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ON FINITE-DIMENSIONAL ABSOLUTE VALUED ALGEBRAS

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This thesis consists of the following papers, referred to by their Roman numerals:

I. S. Alsaody, Morphisms in the Category of Finite-Dimensional Absolute Valued Algebras, Colloq. Math. 125 (2011), pp. 147–174.
II. S. Alsaody, Corestricted Group Actions and Eight-Dimensional Absolute Valued Algebras, submitted for publication.

Introduction

Preliminaries and History. The aim of this thesis is to contribute to the understanding of finite-dimensional absolute valued algebras, where by understanding we mean obtaining a description and a classification of the algebras, and determining the morphisms between them, and their decompositions.

An *algebra* over a field k is a k-vector space endowed with a bilinear (i.e. distributive) multiplication, which is neither assumed to be associative or commutative, nor to admit a unit element. An algebra A is called *absolute valued* [1] if A is a non-zero algebra over $\mathbb{R}$ equipped with a multiplicative norm $\|\|$, i.e. the norm and the multiplication $\cdot$ satisfy

$$\forall x, y \in A, \quad \|x \cdot y\| = \|x\|\|y\|.$$

Absolute valued algebras hence have no zero divisors, which in finite dimension implies that they are division algebras. The most frequently encountered examples are the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ of the real numbers, the complex numbers, the quaternions, and the octonions, respectively.

These four classical examples also illustrate the historical development of the systematic study of absolute valued algebras, which goes almost a century back in time. Indeed, it was proven by Ostrowski [14] in 1916 that every associative and commutative absolute valued algebra is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, and by Mazur [13] around two decades later that every associative, non-commutative such algebra is isomorphic to $\mathbb{H}$. A major step was taken by Albert, who in [1] from 1947 proved that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are, up to isomorphism, the only finite-dimensional absolute valued algebras admitting a unit element, and that, up to isomorphism, every finite dimensional absolute valued algebra A is an orthogonal isotope of precisely one $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, i.e. $A = \mathbb{A}$ as a vector space, and the multiplication $\cdot$ of A is given by

$$\forall x, y \in A, \quad x \cdot y = f(x)g(y),$$

where f and g are orthogonal linear operators, and juxtaposition denotes multiplication in $\mathbb{A}$. Thus the dimension of an absolute valued algebra, if finite, is 1, 2, 4 or 8. In comparison it is interesting to note that Hopf [10] proved in 1940 that every finite-dimensional real division algebra has dimension 2^n for some $n \in \mathbb{N}$, and the

[1] The terminology *normed algebra* occurs, but is sometimes defined differently.

proof that this dimension is at most 8 was given in 1958 independently by Bott and Milnor [3] and Kervaire [12].[2]

While this thesis is concerned with finite-dimensional algebras, we mention two results that apply to the infinite-dimensional case as well. The first, from 1949, is due to Albert [2], who showed that every algebraic absolute valued algebra is finite-dimensional, where an algebra is said to be *algebraic* if each subalgebra generated by one element has finite dimension. About a decade later, Urbanik and Wright showed in [17] that every absolute valued algebra admitting a unit element is isomorphic to one of the four classical examples, and hence in particular finite-dimensional.

A more detailed historical survey is to be found in [16].

General Structure and Algebras of Dimensions 1 and 2. Consider the category $\mathcal{A}$, in which the objects are all finite-dimensional absolute valued algebras, and the morphisms are the non-zero algebra homomorphisms. This is a full subcategory of the category of finite-dimensional real division algebras, which implies that all morphisms in $\mathcal{A}$ are injective. The objects of $\mathcal{A}$ are, by the above, partitioned as

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_4 \cup \mathcal{A}_8,$$

where $\mathcal{A}_d$ is the subcategory of $\mathcal{A}$ consisting of all d-dimensional absolute valued algebras. A general aim is to classify each $\mathcal{A}_d$ up to isomorphism. It is easy to see that $\mathcal{A}_1$ is classified by $\{\mathbb{R}\}$. For $d > 1$, Darpö and Dieterich showed in [7] that $\mathcal{A}_d$ decomposes as a coproduct

$$\mathcal{A}_d = \coprod_{(i,j)\in C_2^2} \mathcal{A}_d^{ij}.$$

To define the components $\mathcal{A}_d^{ij}$, we need the notion of the *double sign*. Indeed, for each $A \in \mathcal{A}$ and each element $a \in A$, the left and right multiplication maps

$$L_a : A \to A, \quad x \mapsto ax \quad \text{and} \quad R_a : A \to A, \quad x \mapsto xa$$

are non-singular, by definition of a division algebra, and their determinants have well-defined signs. It is shown in [7] that these signs are invariants of A, and hence one may define the *double sign* of A to be the pair $(\mathrm{sgn}(\det L_a), \mathrm{sgn}(\det R_a))$ for any $a \in A$. Then

$$\mathcal{A}_d^{ij} = \{A \in \mathcal{A}_d | \text{ the double sign of } A \text{ is } (i,j)\}.$$

The double sign is perhaps best illustrated by the classification of $\mathcal{A}_2$. In this case, for each $(i,j) \in C_2^2$, $\mathcal{A}_2^{ij}$ is classified by the singleton $\{\mathbb{C}^{ij}\}$, where[3]

$$\mathbb{C}^{++} = \mathbb{C}_{\mathrm{Id},\mathrm{Id}}, \quad \mathbb{C}^{+-} = \mathbb{C}_{\kappa,\mathrm{Id}}, \quad \mathbb{C}^{-+} = \mathbb{C}_{\mathrm{Id},\kappa}, \quad \mathbb{C}^{--} = \mathbb{C}_{\kappa,\kappa},$$

and κ denotes complex conjugation.

Descriptions and 4-Dimensional Algebras. When the dimension d is 4 or 8, the problem of classifying $\mathcal{A}_d$ becomes harder. Since the morphisms in $\mathcal{A}$ are injective, each $\mathcal{A}_d$, and further each $\mathcal{A}_d^{ij}$, is a *groupoid*, i.e. a category where all morphisms are isomorphisms. Thus one may attempt to describe these categories in terms of group actions, as follows. Given a group action

$$\alpha : G \times X \to X, \quad (g,x) \mapsto g \cdot x,$$

[2] The numbers 1, 2, 4 and 8 appear in this context already in the work of Hurwitz [11] from 1898 as the only possible finite dimensions in which real quadratic forms admit composition.

[3] To enhance legibility, we write the elements 1 and -1 of C_2 simply as $+$ and $-$, respectively.

we define the *groupoid arising from* α to be the category ${}_GX$ where the objects are the elements of X, and the morphisms are the elements of G in the sense that for each $x, y \in X$,

$$ {}_GX(x,y) = \{(g,x,y) | g \in G, g \cdot x = y\}. $$

(The action α is implicit in the notation.)

A *description* (in the sense of Dieterich, [8]) of a subcategory $\mathcal{C}$ of $\mathcal{A}$ is then a group action $\alpha : G \times X \to X$ and an equivalence of categories $\mathcal{F} : {}_GX \to \mathcal{C}$; we then say that $\mathcal{C}$ is *described by* ${}_GX$. Having a description transfers the classification problem of $\mathcal{C}$ to the normal form problem for the action α, i.e. the problem of finding a transversal for the orbits of α. For the solution of this problem to be useful in the classification of $\mathcal{A}$, it is important that the subcategory $\mathcal{C}$ be full.

This method was used to obtain a classification of $\mathcal{A}_4$. Using results obtained by Ramírez in [15], Forsberg, in [9], constructed a description of each $\mathcal{A}_4^{ij}$. In all four cases, the group action involved was that of SO_3 on $SO_3 \times SO_3$ by simultaneous conjugation, for which the normal form problem was solved. Moreover, the automorphism groups, which hence are subgroups of SO_3, were computed.

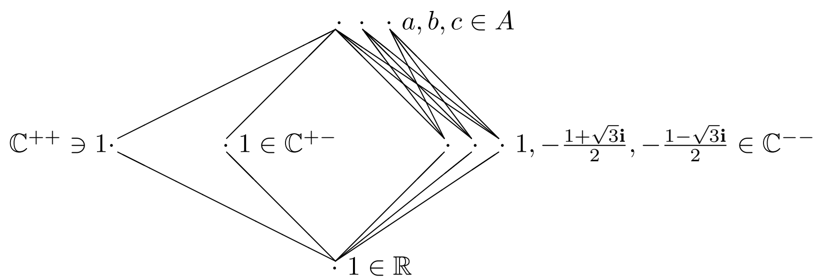
Morphisms and Paper I. At this point, each $\mathcal{A}_d$ with $d \leq 4$ is well understood. To gain a full understanding of the category $\mathcal{A}_{\leq 4}$ of all absolute valued algebras of dimension at most 4, the morphisms between algebras of different dimensions must be investigated as well. This is done in Paper I of this thesis. Injectivity excludes all but three types of morphisms: morphisms from $\mathcal{A}_1$ to $\mathcal{A}_2$ or $\mathcal{A}_4$, and those from $\mathcal{A}_2$ to $\mathcal{A}_4$.

For the first two cases, it suffices to consider the morphisms from $\mathbb{R}$ to A for each absolute valued algebra A of dimension 2 or 4. These morphisms correspond bijectively to the non-zero idempotents of A, which must now be determined. For $A = \mathbb{C}^{ij}$ with $(i,j) \in C_2^2$, this problem is easy and the solution is well known. In Paper I we thus turn our attention to dimension 4, and determine the non-zero idempotents of each $A \in \mathcal{A}_4$. Some idempotents are given explicitly, while others are determined via roots of quintic polynomials. These polynomials are in fact solvable by radicals whenever the double sign is not $(-,-)$. The number of non-zero idempotents in any $A \in \mathcal{A}_4$ is found to be 1, 3, 5, or infinite.

The morphisms from $\mathcal{A}_2$ to $\mathcal{A}_4$ are the embeddings of two-dimensional absolute valued algebras into four-dimensional ones as subalgebras. First, one must determine which four-dimensional algebras admit a given two-dimensional algebra as a subalgebra, which was done in [15]. Then one has to determine the embeddings in the favourable cases, which we do in Paper I by computing the morphisms explicitly.

The morphisms from a 2-dimensional absolute valued algebra to a 4-dimensional one can further be composed with the automorphisms of these algebras. Thus the automorphism groups act on the set of morphisms by composition from either side. We hence use our explicit list of morphisms to determine the number of orbits of these actions for all possible algebras. One result is that whenever the automorphism groups of the domain and codomain act simultaneously from both sides, there is only one orbit, i.e. the action is transitive.

We then investigate the *irreducibility* of the morphisms, i.e. whether a given morphism factors into two morphisms, non of which is an isomorphism. This is only non-trivial in the case of morphisms from $\mathcal{A}_1$ to $\mathcal{A}_4$, and we determine which of these factor via algebras in $\mathcal{A}_2$. Diagrammatically, this may look as follows.



This *morphism quiver* illustrates that the algebra $A \in \mathcal{A}_4$ has three non-zero idempotents a, b and c, each corresponding to a reducible morphism, and further depicts how these morphisms factor via 2-dimensional subalgebras. This particular example depicts a case where all morphisms are reducible; indeed there are cases where none or only some, but not all, of the morphisms are reducible.

Throughout the paper, we note that the algebras of double sign $(-,-)$ behave differently in several aspects. This is remarkable, especially as the four subcategories $\mathcal{A}_4^{ij}$ of $\mathcal{A}_4$ are all equivalent categories.

8-Dimensional Algebras and Paper II. In Paper II we turn our attention to 8-dimensional absolute valued algebras. The classification problem for $\mathcal{A}_8$ is yet unsolved, and has proven quite hard. There exist partial classifications, among which we note the classification of all eight-dimensional absolute valued algebras admitting a one-sided unity or a non-zero central idempotent. This classification was carried out in [6] using a description of each of the full subcategories $\mathcal{A}_8^l$, $\mathcal{A}_8^r$ and $\mathcal{A}_8^c$ of $\mathcal{A}_8$ consisting of all algebras with a left unity, a right unity, and a non-zero central idempotent, respectively. These subcategories were each found to be equivalent to the groupoid arising from the action of $G_2 = \mathrm{Aut}(\mathbb{O})$ on the orthogonal group O_7 by conjugation, and the normal form problem for this action was solved.

A necessary and sufficient condition for two arbitrary eight-dimensional absolute valued algebras to be isomorphic was given in [4]. From this condition we deduce, in Paper II, a description of $\mathcal{A}_8$. We obtain thus an equivalence of categories

$$_{SO_8}\mathcal{O}_8 \to \mathcal{A}_8$$

where $\mathcal{O}_8$ is the quotient group $(O_8 \times O_8)/\{\pm(\mathbf{1},\mathbf{1})\}$. The group action of the special orthogonal group SO_8 involves *triality*, which deserves some explanation. The *Principle of Triality*, due to Elie Cartan ([5], 1925), implies that for each $\phi \in SO_8$ there exists a pair $(\phi_1, \phi_2) \in SO_8 \times SO_8$ of *triality components*, unique up to overall sign, such that

$$\forall x, y \in \mathbb{O}, \quad \phi(xy) = \phi_1(x)\phi_2(y).$$

where juxtaposition is multiplication in $\mathbb{O}$. In general, triality components are difficult to compute, but for $\phi \in G_2$ we simply have $\phi_1 = \phi_2 = \pm\phi$. For algebras with a one-sided unity or a non-zero central idempotent, the action of SO_8 on $\mathcal{O}_8$ in the description reduces to the action of G_2 on O_7 by conjugation, and we recover the above descriptions of $\mathcal{A}_8^l$, $\mathcal{A}_8^r$ and $\mathcal{A}_8^c$.

Inspired by these three examples, we set out to systematically construct full subcategories of $\mathcal{A}_8$ for which the classification problem is simplified. For this, we use the description we have obtained, and consider subgroupoids of $_{SO_8}\mathcal{O}_8$ arising from the corestriction of the action of SO_8 to subsets of $\mathcal{O}_8$. The first question is for which subsets this subgroupoid is full. We address this question in broader generality, considering a groupoid ${}_GX$ arising from an arbitrary group action, and defining, for each $Y \subseteq X$, the *sharp stabilizer* $\mathrm{St}^*(Y)$ to be the largest subgroup of G that stabilizes Y as a set. The question is then for which $Y \subseteq X$ the subgroupoid ${}_{\mathrm{St}^*(Y)}Y$ is full. We give precise conditions for this, and prove some properties of the subgroupoid ${}_{\mathrm{St}^*(Y)}Y$ under these conditions. This provides a method for finding and studying full subcategories of a category for which a description is known.

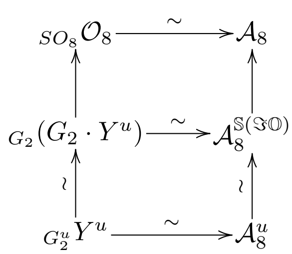
Returning to $\mathcal{A}_8$, we see how $\mathcal{A}_8^l$, $\mathcal{A}_8^r$ and $\mathcal{A}_8^c$ fit into this framework. Then we use the description of $\mathcal{A}_8$ and proceed by the above to construct full subcategories whose morphisms are elements of G_2, which avoids triality computations and thus simplifies classification. More precisely we construct, for each imaginary octonion u of unit length, the subcategory $\mathcal{A}_8^u$ of *left u-reflection algebras*. This is the subcategory of $\mathcal{A}_8$ described by the groupoid ${}_{\mathrm{St}^*(Y^u)}Y^u \subseteq {}_{SO_8}\mathcal{O}_8$, where

$$Y^u = \{[(f, \sigma_u)] \in \mathcal{O}_8 | f(1) = 1\},$$

with σ_u denoting the reflection in the hyperplane $u^\perp$. Then $G_2^u := \mathrm{St}^*(Y^u) \subseteq G_2$. We reduce the classification problem using the explicit classification of $\mathcal{A}_8^l$ from [6], and solve it in some cases. Moreover, we show, using the framework above, that any $\mathcal{A}_8^u$ is equivalent to the larger full subcategory $\mathcal{A}_8^{\mathbb{S}(\Im\mathbb{O})}$ of $\mathcal{A}_8$ with object set

$$\{\mathbb{O}_{f,\sigma_w} \in \mathcal{A}_8 | f(1) = 1, w \text{ is an imaginary octonion of length } 1\}.$$

Indeed we get the following commutative diagram of categories and full functors, where functors marked with $\sim$ are equivalences of categories.



Future Directions. In [4] it is proven that each 8-dimensional absolute valued algebra is isomorphic to an algebra of type $\mathbb{O}_{f,g}$ where f and g are orthogonal maps fixing $1 \in \mathbb{O}$. The previous section thus suggests a tentative method to divide the study of $\mathcal{A}_8$ systematically into smaller parts. Indeed, the Cartan–Dieudonné Theorem implies that every orthogonal operator on $\mathbb{R}^n$ is the product of at most n hyperplane reflections. Thus each eight-dimensional absolute valued algebra is isomorphic to some $\mathbb{O}_{f,g}$ where f and g fix $1 \in \mathbb{O}$ and, say, g is the product of n reflections, where $0 \le n \le 7$. We are now in the following situation:

- Those $\mathbb{O}_{f,g}$ where f and g fix $1 \in \mathbb{O}$ and g is the product of no reflections (hence is the identity) constitute a subset of $\mathcal{A}_8^l$, which was classified in [6].
- Each $\mathbb{O}_{f,g}$ where f and g fix $1 \in \mathbb{O}$ and g is (the product of) one reflection is isomorphic to a left u-reflection algebra for an imaginary octonion u of unit length; such algebras are considered in Paper II.

One may thus attempt to use the methods of Paper II to investigate, for each $2 \le n \le 7$, the set of all algebras $\mathbb{O}_{f,g}$ where f and g fix $1 \in \mathbb{O}$ and g is the product of n reflections. The case of two reflections appears to be treatable in analogy with that of one reflection, the main difference being an increase in the amount of computations. For larger n, the situation is more involved. To begin with, as the number of reflections is not invariant under isomorphism, one must, for each n, exclude such left n-reflection algebras that are isomorphic to left n'-reflection algebras for some $n' < n$. Secondly, for $n \geq 3$, it is not necessarily the case that all morphisms are G_2-morphisms, and hence care must be taken to avoid, or to compute, triality components of different maps. It is the hope of the author to be able to investigate these matters in a forthcoming publication.

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MORPHISMS IN THE CATEGORY OF FINITE-DIMENSIONAL ABSOLUTE VALUED ALGEBRAS

SEIDON ALSAODY (UPPSALA)

Abstract. This is a study of morphisms in the category of finite dimensional absolute valued algebras, whose codomains have dimension four. We begin by citing and transferring a classification of an equivalent category. Thereafter, we give a complete description of morphisms from one-dimensional algebras, partly via solutions of real polynomials, and a complete, explicit description of morphisms from two-dimensional algebras. We then give an account of the reducibility of the morphisms, and for the morphisms from two-dimensional algebras we describe the orbits under the actions of the automorphism groups involved. Parts of these descriptions rely on a suitable choice of a cross-section of four-dimensional absolute valued algebras, and we thus end by providing an explicit means of transferring these results to algebras outside this cross-section.

1. Definitions and background

An *algebra* $A = (A, \cdot)$ over a field k is a vector space A over k equipped with a k-bilinear multiplication $A \times A \to A, (x, y) \mapsto xy = x \cdot y$. Neither associativity nor commutativity is in general assumed. A is called *unital* if it contains an element neutral under multiplication; in that case, such an element is unique, and will be denoted by 1. If A is non-zero, and if for each $a \in A \setminus \{0\}$, the maps $L_a : A \to A, x \mapsto ax$ and $R_a : A \to A, x \mapsto xa$ are bijective, A is called a *division algebra*. This implies that A has no zero divisors and, if the dimension of A is finite, it is equivalent to having no zero divisors.

An algebra A is called *absolute valued* if the vector space is real and equipped with a norm $\| \cdot \|$ such that $\|xy\| = \|x\|\|y\|$ for all $x, y \in A$. By [1] the norm in a finite dimensional absolute valued algebra is uniquely determined by the algebra multiplication if the algebra has finite dimension. The multiplicativity of the norm implies that an absolute valued algebra has no zero divisors and hence, if it is finite dimensional, that it is a division algebra. The class of all finite dimensional absolute valued algebras forms a category $\mathcal{A}$, in which the morphisms are the non-zero algebra homomorphisms. Thus $\mathcal{A}$ is a full subcategory of the category $\mathcal{D}(\mathbb{R})$ of finite dimensional real division algebras. It is known that morphisms in $\mathcal{A}$ respect the norm, and are hence injective. (Injectivity in fact holds for all morphisms in $\mathcal{D}(\mathbb{R})$.)

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Key words and phrases. Absolute valued algebra, division algebra, homomorphism, irreducibility, composition.

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1.1. **Notation.**

1.1.1. *Complex numbers and quaternions.* The real and imaginary part of $A \in \{\mathbb{C}, \mathbb{H}\}$ will be denoted by $\Re A$ and $\Im A$, respectively. We also use the notation $a = \Re(a) + \Im(a)$ for elements $a \in A$. The letters $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote the standard basis of the imaginary space $\Im\mathbb{H}$ of the quaternion algebra $\mathbb{H}$, and $\mathbf{i}$ will also be used as the imaginary unit in $\mathbb{C}$ as confusion is improbable. Complex and quaternion conjugation (negation of the imaginary part) will be denoted by $x \mapsto \overline{x}$ for $x \in \mathbb{C}$ or $x \in \mathbb{H}$. A quaternion with vanishing imaginary part and real part r is simply denoted by r in view of the embedding of $\mathbb{R}$ into $\mathbb{H}$, and the notation $\mathbb{S}(\mathbb{H})$ and $\mathbb{S}(\Im\mathbb{H})$ will be used for the set of quaternions of norm one and the set of purely imaginary quaternions of norm one, respectively. For $p \in \mathbb{S}(\mathbb{H})$ we have $p^{-1} = \overline{p}$, and we will denote the map $x \mapsto pxp^{-1} = px\overline{p}$ by κ_p and refer to it as conjugation by p.

1.1.2. *Other conventions.* Throughout the paper, the abbreviations $\nu_c := \cos\nu$ and $\nu_s := \sin\nu$ will be used to enhance readability, as trigonometric expressions are abundant in many equations, where at the same time the trigonometry itself is of little importance.

Moreover, the elements 1 and -1 of the cyclic group C_2 will often be written simply as $+$ and $-$, respectively. If n is a positive integer, the notation $\underline{n} = \{k \in \mathbb{N} \mid 1 \leq k \leq n\}$ will be used. Square brackets [ ] around a sequence of vectors will denote their span, whereas $\langle , \rangle$ denotes the following inner product of two quaternions: given $x = s_0 + s_1\mathbf{i} + s_2\mathbf{j} + s_3\mathbf{k}$ and $x' = s'_0 + s'_1\mathbf{i} + s'_2\mathbf{j} + s'_3\mathbf{k}$, set $\langle x, x' \rangle = \sum_{i=0}^{3} s_i s'_i$. The norm of the absolute valued algebra $\mathbb{H}$ is then given by $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in \mathbb{H}$.

Finally, given a category $\mathcal{C}$, and objects $A, B \in \mathcal{C}$, the class of morphisms in $\mathcal{C}$ from A to B is denoted $\mathcal{C}(A, B)$. Given a group G acting from the left on a set S, we denote by ${}_G S$ the category whose object class is S, and in which for $x, y \in S$, a morphism from x to y is a triple (x, y, g) such that $g \cdot x = y$. When the objects x and y are clear from context, we will denote such a morphism simply by g to avoid cumbersome notation.

1.2. **History and outline.**

In 1947, Albert characterizes all finite dimensional absolute valued algebras as follows. [1]

Proposition 1.1. *Every absolute valued algebra is isomorphic to an orthogonal isotope $(A, \cdot)$ of a unique $A' \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, i.e. $A = A'$ as a vector space, and the multiplication in A is given by*

$$x \cdot y = f(x)g(y)$$

for all $x, y \in A$, where f and g are linear orthogonal operators on A, and juxtaposition is multiplication in A'.

Moreover, Albert shows in [1] that the norm in A coincides with the norm defined in A'.

Thus the objects of $\mathcal{A}$ are partitioned into four classes according to their dimension, and the class of d-dimensional algebras, $d \in \{1, 2, 4, 8\}$, forms a full subcategory $\mathcal{A}_d$ of $\mathcal{A}$. For $d > 1$ we moreover have the following decomposition due to Darpö and Dieterich [6].

PROPOSITION 1.2. *Let $A \in \mathcal{A}_d$ where $d \in \{2,4,8\}$. For each $a, b \in A \setminus \{0\}$ it holds that* $\mathrm{sgn}(\det(L_a)) = \mathrm{sgn}(\det(L_b))$ *and* $\mathrm{sgn}(\det(R_a)) = \mathrm{sgn}(\det(R_b))$.[1] *The* double sign *of A is the pair $(i,j) \in C_2^2$ where $i = \mathrm{sgn}(\det(L_a))$ and $j = \mathrm{sgn}(\det(R_a))$ for all $a \in A \setminus \{0\}$. Moreover, for all $d \in \{2,4,8\}$, it holds that*

$$\mathcal{A}_d = \coprod_{(i,j)\in C_2^2} \mathcal{A}_d^{ij} \tag{1.1}$$

where $\mathcal{A}_d^{ij}$ is the full subcategory of $\mathcal{A}_d$ formed by all objects having double sign (i,j).

Furthermore, the following has been achieved towards obtaining a complete understanding of the category $\mathcal{A}$.

- A classification of the categories $\mathcal{A}_1$ and $\mathcal{A}_2$, and a complete description of the set $\mathcal{A}(\mathbb{R}, B)$ for $B \in \mathcal{A}_2$.
- A classification of the category ${}_{SO_3}(SO_3 \times SO_3)$, where the action is by simultaneous conjugation, and a proof that this category is equivalent to $\mathcal{A}_4^{kl}$ for any $(k,l) \in C_2^2$. The equivalence is expressed in terms of a category $\mathcal{C}$ and equivalences $\mathcal{F}^{kl} : \mathcal{C} \to \mathcal{A}_4^{kl}$ and $\mathcal{G} : \mathcal{C} \to {}_{SO_3}(SO_3 \times SO_3)$, see [11].
- A description of the automorphism groups in $\mathcal{A}_4$, see [11].
- An explicit description of all those $A \in \mathcal{A}_4$ for which there is a morphism $\phi : C \to A$ for some $C \in \mathcal{A}_2$, see [12].
- Conditions for when two eight-dimensional absolute valued algebras are isomorphic, and an exhaustive list of the algebras in $\mathcal{A}_8$ obtained from algebras in $\mathcal{A}_4$ by a so called duplication process, see [2].[2]
- Partial classifications of the category $\mathcal{A}_8$, see e.g. [4]. These use results on pairs of rotations in Euclidean space, studied in [5] and [8].

In the remainder of this section, the first item in this list will be summarized. Section 2 recollects the results of the second item, and expresses it in terms of a cross-section for $\mathcal{A}_4$. The main results of the present article, and consequences thereof, are given in Section 3, where we investigate morphisms from $\mathbb{R}$ to absolute valued algebras of dimension four, and in Section 4, where the same is done for morphisms from two-dimensional absolute valued algebras. In Section 5 we study the irreducibility of the morphisms of Section 3, and in Section 6 we determine of the number of orbits of $\mathcal{A}(C, A)$ for $C \in \mathcal{A}_2$ and $A \in \mathcal{A}_4$, under the action of the automorphism groups of C and A by composition. The final section supplies technical arguments to carry results that have been obtained for a specific cross-section of $\mathcal{A}_4$ to general four-dimensional absolute valued algebras.

