# TEM wave propagation in a coaxial cable with a step discontinuity on the outer wall 

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#### Abstract

In this report, the propagation of TEM waves along a coaxial waveguide with a step discontinuity on its outer wall is investigated rigorously by applying direct Fourier transform and reducing the problem into the solution of a modified WienerHopf equation. The solution for the field terms are determined in terms of infinite number of unknown coefficients which satisfy an infinite set of linear algebraic equations. These equations are solved numerically and the effect of area ratio is presented graphically at the end of the analysis. Also, the same problem is analyzed by applying mode matching technique and the results of the two approaches are compared. It is observed numerically that the Wiener-Hopf technique provides a better convergence than the mode matching technique.


## 1 Introduction

Electromagnetic wave propagation in waveguides has been an interesting topic and subject to various engineering problems, such as microwave and transmission line measurement techniques, filters, connectors and matching devices, one of which is the measurement setup described in [1], where experiments utilizing electromagnetic pulses have been performed on a 80 km long High Voltage Direct Current (HVDC) power cable in a factory [2] and on the 250 km long Baltic HVDC power cable [3]. Low frequency electromagnetic modeling of this measurement setup is very interesting as there exist many scattering mechanisms such as different inner and outer radii of two connected coaxial

[^0]cables (step discontinuity on the outer and inner walls), different dielectric media, etc. Among these, scattering by step discontinuities in coaxial waveguides has been drawing interest since many decades. It was first studied by Whinnery in 1944 where he obtained an equivalent circuit by placing an admittance at the plane of discontinuity in the case of TM waves [4]. Then in 1998, Mongiardo et. al. analyzed the same problem with generalized network formulation by the use of Green's function [5]. Yu et. al. applied nonuniform FDTD technique to study cascaded circularly symmetric discontinuities on waveguides in 2001 [6]. Finally in 2006, Fallahi and Rashed-Mohassel considered dyadic Green's function approach using the principle of scattering superposition for the problem where there is a step discontinuity on the inner wall [7].

In this report, TEM wave propagation along a coaxial waveguide with a step discontinuity on the outer wall is analyzed rigorously, in order to understand the effect of the area expansion on the scattering phenomenon as part of the above mentioned measurement setup. The analysis is done by considering the same method as in [8] where direct Fourier transform is applied and the problem is reduced into the solution of a modified Wiener-Hopf equation. The solution for the field terms are determined in terms of infinite number of unknown coefficients which satisfy an infinite set of linear algebraic equations. These equations are solved numerically and the effect of area ratio on the reflection and transmission coefficients is presented graphically at the end of the analysis.

The same problem is then analyzed by applying the mode matching technique as described in [9]. This technique has been widely used in previous works involving step discontinuities at waveguides in general $[10,11,12]$ and at coaxial waveguides $[13,14]$ when the discontinuities exist both on inner and outer walls. Following a similar procedure as described in [12], the scattering coefficients are determined and they are compared to the ones obtained via the Wiener-Hopf technique. Besides, it is observed numerically that the Wiener-Hopf technique provides a better convergence than the mode matching technique. The flow of energy is also analyzed in this report.

A time dependence $\exp (-i \omega t)$ with $\omega$ being the angular frequency is assumed and supressed throughout the analysis.

## 2 Formulation of the problem

Consider a semi-infinite coaxial cylindrical waveguide whose inner and outer cylindrical walls are located at $S=\{\rho=a,-\infty<z<0\}$ and $S=\{\rho=b,-\infty<z<0\}$ is connected to another semi-infinite coaxial cylindrical waveguide whose inner and outer cylindrical walls are located at $S=\{\rho=a, 0<z<\infty\}$ and $S=\{\rho=d, 0<z<\infty\}$ (see Figure 1).

Let the incident TEM mode propagating in the positive $z$ direction be given by

$$
\begin{equation*}
H_{\phi}^{i}(\rho, z)=u_{i}(\rho, z)=\frac{\mathrm{e}^{\mathrm{i} k z}}{\rho} \tag{1}
\end{equation*}
$$

where $k$ is the propagation constant which is assumed to have a small positive imaginary part corresponding to a slightly lossy medium. The lossless case can then be obtained


Figure 1: The geometry of the problem.
by letting $\operatorname{Im}(k) \rightarrow 0$ at the end of the analysis. In virtue of the axial symmetry of the problem, all the field components may be expressed in terms of $H_{\phi}(\rho, z)=u(\rho, z)$ as

$$
\begin{equation*}
E_{\rho}=\frac{1}{\mathrm{i} \omega \varepsilon} \frac{\partial}{\partial z} u(\rho, z) \quad \text { and } \quad E_{z}=-\frac{1}{\mathrm{i} \omega \varepsilon} \frac{1}{\rho} \frac{\partial}{\partial \rho}[\rho u(\rho, z)] \tag{2}
\end{equation*}
$$

where the other components of the fields are zero. For the sake of analytical convenience, the total field $u_{T}(\rho, z)$ can be expressed as

$$
u_{T}(\rho, z)=\left\{\begin{array}{cc}
u_{i}(\rho, z)+u_{1}(\rho, z) & , \quad a<\rho<b  \tag{3}\\
u_{2}(\rho, z) H(z) & , \quad b<\rho<d
\end{array}\right.
$$

with $H(z)$ being the Heaviside step function and where $u_{1}(\rho, z)$ and $u_{2}(\rho, z)$ are the scattered fields which satisfy the Helmholtz equation

$$
\begin{equation*}
\left[\frac{\partial^{2}}{\partial \rho^{2}}+\frac{1}{\rho} \frac{\partial}{\partial \rho}+\frac{\partial^{2}}{\partial z^{2}}+\left(k^{2}-\frac{1}{\rho^{2}}\right)\right] u_{j}(\rho, z)=0 \quad, \quad j=1,2 \tag{4}
\end{equation*}
$$

in their domains of validity with the boundary conditions

$$
\begin{align*}
& u_{1}(a, z)+a \frac{\partial}{\partial \rho} u_{1}(a, z)=0, \quad z \in(-\infty, \infty)  \tag{5}\\
& u_{1}(b, z)+b \frac{\partial}{\partial \rho} u_{1}(b, z)=0, \quad z \in(-\infty, 0)  \tag{6}\\
& u_{2}(d, z)+d \frac{\partial}{\partial \rho} u_{2}(d, z)=0, \quad z \in(0, \infty)  \tag{7}\\
& \frac{\partial u(\rho, 0)}{\partial z}=0 \quad, \quad \rho \in(b, d) \tag{8}
\end{align*}
$$

continuity relations

$$
\begin{equation*}
u_{1}(b, z)+b \frac{\partial}{\partial \rho} u_{1}(b, z)=u_{2}(b, z)+b \frac{\partial}{\partial \rho} u_{2}(b, z) \quad, \quad z \in(0, \infty) \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
u_{1}(b, z)+\frac{\mathrm{e}^{\mathrm{i} k z}}{b}=u_{2}(b, z) \quad, \quad z \in(0, \infty) \tag{10}
\end{equation*}
$$

