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Corestricted Group Actions and Eight-Dimensional Absolute Valued Algebras

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ABSTRACT. A condition for when two eight-dimensional absolute valued algebras are isomorphic was given in [4]. We use this condition to deduce a description (in the sense of Dieterich, [9]) of the category of such algebras, and show how previous descriptions of some full subcategories fit in this description. Led by the structure of these examples, we aim at systematically constructing new subcategories whose classification is manageable. To this end we propose, in greater generality, the definition of sharp stabilizers for group actions, and use these to obtain conditions for when certain subcategories of groupoids are full. This we apply to the category of eight-dimensional absolute valued algebras and obtain a class of subcategories, for which we simplify, and partially solve, the classification problem.

1. INTRODUCTION

This paper is concerned with the classification of finite-dimensional absolute valued algebras. An *algebra* over a field k is a vector space A over k equipped with a k-bilinear multiplication $A \times A \to A, (x, y) \mapsto xy$. Neither associativity nor commutativity is in general assumed. A is called *absolute valued* if the vector space is real, non-zero and equipped with a norm $\| \cdot \|$ such that $\|xy\| = \|x\|\|y\|$ for all $x, y \in A$.

Finite-dimensional absolute valued algebras exist only in dimensions 1, 2, 4 and 8; except in dimension 8, they have been classified up to isomorphism, and the morphisms between them have been described (see [2] and [10]). In dimension 8, conditions for when two algebras are isomorphic were obtained in [4], using triality. We formulate these conditions as a description (in the sense of Dieterich, [9]) of the category of eight-dimensional absolute valued algebras in Section 2. Since the classification problem has proved hard, we set out to systematically find suitable full subcategories for which the classification problem is feasible, in particular, where the generally difficult computations in connection with triality are avoided.

As a model, we consider, in Section 3, the full subcategories of eight-dimensional absolute valued algebras with a left unity, a right unity, and a non-zero central idempotent, respectively. These were described and classified in [7], and we embed this description in that of Section 2.

Here, a *description* of a groupoid[1] $\mathcal{C}$ is an equivalence of categories between $\mathcal{C}$ and a groupoid arising from a group action. To construct full subcategories of $\mathcal{C}$,

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[1]By a *groupoid* we understand a (not necessarily small) category where all morphisms are isomorphisms.

we seek conditions under which the group action used in the description induces an action on a subset, which in turn gives a description of a full subcategory of $\mathcal{C}$.[2] As it turns out, this occurs precisely when the subset imposes a certain dichotomy on the group. This is explored in Section 4, where a general framework is obtained, based on stabilizers of subsets with respect to a group action. One of the results there describes how the object class of a groupoid can be partitioned, giving rise to pairwise isomorphic subgroupoids. This simplifies the classification problem and gives additional structural insight.

In Section 5, we examine the subcategories classified in [7] in this new framework. A common feature of these subcategories is the triviality of the triality phenomenon. Using this knowledge, and the structural insight gained in Section 4, we construct and describe a new subcategory of absolute valued algebras in Section 6. We reduce the classification problem for these algebras to a manageable, though somewhat computational, one, and in the final section, we classify some subclasses of such algebras to demonstrate the computations involved.

1.1. **Preliminaries.** By [1], the norm in a finite-dimensional absolute valued algebra is uniquely determined by the algebra multiplication, and multiplicativity of the norm implies that an absolute valued algebra has no zero divisors and hence, if it is finite-dimensional, that it is a division algebra. (We recall that a *division algebra* is a non-zero algebra D such that for each $a \in D \setminus \{0\}$, the left and right multiplication maps $L_a : D \to D, x \mapsto ax$ and $R_a : D \to D, x \mapsto xa$ are bijective. This implies the non-existence of zero divisors and, if D has finite dimension, it is equivalent to it.)

The class of all finite-dimensional absolute valued algebras forms a category $\mathcal{A}$, in which the morphisms are all non-zero algebra homomorphisms. Thus $\mathcal{A}$ is a full subcategory of the category $\mathcal{D}(\mathbb{R})$ of finite-dimensional real division algebras. It is known that morphisms in $\mathcal{D}(\mathbb{R})$ are injective, and morphisms in $\mathcal{A}$ are isometries.

In 1947, Albert characterized all finite-dimensional absolute valued algebras as follows ([1]).

Proposition 1.1. *Every finite-dimensional absolute valued algebra is isomorphic to an orthogonal isotope A of a unique $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, i.e. $A = \mathbb{A}$ as a vector space, and the multiplication $\cdot$ in A is given by*

$$x \cdot y = f(x)g(y)$$

for all $x, y \in A$, where f and g are linear orthogonal operators on A, and juxtaposition denotes multiplication in $\mathbb{A}$.

Moreover, Albert shows that the norm in A coincides with the norm in $\mathbb{A}$.

Thus the objects of $\mathcal{A}$ are partitioned into four classes according to their dimension, and the class of d-dimensional algebras, $d \in \{1, 2, 4, 8\}$, forms a full subcategory $\mathcal{A}_d$ of $\mathcal{A}$. For $d > 1$ we moreover have the following decomposition due to Darpö and Dieterich [8]. [3]

Proposition 1.2. *Let $A \in \mathcal{A}_d$ with $d \in \{2, 4, 8\}$. For any $a, b \in A \setminus \{0\}$,*

$$\operatorname{sgn}(\det(L_a)) = \operatorname{sgn}(\det(L_b)), \quad \operatorname{sgn}(\det(R_a)) = \operatorname{sgn}(\det(R_b)).$$

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[2]It is important that the subcategory be full, in order for a classification of it to be useful in classifying the larger category.

[3]We define the sign function $\operatorname{sgn} : \mathbb{R} \to C_2$ by $\operatorname{sgn}(r) = r/|r|$ if $r \neq 0$, and $\operatorname{sgn}(0) = 1$. The definition at 0, though immaterial here, will be useful in later sections.