1.3. **Basic results.** It is known that $\mathcal{A}_1$ is classified by $\mathbb{R}$, and that every $C \in \mathcal{A}_2$ with double sign $(i,j) \in C_2^2$ is isomorphic to $\mathbb{C}^{ij}$, this being the algebra with underlying vector space $\mathbb{C}$, and multiplication

$$(x, y) \mapsto x_j y_i,$$

[1] The sign function $\mathrm{sgn} : \mathbb{R} \setminus \{0\} \to \{1, -1\}$ is defined by $\mathrm{sgn}(r) = r/|r|$.

[2] The above mentioned duplication process is similar to the construction of doubled eight-dimensional real quadratic division algebras, which are studied in [10]. The category of all real quadratic division algebras is equivalent to the category of dissident triples, due to [7] and [10].

where $\forall c \in \mathbb{C}, c_+ = c$ and $c_- = \overline{c}$, and juxtaposition is multiplication in $\mathbb{C}$.[3]

To describe the morphisms from $\mathbb{R}$ to algebras of dimension two, we recall the following result, which will be important in the coming sections.

PROPOSITION 1.3. *Let A be a finite dimensional absolute valued algebra, and let $\mathrm{Ip}(A)$ be the set of all idempotents in $A \setminus \{0\}$. Then*

(1) $\mathrm{Ip}(A) \neq \emptyset$, and
(2) for each algebra homomorphism $\psi : \mathbb{R} \to A$, $\psi(1)$ is an idempotent, and the map $\psi \mapsto \psi(1)$ defines a one to one correspondence between $\mathcal{A}(\mathbb{R}, A)$ and $\mathrm{Ip}(A)$.

The first item in fact holds for any finite dimensional non-zero real or complex algebra where $x^2 \neq 0$ for each $x \neq 0$ [13], and the second is readily checked. For absolute valued algebras of dimension two, it is known that $\mathrm{Ip}(\mathbb{C}^{ij}) = \{1\}$ for $(i, j) \neq (-, -)$, and $\mathrm{Ip}(\mathbb{C}^{--}) = \{x \in \mathbb{C} \mid x^3 = 1\}$. Hence, the category $\mathcal{A}_{\leq d}$ of absolute valued algebras with dimension at most d is understood for $d = 2$, and we intend to gain the same understanding of $\mathcal{A}_{\leq 4}$.

2. ABSOLUTE VALUED ALGEBRAS OF DIMENSION FOUR

2.1. Introduction.

In view of Proposition 1.2, the category $\mathcal{A}_4$ of four-dimensional absolute valued algebras admits the decomposition

$$\mathcal{A}_4 = \coprod_{(k,l) \in C_2^2} \mathcal{A}_4^{kl} \tag{2.1}$$

where for each $(k, l) \in C_2^2$, $\mathcal{A}_4^{kl}$ consists of all algebras in $\mathcal{A}_4$ with double sign (k, l). Each object in $\mathcal{A}_4$ is isomorphic to an object with multiplication defined in terms of quaternion multiplication as follows. [12]

PROPOSITION 2.1. *For each $A \in \mathcal{A}_4^{kl}$ there exists $A' = (A', \cdot) \in \mathcal{A}_4^{kl}$ and $a, b \in \mathbb{S}(\mathbb{H})$, such that $A \simeq A'$ and the multiplication $\cdot$ is given by*

$$\begin{array}{lll} x \cdot y = axyb & \text{if} & (k,l) = (+,+), \\ x \cdot y = \overline{x}ayb & \text{if} & (k,l) = (+,-), \\ x \cdot y = axb\overline{y} & \text{if} & (k,l) = (-,+), \text{ and} \\ x \cdot y = a\overline{x}\,\overline{y}b & \text{if} & (k,l) = (-,-), \end{array} \tag{2.2}$$

where juxtaposition denotes multiplication in $\mathbb{H}$. Conversely, given any $a, b \in \mathbb{S}(\mathbb{H})$, (2.2) determines the structure of an algebra in $\mathcal{A}_4^{kl}$ for each $(k, l) \in C_2^2$.

An algebra $A' \in \mathcal{A}_4^{kl}$ with multiplication given by (2.2) for some $a, b \in \mathbb{S}(\mathbb{H})$ will be denoted by $\mathbb{H}^{kl}(a, b)$.

2.2. Classification.

It was shown in [11] that for each $(k, l) \in C_2^2$ there are equivalences of categories

$$\mathcal{A}_4^{kl} \xleftarrow{\mathcal{F}^{kl}} {}_E(\mathbb{S}(\mathbb{H}) \times \mathbb{S}(\mathbb{H})) \xrightarrow{\mathcal{G}} {}_{SO_3}(SO_3 \times SO_3) \tag{2.3}$$

where $E = C_2^2 \times (\mathbb{S}(\mathbb{H})/\{1, -1\})$ acts on $\mathbb{S}(\mathbb{H}) \times \mathbb{S}(\mathbb{H})$ by

$$E \times (\mathbb{S}(\mathbb{H}) \times \mathbb{S}(\mathbb{H})) \to \mathbb{S}(\mathbb{H}) \times \mathbb{S}(\mathbb{H}), ((\epsilon, \delta, p\{1, -1\}), (a, b)) \mapsto (\epsilon pa\overline{p}, \delta pb\overline{p}),$$

[3] The notation $\mathbb{C}^{ij}$ is used due to practical advantages over the standard notation $\mathbb{C} = \mathbb{C}^{++}$, ${}^*\mathbb{C} = \mathbb{C}^{+-}$, $\mathbb{C}^* = \mathbb{C}^{-+}$, and $\overset{*}{\mathbb{C}} = \mathbb{C}^{--}$.

and SO_3 acts on $SO_3 \times SO_3$ by simultaneous conjugation

$$SO_3 \times (SO_3 \times SO_3) \to SO_3 \times SO_3, (\rho, (\phi, \psi)) \mapsto (\rho\phi\rho^{-1}, \rho\psi\rho^{-1}).$$

The functors $\mathcal{F}^{kl}$ are defined on objects by $\mathcal{F}^{kl}(a, b) = \mathbb{H}^{kl}(a, b)$, and on morphisms by

$$\mathcal{F}^{kl}(\epsilon, \delta, p\{1, -1\}) = \epsilon\delta\kappa_p.$$

The functor $\mathcal{G}$ is defined on objects by $\mathcal{G}(a, b) = (\kappa_a, \kappa_b)$, and $\mathcal{G}(\epsilon, \delta, p\{1, -1\})$ is the morphism defined by

$$(\phi, \psi) \mapsto (\kappa_p\phi\kappa_{\overline{p}}, \kappa_p\psi\kappa_{\overline{p}})$$

for each $(\phi, \psi) \in SO_3 \times SO_3$. The fact that these constructions are well-defined was shown in [11].

We begin by applying the equivalences of categories to express the classification of ${}_{SO_3}(SO_3 \times SO_3)$, given in [11], as a classification of all four-dimensional absolute valued algebras, i.e. to describe the image of the given cross-section of ${}_{SO_3}(SO_3 \times SO_3)$ under the functor

$$\mathcal{F}^{kl} \circ \mathcal{H}$$

for each $(k, l) \in C_2^2$, where $\mathcal{H}$ is a quasi-inverse functor to $\mathcal{G}$. This is the content of the following result.

Theorem 2.2. *Let $u, v \in \mathbb{S}(\Im\mathbb{H})$ be any two orthogonal elements. Let $(k, l) \in C_2^2$ and $A \in \mathcal{A}_4^{kl}$. Then $A \simeq \mathbb{H}^{kl}(a, b)$ where a, b are given by*

$$a = \alpha_c + \alpha_s u, \quad b = \beta_c + \beta_s(\gamma_c u + \gamma_s v) \tag{2.4}$$

for precisely one triple (α, β, γ) satisfying one of

(1) $(\alpha, \beta, \gamma) \in [0, \pi/2] \times \{0\} \times \{0\}$,
(2) $(\alpha, \beta, \gamma) \in \{0\} \times (0, \pi/2] \times \{0\}$,
(3) $(\alpha, \beta, \gamma) \in (0, \pi/2) \times (0, \pi) \times [0, \pi/2)$,
(4) $(\alpha, \beta, \gamma) \in \{\pi/2\} \times (0, \pi/2] \times [0, \pi/2)$, *or*
(5) $(\alpha, \beta, \gamma) \in (0, \pi/2] \times (0, \pi/2] \times \{\pi/2\}$.

Remark 2.3. Note that in case 1 above, the restriction on γ is for the sake of uniqueness; indeed, when $\beta = 0$, it holds that $b = 1$ for any value of γ. Observe moreover that the five cases are mutually exclusive.

Theorem 2.2 follows from the classification of ${}_{SO_3}(SO_3 \times SO_3)$ and the explicit description of the equivalences of categories (2.3) given in [11] and quoted above. These use the following fact proved in [3]:

Given a quaternion $q = \cos\theta + w \sin\theta$, where $w \in \mathbb{S}(\Im\mathbb{H})$, the map $x \mapsto qx\overline{q}$ is a rotation in $\Im\mathbb{H}$ with axis w and angle of rotation 2θ.

We fix a pair of quaternions $u, v \in \mathbb{S}(\Im\mathbb{H})$ for the sake of definiteness as follows.

Definition 2.4. The set of all $\mathbb{H}^{kl}(a, b) \in \mathcal{A}_4$, with $(k, l) \in C_2^2$ and

$$a = \alpha_c + \alpha_s\mathbf{i}, \quad b = \beta_c + \beta_s(\gamma_c\mathbf{i} + \gamma_s\mathbf{j}) \tag{2.5}$$

with (α, β, γ) as in Theorem 2.2, is called the *canonical cross-section of $\mathcal{A}_4$*.

The particular choice of orthogonal quaternions in Definition 2.4 is made in order to simplify calculations, and will be used throughout.

3. Morphisms from $\mathbb{R}$ to four-dimensional algebras

3.1. **Preparatory results.** We now study morphisms from the unique (up to isomorphism) one-dimensional absolute valued algebra $\mathbb{R}$ to four-dimensional algebras belonging to the canonical cross-section of Definition 2.4, thus acquiring an understanding of $\mathcal{A}(\mathbb{R}, A)$ for each $A \in \mathcal{A}_4$. Moreover, the results of Section 7 below transfer details specific to algebras of the canonical cross-section to any four-dimensional absolute valued algebra given as $\mathbb{H}^{kl}(a, b)$ for some $a, b \in \mathbb{S}(\mathbb{H})$.

By virtue of Proposition 1.3, for each $A \in \mathcal{A}_4$, describing $\mathcal{A}(\mathbb{R}, A)$ amounts to describing all non-zero idempotents in A. Rewriting the equations (2.2) with $y = x$ we thus see that these idempotents are precisely the non-zero solutions to the quaternion equation

$$\begin{aligned} x^2 &= \overline{a}x\overline{b} \quad \text{for } \mathcal{A}_4^{++}, \\ x^2 &= axb \quad \text{for } \mathcal{A}_4^{+-} \text{ and } \mathcal{A}_4^{-+}, \text{ and} \\ \overline{x}^2 &= \overline{a}x\overline{b} \quad \text{for } \mathcal{A}_4^{--}. \end{aligned} \tag{3.1}$$

To simplify the quadratic terms in the above equations, we recall the notion of a quadratic algebra.

Definition 3.1. An algebra A over a field k is called *quadratic* if it is non-zero, unital, and if for each $x \in A$ there exist $\lambda, \mu \in k$ such that

$$x^2 = \lambda x + \mu 1.$$

Calculating x^2 for arbitrary $x \in \mathbb{H}$ proves the following result.

Lemma 3.2. *$\mathbb{H}$ is quadratic and each $x \in \mathbb{H}$ satisfies $x^2 = 2\Re(x)x - \|x\|^2 1$.*

With this in mind, we construct for each real number a set of matrices in $\mathbb{R}^{4\times4}$, to be used as the main tool in investigating non-zero idempotents.

Definition 3.3. Given $a, b \in \mathbb{S}(\mathbb{H})$, and $(k, l) \in C_2^2$, the maps $M_{a,b}^{kl} : \mathbb{R} \to \mathbb{R}^{4\times4}$ are defined by

1. $M_{a,b}^{++}(r) = 2rI - L_{\overline{a}}R_{\overline{b}}$
2. $M_{a,b}^{+-}(r) = M_{a,b}^{-+}(r) = 2rI - L_aR_b$
3. $M_{a,b}^{--}(r) = 2rK - L_{\overline{a}}R_{\overline{b}}$

for all $r \in \mathbb{R}$, where I is the identity matrix in $\mathbb{R}^{4\times4}$ and K the matrix of quaternion conjugation.

Now, due to Lemma 3.2, the following proposition outlines the method that will be used to determine the idempotents. To simplify notation we identify a quaternion $x = r + s_1\mathbf{i} + s_2\mathbf{j} + s_3\mathbf{k}$ with the column matrix $(r, s_1, s_2, s_3)^T$, and use the notation L_c and R_c, $c \in \mathbb{H}$, also for the matrices in the standard basis of left and right multiplication by c, respectively.

Proposition 3.4. *Given $(k, l) \in C_2^2$, and $a, b \in \mathbb{S}(\mathbb{H})$, let $A = \mathbb{H}^{kl}(a, b)$, and let $x = r + s_1\mathbf{i} + s_2\mathbf{j} + s_3\mathbf{k} \in A$. Then*

(1) $x \in \mathrm{Ip}(A)$ if and only if $M_{a,b}^{kl}(r)x = 1$ and $\|x\| = 1$, and
(2) if A belongs to the canonical cross-section, then for each fixed r, the quaternion equation $M_{a,b}^{kl}(r)x = 1$ is equivalent to a linear system of four real equations in the variables $s_i, i \in \underline{3}$.

Proof. We prove the statements for $(k,l)=(+,+)$. The other cases are proven analogously.

(1) We have

$$M^{++}_{a,b}(r)x = 2rx - \overline{a}x\overline{b}.$$

Assume that $x \in \mathbb{H}$ satisfies $M^{++}_{a,b}(r)x = 1$ and $\|x\| = 1$. Then $\overline{a}x\overline{b} = 2rx - 1 = 2\Re(x)x - \|x\|^2 1$, which by virtue of Lemma 3.2 implies the equation (3.1) corresponding to $(k,l)=(+,+)$. Hence x is non-zero and idempotent. Conversely, if x is non-zero and idempotent, then by multiplicativity of the norm, $\|x\|=1$, and

$$M^{++}_{a,b}(r)x = 2rx - \overline{a}x\overline{b} = 2\Re(x)x - \|x\|^2 1 + 1 - \overline{a}x\overline{b} = 1 + x^2 - \overline{a}x\overline{b} = 1$$

where the two rightmost equalities follow from Lemma 3.2 and (3.1).

(2) Writing out the equation componentwise, one obtains

$$\begin{aligned} 2r^2 - 1 &= (\alpha_c\beta_c - \alpha_s\beta_s\gamma_c)r + (\alpha_s\beta_c + \alpha_c\beta_s\gamma_c)s_1 \\ &\quad + \alpha_c\beta_s\gamma_s s_2 + \alpha_s\beta_s\gamma_s s_3 \end{aligned} \tag{3.2}$$

$$\begin{aligned} 2rs_1 &= -(\alpha_s\beta_c + \alpha_c\beta_s\gamma_c)r + (\alpha_c\beta_c - \alpha_s\beta_s\gamma_c)s_1 \\ &\quad - \alpha_s\beta_s\gamma_s s_2 + \alpha_c\beta_s\gamma_s s_3 \end{aligned} \tag{3.3}$$

$$\begin{aligned} 2rs_2 &= -\alpha_c\beta_s\gamma_s r - \alpha_s\beta_s\gamma_s s_1 \\ &\quad + (\alpha_c\beta_c + \alpha_s\beta_s\gamma_c)s_2 + (\alpha_s\beta_c - \alpha_c\beta_s\gamma_c)s_3 \end{aligned} \tag{3.4}$$

$$\begin{aligned} 2rs_3 &= \alpha_s\beta_s\gamma_s r - \alpha_c\beta_s\gamma_s s_1 \\ &\quad + (\alpha_c\beta_s\gamma_c - \alpha_s\beta_c)s_2 + (\alpha_c\beta_c + \alpha_s\beta_s\gamma_c)s_3. \end{aligned} \tag{3.5}$$

Fixing r, this is a linear system in $s_i, i \in \underline{3}$, with real coefficients.

□

3.2. **Description of idempotents.** In order to describe the idempotents in each four-dimensional absolute valued algebra, we split into cases according to the double sign of the algebra, and determine the non-zero idempotents by solving the equations of Proposition 3.4(1) for the double sign in question. The results are presented below. It turns out that the algebras having double sign $(-,-)$ have substantially different properties with respect to idempotents, and therefore we present this case separately. The computations, however, are analogous to those of the other cases.

3.2.1. *Idempotents in $\mathbb{H}^{kl}(a,b)$ with $(k,l) \neq (-,-)$.* In this section, the non-zero idempotents are given either explicitly or in terms of roots of a real polynomial. To begin with, this polynomial, together with a number of other functions to be used, are defined.

Definition 3.5. Given $(k,l) \in C_2^2 \setminus \{(-,-)\}$, let $A = \mathbb{H}^{kl}(a,b)$ be in the canonical cross-section with a,b given in terms of (α,β,γ) by (2.5), and set $\sigma = -kl$. Define $p = p^{kl}_{a,b}, q = q^{kl}_{a,b} \in \mathbb{R}[X]$ and $t_i = t^{kl}_{i,a,b} \in \mathbb{R}(X), i \in \underline{3}$, by

$$p(X) = (4X^3 - 8\alpha_c\beta_c X^2 + (4\alpha_c^2 + 4\beta_c^2 - 3)X + \alpha_s\beta_s\gamma_c - \alpha_c\beta_c)(4X^2 - 1),$$

$$\begin{aligned} q(X) = \alpha_s\beta_s\gamma_s(8X^3 - 4(3\alpha_c\beta_c + \alpha_s\beta_s\gamma_c)X^2 + (4\alpha_c^2 + 4\beta_c^2 - 2)X \\ + \alpha_s\beta_s\gamma_c - \alpha_c\beta_c), \end{aligned}$$

$$t_1(X) = \sigma\alpha_s\beta_s\gamma_s X\frac{(\alpha_s\beta_c + \alpha_c\beta_s\gamma_c)(4X^2+1) - 4(\alpha_c\alpha_s + \beta_c\beta_s\gamma_c)X}{q(X)},$$

$$t_2(X) = \sigma\alpha_s\beta_s^2\gamma_s^2 X\frac{\alpha_c(4X^2+1) - 4\beta_c X}{q(X)}, \quad t_3(X) = \alpha_s^2\beta_s^2\gamma_s^2 X\frac{4X^2-1}{q(X)}.$$

Using Proposition 3.4(1) to determine the non-zero idempotents, we arrive at the following result.

THEOREM 3.6. *Given* $(k,l) \in C_2^2 \setminus \{(-,-)\}$, *let* $A = \mathbb{H}^{kl}(a,b)$ *be in the canonical cross-section with* a, b *given in terms of* (α,β,γ) *by (2.5), and set* $\sigma = -kl$. *Let moreover* p, q *and* t_i, $i \in \underline{3}$, *be given by Definition 3.5.*

(1) *If* $\gamma = 0$, *then* $x = (\alpha+\beta)_c + \sigma(\alpha+\beta)_s\mathbf{i}$ *is the unique isolated non-zero idempotent in* A.
(2) *If* $\gamma = 0$ *and* $\alpha = \beta > \pi/6$, *then the points of the set*

$$\left\{\frac{1}{2} + \sigma\frac{\alpha_c}{2\alpha_s}\mathbf{i} + s_2\mathbf{j} + s_3\mathbf{k} \mid s_2^2 + s_3^2 = 1 - \frac{1}{4\alpha_s^2}\right\}$$

are precisely the non-isolated idempotents in A.
(3) *If* $\gamma \neq 0$ *and* $\alpha_c\beta_c = \alpha_s\beta_s\gamma_c$, *then* $\sigma\beta_c\mathbf{i}/\alpha_s + \sigma\alpha_c\beta_s\gamma_s\mathbf{j} - \alpha_s\beta_s\gamma_s\mathbf{k}$ *is a non-zero idempotent.*
(4) *If* $\gamma \neq 0$, *and* $r \in \mathbb{R}$ *satisfies*

$$p(r) = 0 \neq q(r) \text{ and } r^2 + \sum_{i=1}^{3} t_i(r)^2 = 1,$$

then $r + t_1(r)\mathbf{i} + t_2(r)\mathbf{j} + t_3(r)\mathbf{k}$ *is a non-zero idempotent.*
(5) *Every non-zero idempotent in* A *is given by precisely one of the cases 1–4.*

Proof. We outline the main details of the computations in the case of double sign $(+,+)$, as again the other cases are proven analogously. To this end we solve the equations (3.2)–(3.5) above.

For each fixed r, we take three equations among (3.2)–(3.5); our choice will be (3.3)–(3.5). In the variables $s_i, i \in \underline{3}$, this gives a system of linear equations with coefficient matrix

$$M = \begin{pmatrix} -\alpha_s\beta_s\gamma_s & \alpha_c\beta_c + \alpha_s\beta_s\gamma_c - 2r & \alpha_s\beta_c - \alpha_c\beta_s\gamma_c \\ -\alpha_c\beta_s\gamma_s & \alpha_c\beta_s\gamma_c - \alpha_s\beta_c & \alpha_c\beta_c + \alpha_s\beta_s\gamma_c - 2r \\ \alpha_c\beta_c - \alpha_s\beta_s\gamma_c - 2r & -\alpha_s\beta_s\gamma_s & \alpha_c\beta_s\gamma_s \end{pmatrix}$$

and right hand side

$$N = \begin{pmatrix} \alpha_c\beta_s\gamma_s r \\ -\alpha_s\beta_s\gamma_s r \\ (\alpha_s\beta_c + \alpha_c\beta_s\gamma_c)r \end{pmatrix}.$$

(Here, the order of the equations has been altered for computational simplicity.) We now aim at solving, for each fixed r, the system $Ms = N$, with $s = (s_1, s_2, s_3)^T$, using Gauß–Jordan elimination. Thus we must distinguish those cases for which any of the upper left block determinants of M is zero. The block determinants are all non-zero if and only if $0 \notin \{q(r), m(r)\}$, where $m(r) = \alpha_s\beta_s\gamma_s(\beta_c - 2\alpha_c r)$, and we thus consider separately the cases

(1) $m(r) = 0$,

(2) $m(r) \neq 0$, $q(r) = 0$ and
   i. $n(r) = 0$,
   ii. $n(r) \neq 0$

where $n(r) = \det(M_1\ M_2\ N) = \alpha_s\beta_s\gamma_s r(1-4r^2)$, using the notation M_i for the i^{th} column of M.

In case 1, Gauß–Jordan elimination cannot be completed straight-forwardly, and in case 2.i, the system $Ms = N$ has infinitely many solutions. In both these cases it turns out that the equations (3.2)–(3.5), together with the condition $r^2 + \|s\|^2 = 1$ on the norm, can easily be solved altogether, giving a list L of idempotents for each (α, β, γ). Computations show that L includes the idempotents of Items 1–3 of Theorem 3.6. In case 2.ii, the system $Ms = N$ has no solutions.

If neither case among 1–2.ii holds, then $q(r) \neq 0$ and Gauß–Jordan elimination determines $s_i, i \in \underline{3}$ as $s_i = t_i(r)$, and inserting these into (3.2) gives the equation $p(r) = 0$. For each r that solves this equation and satisfies $r^2 + \|s\|^2 = 1$ it then follows by Proposition 3.4(1) that $r + s_1\mathbf{i} + s_2\mathbf{j} + s_3\mathbf{k}$ is a non-zero idempotent. Moreover, the elements of L that are not given by Items 1–3 are verified to satisfy the conditions of Item 4. This proves Items 4 and 5, and the theorem follows. □

3.2.2. *Idempotents in* $\mathbb{H}^{--}(a,b)$. We proceed similarly in the case of the double sign $(-,-)$.

Definition 3.7. Let $A = \mathbb{H}^{--}(a,b)$ be in the canonical cross-section with a, b given in terms of (α, β, γ) by (2.5). Define $p' = p^{--}_{a,b}, q' = q^{--}_{a,b} \in \mathbb{R}[X]$ and $t'_i = t^{--}_{i,a,b} \in \mathbb{R}(X), i \in \underline{3}$ by

$$p'(X) = 16X^5 + 16(\alpha_c\beta_c + \alpha_s\beta_s\gamma_c)X^4 - 8X^3 - 8(2\alpha_c\beta_c + \alpha_s\beta_s\gamma_c)X^2 + (1 - 4\alpha_c^2 - 4\beta_c^2)X + \alpha_s\beta_s\gamma_c - \alpha_c\beta_c,$$

$$q'(X) = \alpha_s\beta_s\gamma_s(8X^3 + 4(3\alpha_c\beta_c + \alpha_s\beta_s\gamma_c)X^2 + (4\alpha_c^2 + 4\beta_c^2 - 2)X + \alpha_c\beta_c - \alpha_s\beta_s\gamma_c),$$

$$t'_1(X) = \alpha_s\beta_s\gamma_s X \frac{(\alpha_s\beta_c + \alpha_c\beta_s\gamma_c)(4X^2+1) + 4(\alpha_c\alpha_s + \beta_c\beta_s\gamma_c)X}{q(X)},$$

$$t'_2(X) = \alpha_s\beta_s^2\gamma_s^2 X \frac{\alpha_c(4X^2+1) + 4\beta_c X}{q(X)}, \quad t'_3(X) = \alpha_s^2\beta_s^2\gamma_s^2 X \frac{1-4X^2}{q(X)}.$$

We then use Proposition 3.4(1) to determine the idempotents.

Theorem 3.8. *Let $A = \mathbb{H}^{--}(a,b)$ be in the canonical cross-section with a, b given in terms of (α, β, γ) by (2.5). Let moreover p', q' and t'_i, $i \in \underline{3}$, be given by Definition 3.7.*

(1) If $\gamma = 0$ and at least one of α, β is non-zero, then

$$x = \cos\left(\frac{2\pi k + \alpha + \beta}{3}\right) + \sin\left(\frac{2\pi k + \alpha + \beta}{3}\right)\mathbf{i}$$

for $k \in \underline{3}$ are precisely the non-zero idempotents in A.

(2) If $\alpha = \beta = \gamma = 0$, then 1 is the unique isolated non-zero idempotent in A, and the points of the set

$$\left\{ -\frac{1}{2} + s_1\mathbf{i} + s_2\mathbf{j} + s_3\mathbf{k} \mid s_1^2 + s_2^2 + s_3^2 = \frac{3}{4} \right\}$$

are precisely the non-isolated idempotents.

(3) *If* $\gamma \neq 0$ *and* $\alpha_c\beta_c = \alpha_s\beta_s\gamma_c$, *then* $-\beta_c\mathbf{i}/\alpha_s - \alpha_c\beta_s\gamma_s\mathbf{j} - \alpha_s\beta_s\gamma_s\mathbf{k}$ *is a non-zero idempotent.*

(4) *If* $\gamma \neq 0$ *and* $\alpha + \beta = \pi$, *then*

$$\left\{\frac{1}{2} + \frac{\gamma_c + 1}{2\gamma_s}e\mathbf{i} + \frac{e}{2}\mathbf{j} + \frac{\gamma_c - 1}{2\gamma_s}\mathbf{k} \mid e \in \mathbb{R}, e^2 = \frac{\gamma_c - (2\gamma)_c}{\gamma_c + 1}\right\}$$

contains precisely two non-zero idempotents.