To ensure the uniqueness of the mixed boundary-value problem defined by the Helmholtz equation and the conditions (5)-(9), one has to take into account the radiation and edge conditions [9] as well which are

$$
\begin{equation*}
\left(\frac{\partial u}{\partial \rho}-\mathrm{i} k u\right)=\mathcal{O}\left(\rho^{-1 / 2}\right), \quad \rho \rightarrow \infty \tag{11}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{\phi}=\mathcal{O}\left(z^{2 / 3}\right), E_{z}=\mathcal{O}\left(z^{-1 / 3}\right), \quad z \rightarrow 0 \tag{12}
\end{equation*}
$$

respectively.
The Fourier transform of the Helmholtz equation satisfied by $u_{1}(\rho, z)$ with respect to $z$, in the range of $z \in(-\infty, \infty)$ gives

$$
\begin{equation*}
\left[\frac{\partial^{2}}{\partial \rho^{2}}+\frac{1}{\rho} \frac{\partial}{\partial \rho}+\left(K^{2}(\alpha)-\frac{1}{\rho^{2}}\right)\right] F(\rho, \alpha)=0 \tag{13}
\end{equation*}
$$



Figure 2: Complex $\alpha$-plane.
Here $K(\alpha)=\sqrt{k^{2}-\alpha^{2}}$ is the square-root function defined in the complex $\alpha$-plane, cut along $\alpha=k$ to $\alpha=k+\mathrm{i} \infty$ and $\alpha=-k$ to $\alpha=-k-\mathrm{i} \infty$, such that $K(0)=k$ (this choice of branch will be assumed for all square-root functions throughout the paper, see Figure 2), and the Fourier transform is defined by

$$
\begin{equation*}
F(\rho, \alpha)=F_{-}(\rho, \alpha)+F_{+}(\rho, \alpha) \tag{14}
\end{equation*}
$$

where

$$
\begin{equation*}
F_{ \pm}(\rho, \alpha)= \pm \int_{0}^{ \pm \infty} u_{1}(\rho, z) \mathrm{e}^{\mathrm{i} \alpha z} d z \tag{15}
\end{equation*}
$$

Notice that $F_{+}(\rho, \alpha)$ and $F_{-}(\rho, \alpha)$ are unknown functions which are regular in the halfplanes $\operatorname{Im}(\alpha)>\operatorname{Im}(-k)$ and $\operatorname{Im}(\alpha)<\operatorname{Im}(k)$, respectively. The general solution of (13) is determined as

$$
\begin{equation*}
F(\rho, \alpha)=A(\alpha) \mathrm{J}_{1}(K \rho)+B(\alpha) \mathrm{Y}_{1}(K \rho) \tag{16}
\end{equation*}
$$

Here $\mathrm{J}_{1}(K \rho)$ and $\mathrm{Y}_{1}(K \rho)$ are the usual Bessel functions of the first and second kinds, respectively. Applying Fourier transform to the boundary condition (5) yields

$$
\begin{equation*}
B(\alpha)=-A(\alpha) \frac{\mathrm{J}_{0}(K a)}{\mathrm{Y}_{0}(K a)} \tag{17}
\end{equation*}
$$

Substituting (17) into (16), one gets

$$
\begin{equation*}
F(\rho, \alpha)=\frac{A(\alpha)}{\mathrm{Y}_{0}(K a)}\left[\mathrm{J}_{1}(K \rho) \mathrm{Y}_{0}(K a)-\mathrm{J}_{0}(K a) \mathrm{Y}_{1}(K \rho)\right] \tag{18}
\end{equation*}
$$

On the other hand, the Fourier transform of the boundary condition (6) gives

$$
\begin{equation*}
F_{-}(b, \alpha)+b F_{-}^{\prime}(b, \alpha)=0, \tag{19}
\end{equation*}
$$

which allows one to write

$$
\begin{equation*}
A(\alpha)=\frac{\left[F_{+}(b, \alpha)+b F_{+}^{\prime}(b, \alpha)\right] \mathrm{Y}_{0}(K a)}{K b\left[\mathrm{~J}_{0}(K b) \mathrm{Y}_{0}(K a)-\mathrm{J}_{0}(K a) \mathrm{Y}_{0}(K b)\right]} \tag{20}
\end{equation*}
$$

Hence, equation (18) becomes

$$
\begin{equation*}
F(\rho, \alpha)=P_{+}(\alpha) \frac{\left[\mathrm{J}_{1}(K \rho) \mathrm{Y}_{0}(K a)-\mathrm{J}_{0}(K a) \mathrm{Y}_{1}(K \rho)\right]}{K b\left[\mathrm{~J}_{0}(K b) \mathrm{Y}_{0}(K a)-\mathrm{J}_{0}(K a) \mathrm{Y}_{0}(K b)\right]} \tag{21}
\end{equation*}
$$

where $P_{+}(\alpha)$ stands for

$$
\begin{equation*}
P_{+}(\alpha)=F_{+}(b, \alpha)+b F_{+}^{\prime}(b, \alpha) \tag{22}
\end{equation*}
$$

Similarly, the scattered field $u_{2}(\rho, z)$ satisfies the Helmholtz equation in the region $\rho \in$ $(b, d), z \in(0, \infty)$, whose Fourier transform with respect to $z$ yields

$$
\begin{equation*}
\left[\frac{\partial^{2}}{\partial \rho^{2}}+\frac{1}{\rho} \frac{\partial}{\partial \rho}+\left(K^{2}(\alpha)-\frac{1}{\rho^{2}}\right)\right] G_{+}(\rho, \alpha)=-\mathrm{i} \alpha f(\rho), \tag{23}
\end{equation*}
$$

where the boundary condition (8) is taken into account while $G_{+}(\rho, \alpha)$ and $f(\rho)$ stand for

$$
\begin{equation*}
G_{+}(\rho, \alpha)=\int_{0}^{\infty} u_{2}(\rho, z) \mathrm{e}^{\mathrm{i} \alpha z} d \alpha \tag{24}
\end{equation*}
$$

and

$$
\begin{equation*}
f(\rho)=u_{2}(\rho, 0), \tag{25}
\end{equation*}
$$

respectively. A particular solution of (23) can be expressed in terms of the Green's function related to this differential equation, which satisfies

$$
\begin{equation*}
\left[\frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right)+\left(K^{2}(\alpha)-\frac{1}{\rho^{2}}\right)\right] \mathcal{G}(\rho, t, \alpha)=\frac{1}{t} \delta(\rho-t) \quad, \quad \rho, t \in(b, d) \tag{26}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{G}(t+0, t, \alpha)-\mathcal{G}(t-0, t, \alpha)=0 \tag{27}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial}{\partial \rho} \mathcal{G}(t+0, t, \alpha)-\frac{\partial}{\partial \rho} \mathcal{G}(t-0, t, \alpha)=\frac{1}{t}, \tag{28}
\end{equation*}
$$

with the conditions

$$
\begin{equation*}
\mathcal{G}(b, t, \alpha)+b \frac{\partial}{\partial \rho} \mathcal{G}(b, t, \alpha)=0 \tag{29}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{G}(d, t, \alpha)+d \frac{\partial}{\partial \rho} \mathcal{G}(d, t, \alpha)=0 \tag{30}
\end{equation*}
$$