The double sign *of A is the pair $(i,j) \in C_2^2$ where $i = \operatorname{sgn}(\det(L_a))$ and $j = \operatorname{sgn}(\det(R_a))$ for any $a \in A \setminus \{0\}$. Moreover, for all $d \in \{2,4,8\}$,*

$$\mathcal{A}_d = \coprod_{(i,j)\in C_2^2} \mathcal{A}_d^{ij} \tag{1.1}$$

where $\mathcal{A}_d^{ij}$ is the full subcategory of $\mathcal{A}_d$ formed by all algebras having double sign (i,j).

In the classification of finite-dimensional real division algebras, a certain type of categories, more specifically, of groupoids, has proven useful. We recall their definition.

Definition 1.3. Let G be a group, X a set, and $\alpha : G \times X \to X$ a left group action. The *groupoid arising from α* is the category ${}_GX$ with object set X and where, for each $x, y \in X$,

$${}_GX(x,y) = \{(g,x,y) | g \in G, g \cdot x = y\}.$$

It is clear that ${}_GX$ is a groupoid. The group action is implicit in the notation ${}_GX$, and if the domain and codomain of a morphism (g,x,y) are clear from the context, the morphism is simply referred to by g.

Groupoids arising from group actions can, and will in this article, be used to gain an understanding of finite-dimensional absolute valued algebras in the following way, due to [9].

Definition 1.4. Let $d \in \{2,4,8\}$. A *description (in the sense of Dieterich)* of a full subcategory $\mathcal{C} \subseteq \mathcal{A}_d$ is a quadruple $(G,X,\alpha,\mathcal{F})$, where G is a group, X a set, $\alpha : G \times X \to X$ a left group action, and $\mathcal{F} : {}_GX \to \mathcal{C}$ an equivalence of categories.

Once a description is obtained, the problem of classifying $\mathcal{C}$ is transformed to the normal form problem for α, i.e. the problem of finding a transversal for the orbits of α. It is therefore crucial that the quadruple $(G,X,\alpha,\mathcal{F})$ be given explicitly. In [9], descriptions are defined in the more general context of finite-dimensional real division algebras, which we will not need here.

1.2. **Notation.** We use the convention that $0 \in \mathbb{N}$, and use the notation $\mathbb{Z}_+$ for the set $\mathbb{N} \setminus \{0\}$. For each $n \in \mathbb{Z}_+$ we denote by $\underline{n}$ the set $\{1,2,\ldots,n\}$. As $\mathbb{O}$ denotes the algebra of octonions, $\Im\mathbb{O}$ denotes the hyperplane of its purely imaginary elements.

For a vector space V, we denote by $\mathbb{P}(V)$ the projective space of V, whose elements are the lines through the origin in V. An element in $\mathbb{P}(V)$ containing a non-zero vector v will be denoted by $[v]$. More generally, square brackets denote the linear span of a collection of vectors in V. If a basis is given, upper indices will always denote the coordinates of a vector in this basis; hence v^i is the i^{th} coordinate of v.

If V is normed and $U \subseteq V$ a subset, we denote by $\mathbb{S}(U)$ the set of all elements of U having norm 1. Unless otherwise stated, for each $n \in \mathbb{N}$, $\mathbb{R}^{n+1}$ is equipped with the Euclidean norm, and $\mathbb{S}^n = \mathbb{S}(\mathbb{R}^{n+1})$ denotes the unit n-sphere.

The general linear group in dimension n over $\mathbb{R}$ will be denoted $GL_n = GL(\mathbb{R}^n)$, which we identify with $GL_n(\mathbb{R})$ upon endowing $\mathbb{R}^n$ with a standard basis. Analogous notation will be used for its classical subgroups, notably O_n and SO_n. We denote by O_8^1 the group of all $g \in O_8 = O(\mathbb{O})$ fixing $1 \in \mathbb{O}$, and by O_8^+ and O_8^- the set of all $f \in O_8$ with positive and negative determinant, respectively. (The elements of the cyclic group C_2 are written as $+$ and $-$ rather than as 1 and -1.) The notation $\mathbb{I}_n$ will be used for the $n \times n$ identity matrix.

The symbol $\leq$ will be used to denote the subgroup relation. For a group action $\alpha : G \times X \to X$, $g \in G$ and $x, y \in X$, we write $g \cdot x$ for $\alpha(g, x)$, and $x \equiv_\alpha y$ or $x \equiv y$ *with respect to* α to denote that x and y are in the same orbit.

Finally, given a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$, we denote by $\mathcal{F}|_\mathcal{C}$ the restriction of $\mathcal{F}$ to a subcategory $\mathcal{C}$ of $\mathcal{A}$.

1.3. **Triality, G_2 and Cayley Triples.** The study we are about to undertake makes frequent use of two concepts: the principle of triality, and the group G_2. Both have been subject to profound research, which goes far beyond the scope of this paper. The aim of this section is to recall such facts about these concepts that will be needed here, in the form applicable to the problems at hand. For a more general approach, the reader is directed to the literature: both concepts are treated in the overview article of Baez [3], as well as in [6], triality is further treated by Chevalley [5], while G_2 and Cayley triples are dealt with in [12], Chapter 1. Applications of these concepts to absolute valued algebras can be found in [4] and [7], to which we will refer in several places.

We will be concerned with the *principle of triality* as applied to SO_8, which we quote here.

Proposition 1.5. *For each $\phi \in SO_8$ there exist $\phi_1, \phi_2 \in SO_8$ such that for each $x, y \in \mathbb{O}$,*

$$\phi(xy) = \phi_1(x)\phi_2(y).$$

The pair (ϕ_1, ϕ_2) is unique up to (overall) sign.