(5) *If* $\gamma \neq 0$ *and* $\alpha = \beta \geq \pi/6$, *then*

$$\left\{-\frac{1}{2} + \frac{\gamma_c - 1}{2\gamma_s}f\mathbf{i} + \frac{f}{2}\mathbf{j} - \frac{\gamma_c + 1}{2\gamma_s}\mathbf{k} \mid f \in \mathbb{R}, f^2 = \frac{\gamma_c + (2\gamma)_c}{\gamma_c - 1}\right\}$$

contains precisely one non-zero idempotent if $\alpha = \beta = \pi/6$, *and precisely two otherwise.*

(6) *If* $\gamma \neq 0$, *and* $r \in \mathbb{R}$ *satisfies*

$$p'(r) = 0 \neq q'(r) \text{ and } r^2 + \sum_{i=1}^{3} t_i'(r)^2 = 1,$$

then $r + t_1'(r)\mathbf{i} + t_2'(r)\mathbf{j} + t_3'(r)\mathbf{k}$ *is a non-zero idempotent.*

(7) *Every idempotent in* A *is given by precisely one of the cases 1–6.*

The proof is analogous to that of Theorem 3.6.

3.3. **General remarks.** In this section we comment on the results obtained above, partly in the light of the following result from [2].

Proposition 3.9. *The cardinality* $|\operatorname{Ip}(A)|$ *for an absolute valued algebra* A *is either odd or infinite. If it is infinite, then* $\operatorname{Ip}(A)$ *contains a differentiable manifold of positive dimension.*

An open question is posed in [2] asking for an upper bound of the number of non-zero idempotents in an arbitrary absolute valued algebra with finitely many idempotents. We are now able to give a precise answer, along with additional information in the cases where the number of idempotents is infinite.

Proposition 3.10. *If* $A \in \mathcal{A}_4$, *then* $|\operatorname{Ip}(A)| \in \{1, 3, 5, \infty\}$. *All four cases do occur. If* $|\operatorname{Ip}(A)| = \infty$, *then* $\operatorname{Ip}(A)$ *contains precisely one isolated element* x, *and an* n-*sphere with all points equidistant from* x, *and with* $n = 2$ *if* $(k,l) = (-,-)$, *and* $n = 1$ *otherwise.*

Proof. Assume first that A belongs to the canonical cross-section of $\mathcal{A}_4$. The last statement is a refolmulation of items 1 and 2 of Theorems 3.6 and 3.8, respectively, from which it also follows that the case $|\operatorname{Ip}(A)| = \infty$ does occur. Next we show that $|\operatorname{Ip}(A)| < \infty$ implies $|\operatorname{Ip}(A)| \leq 5$.

Assume hence that $|\operatorname{Ip}(A)| < \infty$. If $A = \mathbb{H}^{kl}(a,b)$ with a, b given in terms of (α, β, γ) by (2.5), and $\gamma = 0$, then it follows from Theorems 3.6 and 3.8 that A has three idempotents if $(k,l) = (-,-)$, and a unique idempotent otherwise. If $\gamma \neq 0$, then the number of idempotents equals the sum of the number of roots of the quintic $p_{a,b}^{kl}$ and the number of idempotents given by Item 3 of Theorem 3.6 (if $(k,l) \neq (-,-)$) or Items 3–5 of Theorem 3.8 (if $(k,l) = (-,-)$). However, if r is the real part of m idempotents given by Theorem 3.6(3) or 3.8(3)–(5), then one verifies directly that $(r - X)^m | p_{a,b}^{kl}(X)$ and that $q_{a,b}^{kl}(r) = 0$. Thus r is not the real part of any idempotent given by Theorem 3.6(4) or 3.8(6), and the total number of idempotents does not exceed the number of roots of $p^{kl}_{a,b}$, which is at most five.

Thus by Proposition 3.9, $|\operatorname{Ip}(A)| \in \{1, 3, 5, \infty\}$ for each A in the canonical cross-section. If A does not belong to the canonical cross-section, then there exists A' in the cross-section and an isomorphism $\rho : A' \to A$. The idempotents of A are precisely the images under ρ of the idempotents of A', and $|\operatorname{Ip}(A)| \in \{1, 3, 5, \infty\}$ by the above. If moreover $|\operatorname{Ip}(A')| = \infty$, then the configuration of the idempotents is preserved under ρ, as an isomorphism of absolute valued algebras respects the norm and maps the standard basis to an orthonormal basis in A.

Finally, applying Theorem 3.6 to $\mathbb{H} = \mathbb{H}^{++}(1, 1)$ and $\mathbb{H}^{++}(\mathbf{i}, \mathbf{j})$, and Theorem 3.8 to $\mathbb{H}^{--}(\mathbf{i}, \mathbf{j})$, one obtains that these algebras have 1, 3 and 5 idempotents respectively. This completes the proof. □

Remark 3.11. The proposition in fact answers, for the case of dimension four, another open question in [2], namely it gives the number of connected components of $\operatorname{Ip}(A)$ in an absolute valued algebra A with $|\operatorname{Ip}(A)| = \infty$. This number is hence two for all such four-dimensional algebras.

Regarding the quintic polynomials $p^{kl}_{a,b}$, the reader may have noticed that when $(k, l) = (-, -)$, they were not expressed as products of factors of lower degree. This calls for a comment on the issue of their solvability, which we address here.

Proposition 3.12. *There exist $a, b \in \mathbb{S}(\mathbb{H})$ such that the polynomial $p^{--}_{a,b}$ is not solvable by radicals.*

Proof. Construct the polynomial $p^{--}_{a,b}$ where

$$a = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}, \quad b = \frac{1}{4} + \frac{\sqrt{15}}{4}\mathbf{j}.$$

We then have that $P = 8p^{--}_{a,b}$ is a polynomial with integer coefficents. We first prove that P is irreducible over $\mathbb{Z}$, by verifying that there exist no $l, m, n \in \mathbb{Z}$, no $Q \in \mathbb{Z}[X]$ of degree 4 and no $R \in \mathbb{Z}[X]$ of degree 3 such that $P(X) = (X + l)Q(X)$ or $P(X) = (X^2 + mX + n)R(X)$.[4] A well-known result by Gauß implies that P is then irreducible over $\mathbb{Q}$, and hence clearly so is $p^{--}_{a,b}$.

Dividing $p^{--}_{a,b}$ by its derivative, and using a suitable method for determining the number of real zeros of a polynomial in a given interval, it turns out that $p^{--}_{a,b}$ has precisely three real roots, each of multiplicity one. By Lemma 14.7 in [14], the Galois group over $\mathbb{Q}$ of an irreducible polynomial of prime degree p with rational coefficients, having precisely two non-real roots, is the symmetric group on p elements, and the statement follows. □

The reader may find the statement of the proposition discouraging. In the search for other methods to solve the idempotency problem, the author has examined available literature on solutions of quadratic equations in $\mathbb{H}$. This examination has indicated that equations of the form $x^2 + cxd = 0$, where c and d are given quaternions (cf. (3.1) above), have been little studied, and an explicit method of finding the solutions seems not to be known. In any case, the above results, even

[4]This is done by evaluating both sides of each equation at $X = 0$, and those of the second at $X = 1$, to obtain a finite list of possible values for l, m and n, and then checking that each of these gives a non-zero remainder when $P(X)$ is divided by $X + l$ and $X^2 + mX + n$, respectively.

in the cases where Proposition 3.12 holds, are useful to determine whether a given element is an idempotent or not, or to extract various properties of the idempotents.

4. Morphisms from two-dimensional algebras

In this section we explicitly determine all morphisms from any of the four non-isomorphic two-dimensional absolute valued algebras $\mathbb{C}^{ij}, (i,j) \in C_2^2$, to any algebra in the canonical cross-section of $\mathcal{A}_4$. As in the case of morphisms from $\mathbb{R}$, Section 7 transfers those results of this section which are specific to algebras of the canonical cross-section to any four-dimensional absolute valued algebra given as $\mathbb{H}^{kl}(a,b)$ for some $a, b \in \mathbb{S}(\mathbb{H})$.

4.1. **Preparatory results.** We start with the following general observation.

Proposition 4.1. *Take $\mathbb{C}^{ij} \in \mathcal{A}_2$ and let $A = (A, \cdot)$ be a real algebra with $a \in A$. Then there is at most one algebra homomorphism $\phi : \mathbb{C}^{ij} \to A$ such that $\phi(\mathbf{i}) = a$.*

Proof. Assume that there are ϕ_1 and ϕ_2 such that $\phi_1(\mathbf{i}) = \phi_2(\mathbf{i}) = a$. Then, denoting the multiplication in $\mathbb{C}^{ij}$ by $\circ$, we have, since conjugation is self-inverse, that

$$\phi_1(1) = \phi_1(-\mathbf{ii}) = -\phi_1(\mathbf{i}_j \circ \mathbf{i}_i) = -\phi_1(\mathbf{i}) \cdot \phi_1(\mathbf{i}) = -\phi_2(\mathbf{i}) \cdot \phi_2(\mathbf{i}) = \phi_2(1)$$

where juxtaposition is multiplication in $\mathbb{C}$, and for each $c \in \mathbb{C}^{ij}$, $c_+ = c$ and $c_- = \overline{c}$. Since ϕ_1 and ϕ_2 are linear and the vector space $\mathbb{C}$ is spanned by $\{1, \mathbf{i}\}$, it follows that $\phi_1 = \phi_2$. □

Thus the homomorphisms to be treated in this section are determined by the image of the imaginary unit under them. In computations, however, it is often more convenient to use the following characterization of the morphisms.

Proposition 4.2. *Let $C = \mathbb{C}^{ij}, (i,j) \in C_2^2$, and let $A = (A, \cdot) \in \mathcal{A}_4$. A map $\phi : C \to A$ is an algebra homomorphism if and only if it is linear and the following conditions hold:*

(1) $\phi(1) \cdot \phi(1) = \phi(1)$,
(2) $\phi(1) \cdot \phi(\mathbf{i}) = i\phi(\mathbf{i})$,
(3) $\phi(\mathbf{i}) \cdot \phi(1) = j\phi(\mathbf{i})$ *and*
(4) $\phi(\mathbf{i}) \cdot \phi(\mathbf{i}) = -ij\phi(1)$.

Proof. If ϕ is a homomorphism, then ϕ is linear and respects multiplication. The latter property, together with the definition of the multiplication in $\mathbb{C}^{ij}$, implies the four items above. If ϕ is linear, to show that it is a homomorphism we need only show that it respects the multiplication of the elements of a basis of $\mathbb{C}^{ij}$. Choosing the basis $\{1, \mathbf{i}\}$, this is precisely the content of the four items of the proposition. □

Since morphisms in $\mathcal{A}$ are always injective, the set $\mathcal{A}(\mathbb{C}^{ij}, \mathbb{H}^{kl}(a,b))$ is non-empty if and only if $\mathbb{H}^{kl}(a,b)$ contains a subalgebra isomorphic to $\mathbb{C}^{ij}$. For each $(i,j), (k,l) \in C_2^2$, [12] gives a list of conditions on $a, b \in \mathbb{S}(\mathbb{H})$ that hold if and only if $\mathbb{H}^{kl}(a,b)$ has a subalgebra $D \simeq \mathbb{C}^{ij}$. We present here its explicit concequences for elements in the canonical cross-section.

Proposition 4.3. *Given $(k,l) \in C_2^2$, let $A = \mathbb{H}^{kl}(a,b)$ be in the canonical cross-section with a, b given in terms of (α, β, γ) by (2.5). Then there exists a morphism*

$\phi : \mathbb{C}^{ij} \to A$ precisely when

1. $\gamma = 0$, *if* $(i,j) = (k,l)$,
2. $\alpha = \gamma = \pi/2$,
   or $\alpha = \pi/2, \beta = 0$, *if* $(i,j,k,l) = (+,+,+,-) \vee (i,j) = (+,-) \neq (k,l)$,
3. $\beta = \gamma = \pi/2$
   or $\alpha = 0, \beta = \pi/2$, *if* $(i,j,k,l) = (+,+,-,+) \vee (i,j) = (-,+) \neq (k,l)$,
4. $\alpha = \beta = \pi/2$, *if* $(i,j,k,l) = (+,+,-,-) \vee (i,j) = (-,-) \neq (k,l)$.

The results follow immediately upon applying the conditions in Proposition 3.2 in [12] to the canonical cross-section.

4.2. **Description of morphisms.** Before presenting the complete description of the morphisms, we give the following result, which is meant to give a geometric picture of the set of morphisms from a two-dimensional absolute valued algebra to a four-dimensional.

THEOREM 4.4. *For any $(i,j),(k,l) \in C_2^2$ and any $a,b \in \mathbb{S}(\mathbb{H})$, consider $C = \mathbb{C}^{ij}$ and $A = \mathbb{H}^{kl}(a,b)$. Then either the set $\mathcal{A}(C,A)$ is empty, or the map $\mathcal{A}(C,A) \to A, \phi \mapsto \phi(\mathbf{i})$ induces a bijection*

$$\mathcal{A}(C,A) \to \bigsqcup_{\mu=1}^{m} \mathbb{S}^n$$

where $m \in \{1,3\}$ is the number of non-zero idempotents in C, and $n \in \{0,1,2\}$ satisfies

$$n = \begin{cases} 0 & \text{if } \dim[\Im(a),\Im(b)] = 1 \ \wedge (i,j) = (k,l), \\ 2 - \dim[\Im(a),\Im(b)] & \text{otherwise.} \end{cases}$$

REMARK 4.5. The statement that the map $\phi \mapsto \phi(\mathbf{i})$ induces the bijection here means that the image of this map consists of m disjoint n-spheres embedded in A, and the bijection is obtained by identifying this image with $\bigsqcup_{\mu=1}^{m} \mathbb{S}^n$ in a natural way. The theorem follows from the description of the morphisms from each $\mathbb{C}^{ij}$ to each $A = \mathbb{H}^{kl}(a,b)$ in the canonical cross-section, given below, and holds for arbitrary $\mathbb{H}^{kl}(a,b)$ due to the properties of isomorphisms in $\mathcal{A}_4$ given in [12] and quoted in Proposition 7.1 below.

REMARK 4.6. Section 6 below deals with the orbits of the actions of the automorphism groups of C and A on $\mathcal{A}(C,A)$ by composition. We will briefly return to the above theorem and comment on it in the light of the results obtained there.

We now give the description of the morphisms to algebras in the canonical cross-section, divided into three parts according to the value of $\dim[\Im(a),\Im(b)]$.

PROPOSITION 4.7. *Let $C = \mathbb{C}^{ij}$ and let $A = \mathbb{H}^{kl}(a,b)$ be in the canocinal cross-section with $\dim[\Im(a),\Im(b)] = 0$. Then*

$$\mathcal{A}(C,A) \neq \emptyset \iff (i,j) = (k,l).$$

In that case $\phi \in \mathcal{A}(C,A)$ if and only if

$$\phi(\mathbf{i}) = \sin\frac{2\pi\mu}{m} + u\cos\frac{2\pi\mu}{m}$$

for some $u \in \mathbb{S}(\Im\mathbb{H})$ and $\mu \in \underline{m}$, where $m = |\operatorname{Ip}(C)|$.

Proposition 4.8. *Let $C = \mathbb{C}^{ij}$ and let $A = \mathbb{H}^{kl}(a,b)$ be in the canonical cross-section with* $\dim[\Im(a),\Im(b)] = 1$ *and* $(i,j) \neq (k,l)$*. If* $\mathcal{A}(C,A) \neq \emptyset$*, then* $\phi \in \mathcal{A}(C,A)$ *if and only if*

$$\phi(\mathbf{i}) = \sin\frac{2\pi\mu}{m} + u\cos\frac{2\pi\mu}{m}$$

for some $u \in \mathbb{S}(\Im\mathbb{H}) \cap [\Im(a),\Im(b)]^\perp$ *and* $\mu \in \underline{m}$*, where* $m = |\operatorname{Ip}(C)|$*.*

Proposition 4.9. *Let $C = \mathbb{C}^{ij}$ and let $A = \mathbb{H}^{kl}(a,b)$ be in the canonical cross-section with either* $\dim[\Im(a),\Im(b)] = 1$ *and* $(i,j) = (k,l)$*, or* $\dim[\Im(a),\Im(b)] = 0$*. If* $\mathcal{A}(C,A) \neq \emptyset$*, then* $\phi \in \mathcal{A}(C,A)$ *if and only if*

$$\phi(\mathbf{i}) = \pm\Big[v\sin\Big(\frac{\alpha+\beta-\gamma+2\pi\mu}{k}\Big) + w\cos\Big(\frac{\alpha+\beta-\gamma+2\pi\mu}{m}\Big)\Big]$$

for some $\mu \in \underline{m}$*, where* $m = |\operatorname{Ip}(C)|$*,* a, b *are given in terms of* (α,β,γ) *by (2.5), and the pair* (v,w) *is given by Table 1.*

(k,l)	$(i,j)=(+,+)$	$(i,j)=(+,-)$	$(i,j)=(-,+)$	$(i,j)=(-,-)$
$(+,+)$	$(1,\mathbf{i})$	$(\mathbf{i},-\mathbf{k})$	$(\mathbf{j},-\mathbf{k})$	$(1,-\mathbf{k})$
$(+,-)$	$(\mathbf{i},\mathbf{k})$	$(1,-\mathbf{i})$	$(\mathbf{j},\mathbf{k})$	$(1,-\mathbf{k})$
$(-,+)$	$(\mathbf{j},\mathbf{k})$	$(\mathbf{i},\mathbf{k})$	$(1,-\mathbf{i})$	$(1,-\mathbf{k})$
$(-,-)$	$(1,\mathbf{k})$	$(\mathbf{i},-\mathbf{k})$	$(\mathbf{j},-\mathbf{k})$	$(1,-\mathbf{i})$

Table 1. The pair (v,w) of Proposition 4.9.

The proofs of Propositions 4.7–4.9 are computationally heavy; we give an outline of the general ideas, and illustrate the computations by an example.

Outline of proof. Take $A \in \mathcal{A}_4$ in the canonical cross-section that satisfies any of the conditions of Proposition 4.3. We first determine the idempotents of A by applying Theorem 3.6 or 3.8. It turns out that under the conditions of Proposition 4.3, the computations are straight-forward as the roots of the polynomials p_{ab}^{kl} of Theorems 3.6 and 3.8 are easily found. Take now $C = \mathbb{C}^{ij}$ for some $(i,j) \in C_2^2$. According to Item 1 of Proposition 4.2, the set $\{\phi(1) \mid \phi \in \mathcal{A}(C,A)\}$ is a subset of the set of all non-zero idempotents of A. Due to Proposition 4.2.(2)–(4), to each non-zero idempotent y we solve the equations

$$y \cdot x = ix, x \cdot y = jx, x \cdot x = -ijy \tag{4.1}$$

for x. For each solution x there then exists $\phi \in \mathcal{A}(C,A)$ with $\phi(\mathbf{i}) = x$ and $\phi(1) = y$. (If there exist no solutions, then y is not the image of 1 under any morphism in $\mathcal{A}(C,A)$.) Doing this for all idempotents $y \in A$ determines $\mathcal{A}(C,A)$ completely. □

As an example we determine $\mathcal{A}(\mathbb{C}^{+-},\mathbb{H}^{-+}(a,b))$ for $\mathbb{H}^{-+}(a,b))$ in the canonical cross-section with $\gamma \neq 0$.

Example 4.10. The cases with $(i,j) = (+,-)$ and $(k,l) = (-,+)$ fall under Item 2 of Proposition 4.3, where we also have $\beta \neq 0$ as $\gamma \neq 0$. Setting thus $\alpha = \gamma = \pi/2$, we consider Theorem 3.6. The first two items of the theorem give no idempotents, as $\gamma \neq 0$. The third item is applicable, since $\gamma_c = \alpha_c = 0$, and gives the idempotent

$\beta_c\mathbf{i} - \beta_s\mathbf{k}$. In the forth item, we obtain that the roots of p that are not roots of q under the given conditions are $\pm\sqrt{3-4\beta_c^2}/2$ when $\beta \geq \pi/6$, and none otherwise. Evaluating the functions $t_i(r)$ and computing $r^2 + \sum_{i=1}^{3} t_i(r)^2$ for each root r, we find that there are precisely two additional idempotents

$$-\beta_c\mathbf{j} + \frac{1-2\beta_c^2}{2\beta_s}\mathbf{k} \pm \frac{\sqrt{3-4\beta_c^2}}{2}\left(1 - \frac{\beta_c}{\beta_s}\mathbf{j}\right)$$

if $\beta > \pi/6$, and none otherwise.

Next we solve (4.1) for each idempotent y. If $\|x\| \neq 1$, then by multiplicativity of the norm, x does not satisfy the third equation in (4.1). Thus we require $\|x\| = 1$, under which condition Lemma 3.2 implies that (4.1) can be rewritten as

$$ayb = 2\Re(x)x - 1, axb = -xy, axb = yx.$$

This is solved by writing each equation componentwise as a system of real equations. For $y = \beta_c\mathbf{i} - \beta_s\mathbf{k}$, one obtains two solutions $x = \pm(\beta_s\mathbf{i} + \beta_c\mathbf{k})$, while for the other idempotents, no solution exists. Hence for each $\mathbb{H}^{-+}(a,b)$ in the canonical cross-section with $\gamma \neq 0$ we have

$$\phi \in \mathcal{A}(\mathbb{C}^{+-}, \mathbb{H}^{-+}(a,b)) \Longleftrightarrow \phi(\mathbf{i}) \in \{\pm(\beta_s\mathbf{i} + \beta_c\mathbf{k})\}.$$

5. Irreducibility

5.1. **Definition and background.** A natural question to ask once a class of morphisms has been described is whether the morphisms are irreducible. To begin with, we quote the definition of irreducibility for division algebras. Recall, to this end, that over any field k the finite dimensional division algebras form a category $\mathcal{D}(k)$, in which the morphisms are the non-zero algebra morphisms. The following definition is due to Dieterich [9].

Definition 5.1. Let A and B be finite dimensional division algebras over a field k. A morphism $\psi : A \to B$ in $\mathcal{D}(k)$ is *irreducible* if it is not an isomorphism and if for any pair (ψ_1, ψ_2) of morphisms in $\mathcal{D}(k)$ such that $\psi = \psi_2\psi_1$, either ψ_1 is an isomorphism or ψ_2 is an isomorphism. ψ is *reducible* if it is not an isomorphism and not irreducible.

An immediate consequence of the definition, and the injectivity of the morphisms in $\mathcal{D}(k)$, is the following proposition.

Proposition 5.2. *Let A and B be finite dimensional division algebras over a field k. Then there exists a reducible morphism $\psi : A \to B$ only if there is a subalgebra $C \subset B$ such that* $\dim A < \dim C < \dim B$.

For $A, B \in \mathcal{A}_{\leq 4}$ this implies that all morphisms $A \to B$ are irreducible in case $\dim A = 2$ or $\dim B = 2$. It remains to consider the morphisms $\mathbb{R} \to B$ where $\dim B = 4$ and B has a two-dimensional subalgebra. As indicated in the outlined proof of Propositions 4.7–4.9, for such algebras that moreover belong to the canonical cross-section it is straight-forward to determine the idempotents explicitly, and this will be used here to investigate the reducibility of the corresponding morphisms.

5.2. **Morphisms from $\mathbb{R}$ to $\mathbb{H}^{kl}(a,b)$ with $(k,l) \neq (-,-)$.** Without further ado, we describe the irreducibility of the morphisms from $\mathbb{R}$ to $\mathbb{H}^{kl}(a,b)$. Note that if $\mathbb{H}^{kl}(a,b)$ has a subalgebra isomorphic to $\mathbb{C}^{ij}$ for some $(i,j) \in C_2^2$, then a morphism from $\mathbb{R}$ to $\mathbb{H}^{kl}(a,b)$ factors over $\mathbb{C}^{ij}$ if and only if it factors over each subalgebra of $\mathbb{H}^{kl}(a,b)$ isomorphic to $\mathbb{C}^{ij}$. In the following, we will use these two equivalent formulations interchangeably.

Proposition 5.3. *Given $(k,l) \in C_2^2 \setminus \{(-,-)\}$, let $A = \mathbb{H}^{kl}(a,b)$ with $a, b \in \mathbb{S}(\mathbb{H})$ such that A contains a two-dimensional subalgebra.*[5]

(1) If a and b are purely imaginary and orthogonal, then A has a subalgebra isomorphic to $\mathbb{C}^{ij}$ for each $(i,j) \neq (k,l)$, and none isomorphic to $\mathbb{C}^{kl}$, and there are precisely three morphisms $\mathbb{R} \to A$. All of these are reducible and factor over $\mathbb{C}^{--}$, and precisely one factors over each subalgebra.

(2) i. If a and b are purely imaginary and proportional, then A has precisely two isomorphism types of two-dimensional subalgebras, and there are uncountably many morphisms $\mathbb{R} \to A$. All of these are reducible and factor over $\mathbb{C}^{--}$, and only the unique morphism corresponding to the isolated non-zero idempotent in A factors over each subalgebra.

ii. If one of a and b is real and the other purely imaginary, then A has precisely two isomorphism types of two-dimensional subalgebras, and there is precisely one morphism $\mathbb{R} \to A$. This unique morphism is reducible and factors over each subalgebra.

(3) Otherwise, A has precisely one two-dimensional subalgebra, up to isomorphism. Moreover,

i. if a and b are proportional with $1/2 < \|\Im(a)\| = \|\Im(b)\| < 1$, then there are uncountably many morphisms $\mathbb{R} \to A$. The unique morphism corresponding to the isolated non-zero idempotent in A is reducible, and all other morphisms are irreducible.

ii. if a and b are orthogonal, one is purely imaginary, and the other has imaginary part z, $1/2 < \|z\| < 1$, then there are precisely three morphisms $\mathbb{R} \to A$, and precisely one of these is reducible.

iii. in all other cases, there are precisely three morphisms $\mathbb{R} \to A$ if both a and b are purely imaginary, and precisely one if not. All of these are reducible.

Proof. A morphism $\psi : \mathbb{R} \to A$ is reducible if and only if there exists a subalgebra $C \subset A$ of dimension two, and $\phi : C \to A$, such that $\psi(1) = \phi(z)$ for an idempotent $z \in C$. The result follows for A in the canonical cross-section by checking, for each $\psi : \mathbb{R} \to A$ and $C = \mathbb{C}^{ij}$, whether or not this condition is satisfied. If $\mathbb{H}^{kl}(c,d)$ is not in the cross-section, then evidently it has the same number of subalgebras and morphisms as its representative, and the morphisms factor in the same way. In addition, the conditions on isomorphisms in $\mathcal{A}_4$ quoted in Proposition 7.1 below imply that if $\mathbb{H}^{kl}(c,d) \simeq \mathbb{H}^{kl}(a,b)$, then $\|\Im(c)\| = \|\Im(a)\|$ and $\|\Im(d)\| = \|\Im(b)\|$, and moreover $|\langle c,d\rangle| = |\langle a,b\rangle|$. Hence $\mathbb{H}^{kl}(c,d)$ satisfies the same condition among 1–3.iii as does $\mathbb{H}^{kl}(a,b)$, and the proof is complete. □

[5] In other words, assume that A satisfies the conditions of Proposition 3.2 in [12]. If A is in the canonical cross-section, this is equivalent to Proposition 4.3 above.

Note how the isolated idempotents differ in nature whenever there are infinitely many morphisms, and how the magnitude of the imaginary part is of importance in some cases.

5.3. **Morphisms from $\mathbb{R}$ to $\mathbb{H}^{--}(a,b)$.** The case of double sign $(-,-)$ exhibits, as the reader may have assumed, several fundamental differences.