The general solution of (26) is in the form of

$$
\mathcal{G}(\rho, t, \alpha)=\left\{\begin{array}{ll}
A \mathrm{~J}_{1}(K \rho)+B \mathrm{Y}_{1}(K \rho) & , \quad b<\rho<t  \tag{31}\\
C \mathrm{~J}_{1}(K \rho)+D \mathrm{Y}_{1}(K \rho) & , \quad t<\rho<d
\end{array} .\right.
$$

Taking into account the equations given in (27-30), the Green's function is determined as

$$
\begin{equation*}
\mathcal{G}(\rho, t, \alpha)=\frac{\mathcal{Q}(\rho, t, \alpha)}{M(b, d, \alpha)} \tag{32}
\end{equation*}
$$

with

$$
\mathcal{Q}(\rho, t, \alpha)=\frac{\pi}{2} \begin{cases}L(t, d, \alpha) L(\rho, b, \alpha) & , \quad b<\rho<t  \tag{33}\\ L(t, b, \alpha) L(\rho, d, \alpha) & , \quad t<\rho<d\end{cases}
$$

and

$$
\begin{equation*}
M\left(\rho_{1}, \rho_{2}, \alpha\right)=\mathrm{J}_{0}\left(K \rho_{1}\right) \mathrm{Y}_{0}\left(K \rho_{2}\right)-\mathrm{J}_{0}\left(K \rho_{2}\right) \mathrm{Y}_{0}\left(K \rho_{1}\right) \tag{34}
\end{equation*}
$$

In (33), $L\left(\rho_{1}, \rho_{2}, \alpha\right)$ stands for

$$
\begin{equation*}
L\left(\rho_{1}, \rho_{2}, \alpha\right)=\mathrm{J}_{1}\left(K \rho_{1}\right) \mathrm{Y}_{0}\left(K \rho_{2}\right)-\mathrm{J}_{0}\left(K \rho_{2}\right) \mathrm{Y}_{1}\left(K \rho_{1}\right) \tag{35}
\end{equation*}
$$

Hence, the general solution of (23) can be expressed as

$$
\begin{align*}
& G_{+}(\rho, \alpha)=\frac{1}{M(b, d, \alpha)}\left\{C(\alpha)\left[\mathrm{J}_{1}(K \rho) \mathrm{Y}_{0}(K d)-\mathrm{J}_{0}(K d) \mathrm{Y}_{1}(K \rho)\right]\right. \\
&\left.-\mathrm{i} \alpha \int_{b}^{d} f(t) \mathcal{Q}(t, \rho, \alpha) t d t\right\} . \tag{36}
\end{align*}
$$

The Fourier transform of the continuity relation (9) yields

$$
\begin{equation*}
C(\alpha)=\frac{P_{+}(\alpha)}{K b} \tag{37}
\end{equation*}
$$

and (36) becomes

$$
\begin{align*}
& G_{+}(\rho, \alpha)=\frac{1}{K^{2} b M(b, d, \alpha)}\left\{P_{+}(\alpha) K\left[\mathrm{~J}_{1}(K \rho) \mathrm{Y}_{0}(K d)-\mathrm{J}_{0}(K d) \mathrm{Y}_{1}(K \rho)\right]\right. \\
&\left.-\mathrm{i} \alpha b \int_{b}^{d} f(t) K^{2} \mathcal{Q}(t, \rho, \alpha) t d t\right\} . \tag{38}
\end{align*}
$$

Although the left-hand side of (38) at $\rho=b$ is a regular function of $\alpha$ in the upper half-plane, the regularity of the right-hand side is violated by the presence of simple poles occurring at the zeros of $K^{2} M(b, d, \alpha)$ lying in the upper half of the complex $\alpha$-plane, namely, at $\alpha=\gamma_{m}$ 's $(m=0,1,2, \ldots)$. These poles can be eliminated by imposing that their residues are zero. This gives

$$
\begin{equation*}
f_{0}=\frac{P_{+}(k)}{\mathrm{i} k b \log (d / b)} \tag{39}
\end{equation*}
$$

and

$$
\begin{equation*}
f_{m}=\frac{2}{\mathrm{i} \gamma_{m} b} \frac{\mathrm{Y}_{0}\left(\Gamma_{m} d\right) \mathrm{Y}_{0}\left(\Gamma_{m} b\right)}{\left[\mathrm{Y}_{0}^{2}\left(\Gamma_{m} b\right)-\mathrm{Y}_{0}^{2}\left(\Gamma_{m} d\right)\right]} P_{+}\left(\gamma_{m}\right) \quad, \quad m=1,2, \ldots \tag{40}
\end{equation*}
$$

with

$$
\begin{gather*}
f_{0}=\frac{1}{\log (d / b)} \int_{b}^{d} f(t) d t  \tag{41}\\
f_{m}=\frac{2 \mathrm{Y}_{0}^{2}\left(\Gamma_{m} b\right)}{\left[\mathrm{Y}_{0}^{2}\left(\Gamma_{m} b\right)-\mathrm{Y}_{0}^{2}\left(\Gamma_{m} d\right)\right]} \int_{b}^{d} f(t)\left[\frac{\pi}{2} \Gamma_{m} L\left(t, d, \gamma_{m}\right)\right] t d t \quad, \quad m=1,2, \ldots \tag{42}
\end{gather*}
$$

and

$$
\Gamma_{m}=\sqrt{k^{2}-\gamma_{m}^{2}}, m=1,2, \ldots
$$

Owing to (39-42), $f(\rho)$ can be expanded into Fourier-Bessel series as

$$
\begin{equation*}
f(\rho)=\frac{f_{0}}{\rho}+\sum_{m=1}^{\infty} f_{m}\left\{\frac{\pi}{2} \Gamma_{m}\left[\mathrm{~J}_{1}\left(\Gamma_{m} \rho\right) \mathrm{Y}_{0}\left(\Gamma_{m} d\right)-\mathrm{J}_{0}\left(\Gamma_{m} d\right) \mathrm{Y}_{1}\left(\Gamma_{m} \rho\right)\right]\right\} . \tag{43}
\end{equation*}
$$