Thus there exist two *triality pairs* $\pm(\phi_1, \phi_2)$ for each $\phi \in SO_8$. Moreover, triality respects composition, i.e. if $\phi, \psi \in SO_8$, and (ϕ_1, ϕ_2) and (ψ_1, ψ_2) are triality pairs for ϕ and ψ, respectively, then $(\phi_1\psi_1, \phi_2\psi_2)$ is a triality pair for the product $\phi\psi$, since for any $x, y \in \mathbb{O}$,

$$\phi_1\psi_1(x)\phi_2\psi_2(y) = \phi(\psi_1(x)\psi_2(y)) = \phi\psi(xy).$$

As $(\mathrm{Id}, \mathrm{Id})$ is a triality pair for $\mathrm{Id} \in SO_8$, we deduce that $(\phi_1^{-1}, \phi_2^{-1})$ is a triality pair for ϕ^{-1}. We moreover have the identities

$$\phi_1 = R_{\phi_2(1)^{-1}}\phi, \quad \phi_2 = L_{\phi_1(1)^{-1}}\phi,$$

where $\phi_i(1)^{-1}$ is not to be confused with $\phi_i^{-1}(1)$ for $i \in \underline{2}$.

Every automorphism of $\mathbb{O}$ has determinant 1. Thus $\mathrm{Aut}(\mathbb{O}) \leq SO_8$, and an element $\phi \in SO_8$ is an automorphism of $\mathbb{O}$ precisely when (ϕ, ϕ) is a triality pair for ϕ. We then say that the triality components of ϕ are *trivial.* The group $\mathrm{Aut}(\mathbb{O})$ is the Lie group G_2. It has dimension 14, and is thus the smallest of the exceptional Lie groups. G_2 may equivalently be characterized as the set of all $\phi \in SO_8$ such that

$$\phi(1) = \phi_1(1) = \phi_2(1) = 1$$

for some triality pair (ϕ_1, ϕ_2) of ϕ. The identity $\phi(1) = \phi_1(1)\phi_2(1)$ shows that if any two of $\phi(1)$, $\phi_1(1)$ and $\phi_2(1)$ equal 1, then so does the third. Since the group of all $\phi \in SO_8$ such that $\phi(1) = 1$ is isomorphic to SO_7, one may view G_2 as a subgroup of SO_7, which we will sometimes do for notational convenience.

Another way to characterize G_2 is via Cayley triples. [4]

[4]Named in honour of Arthur Cayley, 1821–1895.

Definition 1.6. A *Cayley triple* is an orthonormal triple $(u, v, z) \in (\Im\mathbb{O})^3$ such that $z \perp uv$.

Some fundamental facts about Cayley triples are given in the following well-known result.

Proposition 1.7. *Let $(u, v, z) \in (\Im\mathbb{O})^3$ be a Cayley triple.*

(i) The algebra $\mathbb{O}$ is generated by (u, v, z).
(ii) $(1, u, v, uv, z, uz, vz, (uv)z)$ is an orthonormal basis of $\mathbb{O}$, called the basis induced by (u, v, z).
(iii) The group G_2 corresponds bijectively to the set of all Cayley triples, the bijection being given by $\phi \mapsto (\phi(u), \phi(v), \phi(z))$ for all $\phi \in G_2$.

Cayley triples and induced bases will be used in the computations of Section 7.

2. Description of $\mathcal{A}_8$

Conditions for when two finite-dimensional absolute valued algebras are isomorphic are given in [4]. In this section we deduce from this a description of $\mathcal{A}_8$. In other words, we establish an equivalence of categories from a groupoid arising from a group action to $\mathcal{A}_8$. To begin with, we introduce the action, for which we define the quotient group

$$\mathcal{O}_8 = (O_8 \times O_8)/\{\pm(\mathrm{Id}, \mathrm{Id})\}$$

and write $[f, g]$ for the coset of $(f, g) \in O_8 \times O_8$.

Proposition 2.1. *The map $\tau : SO_8 \times \mathcal{O}_8 \to \mathcal{O}_8$ defined by*

$$(\phi, [f, g]) \mapsto \phi \cdot [f, g] = [\phi_1 f \phi^{-1}, \phi_2 g \phi^{-1}],$$

where (ϕ_1, ϕ_2) is any of the two triality pairs of ϕ, is a left group action.

This action will be called the *triality action*, and we say that SO_8 acts *by triality.*

Proof. Note, at first, that the map is well-defined, since the two triality pairs of ϕ, as well as the two representatives of $[f, g]$, are equal up to overall sign. The identity axiom for group actions holds since $(\mathrm{Id}, \mathrm{Id})$ is a triality pair for $\mathrm{Id} \in SO_8$. For the product axiom, take $\phi, \psi \in SO_8$. Then for any triality pair (ϕ_1, ϕ_2) of ϕ and (ψ_1, ψ_2) of ψ

$$\begin{aligned} \phi \cdot (\psi \cdot [f, g]) &= \phi \cdot [\psi_1 f \psi^{-1}, \psi_2 g \psi^{-1}] \\ &= [\phi_1 \psi_1 f \psi^{-1} \phi^{-1}, \phi_2 \psi_2 g \psi^{-1} \phi^{-1}] = \phi\psi \cdot [f, g], \end{aligned}$$

where the rightmost equality holds since triality respects composition. □

Remark 2.2. Note that for each $h \in GL_8$ we have $\det(h) = \det(-h)$. Thus the quotient in Proposition 2.1 respects the sign of the determinant of each of f and g, i.e. if $(f, g) \in O_8^j \times O_8^i$ for some $(i, j) \in C_2^2$, then $(-f, -g) \in O_8^j \times O_8^i$. Thus $\mathcal{O}_8$ decomposes into four subsets,

$$\mathcal{O}_8^{ij} := \{[f, g] \in \mathcal{O}_8 | \det(f) = j, \det(g) = i\}, \quad (i, j) \in C_2^2.$$

Also note that the group action preserves the pair $(\det(f), \det(g))$ as well, i.e. if $[f, g] \in \mathcal{O}_8^{ij}$ for some $(i, j) \in C_2^2$, then $\phi \cdot [f, g] \in \mathcal{O}_8^{ij}$ for all $\phi \in SO_8$. Hence SO_8

acts by triality on $\mathcal{O}_8^{ij}$ for each $(i,j) \in C_2^2$, and for the groupoid arising from the triality action, we have the coproduct decomposition

$$_{SO_8}\mathcal{O}_8 = \coprod_{(i,j)\in C_2^2} {}_{SO_8}\mathcal{O}_8^{ij}.$$

The seemingly reversed order of i and j in the notation is used for coherence with the double sign defined in Proposition 1.2, as will be clear in the next theorem, which establishes the equivalences of categories.