PROPOSITION 5.4. *Let $A = \mathbb{H}^{--}(a,b)$ with $a, b \in \mathbb{S}(\mathbb{H})$ such that A contains a two-dimensional subalgebra.*

(1) *If a and b are purely imaginary and orthogonal, then A has a subalgebra isomorphic to $\mathbb{C}^{ij}$ for each $(i,j) \neq (-,-)$, and none isomorphic to $\mathbb{C}^{--}$, and there are precisely five morphisms $\mathbb{R} \to A$. Of these morphisms precisely one factors over each subalgebra, and all others are irreducible.*
(2) *If a and b are purely imaginary and proportional, or if one of a and b is real and the other purely imaginary, then A has precisely two isomorphism types of two-dimensional subalgebras, and there are precisely three morphisms $\mathbb{R} \to A$. All of these are reducible and factor over $\mathbb{C}^{--}$, and precisely one factors over each subalgebra.*
(3) *Otherwise, A has precisely one two-dimensional subalgebra, up to isomorphism. Moreover,*
   i. if a and b are real, then there are uncountably many morphisms $\mathbb{R} \to A$. All of these are reducible.
   ii. if a and b are purely imaginary and neither proportional nor orthogonal, then there are precisely five morphisms $\mathbb{R} \to A$ when $0 < |\langle a, b\rangle| < 1/2$ and precisely three when $1/2 \leq |\langle a, b\rangle| < 1$. In both cases precisely one of these is reducible.
   iii. if a and b are orthogonal, one is purely imaginary, and the other having real part r, then there are precisely five morphisms $\mathbb{R} \to A$ when $0 < |r| < 1/2$ and precisely three when $1/2 \leq |r| < 1$. In both cases precisely one of these is reducible.
   iv. in all other cases, there are precisely three morphisms $\mathbb{R} \to A$. All of these are reducible.

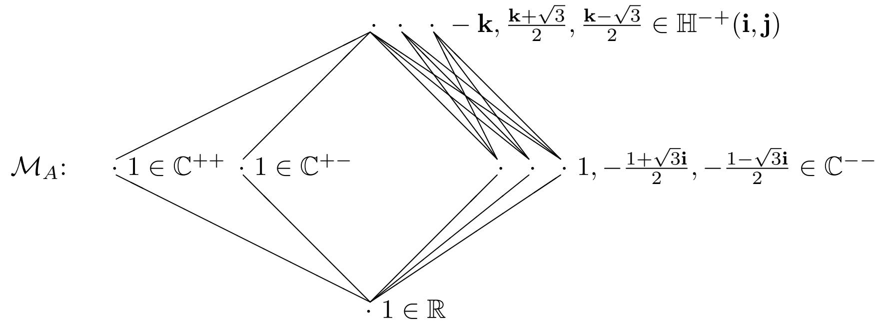
The proof is analogous to that of Proposition 5.3.

5.4. **Morphism quivers.** From Propositions 4.3, 5.3 and 5.4 we extract the following partitioning of the object class of $\mathcal{A}_4$.

COROLLARY 5.5. *For each $(k,l) \in C_2^2$, there exist uncountably many isomorphism classes of objects $A \in \mathcal{A}_4^{kl}$ such that each morphism $\psi : \mathbb{R} \to A$ is irreducible, uncountably many isomorphism classes of objects $A' \in \mathcal{A}_4^{kl}$ such that there is an irreducible morphism $\psi' : \mathbb{R} \to A'$, and a reducible morphism $\psi^* : \mathbb{R} \to A'$, and uncountably many isomorphism classes of objects $A'' \in \mathcal{A}_4^{kl}$ such that each morphism $\psi'' : \mathbb{R} \to A''$ is reducible.*

One may further combine Propositions 5.3 and 5.4 with the descriptions of morphisms from one- and two-dimensional to four-dimensional absolute valued algebras, which were given in Sections 3 and 4. In doing so, one obtains a complete picture not only of whether the morphisms from dimension one are reducible or not, but also of the morphisms from dimension two over which the reducible morphisms factor. A way to visualize this is by means of a quiver, the *morphism quiver* $\mathcal{M}_A$, for each four-dimensional absolute valued algebra A. The nodes of the quiver $\mathcal{M}_A$ are the non-zero idempotents of all canonical representatives of all subalgebras of A, and there exists an arrow from a node $n_1 \in B_1$ to a node $n_2 \in B_2$ if and only if there is an irreducible morphism $\phi : B_1 \to B_2$ such that $\phi(n_1) = n_2$.

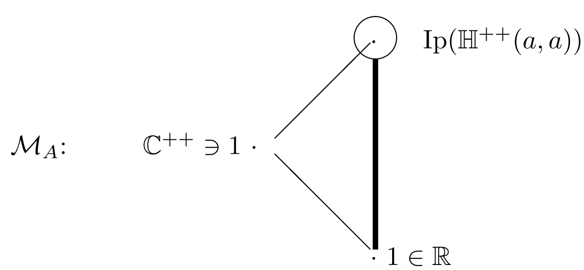
EXAMPLE 5.6. Let $A = \mathbf{H}^{-+}(\mathbf{i}, \mathbf{j})$. Then A satisfies the conditions of part 1 of Proposition 5.3, and we obtain the following quiver.



Each arrow is drawn as a line segment, for visibility, and understood to be directed upwards.

Note that each morphism $\phi : D_1 \to D_2$, where D_1 and D_2 are division algebras over a given field, maps the idempotents of D_1 injectively to the idempotents of D_2. The morphism quiver does, as seen from Example 2, not encode which non-zero idempotent $y \in A$ satisfies $y = \phi(x)$ for a given morphism $\phi : \mathbb{C}^{ij} \to A$ and a given non-zero idempotent $x \in \mathbb{C}^{ij}$, in case there is more than one possibility. Its purpose is to show, for each non-zero idempotent $y \in A$, all possible paths from $1 \in \mathbb{R}$ to y, i.e. all possible factorizations of the morphism corresponding to y into irreducible morphisms.

EXAMPLE 5.7. Let $A = \mathbf{H}^{++}(a, a)$ where $a = \alpha_c + \alpha_s\mathbf{i}$ and $\pi/3 < \alpha < \pi/2$, so that A falls under Item 3.1 of Proposition 5.3, and $\mathrm{Ip}(A)$ consists of an isolated point and a circle. The morphism quiver is as follows.



The thickened line segment here means that there is one arrow from $1 \in \mathbb{R}$ to each point on the circle.

Apart from these examples, there are several more different quivers for different $A \in \mathcal{A}_4$. The interested reader will have no difficulty to construct these for other algebras in $\mathcal{A}_4$.

6. Action of automorphism groups

The above description of morphisms $\phi \in \mathcal{A}(C, A)$ for $C, A \in \mathcal{A}_{\leq 4}$ was done without regard to the automorphisms of C and A. Since for any $\sigma \in \mathrm{Aut}(C)$ and $\tau \in \mathrm{Aut}(A)$ we have $\phi\sigma, \tau\phi \in \mathcal{A}(C, A)$, the automorphism groups of C and A act from the right by precomposition and from the left by postcomposition, respectively. In this section we will consider these two group actions, and determine the number of their orbits. In this context it is natural to also study the left group action

$$(\mathrm{Aut}(C) \times \mathrm{Aut}(A)) \times \mathcal{A}(C, A) \to \mathcal{A}(C, A), ((\sigma, \tau), \phi) \mapsto \tau\phi\sigma^{-1}$$

by pre- and postcomposition. The aim of this section is to understand to what extent the properties of the sets $\mathcal{A}(C, A)$ depend on the automorphism groups, and, in a sense, how closely linked the morphisms in $\mathcal{A}(C, A)$ are to each other. We will consider the cases where $C \in \mathcal{A}_2$ and $A \in \mathcal{A}_4$, as for these cases we have an explicit description of $\mathcal{A}(C, A)$. We start by recalling the structure of the automorphism groups themselves.

6.1. **The automorphism groups in $\mathcal{A}_{\leq 4}$.** For dimensions 1 and 2, we have the following well-known facts.

Proposition 6.1. *Let $C \in \mathcal{A}_{\leq 2}$.*

(1) If $C = \mathbb{R}$, then $\mathrm{Aut}(C)$ is trivial.
(2) If $C = \mathbb{C}^{ij}$ with $(i, j) \neq (-, -)$, then $\mathrm{Aut}(C)$ is generated by complex conjugation.
(3) If $C = \mathbb{C}^{--}$, then $\mathrm{Aut}(C)$ is generated by complex conjugation and rotation by an angle of $2\pi/3$.

Thus for $C \in \mathcal{A}_2$, $\mathrm{Aut}(C)$ has 2 or 6 elements. For dimension 4, the automorphism groups are described for the category ${}_{SO_3}(SO_3 \times SO_3)$ in [11]. Applying the equivalences of categories in (2.3) to this description gives the following description of the automorphism groups of algebras in the canonical cross-section of $\mathcal{A}_4$.

Proposition 6.2. *Let $A = \mathbb{H}^{kl}(a, b) \in \mathcal{A}_4$ be in the canonical cross-section.*

(1) If $\dim[\Im(a), \Im(b)] = 0$, then $\mathrm{Aut}(A) = \{\kappa_p \mid p = \theta_c + \theta_s q; \theta \in [0, \pi), q \in \mathbb{S}(\Im\mathbb{H})\}$.
(2) If $\dim[\Im(a), \Im(b)] = 1$, let $u \in \mathbb{S}(\Im\mathbb{H})$ be a basis vector of $[\Im(a), \Im(b)]$. Then if
  i. at least one of a, b is neither real nor purely imaginary, then $\mathrm{Aut}(A) = \{\kappa_p \mid p = \theta_c + \theta_s u; \theta \in [0, \pi)\}$.
  ii. each of a, b is either real or purely imaginary, then $\mathrm{Aut}(A) = \{\kappa_p \mid p = \theta_c + \theta_s u; \theta \in [0, \pi)\} \cup \{\epsilon\kappa_q \mid q \in \mathbb{S}(\Im\mathbb{H}) \cap u^\perp\}$, where $\epsilon = 1$ if both a and b are purely imaginary, and -1 otherwise.
(3) If $\dim[\Im(a), \Im(b)] = 2$ and
  i. either non of a, b is purely imaginary, or precisely one of a, b is purely imaginary and $\Im(a), \Im(b)$ are not orthogonal, then $\mathrm{Aut}(A)$ is trivial.
  ii. precisely one of a, b is purely imaginary and $\Im(a), \Im(b)$ are orthogonal, then $\mathrm{Aut}(A) = \{\mathrm{Id}, -\kappa_v\}$ where $v \in \mathbb{S}(\Im\mathbb{H})$ is a basis vector of the imaginary part of the non-purely imaginary element in $\{a, b\}$.
  iii. a, b are both purely imaginary and not orthogonal, then $\mathrm{Aut}(A) = \{\mathrm{Id}, \kappa_w\}$, where $w \in \mathbb{S}(\Im\mathbb{H})$ is orthogonal to a and b.

iv. *a, b are both purely imaginary and orthogonal, then* $\mathrm{Aut}(A) = \{\mathrm{Id}, -\kappa_a, -\kappa_b, \kappa_w\}$, *where $w \in \mathbb{S}(\Im\mathbb{H})$ is orthogonal to a and b.*

REMARK 6.3. If $A = \mathbb{H}^{kl}(c, d) \simeq \mathbb{H}^{kl}(a, b) = A'$, where A' is in the canonical cross-section and A is not, then due to the properties of isomorphisms quoted in Proposition 7.1 below, A satisfies the conditions for the same item among 1–3.iv as does A'. Obviously $\mathrm{Aut}(A) \simeq \mathrm{Aut}(A')$, but the explicit description of $\mathrm{Aut}(A)$ may differ from that given above for $\mathrm{Aut}(A')$.

6.2. **Orbits of the actions.** We now use the results of Section 6.1 to determine the number of orbits of the three group actions given above on the set $\mathcal{A}(C, A)$ for all $C \in \mathcal{A}_2$ and $A \in \mathcal{A}_4$. We thus denote by n_C the number of orbits of the right action of $\mathrm{Aut}(C)$ by precomposition, by n_A the number of orbits of the left action of $\mathrm{Aut}(A)$ by postcomposition, and by n_{CA} the number of orbits of the left action of $\mathrm{Aut}(C) \times \mathrm{Aut}(A)$ by pre- and postcomposition.

PROPOSITION 6.4. *Let $C \in \mathcal{A}_2$ and $A \in \mathcal{A}_4$. Then $n_{CA} = 1$ and the pair (n_C, n_A) attains one of*

$$(1, 1), (1, 2), (1, 3), (1, 6), (\infty, 1), (\infty, 3).$$

All of these pairs do occur for suitable $C \in \mathcal{A}_2$ and $A \in \mathcal{A}_4$.

Proof. Using Propositions 6.1 and 6.2, for each $C = \mathbb{C}^{ij}$ and each A in the canonical cross-section, the set $\mathcal{A}(C, A)$ can be partitioned into the equivalence classes of one of the following equivalence relations

(1) $\phi \sim_1 \psi \Leftrightarrow \exists \sigma \in \mathrm{Aut}(C); \psi = \phi\sigma$,
(2) $\phi \sim_2 \psi \Leftrightarrow \exists \tau \in \mathrm{Aut}(A); \psi = \tau\phi$.

Computing the number of equivalence classes of each relation gives the pair (n_C, n_A).

If either $n_C = 1$ or $n_A = 1$, then $n_{CA} = 1$. If not, then by the previous step, $(n_C, n_A) = (\infty, 3)$. Denoting the three $\mathrm{Aut}(A)$-orbits by $\omega_i, i \in \underline{3}$, taking an arbitrary $\phi \in \omega_1$, and precomposing ϕ by each of the (at most) six elements in $\mathrm{Aut}(C)$, one finds that there exist $\rho, \sigma \in \mathrm{Aut}(C)$ such that $\phi\rho \in \omega_2$ and $\phi\sigma \in \omega_3$. Hence there is one single orbit.

For algebras not in the canonical cross-section, the result holds by applying the above to their canonical representatives, as the number of orbits of any of the three actions involved is preserved under isomorphism. □

The computations of the proof of Proposition 6.4, together with Remark 6.3, in fact prove the following statements.

PROPOSITION 6.5. *Let $C \in \mathcal{A}_2$ and $A \in \mathcal{A}_4$.*

1. *The number n_C of orbits of the right action of $\mathrm{Aut}(C)$ by precomposition is 1 if $\mathcal{A}(C, A)$ is finite, and ∞ otherwise.*
2. *The number n_A of orbits of the left action of $\mathrm{Aut}(A)$ by postcomposition equals $|\mathrm{Ip}(C)|$ if A's representative in the canonical cross-section satisfies 1, 2.ii or 3.iv of Proposition 6.2, and $2|\mathrm{Ip}(C)|$ if it satisfies 2.i, 3.ii or 3.iii. In particular, if $\mathcal{A}(C, A)$ is infinite, then $n_A = |\mathrm{Ip}(C)|$.*

Case 3.i does not occur for those four-dimensional algebras that have two-dimensional subalgebras.

Proposition 6.5 partly explains the geometric situation presented in Theorem 4.4. Namely, when $n \in \{1,2\}$, each n-sphere corresponds to an orbit of the $\mathrm{Aut}(A)$-action, while each orbit of the $\mathrm{Aut}(C)$-action consists of a pair of points on each n-sphere. For the $\mathrm{Aut}(C)$-action the same holds in the case $n = 0$, whence all morphisms belong to the same orbit of this action.

7. Isomorphisms to the canonical cross-section

Some results above were only formulated for algebras in the canonical cross-section. In order to extend these to more general objects, either the descriptions have to be generalized, or, given $A \in \mathcal{A}_4$, one has to explicitly construct an isomorphism to the algebra in the canonical cross-section isomorphic to A.

The first approach involves computational difficulties, as the computations of the morphisms, conducted in Sections 3 and 4 above, have depended strongly on the simplifications associated with the particular choice of a cross-section. We therefore devote this section to the second approach; hence, given an algebra $A = \mathbb{H}^{kl}(c,d) \in \mathcal{A}_4$ with arbitrary $c,d \in \mathbb{S}(\mathbb{H})$, we determine its representative in the canonical cross-section, and construct an isomorphism. We begin by the following result from [12], supplemented in [11].

Proposition 7.1. *Two four-dimensional absolute valued algebras $\mathbb{H}^{kl}(a,b)$ and $\mathbb{H}^{k'l'}(c,d)$, with $a,b,c,d \in \mathbb{S}(\mathbb{H})$, are isomorphic if and only if $(k',l') = (k,l)$ and there exists $p \in \mathbb{S}(\mathbb{H})$ and $(\epsilon,\delta) \in C_2^2$ such that $c = \epsilon p a \overline{p}$ and $d = \delta p b \overline{p}$. In that case, every isomorphism $\psi : \mathbb{H}^{kl}(a,b) \to \mathbb{H}^{kl}(c,d)$ is of the form $x \mapsto \epsilon\delta p x \overline{p}$.*

Note that Proposition 7.1 is not constructive, as p is not given explicitly. We begin our explicit construction by determining the representatives in the cross-section.

Lemma 7.2. *Let $A = \mathbb{H}^{kl}(c,d)$ with $c,d \in \mathbb{S}(\mathbb{H})$ be given. Then the representative of A in the canonical cross-section is $\mathbb{H}^{kl}(a,b)$, with a,b given in terms of (α,β,γ) by (2.5), where*

(1) α is determined uniquely by $\alpha_c = |\Re(c)|$,

(2) β is determined uniquely by

$$\beta_c = \begin{cases} \operatorname{sgn}(\Re(c))\operatorname{sgn}(\langle \Im(c), \Im(d)\rangle)\Re(d) & \text{if } 0 \notin \{\Re(c), \langle \Im(c), \Im(d)\rangle\}, \\ |\Re(d)| & \text{otherwise, and} \end{cases}$$

(3) γ is then determined uniquely by $\alpha_s\beta_s\gamma_c = |\langle \Im(c), \Im(d)\rangle|$ if $0 \notin \{\alpha,\beta\}$, and $\gamma = 0$ otherwise.

Proof. By Proposition 7.1, there exists $p \in \mathbb{S}(\mathbb{H})$ such that $c = \epsilon p a \overline{p}$ and $d = \delta p b \overline{p}$ for some $(\epsilon,\delta) \in C_2^2$. Since conjugation by p preserves the real part of a quaternion, we have $\alpha_c = \epsilon\Re(c)$ and $\beta_c = \delta\Re(d)$, hence $|\alpha_c| = |\Re(c)|$ and $|\beta_c| = |\Re(d)|$. By Theorem 2.2, $0 \le \alpha \le \pi/2$, whence α_c is non-negative and determines α, which proves 1.

As for 2, the inner product on $\mathbb{H}$ is preserved under conjugation by a unit norm quaternion, and hence $\langle \Im(c), \Im(d)\rangle = \epsilon\delta\alpha_s\beta_s\gamma_c$. Suppose that $\Re(c) \neq 0$ and $\langle \Im(c), \Im(d)\rangle \neq 0$. Then $\epsilon = \operatorname{sgn}(\Re(c))$ by the above, and $\alpha_s\beta_s\gamma_c \neq 0$, hence positive by Theorem 2.2. Now $\langle \Im(c), \Im(d)\rangle = \epsilon\delta\alpha_s\beta_s\gamma_c$ impies that $\delta = \operatorname{sgn}(\Re(c))\operatorname{sgn}(\langle \Im(c), \Im(d)\rangle)$, and by the above $\beta_c = \delta\Re(d)$.

If $\Re(c) = 0$ or $\langle \Im(c), \Im(d) \rangle = 0$, i.e. if $\alpha_c = 0$ or $\alpha_s\beta_s\gamma_c = 0$, then Theorem 2.2 implies that $0 \le \beta \le \pi/2$, and thus β_c is non-negative. Since in all cases $0 \le \beta \le \pi$, β_c determines β completely.

Regarding 3, if any of α or β vanishes, then so does γ by Theorem 2.2. If both α and β are non-zero, then $\langle \Im(c), \Im(d) \rangle = \epsilon\delta\alpha_s\beta_s\gamma_c$ determines γ_c up to sign. Finally, $0 \le \gamma \le \pi/2$ implies that γ_c is non-negative and determines γ completely. □

The construction of the isomorphisms relies on the following detail.

LEMMA 7.3. *Assume that two quaternions $x = s_1\mathbf{i} + s_2\mathbf{j} + s_3\mathbf{k}$ and $y = t\mathbf{i}, t > 0$ satisfy $\|x\| = \|y\| \le 1$. Then*

(1) if $s_2 = s_3 = 0$, then $|s_1| = |t|$; if $s_1 = t$, then $p = 1$ satisfies $x = py\overline{p}$, and if $s_1 = -t$, then $p = \mathbf{j}$ satisfies $x = py\overline{p}$;

(2) otherwise

$$p = \sqrt{\frac{t+s_1}{2t}} - s_3\sqrt{\frac{t-s_1}{2t(s_2^2+s_3^2)}}\mathbf{j} + s_2\sqrt{\frac{t-s_1}{2t(s_2^2+s_3^2)}}\mathbf{k}$$

satisfies $x = py\overline{p}$.

Note that in 2, p is well-defined as $t \pm s_1 \ge 0$ follows from $\|x\| = \|y\|$, and $s_2^2 + s_3^2 \ne 0$.

Proof. Since $\|x\| = \|y\|$, there exists a rotation in $\Im\mathbb{H}$ that takes y to x. Computing the angle and axis of the rotation by elementary linear algebra, the result follows from the fact that if $u \in \mathbb{S}(\Im\mathbb{H})$, then $q = \theta_c + \theta_s u$ satisfies that $z \mapsto qz\overline{q}$ is a rotation with angle 2θ around u. (The claim can also be verified by direct computation.) □

Once the representative of $A = \mathbb{H}^{kl}(c, d)$ in the canonical cross-section has been determined by Lemma 7.2, the following proposition gives an explicit construction of an isomorphism to A from its representative.

PROPOSITION 7.4. *Let $A = \mathbb{H}^{kl}(c, d)$ with $c, d \in \mathbb{S}(\mathbb{H})$ and let $\mathbb{H}^{kl}(a, b)$ be the representative of A in the canonical cross-section, with a, b given in terms of (α, β, γ) by (2.5).*

If $0 \in \{\alpha, \beta, \gamma\}$, then the map $\rho : \mathbb{H}^{kl}(a, b) \to A, z \mapsto \epsilon\delta pz\overline{p}$ is an isomorphism, where $(\epsilon, \delta) \in C_2^2$ and $p \in \mathbb{S}(\mathbb{H})$ are given as follows.

(1) If $\alpha = \beta = 0$, then $\epsilon = \operatorname{sgn}(\Re(c))$, $\delta = \operatorname{sgn}(\Re(d))$, and $p = 1$.

(2) If $\alpha = 0$ and $\beta \ne 0$, then $\epsilon = \operatorname{sgn}(\Re(c))$, and
  - *i. if $\beta \ne \pi/2$, then $\delta = \operatorname{sgn}(\Re(d))$,*
  - *ii. if $\beta = \pi/2$, then δ can be chosen freely,*

  and p is given by Lemma 7.3 upon setting $y = \Im(b)$ and $x = \delta\Im(d)$.

(3) If $\alpha \ne 0$ and $\beta = 0$, then $\delta = \operatorname{sgn}(\Re(d))$, and
  - *i. if $\alpha \ne \pi/2$, then $\epsilon = \operatorname{sgn}(\Re(c))$,*
  - *ii. if $\alpha = \pi/2$, then ϵ can be chosen freely,*

  and p is given by Lemma 7.3 upon setting $y = \Im(a)$ and $x = \epsilon\Im(c)$.

(4) If $0 \notin \{\alpha, \beta\}$ and $\gamma = 0$, then
  - *i. if $\alpha \ne \pi/2$, then $\epsilon = \operatorname{sgn}(\Re(c))$ and $\delta = \epsilon \operatorname{sgn}(\langle \Im(c), \Im(d) \rangle)$,*
  - *ii. if $\alpha = \pi/2$ and $\beta \ne \pi/2$, then $\delta = \operatorname{sgn}(\Re(d))$ and $\epsilon = \delta \operatorname{sgn}(\langle \Im(c), \Im(d) \rangle)$,*
  - *iii. if $\alpha = \beta = \pi/2$, then ϵ can be chosen freely, and $\delta = \epsilon \operatorname{sgn}(\langle \Im(c), \Im(d) \rangle)$,*

  and p is given by Lemma 7.3 upon setting $y = \Im(a)$ and $x = \epsilon\Im(c)$.

If $0 \notin \{\alpha, \beta, \gamma\}$, then $\rho : \mathbb{H}^{kl}(a,b) \to A$, defined by

$$\mathbf{i} \mapsto \frac{\delta \Im(c)}{\alpha_s}, \quad \mathbf{j} \mapsto \frac{\epsilon \alpha_s \Im(d) - \delta \beta_s \gamma_c \Im(c)}{\alpha_s \beta_s \gamma_s}$$

is an isomorphism, where

(1) if $\alpha \neq \pi/2$, then $\epsilon = \operatorname{sgn}(\Re(c))$ and
  - *i. if $\beta \neq \pi/2$, then $\delta = \operatorname{sgn}(\Re(d)) \operatorname{sgn}(\beta_c)$,*
  - *ii. if $\beta = \pi/2$ and $\gamma \neq \pi/2$, then $\delta = \epsilon \operatorname{sgn}(\langle \Im(c), \Im(d) \rangle)$,*
  - *iii. if $\beta = \gamma = \pi/2$, then δ can be chosen freely;*

(2) if $\alpha = \pi/2 \notin \{\beta, \gamma\}$, then $\delta = \operatorname{sgn}(\Re(d))$ and $\epsilon = \delta \operatorname{sgn}(\langle \Im(c), \Im(d) \rangle)$;

(3) if $\alpha = \pi/2$ and $\pi/2 \in \{\beta, \gamma\}$, then ϵ can be chosen freely, and
  - *i. if $\beta = \pi/2 \neq \gamma$, then $\delta = \epsilon \operatorname{sgn}(\langle \Im(c), \Im(d) \rangle)$,*
  - *ii. if $\gamma = \pi/2 \neq \beta$, then $\delta = \operatorname{sgn}(\Re(d))$,*
  - *iii. if $\beta = \gamma = \pi/2$, then δ can be chosen freely.*

REMARK 7.5. The fact that conjugation by a unit norm quaternion preserves the real part and inner product implies that $\Re(c)$, $\Re(d)$ and $\langle \Im(c), \Im(d) \rangle$ are non-zero whenever this is required for the sign function to be defined, and that Lemma 7.3 is applicable wherever claimed.

Proof. To prove the statements where $0 \in \{\alpha, \beta, \gamma\}$ it suffices, by Proposition 7.1, to check that the given ϵ, δ and p satisfy $c = \epsilon p a \overline{p}$ and $d = \delta p b \overline{p}$, which is straightforward.

For the cases where $0 \notin \{\alpha, \beta, \gamma\}$, we instead use that by Proposition 7.1 there exist such ϵ, δ and p, and that an isomorphism is given by $z \mapsto \epsilon \delta p z \overline{p}$. The image of $\mathbf{i}$ under this isomorphism is then determined by $\Im(c) = \epsilon p \Im(a) \overline{p}$, since $\Im(a) = \alpha_s \mathbf{i}$ with $\alpha_s \neq 0$. This, together with $\Im(d) = \epsilon p \Im(b) \overline{p}$, determines the image of $\mathbf{j}$ since $\Im(b) = \beta_s \gamma_c \mathbf{i} + \beta_s \gamma_s \mathbf{j}$ with $\beta_s \gamma_s \neq 0$. The listed values of ϵ and δ are readily checked. Since by Theorem 2.2 there are no more cases, the proof is complete. □

Note how the construction of an isomorphism involves a number of choices, and different choices may give different isomorphisms. This is of no importance in this context, as any morphism ϕ from an absolute valued algebra C to an algebra $A = \mathbb{H}^{kl}(c,d) \simeq \mathbb{H}^{kl}(a,b) = A'$ factors uniquely over any isomorphism $\rho : A' \to A$.