Now, substituting (38) into the Fourier transform of the continuity relation (10) and making use of (14), (16) and (21), one gets

$$
\begin{equation*}
\frac{N(\alpha)}{\left(k^{2}-\alpha^{2}\right)} P_{+}(\alpha)+b P_{-}(\alpha)=\frac{\mathrm{i}}{(\alpha+k)}+\frac{\mathrm{i} \alpha b}{M(b, d, \alpha)} \int_{b}^{d} f(t) \mathcal{Q}(t, b, \alpha) t d t \tag{44}
\end{equation*}
$$

with

$$
\begin{equation*}
N(\alpha)=\frac{2}{\pi b} \frac{M(a, d, \alpha)}{M(b, d, \alpha) M(a, b, \alpha)} \tag{45}
\end{equation*}
$$

and

$$
\begin{equation*}
P_{-}(\alpha)=F_{-}(b, \alpha) \tag{46}
\end{equation*}
$$

Incorporating (43) into (44), we obtain after term by term integration

$$
\begin{equation*}
\frac{N(\alpha)}{\left(k^{2}-\alpha^{2}\right)} P_{+}(\alpha)+b F_{-}(b, \alpha)=\frac{\mathrm{i}}{(\alpha+k)}+\frac{\mathrm{i} \alpha f_{0}}{\left(k^{2}-\alpha^{2}\right)}-\mathrm{i} \sum_{m=1}^{\infty} \frac{\alpha f_{m}}{\left(\alpha^{2}-\gamma_{m}^{2}\right)} \frac{\mathrm{Y}_{0}\left(\Gamma_{m} d\right)}{\mathrm{Y}_{0}\left(\Gamma_{m} b\right)} . \tag{47}
\end{equation*}
$$

Equation (47) is nothing but the modified Wiener-Hopf equation to be solved. Applying classical Wiener-Hopf procedure, one determines

$$
\begin{align*}
& P_{+}(\alpha)=\frac{2 \mathrm{i} k}{N_{+}(\alpha) N_{+}(k)}-\frac{\mathrm{i} k f_{0}}{N_{+}(\alpha) N_{+}(k)} \\
&-\frac{\mathrm{i}}{2} \sum_{m=1}^{\infty} \frac{f_{m}}{\left(\alpha+\gamma_{m}\right)} \frac{\mathrm{Y}_{0}\left(\Gamma_{m} d\right)}{\mathrm{Y}_{0}\left(\Gamma_{m} b\right)} \frac{(k+\alpha)\left(k+\gamma_{m}\right)}{N_{+}(\alpha) N_{+}\left(\gamma_{m}\right)}, \tag{48}
\end{align*}
$$

with $N_{+}(\alpha)$ being the usual split function

$$
\begin{equation*}
N_{+}(\alpha)=\sqrt{\frac{2}{\pi b}} \frac{M_{+}(a, d, \alpha)}{M_{+}(b, d, \alpha) M_{+}(a, b, \alpha)}, \tag{49}
\end{equation*}
$$

where the factorization of $M\left(\rho_{1}, \rho_{2}, \alpha\right)$ is done by following the procedure described in [15]. The unknown coefficients $f_{m}(m=0,1,2, \ldots)$ can be calculated by taking into account equations (39), (40) and (48) simultaneously, which yields the algebraic system of equations

$$
\begin{equation*}
\mathbf{A f}=\mathbf{B} \tag{50}
\end{equation*}
$$

where the elements of $\mathbf{A}$, namely $A_{m n}$ are

$$
\begin{gather*}
A_{00}=\left[N_{+}(k) b \log (d / b)+\frac{1}{N_{+}(k)}\right] \\
A_{m 0}=\frac{2 k \chi_{m}}{N_{+}(k)}, m=1,2, \ldots \\
A_{0 n}=\frac{\mathrm{Y}_{0}\left(\Gamma_{n} d\right)}{N_{+}\left(\gamma_{n}\right) \mathrm{Y}_{0}\left(\Gamma_{n} b\right)}, n=1,2, \ldots \\
A_{m n}=\left\{\begin{array}{cc}
1+\chi_{m} \frac{\mathrm{Y}_{0}\left(\Gamma_{m} d\right)}{\mathrm{Y}_{0}\left(\Gamma_{m} b\right)} \frac{\left(k+\gamma_{m}\right)^{2}}{2 \gamma_{m} N_{+}\left(\gamma_{m}\right)} \quad, \quad n=m \\
\frac{\left(k+\gamma_{m}\right) \chi_{m}}{\left(\gamma_{m}+\gamma_{n}\right)} \frac{\mathrm{Y}_{0}\left(\Gamma_{n} d\right)}{\mathrm{Y}_{0}\left(\Gamma_{n} b\right)} \frac{\left(k+\gamma_{n}\right)}{N_{+}\left(\gamma_{n}\right)} \quad, \quad n \neq m
\end{array}, m, n=1,2, . .\right. \tag{51}
\end{gather*}
$$

and the elements of $\mathbf{B}$, namely $B_{m}$ are

$$
B_{m}=\left\{\begin{array}{cc}
\frac{2}{N_{+}(k)}, & m=0  \tag{52}\\
\frac{4 k \chi_{m}}{N_{+}(k)} & , \quad m=1,2, \ldots
\end{array}\right.
$$

with

$$
\begin{equation*}
\chi_{m}=\frac{\mathrm{Y}_{0}\left(\Gamma_{m} d\right) \mathrm{Y}_{0}\left(\Gamma_{m} b\right)}{\left[\mathrm{Y}_{0}^{2}\left(\Gamma_{m} b\right)-\mathrm{Y}_{0}^{2}\left(\Gamma_{m} d\right)\right]} \frac{1}{\gamma_{m} b N_{+}\left(\gamma_{m}\right)} \tag{53}
\end{equation*}
$$

## 3 Analysis of the fields

The scattered field $u_{1}(\rho, z)$ can be determined by solving the inverse Fourier transform integral

$$
\begin{equation*}
u_{1}(\rho, z)=\frac{1}{2 \pi} \int_{\mathcal{L}} F(\rho, \alpha) \mathrm{e}^{-\mathrm{i} \alpha z} d \alpha \tag{54}
\end{equation*}
$$

where $\mathcal{L}$ is depicted in Figure 2. Considering (21) and (48), the above integral becomes

$$
\begin{align*}
& u_{1}(\rho, z)=\frac{\mathrm{i}}{2 \pi} \frac{2}{\pi b} \int_{\mathcal{L}}\{ \frac{2 k}{N_{+}(\alpha) N_{+}(k)}-\frac{k f_{0}}{N_{+}(\alpha) N_{+}(k)} \\
&\left.-\frac{1}{2} \sum_{m=1}^{\infty} \frac{f_{m}}{\left(\alpha+\gamma_{m}\right)} \frac{\mathrm{Y}_{0}\left(\Gamma_{m} d\right)}{\mathrm{Y}_{0}\left(\Gamma_{m} b\right)} \frac{(k+\alpha)\left(k+\gamma_{m}\right)}{N_{+}(\alpha) N_{+}\left(\gamma_{m}\right)}\right\} \\
& \times \frac{\frac{\pi}{2} K\left[\mathrm{~J}_{1}(K \rho) \mathrm{Y}_{0}(K a)-\mathrm{J}_{0}(K a) \mathrm{Y}_{1}(K \rho)\right]}{\left(k^{2}-\alpha^{2}\right) M(b, a, \alpha)} \mathrm{e}^{-\mathrm{i} \alpha z} d \alpha . \tag{55}
\end{align*}
$$