Theorem 2.3. *Let $(i,j) \in C_2^2$, and let ${}_{SO_8}\mathcal{O}_8^{ij}$ be the groupoid arising from the triality action of SO_8 on $\mathcal{O}_8^{ij}$. Then*

$$\mathcal{F}^{ij} : {}_{SO_8}\mathcal{O}_8^{ij} \to \mathcal{A}_8^{ij},$$

defined on objects by $\mathcal{F}^{ij}([f,g]) = \mathbb{O}_{f,g}$ and on morphisms by $\mathcal{F}^{ij}(\phi) = \phi$, is an equivalence of categories.

Proof. The maps $\mathcal{F}^{ij}$ are independent of the representative of $[f,g]$ since $\mathbb{O}_{-f,-g} = \mathbb{O}_{f,g}$. They are well-defined by the definition of the double sign and by Remark 2.2, noting that for each $a \in \mathbb{O}_{f,g} \setminus \{0\}$,

$$\operatorname{sgn}(\det(L_a)) = \operatorname{sgn}(\det(g)), \quad \operatorname{sgn}(\det(R_a)) = \operatorname{sgn}(\det(f)).$$

Functoriality follows from the axioms of a group action. Finally, each $\mathcal{F}^{ij}$ is dense by Propositions 1.1 and 1.2, faithful by construction, and full by [4].[5] □

Remark 2.4. We will use $\mathcal{F}$ to denote the functor ${}_{SO_8}\mathcal{O}_8 \to \mathcal{A}_8$ defined by $\mathcal{F}|_{{}_{SO_8}\mathcal{O}_8^{ij}} = \mathcal{F}^{ij}$ for each $(i,j) \in C_2^2$.

Summarizing, $(SO_8, \mathcal{O}_8, \tau, \mathcal{F})$ is a description of $\mathcal{A}_8$, and, more specifically, for each $(i,j) \in C_2^2$, $(SO_8, \mathcal{O}_8^{ij}, \tau_{ij}, \mathcal{F}^{ij})$ is a description of $\mathcal{A}_8^{ij}$, where τ_{ij} is the triality action of SO_8 on $\mathcal{O}_8^{ij}$ from Remark 2.2.

In fact, the result in [4] mentioned above contains more information, namely that for each $f, g \in O_8$ there exist $f', g' \in O_8^1$ such that $\mathbb{O}_{f,g} \simeq \mathbb{O}_{f',g'}$, i.e. that $\mathcal{F}(\mathcal{C}_8^1)$ is dense in $\mathcal{A}_8$, where $\mathcal{C}_8^1 \subseteq {}_{SO_8}\mathcal{O}_8$ is the full subcategory whose object set $\mathcal{O}_8^1$ consists of all $[f,g] \in \mathcal{O}_8$ with $f, g \in O_8^1$. [6]

However, this cannot be used to replace the above description of $\mathcal{A}_8$ with a description using $\mathcal{O}_8^1$ instead of $\mathcal{O}_8$, in the sense that there is no subgroup $H \leq SO_8$ such that $\mathcal{C}^1 = {}_H\mathcal{O}_8^1$. Proving this makes use of the techniques to be developed in Section 4 below, and the proof can be found in Appendix A.

We will apply Theorem 2.3 in various contexts. First, let us review a classification of some full subcategories of $\mathcal{A}_8$ in view of the above description.

3. Algebras having a Non-zero Central Idempotent or a One-sided Unity

Consider the three full subcategories $\mathcal{A}_8^l$, $\mathcal{A}_8^r$ and $\mathcal{A}_8^c$ of $\mathcal{A}_8$, the object classes of which are

$$\mathcal{A}_8^l = \{A \in \mathcal{A}_8 | \exists u \in A, \forall x \in A, ux = x\},$$
$$\mathcal{A}_8^r = \{A \in \mathcal{A}_8 | \exists u \in A, \forall x \in A, xu = x\},$$

[5] In [4], this is part of Theorem 4.3 and the remarks preceding it.

[6] Note that the quotient map $O_8 \times O_8 \to \mathcal{O}_8, (f,g) \mapsto [f,g]$ is injective on $O_8^1 \times O_8^1$.

$$\mathcal{A}_8^c = \{A \in \mathcal{A}_8 | \exists u \in Z(A) \setminus \{0\}, u^2 = u\},$$

and consist of all algebras with a left unity, a right unity, and a non-zero central idempotent, respectively. Here, the centre $Z(B)$ of an algebra B is defined by

$$Z(B) = \{z \in B | \forall b \in B, zb = bz\},$$

and an element is called *central* if it belongs to the centre. The three categories defined above are studied in [7], where a classification of them is obtained. We will return to the classification later; in the present section, we direct our attention to the descriptions of the three categories, given in the following result from [7].

Proposition 3.1. *For each $x \in \{l, r, c\}$, $\mathcal{A}_8^x$ is equivalent to the category ${}_{G_2}O_8^1$, where G_2 acts by conjugation. The equivalences are given by*

$$\begin{array}{lll} \mathcal{F}^l : {}_{G_2}O_8^1 \to \mathcal{A}_8^l, & \mathcal{F}^l(f) = \mathbb{O}_{f,\mathrm{Id}}, & \mathcal{F}^l(\phi) = \phi, \\ \mathcal{F}^r : {}_{G_2}O_8^1 \to \mathcal{A}_8^r, & \mathcal{F}^r(f) = \mathbb{O}_{\mathrm{Id},f}, & \mathcal{F}^r(\phi) = \phi, \\ \mathcal{F}^c : {}_{G_2}O_8^1 \to \mathcal{A}_8^c, & \mathcal{F}^c(f) = \mathbb{O}_{f,f}, & \mathcal{F}^c(\phi) = \phi, \end{array}$$

for each $f \in O_8^1$ and $\phi \in G_2$.