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CORESTRICTED GROUP ACTIONS AND EIGHT-DIMENSIONAL ABSOLUTE VALUED ALGEBRAS

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ABSTRACT. A condition for when two eight-dimensional absolute valued algebras are isomorphic was given in [4]. We use this condition to deduce a description (in the sense of Dieterich, [9]) of the category of such algebras, and show how previous descriptions of some full subcategories fit in this description. Led by the structure of these examples, we aim at systematically constructing new subcategories whose classification is manageable. To this end we propose, in greater generality, the definition of sharp stabilizers for group actions, and use these to obtain conditions for when certain subcategories of groupoids are full. This we apply to the category of eight-dimensional absolute valued algebras and obtain a class of subcategories, for which we simplify, and partially solve, the classification problem.

1. INTRODUCTION

This paper is concerned with the classification of finite-dimensional absolute valued algebras. An *algebra* over a field k is a vector space A over k equipped with a k-bilinear multiplication $A \times A \to A, (x, y) \mapsto xy$. Neither associativity nor commutativity is in general assumed. A is called *absolute valued* if the vector space is real, non-zero and equipped with a norm $\| \cdot \|$ such that $\|xy\| = \|x\|\|y\|$ for all $x, y \in A$.

Finite-dimensional absolute valued algebras exist only in dimensions 1, 2, 4 and 8; except in dimension 8, they have been classified up to isomorphism, and the morphisms between them have been described (see [2] and [10]). In dimension 8, conditions for when two algebras are isomorphic were obtained in [4], using triality. We formulate these conditions as a description (in the sense of Dieterich, [9]) of the category of eight-dimensional absolute valued algebras in Section 2. Since the classification problem has proved hard, we set out to systematically find suitable full subcategories for which the classification problem is feasible, in particular, where the generally difficult computations in connection with triality are avoided.

As a model, we consider, in Section 3, the full subcategories of eight-dimensional absolute valued algebras with a left unity, a right unity, and a non-zero central idempotent, respectively. These were described and classified in [7], and we embed this description in that of Section 2.

Here, a *description* of a groupoid[1] $\mathcal{C}$ is an equivalence of categories between $\mathcal{C}$ and a groupoid arising from a group action. To construct full subcategories of $\mathcal{C}$,

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[1]By a *groupoid* we understand a (not necessarily small) category where all morphisms are isomorphisms.

we seek conditions under which the group action used in the description induces an action on a subset, which in turn gives a description of a full subcategory of $\mathcal{C}$.[2] As it turns out, this occurs precisely when the subset imposes a certain dichotomy on the group. This is explored in Section 4, where a general framework is obtained, based on stabilizers of subsets with respect to a group action. One of the results there describes how the object class of a groupoid can be partitioned, giving rise to pairwise isomorphic subgroupoids. This simplifies the classification problem and gives additional structural insight.

In Section 5, we examine the subcategories classified in [7] in this new framework. A common feature of these subcategories is the triviality of the triality phenomenon. Using this knowledge, and the structural insight gained in Section 4, we construct and describe a new subcategory of absolute valued algebras in Section 6. We reduce the classification problem for these algebras to a manageable, though somewhat computational, one, and in the final section, we classify some subclasses of such algebras to demonstrate the computations involved.

1.1. **Preliminaries.** By [1], the norm in a finite-dimensional absolute valued algebra is uniquely determined by the algebra multiplication, and multiplicativity of the norm implies that an absolute valued algebra has no zero divisors and hence, if it is finite-dimensional, that it is a division algebra. (We recall that a *division algebra* is a non-zero algebra D such that for each $a \in D \setminus \{0\}$, the left and right multiplication maps $L_a : D \to D, x \mapsto ax$ and $R_a : D \to D, x \mapsto xa$ are bijective. This implies the non-existence of zero divisors and, if D has finite dimension, it is equivalent to it.)

The class of all finite-dimensional absolute valued algebras forms a category $\mathcal{A}$, in which the morphisms are all non-zero algebra homomorphisms. Thus $\mathcal{A}$ is a full subcategory of the category $\mathcal{D}(\mathbb{R})$ of finite-dimensional real division algebras. It is known that morphisms in $\mathcal{D}(\mathbb{R})$ are injective, and morphisms in $\mathcal{A}$ are isometries.

In 1947, Albert characterized all finite-dimensional absolute valued algebras as follows ([1]).

Proposition 1.1. *Every finite-dimensional absolute valued algebra is isomorphic to an orthogonal isotope A of a unique $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, i.e. $A = \mathbb{A}$ as a vector space, and the multiplication $\cdot$ in A is given by*

$$x \cdot y = f(x)g(y)$$

for all $x, y \in A$, where f and g are linear orthogonal operators on A, and juxtaposition denotes multiplication in $\mathbb{A}$.

Moreover, Albert shows that the norm in A coincides with the norm in $\mathbb{A}$.

Thus the objects of $\mathcal{A}$ are partitioned into four classes according to their dimension, and the class of d-dimensional algebras, $d \in \{1, 2, 4, 8\}$, forms a full subcategory $\mathcal{A}_d$ of $\mathcal{A}$. For $d > 1$ we moreover have the following decomposition due to Darpö and Dieterich [8]. [3]

Proposition 1.2. *Let $A \in \mathcal{A}_d$ with $d \in \{2, 4, 8\}$. For any $a, b \in A \setminus \{0\}$,*

$$\operatorname{sgn}(\det(L_a)) = \operatorname{sgn}(\det(L_b)), \quad \operatorname{sgn}(\det(R_a)) = \operatorname{sgn}(\det(R_b)).$$

[2] It is important that the subcategory be full, in order for a classification of it to be useful in classifying the larger category.

[3] We define the sign function $\operatorname{sgn} : \mathbb{R} \to C_2$ by $\operatorname{sgn}(r) = r/|r|$ if $r \neq 0$, and $\operatorname{sgn}(0) = 1$. The definition at 0, though immaterial here, will be useful in later sections.

The double sign *of A is the pair $(i,j) \in C_2^2$ where $i = \mathrm{sgn}(\det(L_a))$ and $j = \mathrm{sgn}(\det(R_a))$ for any $a \in A \setminus \{0\}$. Moreover, for all $d \in \{2,4,8\}$,*

$$\mathcal{A}_d = \coprod_{(i,j)\in C_2^2} \mathcal{A}_d^{ij} \tag{1.1}$$

where $\mathcal{A}_d^{ij}$ is the full subcategory of $\mathcal{A}_d$ formed by all algebras having double sign (i,j).

In the classification of finite-dimensional real division algebras, a certain type of categories, more specifically, of groupoids, has proven useful. We recall their definition.

Definition 1.3. Let G be a group, X a set, and $\alpha : G \times X \to X$ a left group action. The *groupoid arising from* α is the category ${}_GX$ with object set X and where, for each $x, y \in X$,

$${}_GX(x,y) = \{(g,x,y)|g \in G, g \cdot x = y\}.$$

It is clear that ${}_GX$ is a groupoid. The group action is implicit in the notation ${}_GX$, and if the domain and codomain of a morphism (g,x,y) are clear from the context, the morphism is simply referred to by g.

Groupoids arising from group actions can, and will in this article, be used to gain an understanding of finite-dimensional absolute valued algebras in the following way, due to [9].

Definition 1.4. Let $d \in \{2,4,8\}$. A *description (in the sense of Dieterich)* of a full subcategory $\mathcal{C} \subseteq \mathcal{A}_d$ is a quadruple $(G,X,\alpha,\mathcal{F})$, where G is a group, X a set, $\alpha : G \times X \to X$ a left group action, and $\mathcal{F} : {}_GX \to \mathcal{C}$ an equivalence of categories.

Once a description is obtained, the problem of classifying $\mathcal{C}$ is transformed to the normal form problem for α, i.e. the problem of finding a transversal for the orbits of α. It is therefore crucial that the quadruple $(G,X,\alpha,\mathcal{F})$ be given explicitly. In [9], descriptions are defined in the more general context of finite-dimensional real division algebras, which we will not need here.

1.2. **Notation.** We use the convention that $0 \in \mathbb{N}$, and use the notation $\mathbb{Z}_+$ for the set $\mathbb{N}\setminus\{0\}$. For each $n \in \mathbb{Z}_+$ we denote by $\underline{n}$ the set $\{1,2,\dots,n\}$. As $\mathbb{O}$ denotes the algebra of octonions, $\Im\mathbb{O}$ denotes the hyperplane of its purely imaginary elements.

For a vector space V, we denote by $\mathbb{P}(V)$ the projective space of V, whose elements are the lines through the origin in V. An element in $\mathbb{P}(V)$ containing a non-zero vector v will be denoted by $[v]$. More generally, square brackets denote the linear span of a collection of vectors in V. If a basis is given, upper indices will always denote the coordinates of a vector in this basis; hence v^i is the i^{th} coordinate of v.

If V is normed and $U \subseteq V$ a subset, we denote by $\mathbb{S}(U)$ the set of all elements of U having norm 1. Unless otherwise stated, for each $n \in \mathbb{N}$, $\mathbb{R}^{n+1}$ is equipped with the Euclidean norm, and $\mathbb{S}^n = \mathbb{S}(\mathbb{R}^{n+1})$ denotes the unit n-sphere.

The general linear group in dimension n over $\mathbb{R}$ will be denoted $GL_n = GL(\mathbb{R}^n)$, which we identify with $GL_n(\mathbb{R})$ upon endowing $\mathbb{R}^n$ with a standard basis. Analogous notation will be used for its classical subgroups, notably O_n and SO_n. We denote by O_8^1 the group of all $g \in O_8 = O(\mathbb{O})$ fixing $1 \in \mathbb{O}$, and by O_8^+ and O_8^- the set of all $f \in O_8$ with positive and negative determinant, respectively. (The elements of the cyclic group C_2 are written as $+$ and $-$ rather than as 1 and -1.) The notation $\mathbb{I}_n$ will be used for the $n \times n$ identity matrix.

The symbol $\leq$ will be used to denote the subgroup relation. For a group action $\alpha : G \times X \to X$, $g \in G$ and $x, y \in X$, we write $g \cdot x$ for $\alpha(g, x)$, and $x \equiv_\alpha y$ or $x \equiv y$ *with respect to* α to denote that x and y are in the same orbit.

Finally, given a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$, we denote by $\mathcal{F}|_\mathcal{C}$ the restriction of $\mathcal{F}$ to a subcategory $\mathcal{C}$ of $\mathcal{A}$.

1.3. **Triality, G_2 and Cayley Triples.** The study we are about to undertake makes frequent use of two concepts: the principle of triality, and the group G_2. Both have been subject to profound research, which goes far beyond the scope of this paper. The aim of this section is to recall such facts about these concepts that will be needed here, in the form applicable to the problems at hand. For a more general approach, the reader is directed to the literature: both concepts are treated in the overview article of Baez [3], as well as in [6], triality is further treated by Chevalley [5], while G_2 and Cayley triples are dealt with in [12], Chapter 1. Applications of these concepts to absolute valued algebras can be found in [4] and [7], to which we will refer in several places.

We will be concerned with the *principle of triality* as applied to SO_8, which we quote here.

Proposition 1.5. *For each $\phi \in SO_8$ there exist $\phi_1, \phi_2 \in SO_8$ such that for each $x, y \in \mathbb{O}$,*

$$\phi(xy) = \phi_1(x)\phi_2(y).$$

The pair (ϕ_1, ϕ_2) is unique up to (overall) sign.

Thus there exist two *triality pairs* $\pm(\phi_1, \phi_2)$ for each $\phi \in SO_8$. Moreover, triality respects composition, i.e. if $\phi, \psi \in SO_8$, and (ϕ_1, ϕ_2) and (ψ_1, ψ_2) are triality pairs for ϕ and ψ, respectively, then $(\phi_1\psi_1, \phi_2\psi_2)$ is a triality pair for the product $\phi\psi$, since for any $x, y \in \mathbb{O}$,

$$\phi_1\psi_1(x)\phi_2\psi_2(y) = \phi(\psi_1(x)\psi_2(y)) = \phi\psi(xy).$$

As $(\mathrm{Id}, \mathrm{Id})$ is a triality pair for $\mathrm{Id} \in SO_8$, we deduce that $(\phi_1^{-1}, \phi_2^{-1})$ is a triality pair for ϕ^{-1}. We moreover have the identities

$$\phi_1 = R_{\phi_2(1)^{-1}}\phi, \quad \phi_2 = L_{\phi_1(1)^{-1}}\phi,$$

where $\phi_i(1)^{-1}$ is not to be confused with $\phi_i^{-1}(1)$ for $i \in \underline{2}$.

Every automorphism of $\mathbb{O}$ has determinant 1. Thus $\mathrm{Aut}(\mathbb{O}) \leq SO_8$, and an element $\phi \in SO_8$ is an automorphism of $\mathbb{O}$ precisely when (ϕ, ϕ) is a triality pair for ϕ. We then say that the triality components of ϕ are *trivial*. The group $\mathrm{Aut}(\mathbb{O})$ is the Lie group G_2. It has dimension 14, and is thus the smallest of the exceptional Lie groups. G_2 may equivalently be characterized as the set of all $\phi \in SO_8$ such that

$$\phi(1) = \phi_1(1) = \phi_2(1) = 1$$

for some triality pair (ϕ_1, ϕ_2) of ϕ. The identity $\phi(1) = \phi_1(1)\phi_2(1)$ shows that if any two of $\phi(1)$, $\phi_1(1)$ and $\phi_2(1)$ equal 1, then so does the third. Since the group of all $\phi \in SO_8$ such that $\phi(1) = 1$ is isomorphic to SO_7, one may view G_2 as a subgroup of SO_7, which we will sometimes do for notational convenience.

Another way to characterize G_2 is via Cayley triples. [4]

[4] Named in honour of Arthur Cayley, 1821–1895.

Definition 1.6. A *Cayley triple* is an orthonormal triple $(u, v, z) \in (\Im\mathbb{O})^3$ such that $z \perp uv$.

Some fundamental facts about Cayley triples are given in the following well-known result.

Proposition 1.7. *Let $(u, v, z) \in (\Im\mathbb{O})^3$ be a Cayley triple.*

(i) The algebra $\mathbb{O}$ is generated by (u, v, z).
(ii) $(1, u, v, uv, z, uz, vz, (uv)z)$ is an orthonormal basis of $\mathbb{O}$, called the basis induced by *(u, v, z).*
(iii) The group G_2 corresponds bijectively to the set of all Cayley triples, the bijection being given by $\phi \mapsto (\phi(u), \phi(v), \phi(z))$ for all $\phi \in G_2$.

Cayley triples and induced bases will be used in the computations of Section 7.

2. Description of $\mathcal{A}_8$

Conditions for when two finite-dimensional absolute valued algebras are isomorphic are given in [4]. In this section we deduce from this a description of $\mathcal{A}_8$. In other words, we establish an equivalence of categories from a groupoid arising from a group action to $\mathcal{A}_8$. To begin with, we introduce the action, for which we define the quotient group

$$\mathcal{O}_8 = (O_8 \times O_8)/\{\pm(\mathrm{Id}, \mathrm{Id})\}$$

and write $[f, g]$ for the coset of $(f, g) \in O_8 \times O_8$.

Proposition 2.1. *The map $\tau : SO_8 \times \mathcal{O}_8 \to \mathcal{O}_8$ defined by*

$$(\phi, [f, g]) \mapsto \phi \cdot [f, g] = [\phi_1 f \phi^{-1}, \phi_2 g \phi^{-1}],$$

where (ϕ_1, ϕ_2) is any of the two triality pairs of ϕ, is a left group action.

This action will be called the *triality action*, and we say that SO_8 acts *by triality*.

Proof. Note, at first, that the map is well-defined, since the two triality pairs of ϕ, as well as the two representatives of $[f, g]$, are equal up to overall sign. The identity axiom for group actions holds since $(\mathrm{Id}, \mathrm{Id})$ is a triality pair for $\mathrm{Id} \in SO_8$. For the product axiom, take $\phi, \psi \in SO_8$. Then for any triality pair (ϕ_1, ϕ_2) of ϕ and (ψ_1, ψ_2) of ψ

$$\begin{aligned} \phi \cdot (\psi \cdot [f, g]) &= \phi \cdot [\psi_1 f \psi^{-1}, \psi_2 g \psi^{-1}] \\ &= [\phi_1 \psi_1 f \psi^{-1} \phi^{-1}, \phi_2 \psi_2 g \psi^{-1} \phi^{-1}] = \phi\psi \cdot [f, g], \end{aligned}$$

where the rightmost equality holds since triality respects composition. □

Remark 2.2. Note that for each $h \in GL_8$ we have $\det(h) = \det(-h)$. Thus the quotient in Proposition 2.1 respects the sign of the determinant of each of f and g, i.e. if $(f, g) \in O_8^j \times O_8^i$ for some $(i, j) \in C_2^2$, then $(-f, -g) \in O_8^j \times O_8^i$. Thus $\mathcal{O}_8$ decomposes into four subsets,

$$\mathcal{O}_8^{ij} := \{[f, g] \in \mathcal{O}_8 | \det(f) = j, \det(g) = i\}, \quad (i, j) \in C_2^2.$$

Also note that the group action preserves the pair $(\det(f), \det(g))$ as well, i.e. if $[f, g] \in \mathcal{O}_8^{ij}$ for some $(i, j) \in C_2^2$, then $\phi \cdot [f, g] \in \mathcal{O}_8^{ij}$ for all $\phi \in SO_8$. Hence SO_8

acts by triality on $\mathcal{O}_8^{ij}$ for each $(i,j) \in C_2^2$, and for the groupoid arising from the triality action, we have the coproduct decomposition

$$_{SO_8}\mathcal{O}_8 = \coprod_{(i,j)\in C_2^2} {}_{SO_8}\mathcal{O}_8^{ij}.$$

The seemingly reversed order of i and j in the notation is used for coherence with the double sign defined in Proposition 1.2, as will be clear in the next theorem, which establishes the equivalences of categories.

Theorem 2.3. *Let $(i,j) \in C_2^2$, and let ${}_{SO_8}\mathcal{O}_8^{ij}$ be the groupoid arising from the triality action of SO_8 on $\mathcal{O}_8^{ij}$. Then*

$$\mathcal{F}^{ij} : {}_{SO_8}\mathcal{O}_8^{ij} \to \mathcal{A}_8^{ij},$$

defined on objects by $\mathcal{F}^{ij}([f,g]) = \mathbb{O}_{f,g}$ and on morphisms by $\mathcal{F}^{ij}(\phi) = \phi$, is an equivalence of categories.

Proof. The maps $\mathcal{F}^{ij}$ are independent of the representative of $[f,g]$ since $\mathbb{O}_{-f,-g} = \mathbb{O}_{f,g}$. They are well-defined by the definition of the double sign and by Remark 2.2, noting that for each $a \in \mathbb{O}_{f,g} \setminus \{0\}$,

$$\mathrm{sgn}(\det(L_a)) = \mathrm{sgn}(\det(g)), \quad \mathrm{sgn}(\det(R_a)) = \mathrm{sgn}(\det(f)).$$

Functoriality follows from the axioms of a group action. Finally, each $\mathcal{F}^{ij}$ is dense by Propositions 1.1 and 1.2, faithful by construction, and full by [4].[5] □

Remark 2.4. We will use $\mathcal{F}$ to denote the functor ${}_{SO_8}\mathcal{O}_8 \to \mathcal{A}_8$ defined by $\mathcal{F}|_{{}_{SO_8}\mathcal{O}_8^{ij}} = \mathcal{F}^{ij}$ for each $(i,j) \in C_2^2$.

Summarizing, $(SO_8, \mathcal{O}_8, \tau, \mathcal{F})$ is a description of $\mathcal{A}_8$, and, more specifically, for each $(i,j) \in C_2^2$, $(SO_8, \mathcal{O}_8^{ij}, \tau_{ij}, \mathcal{F}^{ij})$ is a description of $\mathcal{A}_8^{ij}$, where τ_{ij} is the triality action of SO_8 on $\mathcal{O}_8^{ij}$ from Remark 2.2.

In fact, the result in [4] mentioned above contains more information, namely that for each $f, g \in O_8$ there exist $f', g' \in O_8^1$ such that $\mathbb{O}_{f,g} \simeq \mathbb{O}_{f',g'}$, i.e. that $\mathcal{F}(\mathcal{C}_8^1)$ is dense in $\mathcal{A}_8$, where $\mathcal{C}_8^1 \subseteq {}_{SO_8}\mathcal{O}_8$ is the full subcategory whose object set $\mathcal{O}_8^1$ consists of all $[f,g] \in \mathcal{O}_8$ with $f, g \in O_8^1$. [6]

However, this cannot be used to replace the above description of $\mathcal{A}_8$ with a description using $\mathcal{O}_8^1$ instead of $\mathcal{O}_8$, in the sense that there is no subgroup $H \le SO_8$ such that $\mathcal{C}^1 = {}_H\mathcal{O}_8^1$. Proving this makes use of the techniques to be developed in Section 4 below, and the proof can be found in Appendix A.

We will apply Theorem 2.3 in various contexts. First, let us review a classification of some full subcategories of $\mathcal{A}_8$ in view of the above description.

3. Algebras having a Non-zero Central Idempotent or a One-sided Unity

Consider the three full subcategories $\mathcal{A}_8^l$, $\mathcal{A}_8^r$ and $\mathcal{A}_8^c$ of $\mathcal{A}_8$, the object classes of which are

$$\mathcal{A}_8^l = \{A \in \mathcal{A}_8 | \exists u \in A, \forall x \in A, ux = x\},$$
$$\mathcal{A}_8^r = \{A \in \mathcal{A}_8 | \exists u \in A, \forall x \in A, xu = x\},$$

[5]In [4], this is part of Theorem 4.3 and the remarks preceding it.

[6]Note that the quotient map $O_8 \times O_8 \to \mathcal{O}_8, (f,g) \mapsto [f,g]$ is injective on $O_8^1 \times O_8^1$.

$$\mathcal{A}_8^c = \{A \in \mathcal{A}_8 | \exists u \in Z(A) \setminus \{0\}, u^2 = u\},$$

and consist of all algebras with a left unity, a right unity, and a non-zero central idempotent, respectively. Here, the centre $Z(B)$ of an algebra B is defined by

$$Z(B) = \{z \in B | \forall b \in B, zb = bz\},$$

and an element is called *central* if it belongs to the centre. The three categories defined above are studied in [7], where a classification of them is obtained. We will return to the classification later; in the present section, we direct our attention to the descriptions of the three categories, given in the following result from [7].

Proposition 3.1. *For each $x \in \{l, r, c\}$, $\mathcal{A}_8^x$ is equivalent to the category ${}_{G_2}O_8^1$, where G_2 acts by conjugation. The equivalences are given by*

$$\begin{array}{lll} \mathcal{F}^l : {}_{G_2}O_8^1 \to \mathcal{A}_8^l, & \mathcal{F}^l(f) = \mathbb{O}_{f,\mathrm{Id}}, & \mathcal{F}^l(\phi) = \phi, \\ \mathcal{F}^r : {}_{G_2}O_8^1 \to \mathcal{A}_8^r, & \mathcal{F}^r(f) = \mathbb{O}_{\mathrm{Id},f}, & \mathcal{F}^r(\phi) = \phi, \\ \mathcal{F}^c : {}_{G_2}O_8^1 \to \mathcal{A}_8^c, & \mathcal{F}^c(f) = \mathbb{O}_{f,f}, & \mathcal{F}^c(\phi) = \phi, \end{array}$$

for each $f \in O_8^1$ and $\phi \in G_2$.

The following proposition implies that this description of $\mathcal{A}_8^x, x \in \{l, r, c\}$ is an instance of the description of $\mathcal{A}_8$ of Section 2.

Proposition 3.2. *Let $x \in \{l, r, c\}$, and define $\mathcal{G}^x : {}_{G_2}O_8^1 \to {}_{SO_8}\mathcal{O}_8$ on objects by*

$$\mathcal{G}^l(f) = [f, \mathrm{Id}], \quad \mathcal{G}^r(f) = [\mathrm{Id}, f], \quad \mathcal{G}^c(f) = [f, f],$$

for each $f \in O_8^1$, and on morphisms by $\mathcal{G}^x(\phi) = \phi$ for each $x \in \{l, r, c\}$ and $\phi \in G_2$. Then for each $x \in \{l, r, c\}$, $\mathcal{G}^x$ is a full and faithful functor, which is moreover injective on the set of objects.

Proof. For each $x \in \{l, r, c\}$, $\mathcal{G}^x$ is well defined, as for any $\phi \in G_2$, we have $\phi(1) = 1$ and (ϕ, ϕ) is a triality pair for ϕ; thus for each $i \in \underline{2}$ and each $f \in O_8^1$,

$$\phi_i \,\mathrm{Id}\, \phi^{-1} = \phi \,\mathrm{Id}\, \phi^{-1} = \mathrm{Id}, \quad \phi_i f \phi^{-1}(1) = \phi f \phi^{-1}(1) = 1,$$

whence $\mathcal{G}^x(\phi)(\mathcal{G}^x(f)) = \mathcal{G}^x(\phi(f))$. Functoriality is obvious, and $\mathcal{G}^x$ is faithful by definition. For injectivity, let first $x = c$, and assume $[f, f] = [f', f']$ for some $f, f' \in O_8^1$. Then $(f, f) = \pm(f', f')$. But $f(1) = f'(1) = 1$, thus $f \neq -f'$, implying $f = f'$. The other cases are similar. To show fullness, take any $f, f' \in O_8^1$, and let $\phi : \mathcal{G}^x(f) \to \mathcal{G}^x(f')$ be any morphism. We need show that $\phi \in G_2$, and we will do this case by case.

Assume first that $x = l$. Then by Proposition 2.1 we have, for one triality pair (ϕ_1, ϕ_2) of ϕ, that

$$\phi_1 f \phi^{-1} = f' \quad \text{and} \quad \phi_2 \,\mathrm{Id}\, \phi^{-1} = \mathrm{Id}\,.$$

From the second equality we then have $\phi_2 = \phi$, thus $\phi(1) = \phi_1(1)\phi(1)$, implying that ϕ_1 fixes $1 \in \mathbb{O}$. Rewriting the first equation as $\phi = f'^{-1}\phi_1 f$, we note that the right hand side fixes 1, and hence $\phi_2(1) = \phi(1) = 1$. Thus $\phi \in G_2$. If instead $x = r$, then the above argument, with ϕ_1 and ϕ_2 interchanged, implies that $\phi \in G_2$. Finally if $x = c$, then

$$\phi_1 f \phi^{-1} = f' \quad \text{and} \quad \phi_2 f \phi^{-1} = f',$$

for one triality pair (ϕ_1, ϕ_2) of ϕ, implying that $\phi_1 = \phi_2$. Take any $y \in \mathbb{O}$. Since ϕ_1 is bijective there exists $x \in \mathbb{O}$ such that $y = \phi_1(x)$. Hence

$$y\phi_1(1) = \phi_1(x)\phi_1(1) = \phi_1(x)\phi_2(1) = \phi(x) = \phi_1(1)\phi_2(x) = \phi_1(1)\phi_1(x) = \phi_1(1)y$$

and thus $\phi_1(1) \in Z(\mathbb{O}) = \mathbb{R}1$. But $\phi_1 \in SO_8$, whereby $\|\phi_1(1)\| = 1$. Moreover,

$$\phi(1) = \phi_1(1)^2 = 1,$$

and together with $\phi_1(1) = \phi_2(1) = \pm 1$ this implies that $\phi \in G_2$. □

Viewing $\mathcal{G}^x$, $x \in \{l, r, c\}$, as inclusion $O_8^1 \hookrightarrow \mathcal{O}_8$, the above result implies that the description of $\mathcal{A}_8^x$ is obtained directly from the description of $\mathcal{A}_8$, by restricting the latter to O_8^1.