In order to determine the reflected field back to the region $z<0$, the above integral can be evaluated by virtue of Jordan's lemma and the application of the law of residues, yielding the sum of the residues related to the poles occurring at the simple zeros of $\left(k^{2}-\alpha^{2}\right) M(b, a, \alpha)$ lying in the upper half-plane, namely, at $\alpha=\alpha_{n}$ 's $(n=0,1,2, \ldots)$. Denoting the reflected field as
$u_{1}(\rho, z)=R_{0} \frac{\mathrm{e}^{-\mathrm{i} k z}}{\rho}+\sum_{n=1}^{\infty} R_{n} \frac{\pi}{2} K_{n}\left[\mathrm{~J}_{1}\left(K_{n} \rho\right) \mathrm{Y}_{0}\left(K_{n} a\right)-\mathrm{J}_{0}\left(K_{n} a\right) \mathrm{Y}_{1}\left(K_{n} \rho\right)\right] \mathrm{e}^{-\mathrm{i} \alpha_{n} z} \quad, \quad z<0$
with

$$
\begin{equation*}
K_{n}=\sqrt{k^{2}-\alpha_{n}^{2}} \tag{56}
\end{equation*}
$$

the reflection coefficient $R_{n}$ is found to be

$$
\begin{align*}
& R_{n}=\frac{2}{\pi b}\left\{\frac{k f_{0}}{N_{+}\left(\alpha_{n}\right) N_{+}(k)}-\frac{2 k}{N_{+}\left(\alpha_{n}\right) N_{+}(k)}\right. \\
&\left.+\frac{1}{2} \sum_{m=1}^{\infty} \frac{f_{m}}{\left(\alpha_{n}+\gamma_{m}\right)} \frac{Y_{0}\left(\Gamma_{m} d\right)}{\mathrm{Y}_{0}\left(\Gamma_{m} b\right)} \frac{\left(k+\alpha_{n}\right)\left(k+\gamma_{m}\right)}{N_{+}\left(\alpha_{n}\right) N_{+}\left(\gamma_{m}\right)}\right\} \\
& \quad \times \frac{1}{\left[\left(k^{2}-\alpha^{2}\right) M(b, a, \alpha)\right]_{\alpha \rightarrow \alpha_{n}}^{\prime}} \tag{58}
\end{align*}
$$

For the fundamental TEM mode $(n=0)$ reflected back to $z<0$, we find

$$
\begin{equation*}
R_{0}=\frac{1}{2 b \log (b / a) N_{+}(k) N_{+}(k)}\left\{f_{0}-2+N_{+}(k) \sum_{m=1}^{\infty} \frac{f_{m}}{N_{+}\left(\gamma_{m}\right)} \frac{\mathrm{Y}_{0}\left(\Gamma_{m} d\right)}{\mathrm{Y}_{0}\left(\Gamma_{m} b\right)}\right\} \tag{59}
\end{equation*}
$$

In a similar fashion, the transmission coefficient can be determined by evaluating the integral in (55) for $z>0$ to give

$$
\begin{equation*}
u_{2}(\rho, z)=T_{0} \frac{\mathrm{e}^{\mathrm{i} k z}}{\rho}+\sum_{n=1}^{\infty} T_{n} \frac{\pi}{2} \xi_{n}\left[\mathrm{~J}_{1}\left(\xi_{n} \rho\right) \mathrm{Y}_{0}\left(\xi_{n} a\right)-\mathrm{J}_{0}\left(\xi_{n} a\right) \mathrm{Y}_{1}\left(\xi_{n} \rho\right)\right] \mathrm{e}^{\mathrm{i} \beta_{n} z}, \quad z>0 \tag{60}
\end{equation*}
$$

with

$$
\begin{equation*}
\xi_{n}=\sqrt{k^{2}-\beta_{n}^{2}} \tag{61}
\end{equation*}
$$

and $\beta_{n}$ 's are the simple zeros of $\left(k^{2}-\alpha^{2}\right) M(a, d, \alpha)$, where $\beta_{0}=k$. The transmission coefficients $T_{n}$ 's and $T_{0}$ are determined as

$$
\begin{align*}
& T_{n}=\left\{\frac{\left(f_{0}-2\right) k}{N_{+}(k)}+\frac{1}{2} \sum_{m=1}^{\infty} \frac{f_{m}}{\left(\gamma_{m}-\beta_{n}\right)} \frac{\mathrm{Y}_{0}\left(\Gamma_{m} d\right)}{\mathrm{Y}_{0}\left(\Gamma_{m} b\right)} \frac{\left(k-\beta_{n}\right)\left(k+\gamma_{m}\right)}{N_{+}\left(\gamma_{m}\right)}\right\} \\
& \times \frac{M\left(b, d, \beta_{n}\right) N_{+}\left(\beta_{n}\right)}{\left[\left(k^{2}-\alpha^{2}\right) M(a, d, \alpha)\right]_{\alpha \rightarrow-\beta_{n}}^{\prime}} \tag{62}
\end{align*}
$$

and

$$
\begin{equation*}
T_{0}=\left(f_{0}-2\right) \frac{\log (d / b)}{2 \log (d / a)} \tag{63}
\end{equation*}
$$

respectively.

## 4 Mode-Matching Formulation

The geometry of the problem is also suitable for applying mode-matching technique [9]. For this type of formulation, the geometry is divided into two regions as shown in Figure 3 and the $\phi$-component of the total magnetic field at each region is defined as

$$
H_{\phi}^{(T)}=\left\{\begin{array}{cc}
\frac{\mathrm{e}^{\mathrm{i} k z}}{\rho}+u_{1}(\rho, z) & , \quad \rho \in(a, b), z<0 \text { (Region A) }  \tag{64}\\
u_{2}(\rho, z) & , \quad \rho \in(a, d), z>0 \text { (Region B) }
\end{array} .\right.
$$

with

$$
\begin{equation*}
H_{\phi}^{(A)}(\rho, z)=u_{1}(\rho, z)=\sum_{n=0}^{\infty} R_{n} \varphi_{n}(\rho) \mathrm{e}^{-\mathrm{i} \alpha_{n} z} \tag{65}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{\phi}^{(B)}(\rho, z)=u_{2}(\rho, z)=\sum_{n=0}^{\infty} T_{n} \psi_{n}(\rho) \mathrm{e}^{\mathrm{i} \beta_{n} z} \tag{66}
\end{equation*}
$$

where $\alpha_{n}$ 's are the zeros of the function $\left(k^{2}-\alpha^{2}\right) M(a, b, \alpha)$ and

$$
\begin{equation*}
K_{n}=\sqrt{k^{2}-\alpha_{n}^{2}} . \tag{67}
\end{equation*}
$$