The following proposition implies that this description of $\mathcal{A}_8^x, x \in \{l, r, c\}$ is an instance of the description of $\mathcal{A}_8$ of Section 2.

Proposition 3.2. *Let $x \in \{l, r, c\}$, and define $\mathcal{G}^x : {}_{G_2}O_8^1 \to {}_{SO_8}\mathcal{O}_8$ on objects by*

$$\mathcal{G}^l(f) = [f, \mathrm{Id}], \quad \mathcal{G}^r(f) = [\mathrm{Id}, f], \quad \mathcal{G}^c(f) = [f, f],$$

for each $f \in O_8^1$, and on morphisms by $\mathcal{G}^x(\phi) = \phi$ for each $x \in \{l, r, c\}$ and $\phi \in G_2$. Then for each $x \in \{l, r, c\}$, $\mathcal{G}^x$ is a full and faithful functor, which is moreover injective on the set of objects.

Proof. For each $x \in \{l, r, c\}$, $\mathcal{G}^x$ is well defined, as for any $\phi \in G_2$, we have $\phi(1) = 1$ and (ϕ, ϕ) is a triality pair for ϕ; thus for each $i \in \underline{2}$ and each $f \in O_8^1$,

$$\phi_i \,\mathrm{Id}\, \phi^{-1} = \phi \,\mathrm{Id}\, \phi^{-1} = \mathrm{Id}, \quad \phi_i f \phi^{-1}(1) = \phi f \phi^{-1}(1) = 1,$$

whence $\mathcal{G}^x(\phi)(\mathcal{G}^x(f)) = \mathcal{G}^x(\phi(f))$. Functoriality is obvious, and $\mathcal{G}^x$ is faithful by definition. For injectivity, let first $x = c$, and assume $[f, f] = [f', f']$ for some $f, f' \in O_8^1$. Then $(f, f) = \pm(f', f')$. But $f(1) = f'(1) = 1$, thus $f \neq -f'$, implying $f = f'$. The other cases are similar. To show fullness, take any $f, f' \in O_8^1$, and let $\phi : \mathcal{G}^x(f) \to \mathcal{G}^x(f')$ be any morphism. We need show that $\phi \in G_2$, and we will do this case by case.

Assume first that $x = l$. Then by Proposition 2.1 we have, for one triality pair (ϕ_1, ϕ_2) of ϕ, that

$$\phi_1 f \phi^{-1} = f' \quad \text{and} \quad \phi_2 \,\mathrm{Id}\, \phi^{-1} = \mathrm{Id}\,.$$

From the second equality we then have $\phi_2 = \phi$, thus $\phi(1) = \phi_1(1)\phi(1)$, implying that ϕ_1 fixes $1 \in \mathbb{O}$. Rewriting the first equation as $\phi = f'^{-1}\phi_1 f$, we note that the right hand side fixes 1, and hence $\phi_2(1) = \phi(1) = 1$. Thus $\phi \in G_2$. If instead $x = r$, then the above argument, with ϕ_1 and ϕ_2 interchanged, implies that $\phi \in G_2$. Finally if $x = c$, then

$$\phi_1 f \phi^{-1} = f' \quad \text{and} \quad \phi_2 f \phi^{-1} = f',$$

for one triality pair (ϕ_1, ϕ_2) of ϕ, implying that $\phi_1 = \phi_2$. Take any $y \in \mathbb{O}$. Since ϕ_1 is bijective there exists $x \in \mathbb{O}$ such that $y = \phi_1(x)$. Hence

$$y\phi_1(1) = \phi_1(x)\phi_1(1) = \phi_1(x)\phi_2(1) = \phi(x) = \phi_1(1)\phi_2(x) = \phi_1(1)\phi_1(x) = \phi_1(1)y$$

and thus $\phi_1(1) \in Z(\mathbb{O}) = \mathbb{R}1$. But $\phi_1 \in SO_8$, whereby $\|\phi_1(1)\| = 1$. Moreover,

$$\phi(1) = \phi_1(1)^2 = 1,$$

and together with $\phi_1(1) = \phi_2(1) = \pm 1$ this implies that $\phi \in G_2$. □

Viewing $\mathcal{G}^x$, $x \in \{l, r, c\}$, as inclusion $O_8^1 \hookrightarrow \mathcal{O}_8$, the above result implies that the description of $\mathcal{A}_8^x$ is obtained directly from the description of $\mathcal{A}_8$, by restricting the latter to O_8^1.

4. Corestricted Group Actions

In view of Proposition 3.2, one may ask under which conditions a description of a full subcategory $\mathcal{C} \subseteq \mathcal{A}_8$ can be obtained by restricting a description $\mathcal{F} : {}_GX \to \mathcal{A}_8$ of $\mathcal{A}_8$ to a subset $Y \subseteq X$. In precise terms, given the groupoid ${}_GX$ arising from a left action α of a group G on a set X, and given a subset $Y \subseteq X$, we seek conditions on Y under which there exists a subgroup $H \leq G$ such that

- the restriction of α to $H \times Y$ admits a corestriction to Y, i.e. α induces a group action $H \times Y \to Y$, and
- the groupoid ${}_HY$ arising from this action is full in ${}_GX$.

In this section, we will answer this question in the general setting, and derive some structural consequences.

4.1. Definitions and General Properties.

The constructions of this section use the notion of a stabilizer of a subset under a group action. This notion, which we will now develop, generalizes the familiar notion of a stabilizer of a single element. We will use left actions throughout; however, all definitions can be analogously made for right actions, and all results apply, *mutatis mutandis*, to these as well.

Definition 4.1. Let $\alpha : G \times X \to X$ be a group action. An element $g \in G$ is said to *stabilize* a subset $Y \subseteq X$ if $g \cdot y \in Y$ for each $y \in Y$. A subgroup $H \leq G$ *stabilizes* Y if each $h \in H$ stabilizes Y.