4. Corestricted Group Actions

In view of Proposition 3.2, one may ask under which conditions a description of a full subcategory $\mathcal{C} \subseteq \mathcal{A}_8$ can be obtained by restricting a description $\mathcal{F} : {}_GX \to \mathcal{A}_8$ of $\mathcal{A}_8$ to a subset $Y \subseteq X$. In precise terms, given the groupoid ${}_GX$ arising from a left action α of a group G on a set X, and given a subset $Y \subseteq X$, we seek conditions on Y under which there exists a subgroup $H \leq G$ such that

- the restriction of α to $H \times Y$ admits a corestriction to Y, i.e. α induces a group action $H \times Y \to Y$, and
- the groupoid ${}_HY$ arising from this action is full in ${}_GX$.

In this section, we will answer this question in the general setting, and derive some structural consequences.

4.1. Definitions and General Properties.

The constructions of this section use the notion of a stabilizer of a subset under a group action. This notion, which we will now develop, generalizes the familiar notion of a stabilizer of a single element. We will use left actions throughout; however, all definitions can be analogously made for right actions, and all results apply, *mutatis mutandis*, to these as well.

Definition 4.1. Let $\alpha : G \times X \to X$ be a group action. An element $g \in G$ is said to *stabilize* a subset $Y \subseteq X$ if $g \cdot y \in Y$ for each $y \in Y$. A subgroup $H \leq G$ *stabilizes* Y if each $h \in H$ stabilizes Y.

Note that the set of all elements of G that stabilize $Y \subseteq X$ is in general not a subgroup of G, since it may happen that the inverse of a stabilizing element is not stabilizing itself.

Example 4.2. Consider the action of the group $(\mathbb{Z}, +)$ on itself by addition. The set of all elements that stabilize $\mathbb{N} \subset \mathbb{Z}$ is $\mathbb{N}$ itself, which does not contain the inverse of any of its non-zero elements.

We define the following subsets.

Definition 4.3. Let $\alpha : G \times X \to X$ be a group action, and let $Y \subseteq X$. The *stabilizer* of Y (*with respect to* α) is the set

$$\mathrm{St}(Y) = \{g \in G | \forall y \in Y : g \cdot y \in Y\},$$

the *sharp stabilizer* of Y is the set

$$\mathrm{St}^*(Y) = \{g \in G | \forall y \in Y : g \cdot y \in Y \wedge g^{-1} \cdot y \in Y\},$$

and the *destabilizer* of Y is the set

$$\mathrm{Dest}(Y) = \{g \in G | \forall y \in Y : g \cdot y \notin Y\}.$$

Remark 4.4. Equivalently, we have that

$$\begin{aligned}\mathrm{St}(Y) &= \{g \in G | g \cdot Y \subseteq Y\}, \\ \mathrm{St}^*(Y) &= \{g \in G | g \cdot Y = Y\}, \text{ and} \\ \mathrm{Dest}(Y) &= \{g \in G | g \cdot Y \cap Y = \emptyset\}.\end{aligned}$$

When necessary, we will write $\mathrm{St}_\alpha(Y)$ etc. to emphasize the group action. If Y is a singleton set $\{y\}$, we will omit the set brackets in the above notation.

In the above example we saw that $\mathrm{St}(\mathbb{N}) = \mathbb{N}$. Moreover, $\mathrm{St}^*(\mathbb{N}) = \{0\}$, and $\mathrm{Dest}(\mathbb{N}) = \emptyset$. In general, we have the following facts.

Proposition 4.5. *Let $\alpha : G \times X \to X$ be a group action, and let $Y \subseteq X$. Then*

(i) $\mathrm{St}(Y)$ *is a submonoid of G,*
(ii) $\mathrm{St}^*(Y)$ *is a subgroup of G, and* $\mathrm{St}^*(Y) = \mathrm{St}(Y) \cap \mathrm{St}(Y)^{-1}$,
(iii) if $H \leq G$ satisfies $H \subseteq \mathrm{St}(Y)$, then $H \leq \mathrm{St}^*(Y)$, and
(iv) if Y is finite, then $\mathrm{St}^*(Y) = \mathrm{St}(Y)$.

Here we have used the notation $Z^{-1} = \{z^{-1} | z \in Z\}$ for a subset $Z \subseteq G$. Note that statement (iv) generalizes the fact that the stabilizer of one element is a group.

Proof. (i) and (ii) are immediate from the definitions and the axioms of a group action. (iii) holds since if $g \in H \subseteq \mathrm{St}(Y)$, then $g^{-1} \in H \subseteq \mathrm{St}(Y)$, whence $g \in \mathrm{St}(Y) \cap \mathrm{St}(Y)^{-1}$. To prove (iv), assume that Y is finite and let $g \in \mathrm{St}(Y)$. Then $g \cdot Y \subseteq Y$ and $|g \cdot Y| = |Y|$, whence $g \cdot Y = Y$, and thus $g \in \mathrm{St}^*(Y)$. □

Remark 4.6. Item (iii) implies that $\mathrm{St}^*(Y)$ is maximal in the sense that it is the largest subgroup of G contained in $\mathrm{St}(Y)$.

We conclude from Proposition 4.5 that the map

$$\mathrm{St}^*(Y) \times Y \to Y, (g, y) \mapsto g \cdot y$$

is an action of the group $\mathrm{St}^*(Y)$ on Y, the so-called *corestriction of $\alpha : G \times X \to X$ from X to Y.*

4.2. **Conditions for Full Subgroupoids.** Let $\alpha : G \times X \to X$ be a group action. By the above, each subset $Y \subseteq X$ gives rise to a subcategory

$${}_{\mathrm{St}^*(Y)}Y \subseteq {}_G X.$$

As mentioned above, we wish to determine for which subsets $Y \subseteq X$ there exists $H \leq G$ such that ${}_H Y$ is a full subcategory of ${}_G X$. Note that ${}_H Y$ is defined if and only if $H \subseteq \mathrm{St}(Y)$. Thus by Remark 4.6, such a subgroup exists if and only if ${}_{\mathrm{St}^*(Y)}Y \subseteq {}_G X$ is full. The following theorem, which is the main result of this section, determines when this holds.

Theorem 4.7. *Let $\alpha : G \times X \to X$ be a group action, and let $\emptyset \neq Y \subseteq X$. Then the following statements are equivalent.*

(i) The inclusion functor $\mathcal{I} : {}_{\mathrm{St}^*(Y)}Y \hookrightarrow {}_G X$ is full.
(ii) $G = \mathrm{St}^*(Y) \sqcup \mathrm{Dest}(Y)$.
(iii) $G = \mathrm{St}(Y) \sqcup \mathrm{Dest}(Y)$.
(iv) For any $g, h \in G$, either $g \cdot Y = h \cdot Y$ or $g \cdot Y \cap h \cdot Y = \emptyset$.
(v) The collection $\pi = \{g \cdot Y | g \in G\}$ is a partition of $G \cdot Y \subseteq X$.
(vi) The inclusion functor $\mathcal{I}' : {}_{\mathrm{St}^*(Y)}Y \hookrightarrow {}_G(G \cdot Y)$ is an equivalence of categories.

If any, hence all, of the above holds, then there is a bijection $\rho : G/\operatorname{St}^*(Y) \to \pi$ *between the left cosets of* $\operatorname{St}^*(Y)$ *and the classes of* π*, given by* $\overline{g} := g\operatorname{St}^*(Y) \mapsto g\cdot Y$*.*

Note that the two sets $\operatorname{St}(Y)$ and $\operatorname{Dest}(Y)$ are disjoint whenever $Y \neq \emptyset$. The principal statements in (ii) and (iii) are that the complement of the union is empty, which is true e.g. for stabilizers of singleton sets, but otherwise not in general (cf. Example 4.2).

Proof. (i) $\Longrightarrow$ (ii): Let $h \in G \setminus \operatorname{Dest}(Y)$. Then there exist $y, y' \in Y$ with $h \cdot y = y'$, whence $(h, y, y') \in {}_GX(y, y')$. Since $\mathcal{I}$ is full we then have $(h, y, y') \in {}_{\operatorname{St}^*(Y)}Y(y, y')$, and hence $h \in \operatorname{St}^*(Y)$.

(ii) $\Longrightarrow$ (i): Take any $y, y' \in Y$ and $(h, y, y') \in {}_GX(y, y')$. Then $h \cdot y = y' \in Y$, whence h does not belong to $\operatorname{Dest}(Y)$. Hence $h \in \operatorname{St}^*(Y)$ by hypothesis, which implies that $(h, y, y') = \mathcal{I}(h, y, y')$, and thus $\mathcal{I}$ is full.

(ii) $\Longrightarrow$ (iii): Take $g \in G \setminus \operatorname{Dest}(Y)$. Then by hypothesis $g \in \operatorname{St}^*(Y) \subseteq \operatorname{St}(Y)$.

(iii) $\Longrightarrow$ (ii): Take $h \in \operatorname{St}(Y)$. Then for any $y \in Y$ we have $y' := h \cdot y \in Y$. But then $h^{-1} \cdot y' = y$, and in particular h^{-1} does not belong to $\operatorname{Dest}(Y)$. Thus by hypothesis $h^{-1} \in \operatorname{St}(Y)$, and hence $h \in \operatorname{St}^*(Y)$. Since h was an arbitrary element of $\operatorname{St}(Y)$ this shows that $\operatorname{St}(Y) = \operatorname{St}^*(Y)$, whence $G = \operatorname{St}^*(Y) \sqcup \operatorname{Dest}(Y)$.

(ii) $\Longrightarrow$ (iv): Let $g, h \in G$ and assume that $g \cdot Y \cap h \cdot Y \neq \emptyset$. Then there exist $y, y' \in Y$ such that $g \cdot y = h \cdot y'$, and thus $h^{-1}g \cdot y = y'$. This excludes the possibility that $h^{-1}g \in \operatorname{Dest}(Y)$, and thus by assumption $h^{-1}g \in \operatorname{St}^*(Y)$. Thus for all $z \in Y$, $z' := h^{-1}g \cdot z \in Y$, implying $g \cdot z = h \cdot z' \in h \cdot Y$, whence $g \cdot Y \subseteq h \cdot Y$.

Since $\operatorname{St}^*(Y)$ is a group we also have $g^{-1}h = (h^{-1}g)^{-1} \in \operatorname{St}^*(Y)$, and repeating the above argument with g and h interchanged we get $h \cdot Y \subseteq g \cdot Y$ as well.

(iv) $\Longrightarrow$ (ii): Let $g \in G$ be arbitrary and set $h = e$, the identity element of G. Then by assumption either $g \cdot Y = Y$, which implies that $g \in \operatorname{St}^*(Y)$, or else $g \cdot Y \cap Y = \emptyset$ and $g \in \operatorname{Dest}(Y)$.

(iv) $\Longleftrightarrow$ (v): By definition of a partition, it follows that these are two reformulations of the same statement, since each $x \in G\cdot Y$ is contained in $g\cdot Y$ for some $g \in G$.

(i) $\Longleftrightarrow$ (vi): The functor $\mathcal{I}'$ is faithful, being an inclusion, and dense by definition of $G \cdot Y$. It is full if and only if for each $y, y' \in Y$ and each $g \in G$, $g \cdot y = y' \Rightarrow g \in \operatorname{St}^*(Y)$. This is equivalent to $\mathcal{I}$ being full.

Finally, we consider the map ρ. To begin with, if $\overline{g} = \overline{h}$, then there exists $j \in \operatorname{St}^*(Y)$ such that $h = gj$. Then the definition of $\operatorname{St}^*(Y)$ implies that $j \cdot Y = Y$, whence $h \cdot Y = g \cdot (j \cdot Y) = g \cdot Y$. Thus ρ is well-defined. It is surjective since for each class C of π there exists, by definition, an element $g \in G$ such that $C = g \cdot Y$, and hence $C = \rho(\overline{g})$. To show injectivity, assume $\rho(\overline{g}) = \rho(\overline{h})$, i.e $g \cdot Y = h \cdot Y$, for some $g, h \in G$, implying that $Y = g^{-1}h \cdot Y$. Thus $g^{-1}h \in \operatorname{St}^*(Y)$, and since $h = g(g^{-1}h)$, we obtain $\overline{g} = \overline{h}$, which completes the proof. □

Definition 4.8. Let $\alpha : G \times X \to X$ be a group action. A subset $Y \subseteq X$ is called *full (with respect to* α*)* if it is non-empty and satisfies the equivalent conditions of Theorem 4.7.

We observe the following from the above proof.

Corollary 4.9. *Let $\alpha : G \times X \to X$ be a group action, and let $Y \subseteq X$. If Y is full, then* $\mathrm{St}^*(Y) = \mathrm{St}(Y)$*.*

The converse is in general false, as the following example shows.

Example 4.10. Consider the action of $(\mathbb{Z}, +)$ on itself by addition. For $Y = \{0, 1\}$ we have $\mathrm{St}^*(Y) = \mathrm{St}(Y) = \{0\}$, but Y is not full, since $1 \notin \mathrm{St}(Y) \cup \mathrm{Dest}(Y)$.

Note, however, that singleton subsets are always full.

In case $\mathrm{St}^*(Y)$ is a normal subgroup of G, the bijection ρ of Theorem 4.7 induces a group structure on π (where the identity element is the class Y). We will however not need this in the sequel.

4.3. **Consequences for the Structure of ${}_GX$.** We now derive some insight into the structure of the groupoid ${}_GX$ from the above theorem, starting with a basic observation.

Lemma 4.11. *Let $\alpha : G \times X \to X$ be a group action, and let $\emptyset \neq Y \subseteq X$. For each $g \in G$, there is a bijection*

$$\lambda_g : Y \to g \cdot Y, \quad y \mapsto g \cdot y$$

and a group isomorphism

$$\kappa_g : \mathrm{St}^*(Y) \to \mathrm{St}^*(g \cdot Y), \quad h \mapsto ghg^{-1}$$

such that for each $y \in Y$ and each $j \in \mathrm{St}^*(Y)$,

$$\kappa_g(j) \cdot \lambda_g(y) = \lambda_g(j \cdot y).$$

Proof. Let $g \in G$. By definition of $g \cdot Y$, λ_g is well-defined and surjective, and it is injective by the axioms of a group action (with inverse $\lambda_{g^{-1}}$).

As for κ_g, we have, for any $j \in \mathrm{St}^*(Y)$, that

$$\kappa_g(j) \cdot (g \cdot Y) = gjg^{-1} \cdot (g \cdot Y) = gj \cdot Y = g \cdot Y,$$

whence κ_g is well-defined. It is then a homomorphism of groups, with inverse homomorphism $\kappa_g^{-1} = \kappa_{g^{-1}} : \mathrm{St}^*(g \cdot Y) \to \mathrm{St}^*(Y)$.

To prove the final statement we note that for any $y \in Y$ and any $j \in \mathrm{St}^*(Y)$,

$$\kappa_g(j) \cdot \lambda_g(y) = gjg^{-1} \cdot (g \cdot y) = gj \cdot y = \lambda_g(j \cdot y),$$

and the proof is complete. □

Remark 4.12. The above lemma may be formulated in the language of *isomorphisms of group actions*. Given two group actions $\alpha_1 : G_1 \times S_1 \to S_1$ and $\alpha_2 : G_2 \times S_2 \to S_2$, a homomorphism $\Theta : \alpha_1 \to \alpha_2$ is a pair (Σ, Γ) where $\Sigma : S_1 \to S_2$ is a function, and $\Gamma : G_1 \to G_2$ is a group homomorphism, such that the following diagram

$$\begin{array}{ccc} G_1 \times S_1 & \xrightarrow{\alpha_1} & S_1 \\ {\scriptstyle \Gamma \times \Sigma}\downarrow & & \downarrow{\scriptstyle \Sigma} \\ G_2 \times S_2 & \xrightarrow{\alpha_2} & S_2 \end{array}$$

commutes. Group actions then form a category, where the morphisms are the homomorphisms of group actions. Lemma 4.11 thus states that for each $g \in G$, the pair (λ_g, κ_g) is an isomorphism of group actions from the corestriction of α to Y to the corestriction of α to $g \cdot Y$.

On the level of groups, Lemma 4.11 has the following corollary.

Corollary 4.13. *Let $\alpha : G \times X \to X$ be a group action, and let $Y \subseteq X$ be full. Then for any $g \in G$, the following holds.*

(i) $G = \mathrm{St}^*(g \cdot Y) \sqcup \mathrm{Dest}(g \cdot Y)$.

(ii) The subgroups $\mathrm{St}^*(Y)$ and $\mathrm{St}^*(g \cdot Y)$ of G are conjugate; more precisely

$$\mathrm{St}^*(g \cdot Y) = g\,\mathrm{St}^*(Y)g^{-1}.$$

Proof. To prove (i), let $h \in G \setminus \mathrm{Dest}(g \cdot Y)$. Then there exist $y, y' \in Y$ such that $hg \cdot y = g \cdot y'$, which implies that $g^{-1}hg \cdot y = y'$. Thus $\kappa_g^{-1}(h) \notin \mathrm{Dest}(Y)$, whence $\kappa_g^{-1}(h) \in \mathrm{St}^*(Y)$ and $h \in \mathrm{St}^*(g \cdot Y)$ by Lemma 4.11. (ii) is a reformulation of the bijectivity of κ_g. □

From item (ii) it follows that for any $g, h \in G$,

$$\mathrm{St}^*(h \cdot Y) = (hg^{-1})\,\mathrm{St}^*(g \cdot Y)(hg^{-1})^{-1}.$$

This is used in the following corollary, on the level of groupoids.

Corollary 4.14. *Let $\alpha : G \times X \to X$ be a group action, and let $Y \subseteq X$ be full. Then for any $g, h \in G$, the following holds.*

(i) The category ${}_{\mathrm{St}^*(g\cdot Y)}(g \cdot Y)$ is a full subcategory of ${}_GX$.

(ii) The functor $\mathcal{T}_{hg} : {}_{\mathrm{St}^*(g\cdot Y)}(g \cdot Y) \to {}_{\mathrm{St}^*(h\cdot Y)}(h \cdot Y)$, defined on objects and morphisms by

$$\mathcal{T}_{hg}(x) = \lambda_{hg^{-1}}(x), \quad \mathcal{T}_{hg}(k) = \kappa_{hg^{-1}}(k),$$

respectively, is an isomorphism of categories.

Proof. (i) follows from item (i) of Corollary 4.13 together with Theorem 4.7. For (ii), Lemma 4.11 implies that $\mathcal{T}_{hg}$ is indeed a functor for each $g, h \in G$. To show that for each $g, h \in G$, $\mathcal{T}_{gh}\mathcal{T}_{hg}$ is the identity functor on ${}_{\mathrm{St}^*(g\cdot Y)}g \cdot Y$, take any $x \in g \cdot Y$. Then

$$\mathcal{T}_{gh}\mathcal{T}_{hg}(x) = \lambda_{gh^{-1}}\lambda_{hg^{-1}}(x) = gh^{-1}hg^{-1} \cdot x = x,$$

and for each morphism $k \in \mathrm{St}^*(g \cdot Y)$,

$$\mathcal{T}_{gh}\mathcal{T}_{hg}(k) = \kappa_{gh^{-1}}\kappa_{hg^{-1}}(k) = gh^{-1}hg^{-1}kgh^{-1}hg^{-1} = k.$$

This completes the proof. □

Thus for each $g \in G$, the full subcategory of ${}_GX$ with object set $g \cdot Y$ is isomorphic to ${}_{\mathrm{St}^*(Y)}Y$. In view of Remark 4.12, this can be expressed by saying that groupoids arising from isomorphic group actions are isomorphic.

5. Applications to $\mathcal{A}_8$

We now apply the above to the setting of Section 3. In the light of Section 4, Proposition 3.2 may then be restated, in terms of groups and group actions, as follows.

Proposition 5.1. *For each $x \in \{l, r, c\}$, let $Y^x = \{\mathcal{G}^x(f) | f \in O_8^1\}$. With respect to the triality action,*

$$\mathrm{St}^*(Y^x) = \mathrm{St}(Y^x) = G_2$$

and $SO_8 = G_2 \sqcup \mathrm{Dest}(Y^x)$.

The definition of the functor $\mathcal{G}^x : {}_{G_2}O_8^1 \to {}_{SO_8}\mathcal{O}_8$, $x \in \{l, r, c\}$, was given in Proposition 3.2.

Proof. Let $x \in \{l, r, c\}$. Then $G_2 \subseteq \mathrm{St}(Y^x)$ by definition of $\mathcal{G}^x$. If $\phi \in SO_8$ stabilizes Y^x, then for all $f \in O_8^1$ there exists $f' \in O_8^1$ such that $\phi(\mathcal{G}^x(f)) = \mathcal{G}^x(f')$. Now fullness of $\mathcal{G}^x$ implies that $\phi \in G_2$. This implies *a fortiori* that $\mathrm{St}(Y^x) \subseteq G_2$. Altogether $G_2 = \mathrm{St}(Y^x)$, whence $G_2 = \mathrm{St}^*(Y^x)$, being a group. The dichotomy of SO_8 then follows from Theorem 4.7 as $\mathcal{G}^x$ is full, and the proof is complete. □

Our aim is now to use the methods of Section 4 in order to extend the understanding of $\mathcal{A}_8$ beyond $\mathcal{A}_8^l$, $\mathcal{A}_8^r$ and $\mathcal{A}_8^c$. We choose $\mathcal{A}_8^l$ as a point of depart for this extension. ($\mathcal{A}_8^r$ would have been an equally suitable choice, while $\mathcal{A}_8^c$ would have caused some difficulties, at least computationally, as the reader may see in the upcoming sections.)

We have seen that the category $\mathcal{A}_8^l$ is equivalent to the full subcategory of ${}_{SO_8}\mathcal{O}_8$ whose object set is

$$Y^l = \{[h, \mathrm{Id}] | h \in O_8^1\}. \tag{5.1}$$

Before continuing, we determine the set $SO_8 \cdot Y^l$ of all $[f, g] \in \mathcal{O}_8$ that are isomorphic to some element of Y^l, i.e. such that $\mathbb{O}_{f,g} \in \mathcal{A}_8^l$. Theorem 4.7 then provides additional structural information by describing how this set is partitioned into certain classes.

Proposition 5.2. *Let $f, g \in O_8$, and denote by $\cdot$ the action of SO_8 on $\mathcal{O}_8$ by triality.*

(i) $[f, g] \in SO_8 \cdot Y^l$ *if and only if* $g = L_a$ *for some* $a \in \mathbb{S}(\mathbb{O})$.

(ii) The set

$$\{[f, L_a] \in \mathcal{O}_8 | f \in O_8, a \in \mathbb{S}(\mathbb{O})\}$$

is partitioned into classes $\phi \cdot Y^l, \phi \in SO_8$.

(iii) For any $f, f' \in O_8$ *and* $a, a' \in \mathbb{S}(\mathbb{O})$, $[f, L_a]$ *is in the same class as* $[f', L_{a'}]$ *if and only if* $(a, f^{-1}(a^{-1})) = (\pm a', f'^{-1}(a'^{-1}))$.

The full subcategories generated by the classes of this partition are, by Corollary 4.14, all isomorphic to ${}_{G_2}Y^l$.

Proof. If $[f, g] \in SO_8 \cdot Y^l$, then there exist $\phi \in SO_8$ and $h \in O_8^1$ such that

$$[f, g] = [\phi_1 h \phi^{-1}, \phi_2 \,\mathrm{Id}\, \phi^{-1}] = [R_{\phi_2(1)^{-1}} \phi h \phi^{-1}, L_{\phi_1(1)^{-1}} \phi \,\mathrm{Id}\, \phi^{-1}]$$

whence $g = L_a$ for some $a \in \{\pm \phi_1(1)^{-1}\}$.

Conversely, assume that $g = L_a$ for some $a \in \mathbb{S}(\mathbb{O})$. Setting $u = f^{-1}(a^{-1})$ and denoting multiplication in $\mathbb{O}_{f,g}$ by $*$, we have, for each $x \in \mathbb{O}_{f,g}$, that

$$u * x = f(u) L_a(x) = f(f^{-1}(a^{-1})) L_a(x) = a^{-1}(ax) = x,$$

where juxtaposition denotes multiplication in $\mathbb{O}$, and the last equality holds since $\mathbb{O} \setminus \{0\}$ is a Moufang loop.[7] Thus u is a left unity, and $\mathbb{O}_{f,g} \in \mathcal{A}_8^l$, implying that $[f, g] \in SO_8 \cdot Y^l$ by Theorem 2.3. This proves (i), from which (ii) follows by Theorem 4.7 and Proposition 5.1.

To prove (iii), take $f \in O_8$ and $a \in \mathbb{S}(\mathbb{O})$. Then there exist $\phi \in SO_8$ and $h \in O_8^1$ such that $[f, L_a] = \phi \cdot [h, \mathrm{Id}]$. Fix a triality pair (ϕ_1, ϕ_2) of ϕ. Then

$$f = \epsilon \phi_1 h \phi^{-1} \quad \text{and} \quad L_a = \epsilon \phi_2 \phi^{-1} = \epsilon L_{\phi_1(1)^{-1}}$$

[7]Indeed, for each Moufang loop M we have $y^{-1}(yz) = z = (zy)y^{-1}$ for all $y, z \in M$.

for some $\epsilon \in C_2$, which is equivalent to

$$\phi = \epsilon f^{-1}\phi_1 h \quad \text{and} \quad \phi_1(1) = \epsilon a^{-1}. \tag{5.2}$$

Take now $f' \in O_8$ and $a' \in \mathbb{S}(\mathbb{O})$.

If $[f', L_{a'}]$ is in the same class as $[f, L_a]$, then as in (5.2) we have

$$\phi = \epsilon' f'^{-1}\phi_1 h' \quad \text{and} \quad \phi_1(1) = \epsilon' a'^{-1}$$

for some $h' \in O_8^1$ and $\epsilon' \in C_2$. We thus have $a' = \pm a$ and $\phi_1 h'(1) = \epsilon' f'\phi(1)$, and using the expression of ϕ in (5.2), we get

$$\phi_1 h'(1) = \epsilon\epsilon' f' f^{-1}\phi_1 h(1) = \epsilon\epsilon' f' f^{-1}\phi_1(1) = \epsilon' f' f^{-1}(a^{-1}).$$

As $h'(1) = 1$, the left hand side is $\epsilon' a'^{-1}$. Applying f'^{-1} to both sides we get

$$f^{-1}(a^{-1}) = f'^{-1}(a'^{-1}).$$

Conversely, assume that $(a, f^{-1}(a^{-1})) = (\epsilon'' a', f'^{-1}(a'^{-1}))$ for some $\epsilon'' \in C_2$. Then (5.2) implies that $\epsilon\epsilon'' a'^{-1} = \phi_1(1)$. Furthermore, $h' := \epsilon\epsilon''\phi_1^{-1} f'\phi \in O_8^1$, since, by (5.2) and the fact that $h \in O_8^1$,

$$h'(1) = \epsilon\epsilon''\phi_1^{-1} f'\phi(1) = \epsilon''\phi_1^{-1} f' f^{-1}\phi_1 h(1) = \epsilon''\phi_1^{-1} f' f^{-1}(\epsilon a^{-1})$$

and by assumption this is equal to

$$\epsilon''\phi_1^{-1} f' f'^{-1}(\epsilon a'^{-1}) = \phi_1^{-1}(\epsilon\epsilon'' a'^{-1}) = 1.$$

Thus

$$[f', L_{a'}] = [\epsilon\epsilon''\phi_1 h'\phi^{-1}, \epsilon\epsilon'' L_{\phi_1(1)^{-1}}] = [\phi_1 f'\phi^{-1}, L_{\phi_1(1)^{-1}}] \in \phi \cdot Y^l,$$

i.e. $[f', L_{a'}]$ is in the same class as $[f, L_a]$, and the proof is complete. □

6. Left Reflection Algebras

6.1. **Preliminaries.** We now introduce a new class of algebras in $\mathcal{A}_8$, and apply the above framework to it. First is a notational definition.

Definition 6.1. Let V be a Euclidean space and $U \subseteq V$ a subspace. The linear operator $\sigma_U : V \to V$ is defined as reflection in the subspace $U^\perp$, i.e. by

$$\sigma_U(v) = \begin{cases} v & \text{if } v \perp U, \\ -v & \text{if } v \in U. \end{cases}$$

Note that $\sigma_U = \sigma_U^{-1}$, being a symmetric orthogonal operator. In this section we only consider cases where $U = \mathbb{R}u$ for some $u \in V$, in which case we write σ_u instead of $\sigma_{\mathbb{R}u}$ to denote the reflection in the hyperplane $u^\perp$. We note the following basic property.

Lemma 6.2. *For each $u \in \mathbb{S}(\mathbb{O})$ and each $\phi \in SO_8$, $\phi\sigma_u\phi^{-1} = \sigma_{\phi(u)}$.*

Proof. Take any $v \in \mathbb{O}$. If $v = \mu\phi(u)$ for some $\mu \in \mathbb{R}$, then

$$\phi\sigma_u\phi^{-1}(v) = \mu\phi\sigma_u(u) = \mu\phi(-u) = -v,$$

and if $v \perp \phi(u)$, then $\phi^{-1}(v) \perp u$, whence $\sigma_u\phi^{-1}(v) = \phi^{-1}(v)$, and $\phi\sigma_u\phi^{-1}(v) = v$, proving the claim. □

Hyperplane reflections define a class of algebras as follows.

Definition 6.3. Let $u \in \mathbb{S}(\Im\mathbb{O})$. An algebra $\mathbb{O}_{f,g} \in \mathcal{A}_8$ is a *left u-reflection algebra* if $f \in O_8^1$ and $g = \sigma_u$. The full subcategory of $\mathcal{A}_8$ whose objects are all left u-reflection algebras is denoted by $\mathcal{A}_8^u$.

An algebra is called a *left reflection algebra* if it is a left u-reflection algebra for some $u \in \mathbb{S}(\Im\mathbb{O})$. We denote the set of all left reflection algebras by $\mathcal{A}_8^{\mathbb{S}(\Im\mathbb{O})}$. Right reflection algebras may be defined analogously, but will not be used here.

Remark 6.4. The terminology is in analogy with that for algebras with a left unity. Indeed, for each $u \in \mathbb{S}(\Im\mathbb{O})$ and each $\mathbb{O}_{f,g} \in \mathcal{A}_8^u$, the operator σ_u is left multiplication by the element 1.

As regards the class of all absolute valued algebras isomorphic to a left reflection algebra, Dieterich (personal communication, November 2012) has made the following observation.

Proposition 6.5. *The set $\mathcal{A}_8^{\mathbb{S}(\Im\mathbb{O})}$ of all left reflection algebras is dense in the full subcategory*

$$\mathcal{A}_8^R = \{A \in \mathcal{A}_8 | L_e \textit{ is a reflection for some idempotent } e \in A\}$$

of $\mathcal{A}_8$. Moreover, $\mathcal{A}_8^R$ is closed under isomorphisms in $\mathcal{A}_8$.

Thus the objects in $\mathcal{A}_8^R$ are precisely those absolute valued algebras which are isomorphic to a left u-reflection algebra for some $u \in \mathbb{S}(\Im\mathbb{O})$.

Proof. If $A \in \mathcal{A}_8^{\mathbb{S}(\Im\mathbb{O})}$, then $A \in \mathcal{A}_8^u$ for some $u \in \mathbb{S}(\Im\mathbb{O})$, and $L_1 = \sigma_u$ by Remark 6.4. Moreover, 1 is idempotent in A since it is fixed by f and σ_u. Therefore $A \in \mathcal{A}_8^R$.

Conversely, assume that $A \in \mathcal{A}_8^R$ and let $e \in A$ be an idempotent satisfying $L_e = \sigma_v$ with $v \in \mathbb{S}(\mathbb{O})$. Consider the isotope $A_{R_e^{-1}, L_e^{-1}}$ of A. This is a unital algebra with unit element e, and thus there is an isomorphism

$$\phi : A_{R_e^{-1}, L_e^{-1}} \to \mathbb{O}$$

with $\phi(e) = 1$. By Proposition 4.1 in [4], the map

$$\phi : A \to \mathbb{O}_{\phi R_e \phi^{-1}, \phi L_e \phi^{-1}}$$

is then an isomorphism. Furthermore,

$$\mathbb{O}_{\phi R_e \phi^{-1}, \phi L_e \phi^{-1}} = \mathbb{O}_{\phi R_e \phi^{-1}, \phi \sigma_v \phi^{-1}} = \mathbb{O}_{f, \sigma_{\phi(v)}}$$

by Lemma 6.2 and the fact that $\phi \in SO_8$, and with $f = \phi R_e \phi^{-1}$. Now

$$f(1) = \phi R_e \phi^{-1}(1) = \phi R_e(e) = \phi(e^2) = 1$$

and $\phi(v) \in \mathbb{S}(\Im\mathbb{O})$ as $\sigma_v(e) = L_e(e) = e$ implies that $e \perp v$, whence $1 = \phi(e) \perp \phi(v)$. Thus $A \simeq \mathbb{O}_{f,\sigma_{\phi(v)}} \in \mathcal{A}_8^{\mathbb{S}(\Im\mathbb{O})}$.

Finally, if $B \in \mathcal{A}_8$ and $\psi : A \to B$ is an isomorphism, then $\psi(e)$ is idempotent in B, and $L_{\psi(e)} = \sigma_{\psi(v)}$. Thus $\mathcal{A}_8^R$ is closed under isomorphisms in $\mathcal{A}_8$. □

Left reflection algebras have no left unity, as made precise by the following result.

Proposition 6.6. *For any $u \in \mathbb{S}(\Im\mathbb{O})$, there exist no $A \in \mathcal{A}_8^u$ and $B \in \mathcal{A}_8^l$ such that $A \simeq B$.*

Proof. By Remark 6.4, left multiplication by 1 in A has determinant -1, while in B left multiplication by the left unit has determinant 1. By Proposition 1.2, A and B are therefore non-isomorphic. □

We set, for each $u \in \mathbb{S}(\Im\mathbb{O})$,

$$Y^u = \{[f,g] \in \mathcal{O}_8 | \mathbb{O}_{f,g} \in \mathcal{A}_8^u\} = \{[f,g] \in \mathcal{O}_8 | f \in O_8^1, g = \sigma_u\}.$$

This set has the following properties.

Proposition 6.7. *The triality action satisfies the following for each $u \in \mathbb{S}(\Im\mathbb{O})$:*

(i) $\mathrm{St}(Y^u) = G_2^u := \{\phi \in G_2 | \phi([u]) = [u]\} = \{\phi \in G_2 | \phi(u) = \pm u\}$,
(ii) $\mathrm{St}^*(Y^u) = \mathrm{St}(Y^u)$,
(iii) $SO_8 = \mathrm{St}(Y^u) \sqcup \mathrm{Dest}(Y^u)$, *and*
(iv) ${}_{G_2^u}Y^u$ *is a full subcategory of* ${}_{SO_8}\mathcal{O}_8$.

Proof. The set G_2^u is a subgroup of G_2. To prove (i), note that $\phi \in \mathrm{St}(Y^u)$ if and only if there exists a triality pair (ϕ_1, ϕ_2) satisfying the two conditions

$$\phi_1 f \phi^{-1} \in O_8^1, \quad \phi_2 \sigma_u \phi^{-1} = \sigma_u \tag{6.1}$$

for any $f \in O_8^1$. If $\phi \in G_2^u \leq G_2 \leq O_8^1$, then (ϕ, ϕ) is a triality pair satisfying the first condition and, as a simple computation shows, the second as well, whence $\phi \in \mathrm{St}(Y^u)$.

Conversely, assume that (ϕ_1, ϕ_2) is a triality pair satisfying the two conditions in (6.1) for *some* $f \in O_8^1$. (We will only need the existence of one $f \in O_8^1$ such that the conditions are satisfied.) The second condition implies that

$$L_{\phi_1(1)^{-1}} \phi \sigma_u \phi^{-1} = \sigma_u,$$

i.e. $\phi \sigma_u \phi^{-1} \sigma_u = L_{\phi_1(1)}$, and by Lemma 6.2 we then have $\sigma_{\phi(u)} \sigma_u = L_{\phi_1(1)}$. Thus $L_{\phi_1(1)}$ fixes any $x \in [u, \phi(u)]^\perp$, whence $\phi_1(1) = 1$ must hold since $\mathbb{O}$ is a unital division algebra. Hence $\sigma_{\phi(u)} = \sigma_u$ and $\phi(u) = \pm u$. Moreover, $\phi_1(1) = 1$ together with the first condition yields $\phi(1) = 1$. Thus $\phi \in G_2$, and then $\phi \in G_2^u$.

Statement (ii) follows from the fact that G_2^u is a group. As for (iii), assume that $\phi \notin \mathrm{Dest}(Y^u)$. Then there exist $f, f' \in O_8^1$ such that $\phi \cdot [f, \sigma_u] = [f', \sigma_u]$, which implies that the conditions in (6.1) hold for some triality pair of ϕ. By the previous paragraph we get $\phi \in G_2^u$. Finally, (iv) is equivalent to (iii) by Theorem 4.7. □

As Y^u satisfies the equivalent conditions of Theorem 4.7, we may apply the results of Section 4 to it. To begin with, we obtain the following.

Corollary 6.8. *Let $u \in \mathbb{S}(\Im\mathbb{O})$. The functors*

$$\mathcal{G}^u : {}_{G_2^u}O_8^1 \to {}_{G_2^u}Y^u, \quad \mathcal{F}|_{Y^u} : {}_{G_2^u}Y^u \to \mathcal{A}_8^u,$$

where $\mathcal{G}^u$ is defined on objects by $\mathcal{G}^u(f) = [f, \sigma_u]$ and on morphisms by $\mathcal{G}^u(\phi) = \phi$, are equivalences of categories.

Proof. Both functors are well-defined and clearly faithful. Moreover, $\mathcal{G}^u$ is dense by the definition of Y^u and full by construction, while $\mathcal{F}|_{Y^u}$ is dense by the definition of $\mathcal{A}_8^u$ and full by fullness of $\mathcal{F}$ and of ${}_{G_2^u}Y^u$ in ${}_{SO_8}\mathcal{O}_8$. □

For any $u \in \mathbb{S}(\Im\mathbb{O})$, Y^u determines the set $SO_8 \cdot Y^u$ of all $[f,g] \in \mathcal{O}_8$ such that $\mathbb{O}_{f,g}$ is isomorphic to a left u-reflection algebra, i.e. such that $\mathbb{O}_{f,g} \in \mathcal{A}_8^R$. By Theorem 4.7 and Corollary 4.14 we know that $SO_8 \cdot Y^u$ is partitioned into sets generating full subcategories each isomorphic to ${}_{G_2^u}Y^u$. While morphisms in ${}_{G_2^u}Y^u$ have trivial triality components, computing $SO_8 \cdot Y^u$ explicitly does involve triality. Instead, we consider the set $G_2 \cdot Y^u$, in view of the chain of subgroups $G_2^u \leq G_2 \leq SO_8$ and the fact that each $\phi \in G_2$ has trivial triality components.

The following proposition contains an explicit description of $G_2 \cdot Y^u$ and its partition into sets generating pairwise isomorphic subcategories. A motivation to study this set is given in the subsequent remark.

Proposition 6.9. *Let $u \in \mathbb{S}(\Im\mathbb{O})$.*

(i) Under the left action of G_2 on O_8^1 by conjugation, $\mathrm{St}(\sigma_u) = G_2^u$. *Moreover, $G_2 = G_2^u \sqcup \mathrm{Dest}(\sigma_u)$ and $G_2 \cdot \sigma_u = \{\sigma_{u'} | u' \in \mathbb{S}(\Im\mathbb{O})\}$, with $\phi \cdot \sigma_u = \psi \cdot \sigma_u$ if and only if $[\phi(u)] = [\psi(u)]$.*

(ii) Under the triality action,

$$G_2 \cdot Y^u = \bigcup_{u' \in \mathbb{S}(\Im\mathbb{O})} Y^{u'},$$

and $\phi \cdot Y^u = \psi \cdot Y^u \iff [\phi(u)] = [\psi(u)]$.

Thus $G_2 \cdot Y^u = \{[f,g] \in \mathcal{O}_8 | \mathbb{O}_{f,g} \in \mathcal{A}_8^{\mathbb{S}(\Im\mathbb{O})}\}$.

Proof. For (i), the stabilizer is obtained from Lemma 6.2, as $\sigma_{\phi(u)} = \sigma_u$ if and only if $\phi(u) = \pm u$. The decomposition of G_2 holds as $\{\sigma_u\}$ contains precisely one element, and the inclusion $G_2 \cdot \sigma_u \subseteq \{\sigma_{u'} | u' \in \mathbb{S}(\Im\mathbb{O})\}$ holds by Lemma 6.2. Moreover, for any $u' \in \mathbb{S}(\Im\mathbb{O})$ there are $v, v', z, z' \in \mathbb{S}(\Im\mathbb{O})$ such that (u, v, z) and (u', v', z') are Cayley triples. Thus there exists for each $u' \in \mathbb{S}(\Im\mathbb{O})$ a map $\phi \in G_2$ such that $\phi(u) = u'$, and by Lemma 6.2, $\phi \cdot \sigma_u = \sigma_{u'}$. Since u' was arbitrary, this proves the inverse inclusion. Moreover, $\sigma_{\phi(u)} = \sigma_{\psi(u)}$ holds if and only if $\phi(u) = \pm\psi(u)$, which proves the equivalence.

For (ii), the triality action of $G_2 \leq SO_8$ on Y^u is simultaneous conjugation in the sense that for all $\phi \in G_2$,

$$\phi \cdot [f, \sigma_u] = [\phi f \phi^{-1}, \phi \sigma_u \phi^{-1}] = [\phi f \phi^{-1}, \sigma_{\phi(u)}].$$

Thus $G_2 \cdot Y^u \subseteq \bigcup Y^{u'}$. Conversely, for every $u' \in \mathbb{S}(\Im\mathbb{O})$ and $[f, \sigma_{u'}] \in Y^{u'}$ there exists, by (i), $\phi \in G_2$ such that $\phi \cdot \sigma_u = \sigma_{u'}$, and then $f = \phi \cdot (\phi^{-1} f \phi)$. Thus the two sets are equal. The equivalence follows from that in (i). □

Remark 6.10. In view of the comment concluding Section 2, to classify $\mathcal{A}_8$ it suffices to consider such $[f,g] \in \mathcal{O}_8$ where $f, g \in O_8^1$. By the Cartan–Dieudonné Theorem, each element in O_8^1 can be written as a product of at most 7 reflections. Thus Y^l contains all such $[f,g]$ where g is the empty product of reflections, and for any $u \in \mathbb{S}(\Im\mathbb{O})$, Y^u exhausts, by the above proposition, all such $[f,g]$ where g is the product of precisely one reflection.

This is one motivation for attempting to classify all left reflection algebras, and Proposition 6.9 reduces this to the classification problem for the set of left u-reflection algebras for a fixed $u \in \mathbb{S}(\Im\mathbb{O})$.

To summarize the pattern we have followed, given the triality action, we first determined a set of algebras in $\mathcal{A}_8$ corresponding to a full subset Y^u of $\mathcal{O}_8$, then formed the groupoid arising from the corestriction of the triality action to Y^u, and finally constructed a larger set of algebras whose classification reduces to that of the smaller set via the equivalence of categories in Theorem 4.7(vii). We thus have the following commutative diagram of groupoids and full functors, where the $\sim$-labeled arrows are moreover equivalences of categories, and the vertical arrows are inclusions.

$$\begin{array}{ccccc}
 & & {}_{SO_8}\mathcal{O}_8 & \xrightarrow{\sim} & \mathcal{A}_8 \\
 & & \uparrow & & \uparrow \\
 & & {}_{SO_8}(SO_8 \cdot Y^u) & \xrightarrow{\sim} & \mathcal{A}_8^R \\
 & & \wr\uparrow & & \wr\uparrow \\
 & & {}_{G_2}(G_2 \cdot Y^u) & \xrightarrow{\sim} & \mathcal{A}_8^{\mathbb{S}(\Im\mathbb{O})} \\
 & & \wr\uparrow & & \wr\uparrow \\
{}_{G_2^u}O_8^1 & \xrightarrow{\sim} & {}_{G_2^u}Y^u & \xrightarrow{\sim} & \mathcal{A}_8^u
\end{array}$$

6.2. **Reduction of the Classification Problem.** In order to classify all left reflection algebras, it remains, by Corollary 6.8, to solve the classification problem for ${}_{G_2^u}O_8^1$. First we define some group actions to be used below.[8]

Definition 6.11. For any $u \in \mathbb{S}(\Im\mathbb{O})$ and $e \in O_8^1$, the group actions β, γ, γ_u and δ_e are defined by

$$\begin{array}{rll}
\beta: & G_2 \times G_2^u \to G_2, & (\phi, \psi) \mapsto \phi\psi, \\
\gamma: & O_8^1 \times G_2 \to O_8^1, & (f, \phi) \mapsto \kappa_\phi^{-1}(f) = \phi^{-1} f \phi, \\
\gamma_u: & O_8^1 \times G_2^u \to O_8^1, & (f, \psi) \mapsto \kappa_\psi^{-1}(f), \\
\delta_e: & G_2 \times (\mathrm{St}_\gamma(e) \times G_2^u) \to G_2, & (\phi, (\chi, \psi)) \mapsto \chi^{-1}\phi\psi.
\end{array}$$

We will now deal with the classification problem. By Corollary 6.8 , this amounts to solving the normal form problem for the group action γ_u. The normal form problem for γ was solved in [7]. Since $G_2^u \leq G_2$, each γ_u-orbit is contained in a γ-orbit. In this sense, the problem at hand (properly) contains the problem solved in [7], and therefore one may ask if it is possible to use this solution in order to simplify the present problem. This is indeed the case, and the details are given in the following theorem, proven by Dieterich for the actions of Definition 6.11 (personal communication, April 2012). We present here a straight-forward generalization.

Theorem 6.12. *Let F, G and H be groups such that $H \leq G \leq F$, and let $e \in F$. Define the group actions*

$$\begin{array}{rll}
\hat\beta: & G \times H \to G, & (g, h) \mapsto gh, \\
\hat\gamma: & F \times G \to F, & (f, g) \mapsto \kappa_g^{-1}(f) = g^{-1} f g, \\
\hat\gamma_H: & F \times H \to F, & (f, h) \mapsto \kappa_h^{-1}(f), \\
\hat\delta_e: & G \times (\mathrm{St}_{\hat\gamma}(e) \times H) \to G, & (g, (s, h)) \mapsto s^{-1} g h,
\end{array}$$

and let $\hat B \subseteq G$ and $\hat C \subseteq F$ be cross-sections for $\hat\beta$ and $\hat\gamma$, respectively. Then

(i) the set $\{\kappa_g^{-1}(f) | g \in \hat B, f \in \hat C,\}$ exhausts the orbits of $\hat\gamma_H$,

(ii) for any $f, f' \in \hat C$ and $g, g' \in \hat B$, $\kappa_g^{-1}(f) \equiv \kappa_{g'}^{-1}(f')$ with respect to $\hat\gamma_H$ if and only if $f' = f$ and $g \equiv g'$ with respect to $\hat\delta_f$, and

(iii) a cross-section for $\hat\gamma_H$ is given by

$$\coprod_{f \in \hat C} \{\kappa_g^{-1}(f) | g \in \hat B \cap D_f\}$$

[8]The reason that the groups act from the right is that the orbits of β are left cosets, which, as we will see, is required for Theorem 4.7 to be applied directly.

where for each $f \in \hat{C}$, D_f is a cross-section of $\hat{\delta}_f$.

Proof. (i) Let $f' \in F$. Since $\hat{C}$ is a cross-section for $\hat{\gamma}$, there exist $f \in \hat{C}$ and $g' \in G$ such that $f' = \kappa_{g'}^{-1}(f)$. Since $\hat{B}$ is a cross-section for $\hat{\beta}$, there exist $h \in H$ and $g \in \hat{B}$ such that $g' = gh$. This proves (i), since then

$$f' = \kappa_{gh}^{-1}(f) = \kappa_h^{-1}(\kappa_g^{-1}(f)).$$

(ii) By definition, $\kappa_g^{-1}(f) \equiv_{\hat{\gamma}_H} \kappa_{g'}^{-1}(f')$ if and only if $\kappa_{g'}^{-1}(f') = \kappa_h^{-1}(\kappa_g^{-1}(f))$ for some $h \in H$, which is equivalent to

$$f' = \kappa_{ghg'^{-1}}^{-1}(f) \tag{6.2}$$

for some $h \in H$, whence $f \equiv_{\hat{\gamma}} f'$. But f and f' belong to the same cross-section of $\hat{\gamma}$, and hence coincide.

Thus (6.2) implies that $\kappa_g^{-1}(f) \equiv_{\hat{\gamma}_H} \kappa_{g'}^{-1}(f')$ if and only if $f' = f$ and there exists $h \in H$ such that $ghg'^{-1} \in \mathrm{St}_{\hat{\gamma}}(f)$. The latter statement is equivalent to $g' \in \mathrm{St}_{\hat{\gamma}}(f)gh$ for some $h \in H$. Hence

$$\kappa_g^{-1}(f) \equiv_{\hat{\gamma}_H} \kappa_{g'}^{-1}(f') \Longleftrightarrow f' = f \wedge g' \in \mathrm{St}_{\hat{\gamma}}(f)gH,$$

which by definition of $\hat{\delta}_f$ proves (ii).

(iii) By (ii), for each $f \in \hat{C}$, the set $D_f \cap \hat{B}$ is a cross-section of $\{\kappa_g^{-1}(f)|g \in \hat{B}\}$. The claim follows since (ii) moreover implies that $\kappa_g^{-1}(f) \equiv_{\hat{\gamma}_H} \kappa_{g'}^{-1}(f')$ for some $g' \in \hat{B}$ and $f' \in \hat{C}$ only if $f' = f$.

□

We now return to the setting of Definition 6.11, where we apply the above theorem. For the remainder of this section, we fix $u \in \mathbb{S}(\Im\mathbb{O})$ and set

$$(F, G, H) = (O_8^1, G_2, G_2^u) \quad \text{and} \quad \forall e \in O_8^1, (\hat{\beta}, \hat{\gamma}, \hat{\gamma}_H, \hat{\delta}_e) = (\beta, \gamma, \gamma_u, \delta_e).$$

As a cross-section $C \subset O_8^1$ for γ, we will henceforth use the one obtained in [7], which we will partly recall in Section 7. What thus remains towards classifying left reflection algebras is to solve the normal form problem for β and that for δ_f for each $f \in C$. The solution to the first of these problems is obtained by an application of Theorem 4.7.

Proposition 6.13. *There is a bijection between the set G_2/G_2^u of orbits of β and $\mathbb{P}(\Im\mathbb{O})$, given by $\phi G_2^u \mapsto [\phi(u)]$ for all $\phi \in G_2$.*

Proof. We apply Theorem 4.7 to the left action of G_2 on O_8^1 by conjugation. By Proposition 6.9(i), there is thus a bijection ρ_1 from the set G_2/G_2^u of left cosets to the partition of $\{\sigma_{u'}|u' \in \mathbb{S}(\Im\mathbb{O})\}$ into its singleton subsets. Identifying this partition with the set itself, Theorem 4.7 asserts that the bijection is given by $\phi G_2^u \mapsto \phi\sigma_u\phi^{-1}$, and by Lemma 6.2, $\phi\sigma_u\phi^{-1} = \sigma_{\phi(u)}$. Furthermore, the map

$$\rho_2 : \{\sigma_{u'}|u' \in \mathbb{S}(\Im\mathbb{O})\} \to \mathbb{P}(\Im\mathbb{O}), \sigma_{u'} \mapsto [u']$$

is bijective, as each line through the origin in $\Im\mathbb{O}$ determines a unique reflection. This gives a bijection

$$\rho_2 \circ \rho_1 : G_2/G_2^u \to \mathbb{P}(\Im\mathbb{O}), \phi G_2^u \mapsto [\phi(u)],$$

whereby the proof is complete. □

Remark 6.14. Thus, finding a cross-section for β amounts to constructing, for each $\ell \in \mathbb{P}(\Im\mathbb{O})$, a unique representative ϕ_ℓ of the set

$$\{\phi \in G_2 | [\phi(u)] = \ell\}.$$

For the sake of definiteness, we perform this construction explicitly in Appendix B.

Having done so, we have proven the following.

Corollary 6.15. *The set*

$$B = \{\phi_\ell | \ell \in \mathbb{P}(\Im\mathbb{O})\}$$

is a cross-section for β.

It remains now, by Theorem 6.12(iii), to determine for each $f \in C$ when two elements of B are in the same orbit of δ_f, and to find a cross-section of B with respect to δ_f.

Proposition 6.16. *Let $\ell, \ell' \in \mathbb{P}(\Im\mathbb{O})$ and $v, v' \in \mathbb{S}(\Im\mathbb{O})$ be such that $[v] = \ell$, and $[v'] = \ell'$ and let $f \in C$. Then $\phi_\ell \equiv \phi_{\ell'}$ with respect to δ_f if and only if $v \approx_f v'$, where*

$$v \approx_f v' \Longleftrightarrow \exists \chi \in \mathrm{St}_\gamma(f) : \chi(v) = \pm v'. \tag{6.3}$$

Recall that we have fixed $u \in \mathbb{S}(\Im\mathbb{O})$, and that $[\phi_\ell(u)] = \ell$ and $[\phi_{\ell'}(u)] = \ell'$.

Proof. The statement that $\phi_\ell \equiv \phi_{\ell'}$ with respect to δ_f is equivalent to the existence of $\chi \in \mathrm{St}_\gamma(f)$ and $\psi \in G_2^u$ such that $\phi_\ell = \chi^{-1}\phi_{\ell'}\psi$, or, equivalently, $\chi\phi_\ell = \phi_{\ell'}\psi$.

If this holds, then

$$\chi\phi_\ell(u) = \phi_{\ell'}\psi(u) \in \{\phi_{\ell'}(\pm u)\} \subset \ell'.$$

But $\phi_\ell(u)$ is either v or $-v$, and thus $\chi(v) \in \ell'$, i.e. $\chi(v) = \pm v'$ since $\|\chi(v)\| = \|v\|$.

Conversely, if $\chi(v) = \pm v'$ for some $\chi \in \mathrm{St}_\gamma(f)$, then $\chi\phi_\ell(u) = \pm\phi_{\ell'}(u)$, and $\phi_{\ell'}^{-1}\chi\phi_\ell(u) = \pm u$. Thus $\phi_{\ell'}^{-1}\chi\phi_\ell \in G_2^u$, whence there exists $\psi \in G_2^u$ such that $\chi\phi_\ell = \phi_{\ell'}\psi$. This completes the proof. □

Remark 6.17. The condition $v \approx_f v'$ is equivalent to ℓ and ℓ' being in the same orbit of the left action of $\mathrm{St}_\gamma(f)$ on $\mathbb{P}(\Im\mathbb{O})$ by evaluation, i.e. the action defined by $\chi \cdot \ell = \{\chi(w) | w \in \ell\}$ for all $\chi \in \mathrm{St}_\gamma(f)$ and all $\ell \in \mathbb{P}(\Im\mathbb{O})$ This expresses the remainder of the classification problem as a normal form problem; nevertheless, (6.3) is a more suitable expression for computations.

7. Classifying Left Reflection Algebras

Let $u \in \mathbb{S}(\Im\mathbb{O})$ be fixed. We are now ready to compute a cross-section for the action γ_u, which would classify left reflection algebras up to isomorphism. Hence, let C be the cross-section obtained in [7] for the group action γ, and let B be the cross-section obtained in Corollary 6.15 for the action β. [9] In view of Section 6.2, what remains to be done is to compute, for each $f \in C$, a cross-section of B for δ_f. More precisely, this consists of computing $\mathrm{St}_\gamma(f)$ and thence a cross-section D'_f for the relation $\approx_f$ on $\mathbb{S}(\Im\mathbb{O})$ defined in Proposition 6.16. Indeed, by that proposition, the map $D'_f \to B, v \mapsto \phi_{[v]}$, is injective, and its image is a cross-section D_f of δ_f. Our set of representatives will thus correspond bijectively to the desired cross-section.

[9] The group actions used in this section were defined in Definition 6.11.

7.1. **Preliminaries.** We begin by summarizing a few facts from [7]. From now on we identify O_8^1 with $O_7 = O(\Im\mathbb{O})$. It is well known that for each $d \in \mathbb{N}$ and each $f \in O_d$ there exists a basis of $\mathbb{R}^d$ in which the matrix of f is block diagonal, where each block is either 1, -1, or

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some $\theta \in]0,\pi[$.

Given $f \in O_d$, Let n_f^+ and n_f^- be the dimensions of the eigenspaces of f corresponding to the eigenvalues 1 and -1, respectively. Let $R(f)$ be the (finite) set of all $\theta \in]0,\pi[$ such that R_θ is a block in the aforementioned block diagonal matrix, and for each $\theta \in R(f)$, let n_f^θ be the dimension of the *generalized eigenspace* corresponding to θ, i.e. twice the number of blocks R_θ. The set $R(f)$ and the numbers n_f^+, n_f^- and $n_f^\theta, \theta \in R(f)$, are, as indicated in [7], well-defined and basis-independent, hence invariant under conjugation by SO_d. For $d = 7$, they are hence invariant under the action γ of $G_2 \leq SO_7$. In [7], the normal form problem for γ was therefore solved separately for each possible *type*, where the type of $f \in O_7$ is the pair of sets $(\{n_\theta | \theta \in R(f)\}, \{n_f^+, n_f^-\})$. For notational consistency with [7], we write each set in the pair as a list of its elements in decreasing order, and separate the lists by a vertical line. The possible types are thus

$$\begin{array}{llll} (\emptyset|7,0), & (\emptyset|6,1), & (\emptyset|5,2), & (\emptyset|4,3), \\ (2|5,0), & (2|4,1), & (2|3,2), & \\ (4|3,0), & (4|2,1), & (2,2|3,0), & (2,2|2,1), \\ (6|1,0), & (4,2|1,0), & (2,2,2|1,0). & \end{array}$$

For simplicity we will denote, for each type T, the set $\{f \in O_7 | f \text{ is of type } T\}$ simply by T.

In this section, we will construct a cross-section for $\approx_f$ for each $f \in C$ of type

$$(\emptyset|7,0), \quad (\emptyset|6,1), \quad (\emptyset|5,2), \quad (\emptyset|4,3), \quad \text{or} \quad (2|5,0).$$

The remaining cases are more computationally demanding. We will content ourselves with computing the above to give an example; among these, the treatment of type $(\emptyset|4,3)$ relies on explicit computations of elements of G_2, while the other mainly use properties of Cayley triples and some linear algebra.

Remark 7.1. From now on we fix $v, z \in \mathbb{S}(\Im\mathbb{O})$ such that (u, v, z) is a Cayley triple, henceforth referred to as the *standard Cayley triple*. (A brief account of Cayley triples can be found in Section 1.2.) We identify $\Im\mathbb{O}$ with $\mathbb{R}^7$ and write the vectors in the basis induced by the standard Cayley triple. Likewise, we identify the set of all linear operators on $\Im\mathbb{O}$ with $\mathbb{R}^{7\times7}$, with matrices given in this basis, and deal analogously with operators on subspaces. Moreover we identify, for each $d \in \mathbb{Z}_+$, column matrices in $\mathbb{R}^{d\times1}$ with their representations as d-tuples in $\mathbb{R}^d$.

We are now ready to perform the computations.

7.2. **Type** $(\emptyset|7,0)$**.** A cross-section for type $(\emptyset|7,0)$ is given in [7] as follows.

Lemma 7.2. $C_{\emptyset|7,0} := \{\pm\mathbb{I}_7\}$ *is a cross-section of* $(\emptyset|7,0)$ *with respect to* γ.

Clearly, every $\phi \in G_2$ stabilizes the identity, and for every $u' \in \mathbb{S}(\Im\mathbb{O})$ there exists $\phi \in G_2$ such that $\phi(u) = u'$. We thus immediately arrive at the following cross-section for $\approx_f$, $f \in C$.

Proposition 7.3. *For any* $f \in C_{\emptyset|7,0}$*,*

(i) $\mathrm{St}_\gamma(f) = G_2$*, and*
(ii) $D'_f = \{u\}$ *is a cross-section of* $\mathbb{S}(\Im\mathbb{O})$ *with respect to* $\approx_f$*.*

7.3. **Type** $(\emptyset|6,1)$**.** A cross-section for γ is given in [7] as follows.

Lemma 7.4. $C_{\emptyset|6,1} := \left\{ \pm \begin{pmatrix} 1 & \\ & -\mathbb{I}_6 \end{pmatrix} \right\}$ *is a cross-section of* $(\emptyset|6,1)$ *with respect to* γ*.*

We can then compute the following cross-section for $\approx_f$.

Proposition 7.5. *For any* $f \in C_{\emptyset|6,1}$*,*

(i) $\mathrm{St}_\gamma(f) = \left\{ \begin{pmatrix} \pm 1 & \\ & \chi' \end{pmatrix} | \chi' \in O_6 \right\} \cap G_2$*, and*
(ii) $D'_f = \{\xi u + \eta v | (\xi, \eta) \in \mathbb{S}([0,1]^2)\}$ *is a cross-section of* $\mathbb{S}(\Im\mathbb{O})$ *with respect to* $\approx_f$*.*

Proof. By a result from linear algebra (see [11], p. 223), if $\chi \in \mathrm{St}_\gamma(f)$ for some $f \in C_{\emptyset|6,1}$ (i.e. if χ commutes with f), then χ is of the form

$$\left\{ \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} | \chi_1 \in \mathbb{R}, \chi_2 \in \mathbb{R}^{6\times 6} \right\},$$

and the converse holds by direct verification. The fact that $\mathrm{St}_\gamma(f) \subseteq G_2$ then implies that $\chi_1 \in O_1$ and $\chi_2 \in O_6$, proving (i).

To prove (ii), given any $w \in \mathbb{S}(\Im\mathbb{O})$ there exist $\xi, \eta \in [0,1]$ with $\xi^2 + \eta^2 = 1$, and $v' \perp u$ with $\|v'\| = 1$, such that $w = \pm(\xi u + \eta v')$. Moreover there exists $z' \in \mathbb{S}(\Im\mathbb{O})$ such that (u, v', z') is a Cayley triple, and hence there exists $\chi \in G_2$ mapping (u, v, z) to (u, v', z'). By (i), $\chi \in \mathrm{St}_\gamma(f)$, and $\chi(\xi u + \eta v) = \pm w$. Hence D'_f is exhaustive.

To show that D'_f is irredundant, assume that

$$\xi u + \eta v \approx_f \xi' u + \eta' v.$$

for some $(\xi, \eta), (\xi', \eta') \in \mathbb{S}([0,1]^2)$. Then $\chi(\xi u + \eta v) = \xi' u + \eta' v$ for some $\chi \in \mathrm{St}_\gamma(f)$, and from (i) it follows that $|\xi| = |\xi'|$, whence $|\eta| = |\eta'|$, and thus $(\xi, \eta) = (\xi', \eta')$, completing the proof. □

7.4. **Types** $(\emptyset|5,2)$ **and** $(2|5,0)$**.** We treat these types simultaneously due to computational similarities. The cross-sections given in [7] are as follows.

Lemma 7.6. *(i)* $C_{\emptyset|5,2} := \left\{ \pm \begin{pmatrix} \mathbb{I}_2 & \\ & -\mathbb{I}_5 \end{pmatrix} \right\}$ *is a cross-section of* $(\emptyset|5,2)$ *with respect to* γ*.*
(ii) $C_{2|5,0} := \left\{ \pm \begin{pmatrix} R_\theta & \\ & -\mathbb{I}_5 \end{pmatrix} | \theta \in]0, \pi[\right\}$ *is a cross-section of* $(2|5,0)$ *with respect to* γ*.*

Without further ado, we find cross-sections for these types with respect to $\approx_f$.

Proposition 7.7. *For any* $f \in C_{\emptyset|5,2} \cup C_{2|5,0}$*,*

(i) $\mathrm{St}_\gamma(f) = \left\{ \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix} | \chi_1 \in O_2, \chi_2 \in O_5 \right\} \cap G_2$*, and*

(ii) $D'_f = \{\xi u + \eta uv + \zeta z | (\xi, \eta, \zeta) \in \mathbb{S}([0,1]^3)\}$ *is a cross-section of* $\mathbb{S}(\Im\mathbb{O})$ *with respect to* $\approx_f$.

Proof. The proof of (i) is analogous to that for type $(\emptyset|6,1)$. For (ii), take any $w \in \mathbb{S}(\Im\mathbb{O})$. Then there exist $(\xi, \eta, \zeta) \in \mathbb{S}([-1,1]^3)$ and $u', z' \in \mathbb{S}(\Im\mathbb{O})$ satisfying $u' \in [u, v]$ and $z' \perp [u, v, uv]$, such that

$$w = \pm(\xi u' + \eta uv + \zeta z'),$$

and u' and z' can be chosen so that ξ, η and ζ are all positive. By definition of the multiplication in $\mathbb{O}$, there is $v' \in u'^{\perp} \cap [u, v]$ such that $u'v' = uv$. Then (u', v', z') is a Cayley triple, and there exists $\chi \in G_2$ mapping (u, v, z) to (u', v', z'); (i) then implies that $\chi \in \mathrm{St}_\gamma(f)$. Furthermore, $\chi(\xi u + \eta uv + \zeta z) = \pm w$, whence D'_f is exhaustive.

If $\chi(\xi u + \eta uv + \zeta z) = \pm(\xi' u + \eta' uv + \zeta' z)$ for some $\chi \in \mathrm{St}_\gamma(f)$ and $(\xi, \eta, \zeta), (\xi', \eta', \zeta') \in \mathbb{S}([0,1]^3)$, then by the block decomposition in (i) we have $\xi = \xi'$, and

$$\eta\chi(uv) + \zeta\chi(z) = \pm(\eta' uv + \zeta' z). \tag{7.1}$$

The block decomposition further implies that $\chi(u), \chi(v) \in [u, v]$. As $\chi \in G_2$ and the product of any two mutually orthogonal unit vectors in $[u, v]$ belongs to $[uv]$, we have

$$\chi(uv) = \chi(u)\chi(v) = \epsilon uv \tag{7.2}$$

for some $\epsilon \in C_2$, whence (7.1) implies that

$$(\epsilon\eta \mp \eta')uv = \pm\zeta' z - \zeta\chi(z). \tag{7.3}$$

Now by (7.2) along with χ being orthogonal, $uv \perp [z, \chi(z)]$. Thus (7.3) implies that $\epsilon\eta \mp \eta' = 0$, and then $(\xi, \eta, \zeta) = (\xi', \eta', \zeta')$. Thus D'_f is irredundant. □

7.5. **Type** $(\emptyset|4,3)$**.** To begin with, we introduce, for each $\theta \in [0, \pi/2]$, the matrix

$$\hat{R}_\theta = \begin{pmatrix} \mathbb{I}_2 & & \\ & R_\theta & \\ & & \mathbb{I}_3 \end{pmatrix}$$

and the sets

$$C_\theta = \left\{ \pm \hat{R}_\theta^{-1} \begin{pmatrix} \mathbb{I}_3 & \\ & -\mathbb{I}_4 \end{pmatrix} \hat{R}_\theta \right\}$$

and

$$T_\theta = \left\{ \hat{R}_\theta^{-1} \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix} \hat{R}_\theta | \chi_1 \in O_3, \chi_2 \in O_4 \right\} \cap G_2.$$

These are then used to express the cross-section of $(\emptyset|4,3)$ computed in [7].

Lemma 7.8. $C_{\emptyset|4,3} := \bigcup_{\theta \in [0,\pi/2]} C_\theta$ *is a cross-section of* $(\emptyset|4,3)$ *with respect to* γ.

The matrices in C_θ are not block-diagonal with respect to the standard basis when $\theta \neq 0$, and to compute a cross-section for $\approx_f, f \in C_\theta$, we need to express $\mathrm{St}_\gamma(f)$ more explicitly than in the previous cases. The details are given in the following lemma, where $\times$ denotes the (standard) vector product in $\mathbb{R}^3$.

Lemma 7.9. *Let $\theta \in]0, \pi/2]$. Then $T_\theta = T'_\theta$, where*

$$T'_\theta = \left\{ \hat{R}_\theta^{-1} \begin{pmatrix} x_1 & x_2 & \epsilon x_1 \times x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (x_1 \times x_2)^* & \epsilon x_2^* & -\epsilon x_1^* & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \epsilon \end{pmatrix} \hat{R}_\theta | \right.$$
$$\left. x_1 \perp x_2 \in \mathbb{S}^2, \epsilon \in E_\theta \right\},$$

with $x^* = (x^3, x^2, -x^1)$ for each $x \in \mathbb{R}^3$, and $E_\theta = C_2$ if $\theta = \pi/2$, and $\{1\}$ if not.

Proof. If $\chi \in T_\theta \subset G_2$, then χ respects octonion multiplication. Computing $\chi(u)\chi(v)$ thus determines $\chi(uv)$ in terms of the first two columns of χ. The fourth column is then obtained from the properties of G_2 upon analysing the entrywise effect of conjugation by $\hat{R}_\theta$. These determine the remaining columns, and carrying out the computations one sees that $\chi \in T'_\theta$. Conversely, if $\chi \in T'_\theta$, denote by χ_{*j} the j^{th} column of χ. Then $(\chi_{*1}, \chi_{*2}, \chi_{*4})$ is a Cayley triple, inducing, as can be directly verified, the basis $(\chi_{*1}, \dots, \chi_{*7})$. Thus $\chi \in G_2$, and then clearly $\chi \in T_\theta$. □

A cross-section for $\approx_f$ is then given by the following result.

Proposition 7.10. *Let $\theta \in [0, \pi/2]$ and $f \in C_\theta$. Then the following holds.*

(i) $\mathrm{St}_\gamma(f) = T_\theta$.
(ii) If $\theta = 0$, then $D'_f = \{\xi u + \eta z | (\xi, \eta) \in \mathbb{S}([0,1]^2)\}$ is a cross-section of $\mathbb{S}(\Im\mathbb{O})$ with respect to $\approx_f$.
(iii) If $\theta \neq 0$, then

$$D'_f = \{\xi(u \sin\omega + v\cos\omega) + \eta uz + \zeta(uv)z | (\omega, \xi, \eta, \zeta) \in [0, \pi] \times \mathbb{S}([0,1]^3),$$
$$\xi\eta = 0 \Rightarrow \omega = 0, \theta = \pi/2 \Rightarrow \omega \leq \pi/2\}$$

is a cross-section of $\mathbb{S}(\Im\mathbb{O})$ with respect to $\approx_f$.

Proof. (i) follows from the fact that

$$\left\{ \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix} | \chi_1 \in O_3, \chi_2 \in O_4 \right\} \cap G_2$$

is the stabilizer of

$$\begin{pmatrix} \mathbb{I}_3 & \\ & -\mathbb{I}_4 \end{pmatrix},$$

which holds by arguments analogous to those in the proofs of the preceding propositions.

To prove (ii), given any $w \in \mathbb{S}(\Im\mathbb{O})$, we have

$$w = \xi u' + \eta z'$$

for some $(\xi, \eta) \in \mathbb{S}([0,1]^2)$ and $u', z' \in \mathbb{S}(\Im\mathbb{O})$ with $u' \in [u, v, uv]$ and $z' \perp [u, v, uv]$. Now for any $v' \in u'^{\perp} \cap [u, v, uv]$ we have $u'v' \in [u, v, uv]$.[10] Thus for any such v', (u', v', z') is a Cayley triple, and there is $\chi \in G_2$ mapping (u, v, z) to (u', v', z'). Then $\chi(\xi u + \eta z) = w$ and, as $\theta = 0$ implies that $\hat{R}_\theta = \mathbb{I}_7$, (i) gives that $\chi \in \mathrm{St}_\gamma(f)$, whence D'_f is exhaustive.

[10] Indeed, $[u, v, uv]$ is a subalgebra of $\mathbb{O}$ isomorphic to $\mathbb{H}$.

To show that D'_f is irredundant, assume that $\chi(\xi u+\eta z)=\xi' u+\eta' z$ for some $(\xi,\eta),(\xi',\eta')\in\mathbb{S}([0,1]^2)$ and $\chi\in\mathrm{St}_\gamma(f)$. Since $\hat R_\theta=\mathbb{I}_7$, we have

$$\chi=\begin{pmatrix}\chi_1 & \\ & \chi_2\end{pmatrix}$$

with $\chi_1\in O_3$ and $\chi_2\in O_4$. Then $\chi(\xi u+\eta z)=\xi' u+\eta' z$ implies that $(\xi,\eta)=(\xi',\eta')$.

For (iii), given $w\in\mathbb{S}(\Im\mathbb{O})$, we shall construct (ω,ξ,η,ζ) such that $(\xi\sin\omega,\xi\cos\omega,0,0,\eta,0,\zeta)\in D'_f$, and, in view of Lemma 7.9, a matrix

$$X=\begin{pmatrix} x_1 & x_2 & \epsilon x_1\times x_2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & (x_1\times x_2)^* & \epsilon x_2^* & -\epsilon x_1^* & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \epsilon\end{pmatrix},\, x_1\perp x_2\in\mathbb{S}^2,\epsilon\in E_\theta,$$

such that $\chi=\hat R_\theta^{-1}X\hat R_\theta$ maps $(\xi\sin\omega,\xi\cos\omega,0,0,\eta,0,\zeta)$ to $\pm w$. To this end, set

$$y_1=(w^1,w^2,w^3\cos\theta-w^4\sin\theta)\quad\text{and}\quad y_2=(-w^6,w^5,w^3\sin\theta+w^4\cos\theta).$$

Then there are four possible, mutually excluding, cases.

If $y_1=y_2=0$, set $\epsilon=1$ and

$$x_1=(1,0,0),\quad x_2=(0,1,0),\quad(\omega,\xi,\eta,\zeta)=(0,0,0,1).$$

If $y_1\neq0$ and $y_2=0$, take any $z\in\mathbb{S}^2$, $z\perp y_1$, and set $\epsilon=1$, and

$$x_1=z,\quad x_2=\tfrac{\mathrm{sgn}(w^7)}{\|y_1\|}y_1,\quad(\omega,\xi,\eta,\zeta)=(0,\|y_1\|,0,|w^7|).$$

If $y_2\neq0$ and $y_1=\nu y_2$ for some $\nu\in\mathbb{R}$, take any $z\in\mathbb{S}^2$, $z\perp y_2$, and set $\epsilon=\mathrm{sgn}(\nu)$ if $\theta=\pi/2$, and $\epsilon=1$ otherwise, and

$$x_1=z,\quad x_2=\tfrac{\mathrm{sgn}(w^7)}{\|y_2\|}y_2,\quad(\cos\omega,\xi,\eta,\zeta)=(\epsilon\,\mathrm{sgn}(\nu),\|y_1\|,\|y_2\|,|w^7|).$$

If $y_1\times y_2\neq0$, let ϵ and ω be given by

$$\epsilon\cos\omega=\tfrac{\langle y_1,y_2\rangle}{\|y_1\|\|y_2\|},\quad\langle y_1,y_2\rangle=0\Rightarrow\epsilon=1,$$

where $\langle,\rangle$ denotes the Euclidean inner product, and furthermore set

$$x_1=\tfrac{\mathrm{sgn}(w^7)}{\sin\omega}\big(\tfrac{\epsilon}{\|y_1\|}y_1-\cos\omega\tfrac{1}{\|y_2\|}y_2\big)\quad x_2=\tfrac{\mathrm{sgn}(w^7)}{\|y_2\|}y_2$$

and

$$(\xi,\eta,\zeta)=(\|y_1\|,\|y_2\|,|w^7|).$$

Then in all four cases, using the convention that $\mathrm{sgn}(0)=1$, (ω,ξ,η,ζ) is uniquely defined by the condition $(\xi\sin\omega,\xi\cos\omega,0,0,\eta,0,\zeta)\in D'_f$. Moreover, $\chi:=\hat R_\theta^{-1}X\hat R_\theta\in T_\theta$ by Lemma 7.9, and χ maps $(\xi\sin\omega,\xi\cos\omega,0,0,\eta,0,\zeta)$ to $\pm w$. Thus D'_f is exhaustive.

To show that D'_f is irredundant, assume that

$$w:=(\xi\sin\omega,\xi\cos\omega,0,0,\eta,0,\zeta)\approx_f(\xi'\sin\omega',\xi'\cos\omega',0,0,\eta',0,\zeta')=:w',$$

i.e. that there exists $\chi\in\mathrm{St}_\gamma(f)$ such that $\chi(w)=\pm w'$. By (i) and Lemma 7.9, this is equivalent to the existence of $x_1\perp x_2\in\mathbb{S}^2$, $\epsilon\in E_\theta$ and $\delta\in C_2$ such that, componentwise,

$$\begin{array}{rcl}\delta\xi'\sin\omega' &=& \xi(x_1^1\sin\omega + x_2^1\cos\omega)\\ \delta\xi'\cos\omega' &=& \xi(x_1^2\sin\omega + x_2^2\cos\omega)\\ 0 &=& \xi(x_1^3\sin\omega + x_2^3\cos\omega)\cos\theta + \epsilon\eta x_2^3\sin\theta\\ 0 &=& -\xi(x_1^3\sin\omega + x_2^3\cos\omega)\sin\theta + \epsilon\eta x_2^3\cos\theta\\ \delta\eta' &=& \epsilon\eta x_2^2\\ 0 &=& -\epsilon\eta x_2^1\\ \delta\zeta' &=& \epsilon\zeta.\end{array}$$

From the bottom three lines, together with a $\sin\theta$-multiple of the third line added to a $\cos\theta$-multiple of the fourth, we deduce that $|\zeta| = |\zeta'|$ and $|\eta| = |\eta'|$, whence $(\xi,\eta,\zeta) = (\xi',\eta',\zeta')$. Thus $w = w'$ if $\xi\eta = 0$. If $\xi\eta \neq 0$, we get $x_2 = (0,\delta\epsilon,0)$, which implies that $x_1^2 = 0$. Then the second line gives $\cos\omega' = \epsilon\cos\omega$, implying $\omega' = \omega$, which completes the proof. □

8. Conclusion and Future Perspectives

The procedure hitherto employed gives, if completed, an explicit classification of left reflection algebras. By the Cartan–Dieudonné Theorem, each $g \in O_8^1$ is the product of n reflections for some $0 \leq n \leq 7$. The cases $n = 0$ and $n = 1$ having been treated in [7] and above, respectively, one may attempt to use the above techniques to investigate the set of all *left n-reflection algebras*, i.e. algebras $\mathbb{O}_{f,g}$ where $f, g \in O_8^1$ and g is the product of n reflections, $2 \leq n \leq 7$. When doing so, two issues arise for larger n. To begin with, as the number of reflections is not invariant under isomorphism, one must exclude such left n-reflection algebras that are isomorphic to left n'-reflection algebras for some $n' < n$. Secondly, the above work was simplified by the fact that for any $u \in \mathbb{S}(\Im\mathbb{O})$, each left reflection algebra is isomorphic, by a G_2-morphism, to $\mathbb{O}_{f,\sigma_u}$. For $n \geq 3$, the situation becomes increasingly complicated, due to the restrictive properties of G_2.

These generalizations are, however, beyond the scope of this paper, and it is the author's hope to be able to treat them in a forthcoming publication.

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Appendix A. On the Category $\mathcal{C}_8^1$

In this appendix we will show that the full subcategory $\mathcal{C}_8^1 \subseteq {}_{SO_8}\mathcal{O}_8$, defined in the end of Section 2, does not arise from the triality action of any subgroup of SO_8 on $\mathcal{O}_8^1$. In other words we will show that the subcategory ${}_{\mathrm{St}^*(\mathcal{O}_8^1)}\mathcal{O}_8^1 \subseteq {}_{SO_8}\mathcal{O}_8$ is not full. By Theorem 4.7, this is equivalent to showing that $\mathcal{O}_8^1$ is not full in $\mathcal{O}_8$ with respect to the triality action, i.e that there exists $\phi \in SO_8 \setminus (\mathrm{St}(\mathcal{O}_8^1) \cup \mathrm{Dest}(\mathcal{O}_8^1))$.

To this end, fix a Cayley triple $(u, v, z) \in \mathbb{O}^3$, set $\theta = 2\pi/3$, and define $\phi \in SO_8$ as the rotation with angle 2θ in the $(1, u)$-plane, i.e. let ϕ be given by

$$\phi(1) = \cos(2\theta)1 + \sin(2\theta)u, \quad \phi(u) = -\sin(2\theta)1 + \cos(2\theta)u,$$

and $\phi(x) = x$ for each $x \in [1, u]^\perp$. In this case one can easily compute a triality pair (ϕ_1, ϕ_2) of ϕ. Expressed in the basis induced by (u, v, z), this is given by

$$\phi_i = \begin{pmatrix} \cos\theta & -\sin\theta & & & & & & \\ \sin\theta & \cos\theta & & & & & & \\ & & \cos\theta & \epsilon_i \sin\theta & & & & \\ & & -\epsilon_i \sin\theta & \cos\theta & & & & \\ & & & & \cos\theta & \epsilon_i sin\theta & & \\ & & & & -\epsilon_i \sin\theta & \cos\theta & & \\ & & & & & & \cos\theta & -\epsilon_i \sin\theta \\ & & & & & & \epsilon_i \sin\theta & \cos\theta \end{pmatrix}$$

for each $i \in \underline{2}$, where $\epsilon_i = (-1)^i$.

Then $[\mathrm{Id}, \mathrm{Id}] \in \mathcal{O}_8^1$ satisfies $\phi \cdot [\mathrm{Id}, \mathrm{Id}] \notin \mathcal{O}_8^1$, implying that $\phi \notin \mathrm{St}(\mathcal{O}_8^1)$, while $[\sigma_u, \sigma_u] \in \mathcal{O}_8^1$ satisfies $\phi \cdot [\sigma_u, \sigma_u] \in \mathcal{O}_8^1$, and hence $\phi \notin \mathrm{Dest}(\mathcal{O}_8^1)$. This completes the proof.

Appendix B. Constructing Certain Elements in G_2

Let $u \in \mathbb{S}(\Im\mathbb{O})$. In this appendix, we describe, for the sake of fixing a choice, a general and explicit method to construct, for each line ℓ through the origin in $\Im\mathbb{O}$, an element $\phi_\ell \in G_2$ such that $[\phi_\ell(u)] = \ell$. We thus fix a Cayley triple (u, v, z), and let $\mathfrak{B}$ be the basis induced by it (see Remark 7.1). Our aim is then to construct a Cayley triple (u', v', z') such that $[u'] = \ell$. To this end let x be the orthogonal projection onto ℓ of the first element in $\mathfrak{B}$ not orthogonal to ℓ. Let y_1 be the first $y \in \mathfrak{B}$ such that $y \notin \ell$. By performing the Gram-Schmidt procedure on (x, y_1) we obtain an orthonormal set (u', v') such that $[u'] = \ell$. Finally, let y_2 be the first $y \in \mathfrak{B}$ such that $y \notin [u', v', u'v']$. As $u'v' \perp [u', v']$ and $\|u'v'\| = 1$, the Gram-Schmidt procedure applied to $(u', v', u'v', y_2)$ yields an orthonormal quadruple $(u', v', u'v', z')$, with (u', v', z') a Cayley triple as required.

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