$\varphi_{n}(\rho)$ and $\psi_{n}(\rho)$ in (65) and (66) stand for

$$
\begin{equation*}
\varphi_{n}(\rho)=\frac{\pi}{2} K_{n}\left[\mathrm{~J}_{1}\left(K_{n} \rho\right) \mathrm{Y}_{0}\left(K_{n} a\right)-\mathrm{J}_{0}\left(K_{n} a\right) \mathrm{Y}_{1}\left(K_{n} \rho\right)\right] \tag{68}
\end{equation*}
$$

and

$$
\begin{equation*}
\psi_{n}(\rho)=\frac{\pi}{2} \xi_{n}\left[\mathrm{~J}_{1}\left(\xi_{n} \rho\right) \mathrm{Y}_{0}\left(\xi_{n} a\right)-\mathrm{J}_{0}\left(\xi_{n} a\right) \mathrm{Y}_{1}\left(\xi_{n} \rho\right)\right] \tag{69}
\end{equation*}
$$



Figure 3: Regions for mode-matching technique.
respectively.
Hence, $\rho$-components of the electric fields become

$$
\begin{equation*}
E_{\rho}^{(A)}(\rho, z)=\frac{1}{\mathrm{i} \omega \varepsilon} \frac{\partial}{\partial z} u_{1}(\rho, z)=-\sum_{n=0}^{\infty} \frac{\alpha_{n}}{\omega \varepsilon} R_{n} \varphi_{n}(\rho) \mathrm{e}^{-\mathrm{i} \alpha_{n} z} \tag{70}
\end{equation*}
$$

and

$$
\begin{equation*}
E_{\rho}^{(B)}(\rho, z)=\frac{1}{\mathrm{i} \omega \varepsilon} \frac{\partial}{\partial z} u_{2}(\rho, z)=\sum_{n=0}^{\infty} \frac{\beta_{n}}{\omega \varepsilon} T_{n} \psi_{n}(\rho) \mathrm{e}^{\mathrm{i} \beta_{n} z} . \tag{71}
\end{equation*}
$$

Note that the eigenfunctions in the Fourier-Bessel series in (65) and (66) are determined, such that, $u_{1}(\rho, z)$ and $u_{2}(\rho, z)$ satisfy the boundary conditions

$$
\begin{align*}
& u_{1}(a, z)+a \frac{\partial}{\partial \rho} u_{1}(a, z)=0,  \tag{72}\\
& u_{1}(b, z)+b \frac{\partial}{\partial \rho} u_{1}(b, z)=0,  \tag{73}\\
& u_{2}(a, z)+a \frac{\partial}{\partial \rho} u_{2}(a, z)=0 \tag{74}
\end{align*}
$$

and

$$
\begin{equation*}
u_{2}(d, z)+d \frac{\partial}{\partial \rho} u_{2}(d, z)=0 . \tag{75}
\end{equation*}
$$

Following the procedure described in [12], the unknown coefficients in the series expansions are to be determined by taking into account the continuity of the tangential components of the electric and the magnetic fields at $z=0$, namely

$$
E_{\rho}\left(\rho, 0^{+}\right)=\left\{\begin{array}{cl}
E_{\rho}\left(\rho, 0^{-}\right) & , \quad \rho \in(a, b)  \tag{76}\\
0 & , \quad \rho \in(b, d)
\end{array}\right.
$$

and

$$
\begin{equation*}
H_{\phi}\left(\rho, 0^{-}\right)=H_{\phi}\left(\rho, 0^{+}\right), \quad \rho \in(a, b) . \tag{77}
\end{equation*}
$$

Multiplying (76) by $\psi_{m}(\rho)$ and integrating along $\rho \in(a, d)$ yields

$$
\begin{equation*}
\beta_{m} T_{m} Q_{m}^{(1)}=k U_{m}^{(1)}-\sum_{n=0}^{\infty} \alpha_{n} R_{n} \triangle_{m n}^{(1)} \tag{78}
\end{equation*}
$$

with

$$
\begin{equation*}
\triangle_{m n}^{(1)}=\int_{a}^{b} \varphi_{n}(\rho) \psi_{m}(\rho) \rho d \rho, Q_{m}^{(1)}=\int_{a}^{d} \psi_{m}(\rho) \psi_{n}(\rho) \rho d \rho, U_{m}^{(1)}=\int_{a}^{b} \psi_{m}(\rho) d \rho \tag{79}
\end{equation*}
$$

Similarly, multiplying (77) by $\varphi_{m}(\rho)$ and integration along $\rho \in(a, b)$ gives

$$
\begin{equation*}
U_{m}^{(2)}+R_{m} Q_{m}^{(2)}=\sum_{n=0}^{\infty} T_{n} \triangle_{m n}^{(2)}, \tag{80}
\end{equation*}
$$

with

$$
\begin{equation*}
\triangle_{m n}^{(2)}=\int_{a}^{b} \varphi_{m}(\rho) \psi_{n}(\rho) \rho d \rho, Q_{m}^{(2)}=\int_{a}^{b} \varphi_{n}(\rho) \varphi_{m}(\rho) \rho d \rho, U_{m}^{(2)}=\int_{a}^{b} \varphi_{m}(\rho) d \rho \tag{81}
\end{equation*}
$$

As a result, equations

$$
\begin{equation*}
\beta_{m} T_{m} Q_{m}^{(1)}=k U_{m}^{(1)}-\sum_{n=0}^{Q} \alpha_{n} R_{n} \triangle_{m n}^{(1)} \quad, \quad m=0,1,2, \ldots, P \tag{82}
\end{equation*}
$$

and

$$
\begin{equation*}
U_{m}^{(2)}+R_{m} Q_{m}^{(2)}=\sum_{n=0}^{P} T_{n} \triangle_{m n}^{(2)} \quad, \quad m=0,1,2, \ldots, Q \tag{83}
\end{equation*}
$$

completely describe the junction uniquely according to [12] forming a $N \times N$ system with $N=P+Q$. Note that $P$ must be greater than $Q$ because equation (82) involves more information than equation (83). The explicit expressions for $\triangle_{m n}^{(1)}, \triangle_{m n}^{(2)}, Q_{m}^{(1)}, Q_{m}^{(2)}, U_{m}^{(1)}$ and $U_{m}^{(2)}$ are