Note that the set of all elements of G that stabilize $Y \subseteq X$ is in general not a subgroup of G, since it may happen that the inverse of a stabilizing element is not stabilizing itself.

Example 4.2. Consider the action of the group $(\mathbb{Z}, +)$ on itself by addition. The set of all elements that stabilize $\mathbb{N} \subset \mathbb{Z}$ is $\mathbb{N}$ itself, which does not contain the inverse of any of its non-zero elements.

We define the following subsets.

Definition 4.3. Let $\alpha : G \times X \to X$ be a group action, and let $Y \subseteq X$. The *stabilizer* of Y (*with respect to* α) is the set

$$\mathrm{St}(Y) = \{g \in G | \forall y \in Y : g \cdot y \in Y\},$$

the *sharp stabilizer* of Y is the set

$$\mathrm{St}^*(Y) = \{g \in G | \forall y \in Y : g \cdot y \in Y \wedge g^{-1} \cdot y \in Y\},$$

and the *destabilizer* of Y is the set

$$\mathrm{Dest}(Y) = \{g \in G | \forall y \in Y : g \cdot y \notin Y\}.$$

Remark 4.4. Equivalently, we have that

$$\begin{aligned}\mathrm{St}(Y) &= \{g \in G | g \cdot Y \subseteq Y\},\\ \mathrm{St}^*(Y) &= \{g \in G | g \cdot Y = Y\}, \text{ and}\\ \mathrm{Dest}(Y) &= \{g \in G | g \cdot Y \cap Y = \emptyset\}.\end{aligned}$$

When necessary, we will write $\mathrm{St}_\alpha(Y)$ etc. to emphasize the group action. If Y is a singleton set $\{y\}$, we will omit the set brackets in the above notation.

In the above example we saw that $\mathrm{St}(\mathbb{N}) = \mathbb{N}$. Moreover, $\mathrm{St}^*(\mathbb{N}) = \{0\}$, and $\mathrm{Dest}(\mathbb{N}) = \emptyset$. In general, we have the following facts.

Proposition 4.5. *Let $\alpha : G \times X \to X$ be a group action, and let $Y \subseteq X$. Then*

(i) $\mathrm{St}(Y)$ *is a submonoid of* G,
(ii) $\mathrm{St}^*(Y)$ *is a subgroup of* G, *and* $\mathrm{St}^*(Y) = \mathrm{St}(Y) \cap \mathrm{St}(Y)^{-1}$,
(iii) if $H \leq G$ satisfies $H \subseteq \mathrm{St}(Y)$, then $H \leq \mathrm{St}^*(Y)$, and
(iv) if Y is finite, then $\mathrm{St}^*(Y) = \mathrm{St}(Y)$.

Here we have used the notation $Z^{-1} = \{z^{-1} | z \in Z\}$ for a subset $Z \subseteq G$. Note that statement (iv) generalizes the fact that the stabilizer of one element is a group.

Proof. (i) and (ii) are immediate from the definitions and the axioms of a group action. (iii) holds since if $g \in H \subseteq \mathrm{St}(Y)$, then $g^{-1} \in H \subseteq \mathrm{St}(Y)$, whence $g \in \mathrm{St}(Y) \cap \mathrm{St}(Y)^{-1}$. To prove (iv), assume that Y is finite and let $g \in \mathrm{St}(Y)$. Then $g \cdot Y \subseteq Y$ and $|g \cdot Y| = |Y|$, whence $g \cdot Y = Y$, and thus $g \in \mathrm{St}^*(Y)$. □

Remark 4.6. Item (iii) implies that $\mathrm{St}^*(Y)$ is maximal in the sense that it is the largest subgroup of G contained in $\mathrm{St}(Y)$.

We conclude from Proposition 4.5 that the map

$$\mathrm{St}^*(Y) \times Y \to Y, (g, y) \mapsto g \cdot y$$

is an action of the group $\mathrm{St}^*(Y)$ on Y, the so-called *corestriction of* $\alpha : G \times X \to X$ *from X to Y.*

4.2. **Conditions for Full Subgroupoids.** Let $\alpha : G \times X \to X$ be a group action. By the above, each subset $Y \subseteq X$ gives rise to a subcategory

$${}_{\mathrm{St}^*(Y)}Y \subseteq {}_G X.$$

As mentioned above, we wish to determine for which subsets $Y \subseteq X$ there exists $H \leq G$ such that ${}_H Y$ is a full subcategory of ${}_G X$. Note that ${}_H Y$ is defined if and only if $H \subseteq \mathrm{St}(Y)$. Thus by Remark 4.6, such a subgroup exists if and only if ${}_{\mathrm{St}^*(Y)}Y \subseteq {}_G X$ is full. The following theorem, which is the main result of this section, determines when this holds.

Theorem 4.7. *Let $\alpha : G \times X \to X$ be a group action, and let $\emptyset \neq Y \subseteq X$. Then the following statements are equivalent.*

(i) The inclusion functor $\mathcal{I} : {}_{\mathrm{St}^*(Y)}Y \hookrightarrow {}_G X$ is full.
(ii) $G = \mathrm{St}^*(Y) \sqcup \mathrm{Dest}(Y)$.
(iii) $G = \mathrm{St}(Y) \sqcup \mathrm{Dest}(Y)$.
(iv) For any $g, h \in G$, either $g \cdot Y = h \cdot Y$ or $g \cdot Y \cap h \cdot Y = \emptyset$.
(v) The collection $\pi = \{g \cdot Y | g \in G\}$ is a partition of $G \cdot Y \subseteq X$.
(vi) The inclusion functor $\mathcal{I}' : {}_{\mathrm{St}^*(Y)}Y \hookrightarrow {}_G(G \cdot Y)$ is an equivalence of categories.

If any, hence all, of the above holds, then there is a bijection $\rho : G/\operatorname{St}^*(Y) \to \pi$ *between the left cosets of* $\operatorname{St}^*(Y)$ *and the classes of* π*, given by* $\overline{g} := g\operatorname{St}^*(Y) \mapsto g\cdot Y$*.*

Note that the two sets $\operatorname{St}(Y)$ and $\operatorname{Dest}(Y)$ are disjoint whenever $Y \neq \emptyset$. The principal statements in (ii) and (iii) are that the complement of the union is empty, which is true e.g. for stabilizers of singleton sets, but otherwise not in general (cf. Example 4.2).

Proof. (i) $\Longrightarrow$ (ii): Let $h \in G \setminus \operatorname{Dest}(Y)$. Then there exist $y, y' \in Y$ with $h \cdot y = y'$, whence $(h, y, y') \in {}_GX(y, y')$. Since $\mathcal{I}$ is full we then have $(h, y, y') \in {}_{\operatorname{St}^*(Y)}Y(y, y')$, and hence $h \in \operatorname{St}^*(Y)$.

(ii) $\Longrightarrow$ (i): Take any $y, y' \in Y$ and $(h, y, y') \in {}_GX(y, y')$. Then $h \cdot y = y' \in Y$, whence h does not belong to $\operatorname{Dest}(Y)$. Hence $h \in \operatorname{St}^*(Y)$ by hypothesis, which implies that $(h, y, y') = \mathcal{I}(h, y, y')$, and thus $\mathcal{I}$ is full.

(ii) $\Longrightarrow$ (iii): Take $g \in G \setminus \operatorname{Dest}(Y)$. Then by hypothesis $g \in \operatorname{St}^*(Y) \subseteq \operatorname{St}(Y)$.

(iii) $\Longrightarrow$ (ii): Take $h \in \operatorname{St}(Y)$. Then for any $y \in Y$ we have $y' := h \cdot y \in Y$. But then $h^{-1} \cdot y' = y$, and in particular h^{-1} does not belong to $\operatorname{Dest}(Y)$. Thus by hypothesis $h^{-1} \in \operatorname{St}(Y)$, and hence $h \in \operatorname{St}^*(Y)$. Since h was an arbitrary element of $\operatorname{St}(Y)$ this shows that $\operatorname{St}(Y) = \operatorname{St}^*(Y)$, whence $G = \operatorname{St}^*(Y) \sqcup \operatorname{Dest}(Y)$.

(ii) $\Longrightarrow$ (iv): Let $g, h \in G$ and assume that $g \cdot Y \cap h \cdot Y \neq \emptyset$. Then there exist $y, y' \in Y$ such that $g \cdot y = h \cdot y'$, and thus $h^{-1}g \cdot y = y'$. This excludes the possibility that $h^{-1}g \in \operatorname{Dest}(Y)$, and thus by assumption $h^{-1}g \in \operatorname{St}^*(Y)$. Thus for all $z \in Y$, $z' := h^{-1}g \cdot z \in Y$, implying $g \cdot z = h \cdot z' \in h \cdot Y$, whence $g \cdot Y \subseteq h \cdot Y$.

Since $\operatorname{St}^*(Y)$ is a group we also have $g^{-1}h = (h^{-1}g)^{-1} \in \operatorname{St}^*(Y)$, and repeating the above argument with g and h interchanged we get $h \cdot Y \subseteq g \cdot Y$ as well.

(iv) $\Longrightarrow$ (ii): Let $g \in G$ be arbitrary and set $h = e$, the identity element of G. Then by assumption either $g \cdot Y = Y$, which implies that $g \in \operatorname{St}^*(Y)$, or else $g \cdot Y \cap Y = \emptyset$ and $g \in \operatorname{Dest}(Y)$.

(iv) $\Longleftrightarrow$ (v): By definition of a partition, it follows that these are two reformulations of the same statement, since each $x \in G\cdot Y$ is contained in $g\cdot Y$ for some $g \in G$.

(i) $\Longleftrightarrow$ (vi): The functor $\mathcal{I}'$ is faithful, being an inclusion, and dense by definition of $G \cdot Y$. It is full if and only if for each $y, y' \in Y$ and each $g \in G$, $g \cdot y = y' \Rightarrow g \in \operatorname{St}^*(Y)$. This is equivalent to $\mathcal{I}$ being full.

Finally, we consider the map ρ. To begin with, if $\overline{g} = \overline{h}$, then there exists $j \in \operatorname{St}^*(Y)$ such that $h = gj$. Then the definition of $\operatorname{St}^*(Y)$ implies that $j \cdot Y = Y$, whence $h \cdot Y = g \cdot (j \cdot Y) = g \cdot Y$. Thus ρ is well-defined. It is surjective since for each class C of π there exists, by definition, an element $g \in G$ such that $C = g \cdot Y$, and hence $C = \rho(\overline{g})$. To show injectivity, assume $\rho(\overline{g}) = \rho(\overline{h})$, i.e $g \cdot Y = h \cdot Y$, for some $g, h \in G$, implying that $Y = g^{-1}h \cdot Y$. Thus $g^{-1}h \in \operatorname{St}^*(Y)$, and since $h = g(g^{-1}h)$, we obtain $\overline{g} = \overline{h}$, which completes the proof. □

Definition 4.8. Let $\alpha : G \times X \to X$ be a group action. A subset $Y \subseteq X$ is called *full (with respect to* α*)* if it is non-empty and satisfies the equivalent conditions of Theorem 4.7.

We observe the following from the above proof.

Corollary 4.9. *Let $\alpha : G \times X \to X$ be a group action, and let $Y \subseteq X$. If Y is full, then* $\mathrm{St}^*(Y) = \mathrm{St}(Y)$.

The converse is in general false, as the following example shows.

Example 4.10. Consider the action of $(\mathbb{Z}, +)$ on itself by addition. For $Y = \{0, 1\}$ we have $\mathrm{St}^*(Y) = \mathrm{St}(Y) = \{0\}$, but Y is not full, since $1 \notin \mathrm{St}(Y) \cup \mathrm{Dest}(Y)$.

Note, however, that singleton subsets are always full.

In case $\mathrm{St}^*(Y)$ is a normal subgroup of G, the bijection ρ of Theorem 4.7 induces a group structure on π (where the identity element is the class Y). We will however not need this in the sequel.

4.3. **Consequences for the Structure of ${}_GX$.** We now derive some insight into the structure of the groupoid ${}_GX$ from the above theorem, starting with a basic observation.

Lemma 4.11. *Let $\alpha : G \times X \to X$ be a group action, and let $\emptyset \neq Y \subseteq X$. For each $g \in G$, there is a bijection*

$$\lambda_g : Y \to g \cdot Y, \quad y \mapsto g \cdot y$$

and a group isomorphism

$$\kappa_g : \mathrm{St}^*(Y) \to \mathrm{St}^*(g \cdot Y), \quad h \mapsto ghg^{-1}$$

such that for each $y \in Y$ and each $j \in \mathrm{St}^*(Y)$,

$$\kappa_g(j) \cdot \lambda_g(y) = \lambda_g(j \cdot y).$$

Proof. Let $g \in G$. By definition of $g \cdot Y$, λ_g is well-defined and surjective, and it is injective by the axioms of a group action (with inverse $\lambda_{g^{-1}}$).

As for κ_g, we have, for any $j \in \mathrm{St}^*(Y)$, that

$$\kappa_g(j) \cdot (g \cdot Y) = gjg^{-1} \cdot (g \cdot Y) = gj \cdot Y = g \cdot Y,$$

whence κ_g is well-defined. It is then a homomorphism of groups, with inverse homomorphism $\kappa_g^{-1} = \kappa_{g^{-1}} : \mathrm{St}^*(g \cdot Y) \to \mathrm{St}^*(Y)$.

To prove the final statement we note that for any $y \in Y$ and any $j \in \mathrm{St}^*(Y)$,

$$\kappa_g(j) \cdot \lambda_g(y) = gjg^{-1} \cdot (g \cdot y) = gj \cdot y = \lambda_g(j \cdot y),$$

and the proof is complete. □

Remark 4.12. The above lemma may be formulated in the language of *isomorphisms of group actions*. Given two group actions $\alpha_1 : G_1 \times S_1 \to S_1$ and $\alpha_2 : G_2 \times S_2 \to S_2$, a homomorphism $\Theta : \alpha_1 \to \alpha_2$ is a pair (Σ, Γ) where $\Sigma : S_1 \to S_2$ is a function, and $\Gamma : G_1 \to G_2$ is a group homomorphism, such that the following diagram

$$\begin{array}{ccc} G_1 \times S_1 & \xrightarrow{\alpha_1} & S_1 \\ \downarrow{\scriptstyle \Gamma \times \Sigma} & & \downarrow{\scriptstyle \Sigma} \\ G_2 \times S_2 & \xrightarrow{\alpha_2} & S_2 \end{array}$$

commutes. Group actions then form a category, where the morphisms are the homomorphisms of group actions. Lemma 4.11 thus states that for each $g \in G$, the pair (λ_g, κ_g) is an isomorphism of group actions from the corestriction of α to Y to the corestriction of α to $g \cdot Y$.

On the level of groups, Lemma 4.11 has the following corollary.

Corollary 4.13. *Let $\alpha : G \times X \to X$ be a group action, and let $Y \subseteq X$ be full. Then for any $g \in G$, the following holds.*

(i) $G = \mathrm{St}^*(g \cdot Y) \sqcup \mathrm{Dest}(g \cdot Y)$.

(ii) The subgroups $\mathrm{St}^*(Y)$ *and* $\mathrm{St}^*(g \cdot Y)$ *of* G *are conjugate; more precisely*

$$\mathrm{St}^*(g \cdot Y) = g\,\mathrm{St}^*(Y)g^{-1}.$$

Proof. To prove (i), let $h \in G \setminus \mathrm{Dest}(g \cdot Y)$. Then there exist $y, y' \in Y$ such that $hg \cdot y = g \cdot y'$, which implies that $g^{-1}hg \cdot y = y'$. Thus $\kappa_g^{-1}(h) \notin \mathrm{Dest}(Y)$, whence $\kappa_g^{-1}(h) \in \mathrm{St}^*(Y)$ and $h \in \mathrm{St}^*(g \cdot Y)$ by Lemma 4.11. (ii) is a reformulation of the bijectivity of κ_g. □

From item (ii) it follows that for any $g, h \in G$,

$$\mathrm{St}^*(h \cdot Y) = (hg^{-1})\,\mathrm{St}^*(g \cdot Y)(hg^{-1})^{-1}.$$

This is used in the following corollary, on the level of groupoids.

Corollary 4.14. *Let $\alpha : G \times X \to X$ be a group action, and let $Y \subseteq X$ be full. Then for any $g, h \in G$, the following holds.*

(i) The category ${}_{\mathrm{St}^*(g\cdot Y)}(g \cdot Y)$ *is a full subcategory of* ${}_G X$.

(ii) The functor $\mathcal{T}_{hg} : {}_{\mathrm{St}^*(g\cdot Y)}(g \cdot Y) \to {}_{\mathrm{St}^*(h\cdot Y)}(h \cdot Y)$, *defined on objects and morphisms by*

$$\mathcal{T}_{hg}(x) = \lambda_{hg^{-1}}(x), \quad \mathcal{T}_{hg}(k) = \kappa_{hg^{-1}}(k),$$

respectively, is an isomorphism of categories.

Proof. (i) follows from item (i) of Corollary 4.13 together with Theorem 4.7. For (ii), Lemma 4.11 implies that $\mathcal{T}_{hg}$ is indeed a functor for each $g, h \in G$. To show that for each $g, h \in G$, $\mathcal{T}_{gh}\mathcal{T}_{hg}$ is the identity functor on ${}_{\mathrm{St}^*(g\cdot Y)}g \cdot Y$, take any $x \in g \cdot Y$. