e0} > 0.2$) [Tokar and Gary, 1984; Gary and Tokar, 1985].

2.3.2 Electron Plasma Waves

In the case where $\zeta_h > 1$, (2.6) may be approximated as

$$\omega^2 = \omega^2_{pe} (1 + 3k^2 \lambda^2_{Dc} + \omega^2_{ph} (1 + 3k^2 \lambda_{Dh}))$$

(2.9)

which represents the electron plasma mode for a three electron component plasma.

2.3.3 Beam-Driven Waves

When the streaming speed of the beam electrons is sufficiently large, the dispersion relation of the wave mode can be approximated as

$$1 - \frac{\omega^2_{pe}}{\omega^2} - \frac{\omega^2_{ph}}{\omega^2} - \frac{\omega^2_{pb}}{(\omega - ku_d)^2} = 1.$$  

(2.10)

One solution of this dispersion relation is the beam-driven wave mode. When the beam density is significantly smaller than the total density ($n_b \ll n_0$) (2.10) can be expressed as [Baumjohann and Treumann, 1997]:

$$\omega = \frac{ku_d}{1 + n_b/n_0} \approx ku_d.$$  

(2.11)
2.4 Observations of Broadband Electrostatic Noise

Free energy sources such as electron beams have been observed in the polar cap and auroral zone [Cairns and Menietti, 1997], together with the coexistence of warm and cool plasma populations [Winningham et al., 1975]. For such conditions the existence of electron plasma [Matsukiyo et al., 2004] and electron acoustic [Sotnikov et al., 1995] waves were found with electron plasma waves being more important at small wave numbers. The electrostatic waves excited in such a system give rise to the term BEN observed in the auroral region. Some of the earliest BEN observations are shown in Figures 2.3 and 2.5. Along with electron plasma and electron acoustic waves, electromagnetic signatures of fast solitary waves were also observed. These solitary waves are associated with a positive charge or electron hole (phase space hole) moving past the spacecraft at a significant fraction of the speed of the energetic electrons. Figure 2.4 shows the observed electric field signature associated with a solitary wave. The opposite sign of the electric field versus time indicates that the parallel electric field has a bipolar structure. The electron acceleration region is produced by some external source such as a current or voltage that is driven from the magnetosphere. In this region positive clouds (electron holes) develop by drawing energy from the accelerating, up-going electrons. This modifies the velocity distribution to a plateau-like distribution and accelerates electrons earthward. The observed properties of fast solitary waves with regards to the auroral region can be summarized as follows:

- They are observed in the downward current region associated with energetic, up-going electron beams.
- Strong electron modulations are associated with these structures.
- Ion pitch angles increase toward 90° during intense emissions.

Figure 2.3: Frequency-time spectrogram of the electric field intensities obtained from SFA measurements showing impulsive emissions of BEN. The electron cyclotron frequency is indicated by the solid line [Dubouloz et al., 1991].
• Solitary waves are often observed in large-scale regions of depleted density.

• The parallel electric field is bipolar and almost always in the electron beam direction and the perpendicular signature is unipolar.

• Nonlinear waves such as electron acoustic solitons are responsible for BEN above $\omega_{pe}$ [Bharuthram, 1991; Schriver and Ashour-Abdalla, 1987] whereas linear electron acoustic waves account for frequencies below $\omega_{pe}$ [Dubouloz et al., 1991].

Figure 2.4: A fast solitary wave at 0.5 $\mu$s resolution. The right plot is $E_\parallel$ versus $E_\perp$ of the first structure. The speed of this structure was $\sim$2000 km/s anti-earthward [Ergun et al., 1998].
Figure 2.5: (a) Langmuir probe current reflecting the plasma density. A 2 nA current roughly corresponds to 1 cm$^{-3}$ density. (b) Perpendicular electric field in the spin plane and nearly along the spacecraft velocity. (c-d) Parallel and perpendicular electric field power spectral density. (e-f) Electron energy flux versus energy and time versus pitch angle and time. (g-h) Ion energy [Ergun et al., 1998].
3 Particle-in-Cell Simulations

The following section is mainly a review of the work from Birdsall and Langdon [2004] and Hockney and Eastwood [1988]. It will introduce and discuss the basic theory and methods behind PIC simulations. The PIC method refers to a technique used to solve a certain class of partial differential equations. All particles are tracked in continuous phase space and quantities such as densities and currents are computed simultaneously on a grid. PIC has been successfully used in the field of plasma physics to study laser-plasma interactions [Klimo et al., 2010], electron acceleration and ion heating in the auroral ionosphere, magnetohydrodynamics and magnetic reconnection [Semenov et al., 2009].

3.1 Superparticles

The real systems studied are often extremely large in terms of the number of particles they contain. In order to make simulations efficient superparticles are used. A superparticle (or macroparticle) is a computational particle that represents many real particles. The number of real particles corresponding to a superparticle must be chosen such that sufficient statistics can be collected on the particle motion.

3.2 Coulomb’s Law

The electrostatic force on each particle, $i$, can be obtained via Coulomb’s law:

$$ F_i = q_i \sum_{j=1,\neq i}^{N_p} \frac{q_j}{|x_i - x_j|^3}. $$

(3.1)

This is the simplest to code and is the most accurate method. The algorithm scales as $N_p^2$ where $N_p$ is the number of superparticles being simulated and therefore computationally intensive. One also has to deal with the force resulting from very small separations between particles that will accelerate particles to velocities beyond what the boundary conditions of the system can handle. In such a case a force softener may be implemented.

3.3 Layout of Simulation Program

A typical simulation will be structured as follows:

1. All variables are initialised and particles are distributed in phase space according to a distribution function.
2. Forces are calculated on each particle (electrostatic and external forces).
3. A second order differential equation is integrated to update the velocity of each particle based on the forces.
4. The position of each particle is updated.
5. Return to step 2.

### 3.4 Leap-Frog Scheme

A fast and accurate time integration scheme is required for steps 3 and 4 during the particle simulations. Calling an integrator many times during a time step can be very time consuming without winning too much with regards to accuracy. The leap-frog is the most popular integration scheme used in particle simulations. The name comes from the way the algorithm is written where the positions and velocities “leap” over each other (see Figure 3.1). Positions are defined at times $t_i, t_{i+1}, t_{i+2}, \ldots$ in integer time intervals of $dt$, while the velocities are defined at half time intervals $t_{i - \frac{1}{2}}, t_{i+\frac{1}{2}}, t_{i+\frac{3}{2}}$. The integration scheme then reads:

\[
x_i = x_{i-1} + v_{i-\frac{1}{2}} \cdot dt \tag{3.2}
\]
\[
v_{i+\frac{1}{2}} = v_{i-\frac{1}{2}} + a \cdot dt \tag{3.3}
\]

The leap-frog holds for values of the time step such that $\omega_p dt < 2$ but generally it is recommended that $\omega_p dt \leq 0.25$ for integration to be accurate and stable.

### 3.5 Quantizing a System to Solve Field Equations

Consider the differential equations to be solved:

\[
E = -\nabla \phi \tag{3.4}
\]
\[
\nabla \cdot E = \frac{\rho}{\epsilon_0} \tag{3.5}
\]

which are combined to obtain Poisson’s equation

\[
\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \tag{3.6}
\]
Using a grid where grid points are equally spaced the equations may be written in discreet form

\[ E_j = \frac{\phi_{j-1} - \phi_{j+1}}{2\Delta x} \]  
(3.7)

\[ \frac{\phi_{j-1} - 2\phi_j - \phi_{j+1}}{(\Delta x)^2} = \frac{\rho_j}{\epsilon_0} \]  
(3.8)

where \( \rho_j \) is known, to solve for the unknown \( \phi_j \) and then \( E_j \) running from 0 to \( L/\Delta x \) where \( L \) is the length of the system of \( N_G \) points. The cell width \( \Delta x = L/N_G \) determines the spatial resolution of the system and needs to be smaller than the smallest Debye length.

### 3.6 Particle and Force Weighting

To obtain a grid loaded with charge to represent \( \rho_j \) in order to solve (3.8) the charge of each particle needs to be deposited onto the grid. The simplest weighting method is zero-order weighting where the number of particles are counted within distance \( \pm \Delta x/2 \) (one cell width) about the \( j^{th} \) grid point and assign that number to that point, \( N(j) \) (Figure 3.2). In one dimension it is simply \( n_j = N(j)/\Delta x \) and the common name for this weighting is called nearest grid point (NGP) assignment. The effect here is that the superparticle appears to have a rectangular shape with a width of \( \Delta x \).

![Figure 3.2: Illustration of Nearest Grid Point charge assignment.](image)

The superparticle can be thought of as having a collection of finite particles. This is very acceptable for the physics in a plasma where close encounters between particles are rare and the collective behaviour at large distances plays a significant role. When a superparticle passes through a cell boundary it will produce a density and an electric field which are relatively noisy both in space and time. A first-order weighting is considered where two grid points are accessed per particle each time step. Consider a particle having charge \( q_c \). The NGP technique is applied to the particle’s position to locate the left-most grid point of the charge \( X_j \). The portion of charge assigned to grid point \( j \) is

\[ q_j = q \left[ \frac{\Delta x - (x_i - X_j)}{\Delta x} \right] = q \frac{X_{j+1} - x_i}{\Delta x} \]  
(3.9)
and the portion to grid point \( j + 1 \) is

\[
q_{j+1} = q \left[ \frac{x_i - X_j}{\Delta x} \right]
\]  

(3.10)

The net effect is to produce a triangular shape which has width \( 2\Delta x \) (Figure 3.3). As the superparticle moves through the grid, it will contribute to the grid much more smoothly than the zero-order weighting and will thus contribute much less noise. This is the preferred weighting function and is used throughout all the simulations. Higher-order weighting by use of a quadratic or gaussian function can also be used to further reduce the noise in the system at the cost of increased computation.

![Figure 3.3: Illustration of first-order weighting charge assignment.](image)

### 3.7 Analytical Solution to Poisson’s Equation

Consider a system which is periodic over the range of grid points \([1, N_G]\). Dropping the constants from (3.8) the set of equations to be solved is

\[
\begin{align*}
\phi_{N_G} - 2\phi_1 + \phi_2 & = \rho_1 \\
\phi_1 - 2\phi_2 + \phi_3 & = \rho_2 \\
& \vdots \\
\phi_{N_G-2} - 2\phi_{N_G-1} + \phi_{N_G} & = \rho_{N_G-1} \\
\phi_{N_G-1} - 2\phi_{N_G} + \phi_1 & = \rho_{N_G}
\end{align*}
\]  

(3.11)

In this set of equations each potential \( \phi \), appears in three separate equations, once with a multiplier \(-2\) and twice with a multiplier of \(+1\). Also for charge neutrality, the sum of charge density should give zero

\[
\sum_{i=1}^{N_G} \rho_i = 0.
\]  

(3.12)

The end potentials are set to \( \phi_1 = \phi_{N_G} = 0 \). Multiplying the first equation of (3.12) by one, the second by two, the third by three, etc., and add up all the resulting equations, the equation
for $\phi_1$ is then:

$$N_G \phi_1 = \sum_{i=1}^{N_G} i \cdot \rho_i$$  \hspace{1cm} (3.13)

The result is that the first equation in (3.12) will give $\phi_2$, the second $\phi_3$, and so forth. The algorithm in pseudo code for computing the potential at all the grid points will be:

1. Potential at grid point 1:
   $$\phi_1 = 0$$
   for $i = 1$ to $N_G$ do
   $$\phi_i = \phi_1 + i \cdot \rho_i$$
   $$\phi_1 = \phi_1 / N_G$$

2. Potential at grid point 2:
   $$\phi_2 = \rho + 2 \cdot \phi_1$$

3. Potentials at remaining grid point:
   for $i = 3$ to $N_G$ do
   $$\phi_i = \rho_{i-1} + 2 \cdot \phi_{i-1} - \phi_{i-2}$$

### 3.8 Solving Poisson’s Equation in Wave Vector Space

The ability to transform from the spatial domain to wave vector space through the use of the Fast Fourier Transform enables the computation time of Poisson’s equation to be reduced significantly. By transforming $\rho(x)$ and $\phi(x)$ into Fourier space, $\rho(k)$ and $\phi(k)$ are obtained, where $k$ is the wave number in the Fourier transform kernel, $\exp(-ik \cdot x)$. This assumption allows $\rho(k)$ and $\phi(k)$ to be obtained directly from Poisson’s equation in one dimension by replacing $\frac{\partial^2}{\partial x^2}$ by $-k^2$

$$\phi(k) = \frac{\rho(k)}{\varepsilon_0 k^2}.$$  \hspace{1cm} (3.14)

To obtain $E(x)$ the inverse Fourier transform of $\phi(k)$ is taken and (3.5) is applied to the result.

The Fourier transform also enables one to construct a dispersion diagram. This is done by transforming $E(x)$ into $E(k)$ at each time interval. Once a series of $E(k)$ at each time interval is obtained, the Fourier transform is applied in the time domain of $E(x)$ to obtain $E(k)$ which is in the frequency domain. Thus, the classical $\omega$–$k$ diagram can be constructed and used to study wave dispersion and growth rate properties.

### 3.9 Sampling Randomly from a Distribution Function

Consider a probability density function $P(x)$ where $P(x)dx$ represents the probability of finding a random variable $x$ in the interval $x$ to $x + dx$. Since a variable $x$ is certain to lie between the range $-\infty$ to $+\infty$, probability densities are normalised as

$$\int_{-\infty}^{+\infty} P(x)dx = 1.$$  \hspace{1cm} (3.15)
Consider a random variable $x$ to be constructed which is uniformly distributed in the range $x_1$ to $x_2$ so that the probability density of $x$ will be given as

$$P(x) = \begin{cases} 
\frac{1}{(x_2 - x_1)} & \text{if } x_1 \leq x \leq x_2 \\
0 & \text{otherwise}
\end{cases}$$

(3.16)

The method used in this study to sample a random variable from a distribution (Maxwellian and Kappa) is called the rejection method. Suppose a random variable $y$ distributed with density $P_y(y)$ in the range $y_{\text{min}}$ and $y_{\text{max}}$ is desired. Let $\max(P_y)$ be the maximum values of $P_y(y)$ in this range. The rejection method works as follows: The variable $y$ is sampled randomly in the range $y_{\text{min}}$ and $y_{\text{max}}$. $P_y(y)$ is evaluated for each value of $y$. Next a random number $x$ which uniformly distributed in the range 0 to $\max(P_y)$ is generated. Finally, if $P_y(y) < x$ then $x$ is rejected; otherwise accepted. If this prescription is followed then $y$ will be distributed according to $P_y(y)$. A graphical illustration of the rejection method is given in Figure 3.4. The rejection method is easy to implement in a simulation code but not very efficient and therefore time consuming. Since the sampling routine is only called once (during the initial part of the simulation) it is sufficient.

Figure 3.4: The rejection method. Shaded and dotted areas represent rejected and accepted states respectively.

3.10 Elementary Validation of Simulation Code

Two-Particle Test

It is of utmost importance that all parameters within the simulation are normalised to known constants to ensure that the output is sensible and easy to interpret. For example, all time variables are normalised to the reciprocal of the plasma frequency. The two-particle test provides a simple method by which the time scales of oscillations can be validated. Consider
two particles which are initially motionless and are some distance apart other than $L/2$ in an one-dimensional system, where $L$ is the physical length of the system. The particles will start to move towards or away from each other due to the electric potential. The period for which a certain pair of velocity and position for a particle is repeated corresponds to one plasma period $\tau_p = 2\pi/\omega_p$ as shown in Figure 3.5. Figure 3.5 also illustrates the effect of a system with periodic boundary conditions. A particle’s velocity may cause it to overshoot a boundary and enter the system on the opposite side.

![Figure 3.5: Two-particle test with time normalised to $1/\omega_p$. The black and red curves represent the trajectories of different particles.](image)

**Energy Conservation**

For a physical system conservation of energy, momentum and angular momentum is required. Energy calculation is a major component of the simulation validation. Any errors in calculating grid quantities or the lack of accuracy caused by an integration scheme will reflect in the energy diagrams. The total energy in a conservative system is given by the sum of the kinetic and potential energies. The kinetic energy $W$ for a system is given by the sum over all particle kinetic energies

$$W = \sum_{i=1}^{N_p} \frac{1}{2} m v_i^2 \quad (3.17)$$

and the potential energy $U$ is given by the sum of the pair potential energies $\phi_{ij} = \phi(x_i; x_j)$

$$U = \frac{1}{2} \sum_{i=1}^{N_p} \sum_{j=1}^{N_p} \phi_{ij}, \text{ for } i \neq j. \quad (3.18)$$
The potential energy can also be calculated by applying the Fourier transform to (3.18) and taking the summation over all wave number components

\[ U = \frac{1}{2} \sum_{l=0}^{N_{\phi}-1} \left( \frac{k_0}{2\pi} \right) \rho^*(k)\phi(k) \]  

(3.19)

where \( \rho^*(k) = \rho(-k) \) is the complex conjugate of the charge density and \( \phi(k) \) is the potential field. For a periodic system of period length \( L \), the “cell” width is given by \( k_0 = 2\pi/L \). This method reduces the computation time in the calculation of the potential energy, \( U \).

### 3.11 Exciting Modes

A PIC code can excite a certain mode by introducing a perturbation in either the spatial or velocity distribution of the particles in the system. For the spatial perturbation, the particles are initially distributed uniformly (all having zero velocity) and thereafter the specific sinusoidal function \( f_l \) is applied containing information about the desired mode to be excited

\[ f_l = \sin(l \frac{2\pi}{L}) \]  

(3.20)

where \( l \) is the number of modes to be excited. This spatial density perturbation in the system will cause a mode in the spatial electric field to be excited which will oscillate with the plasma frequency. The spatial structure of the electric field will have a symmetrical structure which will oscillate with time. Figure 3.6 shows the electric field structure with time for the \( l = 1 \) mode excited. An averaged portion of the electric field is shown in Figure 3.7. For a system where all particles are initially moving uniformly and particles are evenly spaced, a perturbation in the system (in either space or velocity) can be seen to propagate in the system with time in Figure 3.8 where a portion of the electric field takes the form of a wave packet as shown in Figure 3.9.
Figure 3.6: Contour plot for the spatial electric field versus time. All quantities are dimensionless.

Figure 3.7: Averaged electric field for $x = 45–55$. All quantities are dimensionless.
Figure 3.8: Contour plot for the spatial electric field versus time for a system initially in motion with a spatial perturbation. All quantities are dimensionless.

Figure 3.9: Averaged electric field for $x = 45–55$ of a system initially in motion. All quantities are dimensionless.
4 Summary

4.1 Paper I: Particle-in-Cell simulations of beam-driven electrostatic waves in a plasma

In a plasma containing three electron components, where the velocity distribution of the electrons are described by a Maxwellian distribution (Figure 4.1), three electrostatic modes are excited, namely electron plasma, electron acoustic, and beam-driven waves. These modes have a broad frequency spectrum and have been associated with intense BEN observed in the Earth’s auroral zone [Schriver and Ashour-Abdalla, 1987] and other regions in the magnetosphere [Ashour-Abdalla and Okuda, 1986]. Using a 1D electrostatic PIC simulation with periodic boundary conditions, the characteristics of these electrostatic waves were investigated. A parameter survey was performed by varying the beam parameters such as the streaming velocity, density and temperature. Ions were assumed motionless and therefore kept at fixed positions throughout the simulation. The simulation results compared well with the approximate expressions for the frequencies of the different wave modes.

Figure 4.1: Probability function for the three electron components with the black, red and blue curves representing the cool, hot and beam components respectively.

Since the electron beam has a large streaming velocity the dispersion diagram will be dominated by large amplitude beam-driven waves along with weak signatures of the electron plasma
and electron acoustic modes (Figure 4.2). The main results of Paper I can be summarised as follow:

- As the streaming velocity was decreased and approached the thermal velocity of the hot electrons, the electron plasma and acoustic modes became weakly damped. At this value, comparable to the hot electrons thermal velocity, the beam driven mode and electron acoustic mode were coupled, and below this value the source of free energy disappears. For these values the electron plasma and acoustic modes were damped.

- Increasing the temperature of the beam allowed the beam electrons to mix more freely with the cool and hot electron populations. This caused the beam-driven waves to be damped while the electron plasma and acoustic modes were weakly damped.

- Increasing the beam density caused the electron acoustic mode to be strongly damped while the beam-driven mode was weakly damped especially at low frequencies. When the beam density was decreased, the amplitudes for all the modes became comparable.

Figure 4.2: The $\omega-k$ diagrams at 1200–1302.4$\omega_{pe}t$. EPW, EA, and BE curves indicate the theoretical dispersion relations for the electron plasma, electron acoustic, and beam-driven wave modes, respectively.

A contour representation of the development of the electric field can be seen in Figure 4.3. The solitary waves are represented by the diagonal “ridges” in the diagram. When either the streaming velocity or the beam density is decreased or when the beam temperature is increased, the beam-driven mode is weakly damped while the electron plasma and acoustic modes are more prominent.
Not included in Paper I is the variation of the hot/cool density \((n_h/n_c)\) and temperature \((T_h/T_c)\) ratios. Table 4.1 shows a parameter survey for variation of these parameters. It can be seen from Figure 4.4a that the electron acoustic mode is damped for large values of \(n_h/n_c\). For very small values of \(n_h/n_c\) the electron acoustic mode will translate to a cold electron plasma wave at large wave numbers but will be damped at lower wave numbers (Figure 4.4b). Decreasing the electron temperature ratio \(T_h/T_c\) leads to damping of the electron acoustic mode (Figure 4.4a) and coupling between the electron plasma and beam-driven modes where these modes intersect in the dispersion diagram (Figure 4.4b).

One can therefore conclude that for favourable conditions for the existence of the electron acoustic mode, \(n_h/n_c\) should be close to unity and \(T_h/T_c\) to be large as it was throughout Paper I.

An averaged portion of the electric field, \(x = 45–55L\), is shown in Figure 4.5. A frequency spectrogram is constructed from Figure 4.5 by Fourier transforming the averaged portion with a time window of \(\omega_{pe}t = 40\). The window is advanced to show the frequency spectrum versus time as shown in Figure 4.6. Initially strong signatures of beam-driven waves are seen below the plasma frequency. After the beam is destroyed, at \(\sim 120\omega_{pe}t\), electron plasma waves become the dominant signature in the frequency spectrum. Fourier transforming the electric field over the entire time length of the simulation shows the most prominent frequencies of the simulation (Figure 4.7). The dominant frequency is the electron plasma frequency. The lower discreet frequencies are associated with beam-driven waves which can be attributed to the linear dispersion relation of beam-driven waves. The distorted frequency pattern between \(\sim 0.5–0.8\omega_{pe}t\) is that of electron acoustic waves.

Figure 4.3: Contour plot of the electric field at 0–200\(\omega_{pe}t\). The time is normalised by the plasma period, \(\tau_p\).
For Run 2 it has already been showed (Figure 4.4b) that for low values of $n_h/n_e$ the electron acoustic mode translates into the electron plasma mode and is the only mode observable at the electron plasma frequency. The frequency spectrogram confirms this result and shows the growth in amplitude of the mode with time (Figure 4.8). The Fourier components of the entire simulation show only one peak which is at the electron plasma frequency indicating that electron acoustic and beam-driven waves are damped for this set of parameters (Figure 4.8).

![Dispersion diagrams from parameter survey at 0–102.4\(\omega_{pe}t\).](image)

**Figure 4.4:** Dispersion diagrams from parameter survey at 0–102.4\(\omega_{pe}t\).

The dispersion diagrams obtained from the simulations compared well to theoretical dispersion curves. These results validated the simulation code and were therefore used in Paper II.
Figure 4.5: Average portion of the electric field for $x = 45-55L$ of the reference case.

Figure 4.6: Frequency spectrogram constructed for the reference case.

Figure 4.7: Frequency components of electric field for entire simulation.
Figure 4.8: Frequency spectrogram constructed for Run 2.

Figure 4.9: Frequency components of electric field for entire Run 2.
4.2 Paper II: A simulation approach of high-frequency electrostatic waves found in Saturn’s magnetosphere

Using a PIC simulation, the characteristics of electron plasma and electron acoustic waves were investigated in a plasma containing an ion and two electron components. The electron velocities were modelled by two $\kappa$ distributions. The model applies to the extended plasma sheet region in Saturn’s magnetosphere where the cool and hot electron velocities were found to have low indices, $\kappa_c \approx 2$ and $\kappa_h \approx 4$. The same 1D electrostatic simulation code as in Paper I was used with periodic boundary conditions and fixed ions.

![Diagram of electron plasma and acoustic modes](image)

**Figure 4.10:** The $\omega$–$k$ diagrams at 100–202.4 $\omega_{pe} t$. EPW and EA curves denote the theoretical dispersion relations for the electron plasma and electron acoustic modes respectively.

The electron plasma and acoustic mode were generally found to be weakly excited for the plasma parameters used in all the simulations. The electron plasma mode is seen to be weakly damped around the electron plasma frequency for low wave number ranges while the electron acoustic mode is strongly damped for all wave number ranges although signatures of the electron acoustic mode can be seen along the theoretical dispersion diagram which suggests that this mode can be driven unstable if a source of free energy, such as an electron beam, is present (Figure 4.10).

With regards to Saturn’s magnetosphere (regions $R \sim 7–9 \, R_S$ and $R \sim 12–14 \, R_S$), two electron populations (cool and hot) were found, with the velocity distribution of these populations best fitted by $\kappa$ distributions [Schippers et al., 2008]. The results showed the electron acoustic mode to be strongly damped at all wave numbers while the electron plasma mode was weakly damped at the electron plasma frequency. The region $R \sim 7–9 \, R_S$ favours the
existence of electron acoustic as opposed to the region \( R_S \sim 12–14 R_S \) which is attributed to the large temperature ratio found in the region \( R \sim 7–9 R_S \).

### 4.3 Paper III: Mid-latitude Ionospheric Signature of a weak Solar Flare in Winter

Measurements of the amplitude and phase of Very Low Frequency (VLF) transmitter signals on subionospheric paths were used to determine ionospheric D-region electron density enhancements due to a C4.5 solar flare on 13 December 2007 [Koen and Collier, 2013]. These transmission paths are all in the northern hemisphere with transmitters GQD and GBZ located in Great Britain and the receivers located in Hungary and Romania as shown in Figure 4.11. The unperturbed daytime values for \( H' \) and \( \beta \) were taken from previous studies done by Thomson [1993]. The ionospheric parameters were found by modelling the amplitude and phase of VLF transmitter signals using the Long Wave Propagation Capability (LWPC) code which was developed by the U.S Navy. The model specified in LWPC has an ionosphere which is described by [Wait, 1962; Wait and Spies, 1964]

\[
N(z) = N_0 \exp(-0.15H')\exp(\beta - 0.15(z - H'))
\] (4.1)

where \( N_0 = 1.43 \times 10^{13} \text{m}^{-3} \). The profile is determined by the sharpness parameter, \( \beta \) in \( \text{km}^{-1} \), and the reflection height, \( H' \) in km.

Using LWPC \( H' \) was found to decrease by \( \sim 3.6 \text{km} \) and \( \beta \) to increase by \( \sim 0.04 \text{km}^{-1} \). During this time all the modal minima in the amplitude were found to move towards the transmitters.

The VLF perturbations were found to be delayed by \( \sim 1 \text{ minute} \) with respect to the 0.1–0.8 nm X-ray flux, with the phase nearly linearly proportional to the X-ray flux, in contrast to the logarithmic proportionality previously found for stronger flares [Thomson and Clilverd, 2001; McRae and Thomson, 2004].
Figure 4.12: VLF signal (dashed lines) comparison with GOES X-ray flux (solid lines). Left: 0.05–0.4 nm band. Right: 0.1–0.8 nm band. Figure taken from Koen and Collier [2013].
5 Discussion and Conclusions

Given the main results of Paper I and Paper II, one can draw the conclusion that a beam in a plasma is required to drive the electron acoustic mode unstable. For Paper I a beam-driven plasma was investigated with a large streaming velocity. In such a plasma the dispersion diagram is dominated by the beam-driven mode’s signature, being weakly damped. For such a high streaming velocity phase space holes will be generated which will lead to solitary waves.

A parameter survey of BEN was previously performed by Lu et al. [2005] using a computational kinetic model. However, due to the small number of particles used in their simulation, discrepancies arose between the theoretical dispersion curves [Gary and Tokar, 1985] and their simulation results. These discrepancies were significantly reduced by increasing the number of particles and using an equal number of superparticles as real particles in the simulation in Paper I.

However, as Paper II has shown for electrostatic waves found in Saturn’s magnetosphere, the electrostatic modes are damped over a large wavelength range with only the electron plasma mode observable at large wavelengths. The electron acoustic mode is a stable wave mode and will need some source of free energy to drive the wave mode unstable in order to be properly excited. The results from Paper II compared well to a theoretical study performed by Baluku et al. [2011] with regards to emulating the theoretical dispersion curve. However, even for favourable cool to hot electron parameter ratios a beam will be required to drive the mode unstable. Therefore, further studies of the electron acoustic mode using a $\kappa$ distribution in the presence of a beam are required to fully understand the conditions which lead to instability of the mode.

Paper III investigated the ionospheric signatures of a C4.5 solar flare over mid-latitude propagation paths. Using LWPC $H'$ was found to decrease by $\sim 3.6$ km and $\beta$ to increase by $\sim 0.04$ km$^{-1}$. These modifications in the waveguide parameters of the D-region lead to either an increase or decrease in the amplitude, and a phase advancement of the VLF signal which are consistent with previous studies [McRae and Thomson, 2004]. However, the VLF phase was found nearly proportional to the 0.1–0.8 nm X-ray flux band with a lag of $\sim 1$ minute. This is in contrast with previous studies where the proportionality was found logarithmic [Thomson and Clilverd, 2001; McRae and Thomson, 2004] for flares of much greater magnitude.
References