$$
\begin{gather*}
\triangle_{m n}^{(1)}=\left\{\begin{array}{cc}
\log (b / a) & , \\
0 & m=0, n=0 \\
\frac{\pi}{2} M\left(a, b, \beta_{m}\right) & m \neq 0, n \neq 0 \\
\frac{\pi}{2} \frac{\xi_{m}^{2}}{\left(\xi_{m}^{2}-K_{n}^{2}\right)} \frac{\mathrm{Y}_{0}\left(K_{n} a\right)}{\mathrm{Y}_{0}\left(K_{n} b\right)} M\left(a, b, \beta_{m}\right) & , m \neq 0, n \neq 0
\end{array}\right.  \tag{84}\\
\triangle_{m n}^{(2)}=\left\{\begin{array}{cc}
\log (b / a) & , m=0, n=0 \\
\frac{\pi}{2} M\left(a, b, \beta_{n}\right) & m=0, n \neq 0 \\
0 & m \neq 0, n=0 \\
\frac{\pi}{2} \frac{\xi_{n}^{2}}{\left(\xi_{n}^{2}-K_{m}^{2}\right)} \frac{\mathrm{Y}_{0}\left(K_{m} a\right)}{\mathrm{Y}_{0}\left(K_{m} b\right)} M\left(a, b, \beta_{n}\right) & m \neq 0, n \neq 0
\end{array}\right. \tag{85}
\end{gather*}
$$

$$
\begin{gather*}
Q_{m}^{(1)}=\left\{\begin{array}{cc}
\log (d / a) & , \\
\frac{\left[\mathrm{Y}_{0}^{2}\left(\xi_{m} a\right)-\mathrm{Y}_{0}^{2}\left(\xi_{m} d\right)\right]}{2 \mathrm{Y}_{0}^{2}\left(\xi_{m} d\right)}, & m \neq 0
\end{array},\right.  \tag{86}\\
Q_{m}^{(2)}=\left\{\begin{array}{cc}
\log (b / a) & m=0 \\
\frac{\left[\mathrm{Y}_{0}^{2}\left(K_{m} a\right)-\mathrm{Y}_{0}^{2}\left(K_{m} b\right)\right]}{2 \mathrm{Y}_{0}^{2}\left(K_{m} b\right)}, & m \neq 0
\end{array}\right.  \tag{87}\\
U_{m}^{(1)}=\left\{\begin{array}{cc}
\log (b / a) & m=0 \\
\frac{\pi}{2} M\left(a, b, \beta_{m}\right) & ,
\end{array},\right. \tag{88}
\end{gather*}
$$

and

$$
U_{m}^{(2)}=\left\{\begin{array}{cc}
\log (b / a) & , \quad m=0  \tag{89}\\
0 & , \quad m \neq 0
\end{array}\right.
$$

respectively. Constructing the system of linear equations properly allows one to determine the coefficients $R_{n}$ and $T_{n}$ numerically. Note that, when there is only the fundamental TEM mode propagating, the transmission coefficients calculated via mode matching technique and Wiener-Hopf technique have the relation

$$
\begin{equation*}
T_{0, \text { Wiener }- \text { Hopf }}=T_{0, \text { Mode }- \text { Matching }}-1 \tag{90}
\end{equation*}
$$

due to the difference in defining the total field in two formulations.

## 5 Power Analysis

Considering the mode matching formulation and the frequencies where only the fundamental TEM mode is propagating, $\phi$-component of the magnetic field and $\rho$-component of the electric field are

$$
\begin{equation*}
H_{\phi}^{(A)}=\frac{\mathrm{e}^{\mathrm{i} k z}}{\rho}+R_{0} \frac{\mathrm{e}^{-\mathrm{i} k z}}{\rho}, E_{\rho}^{(A)}=\frac{k}{\omega \varepsilon} \frac{\mathrm{e}^{\mathrm{i} k z}}{\rho}-\frac{k}{\omega \varepsilon} R_{0} \frac{\mathrm{e}^{-\mathrm{i} k z}}{\rho} \tag{91}
\end{equation*}
$$

in region $\mathrm{A}(z<0)$ and

$$
\begin{equation*}
H_{\phi}^{(B)}=T_{0} \frac{\mathrm{e}^{\mathrm{i} k z}}{\rho} \quad, \quad E_{\rho}^{(B)}=\frac{k}{\omega \varepsilon} T_{0} \frac{\mathrm{e}^{\mathrm{i} k z}}{\rho} \tag{92}
\end{equation*}
$$

in region $\mathrm{B}(z>0)$. Considering the general formula for calculating the power

$$
\begin{equation*}
P=\frac{1}{2} \operatorname{Re}\left\{\iint_{S}\left(E_{\rho}^{(A)} \hat{a}_{\rho} \times H_{\phi}^{(A) *} \hat{a}_{\phi}\right)\right\} \cdot \hat{a}_{z} d S \tag{93}
\end{equation*}
$$

the power in regions A and B become

$$
\begin{equation*}
P^{(A)}=\frac{\pi k}{\omega \varepsilon}\left(1-\left|R_{0}\right|^{2}\right) \log (b / a) \tag{94}
\end{equation*}
$$

and

$$
\begin{equation*}
P^{(B)}=\frac{\pi k}{\omega \varepsilon}\left|T_{0}\right|^{2} \log (d / a) \tag{95}
\end{equation*}
$$

respectively. The conservation of energy yields

$$
\begin{equation*}
\left(1-\left|R_{0}\right|^{2}\right) \log (b / a)=\left|T_{0}\right|^{2} \log (d / a) \tag{96}
\end{equation*}
$$

The above relation is observed in the numerical calculations as well.

## 6 Computational Results

In this section, the results obtained via both Wiener-Hopf and Mode-Matching techniques are evaluated numerically and compared to each other and the effect of the ratio of cross-sectional areas of the connected coaxial waveguides is presented graphically. During the numerical analysis, one needs to calculate the zeros of the functions $M(a, b, \alpha)$, $M(a, d, \alpha)$ and $M(b, d, \alpha)$. If $M(a, b, \alpha)$ is analyzed asymptotically for large $|\alpha|$ by taking into account

$$
\begin{equation*}
\mathrm{J}_{0}(z) \simeq \sqrt{\frac{2}{\pi z}} \cos \left(z-\frac{\pi}{4}\right) \text { for large }|z| \tag{97}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{Y}_{0}(z) \simeq \sqrt{\frac{2}{\pi z}} \sin \left(z-\frac{\pi}{4}\right) \text { for large }|z| \tag{98}
\end{equation*}
$$

one gets the eigenvalue equation

$$
\begin{equation*}
\sin \left[K_{m}(b-a)\right]=0 \quad m=1,2, \ldots \tag{99}
\end{equation*}
$$

to give

$$
\begin{equation*}
K_{m} \simeq \frac{m \pi}{(b-a)} \quad m=1,2, \ldots \tag{100}
\end{equation*}
$$

Thus, the zeros of $M(a, b, \alpha)$ are calculated by

$$
\begin{equation*}
\alpha_{m}=\sqrt{k^{2}-K_{m}^{2}} \quad m=1,2, \ldots \tag{101}
\end{equation*}
$$

Similarly, for the function $M(a, d, \alpha)$ one finds

$$
\begin{equation*}
\xi_{m} \simeq \frac{m \pi}{(d-a)} \quad m=1,2, \ldots \tag{102}
\end{equation*}
$$

and

$$
\begin{equation*}
\beta_{m}=\sqrt{k^{2}-\xi_{m}^{2}} \quad m=1,2, \ldots \tag{103}
\end{equation*}
$$

Here $\alpha_{m}$ 's and $\beta_{m}$ 's correspond to the wavenumbers of higher-order modes associated with their indices in the regions A and B , respectively. Hence, the corresponding modes are propagating only when these wavenumbers become purely real. Note that, the righthand sides of the equations (100) and (102) are only the asymptotic expressions for $K_{m}$


Figure 4: Variation of $M(a, b, \alpha)$ with respect to $K(\alpha)$.
and $\xi_{m}$. So these are assumed as the initial values in the Newton's procedure to find the accurate values and the numerical calculations for $a=1 \mathrm{~cm}, b=3 \mathrm{~cm}$ and $d=10 \mathrm{~cm}$ give

$$
f=1.58 \mathrm{GHz}
$$

for the cut-off frequency of the first higher-order mode to be propagating in region B and

$$
f=7.39 \mathrm{GHz}
$$

in region A . For the same values of $a, b$ and $d$, the functions $M(a, b, \alpha)$ and $M(a, d, \alpha)$ are presented in Figures 4 and 5, respectively.

On the other hand, the asymptotic expression for the zeros of $M(b, d, \alpha)$ can be found in a similar fashion to give

$$
\begin{equation*}
\Gamma_{m} \simeq \frac{m \pi}{(d-b)} \quad \text { and } \quad \gamma_{m}=\sqrt{k^{2}-\Gamma_{m}^{2}} \quad, \quad m=1,2, \ldots . \tag{104}
\end{equation*}
$$

Since both Wiener-Hopf and Mode-Matching analyses involve infinite sets of linear algebraic equations, convergence of the solution regarding the truncation number must be illustrated for each technique. Figures 6-9 show the dependence of the magnitudes of the reflection and transmission coefficients $R_{0}$ and $T_{0}$ to the truncation number $N$ at frequencies $f=100 \mathrm{MHz}$ and $f=4 \mathrm{GHz}$ for $a=1 \mathrm{~cm}, b=3 \mathrm{~cm}$ and $d=10 \mathrm{~cm}$. These figure show that the Wiener-Hopf technique provides a better convergence than the Mode-Matching technique. This issue is well-known in literature as the lack of edge conditions in the latter technique causes slower convergence.

The Figures 10-13 illustrate the effect of the ratio of cross-sectional areas of the connected coaxial waveguides for different ranges of frequency on the reflection and transmission coefficients. In these figures, the area ratios defined by

$$
\begin{equation*}
\text { Area ratio }=\frac{S_{2}}{S_{1}}=\frac{\left(d^{2}-a^{2}\right)}{\left(b^{2}-a^{2}\right)} \tag{105}
\end{equation*}
$$



Figure 5: Variation of $M(a, d, \alpha)$ with respect to $K(\alpha)$.


Figure 6: Convergence of $\left|R_{0}\right|$ at $f=100 \mathrm{MHz}$.


Figure 7: Convergence of $\left|T_{0}\right|$ at $f=100 \mathrm{MHz}$.


Figure 8: Convergence of $\left|R_{0}\right|$ at $f=4 \mathrm{GHz}$.


Figure 9: Convergence of $\left|T_{0}\right|$ at $f=4 \mathrm{GHz}$.
are 12.375 for black line, 6.6 for blue line and 4.125 for red line. Also, the return and insertion losses are presented in Figures 14-17 for the same area ratios, which are calculated directly by

$$
\begin{equation*}
R L=-20 \log \left|R_{0}\right| \tag{106}
\end{equation*}
$$

and

$$
\begin{equation*}
I L=-20 \log \left|T_{0}\right| \tag{107}
\end{equation*}
$$

respectively. It can be observed in these figures that the two methods, i.e. WienerHopf technique and Mode-Matching technique, have excellent agreement. Besides, one can conlude that when the area ratio increases, the magnitude of the reflection coefficient also increases while the magnitude of the tranmission coefficient decreases. Consequently, the same trend is observed in the return and insertion losses. This means that when the area ratio is increased, more energy is reflected back and less is transmitted to region B. Figures 10, 12, 14 and 16 are of special interest due to the need of low frequency electromagnetic modeling of the measurement setup described in [1].

Note that, the transmission coefficient in these figures correspond to the one defined in (66), and the relation (90) is taken into account when comparing two methods.

Additionally, the magnitudes of the reflection and the transmission coefficients are presented in Figures 18 and 19 upto 4 GHz , where two additional TM modes are observed to start propagating in region B after their relevant cut-off frequencies are exceeded. When new modes start to propagate, the energy carried by the fundamental TEM mode is obviously decreased dramatically. This can be observed in these figures as well.

## 7 Concluding Remarks

In this report, the propagation of TEM waves along a coaxial waveguide with a step discontinuity on its outer wall is analyzed by applying Wiener-Hopf and Mode-Matching


Figure 10: Magnitude of $R_{0}$ upto $f=10 \mathrm{MHz}$.


Figure 11: Magnitude of $R_{0}$ upto $f=1.5 \mathrm{GHz}$.


Figure 12: Magnitude of $T_{0}$ upto $f=10 \mathrm{MHz}$.


Figure 13: Magnitude of $T_{0}$ upto $f=1.5 \mathrm{GHz}$.


Figure 14: Insertion loss upto $f=10 \mathrm{MHz}$.


Figure 15: Insertion loss upto $f=1.5 \mathrm{GHz}$.


Figure 16: Return loss upto $f=10 \mathrm{MHz}$.


Figure 17: Return loss upto $f=1.5 \mathrm{GHz}$.


Figure 18: Magnitude of $R_{0}$ upto $f=4 G H z$.


Figure 19: Magnitude of $T_{0}$ upto $f=4 G H z$.
techniques, both of which are considered to be rigorous analyses apart from the slower convergence observed in the latter. The agreement of two methods is excellent. The results illustrated in the previous section provide understanding of the effect of a step discontinuity on the outer wall of a coaxial waveguide on the propagation of TEM waves. This type of scattering mechanism is one of the many occuring in the measurement setup described in [1]. In particular, these results illustrate the great importance of an area change in the modeling of connector mismatch at low frequencies. So, in order to have a further development in the electromagnetic modeling of this measurement setup, one can include the other scattering mechanisms and apply similar analyses. This will be the subjects of forthcoming studies by the authors.

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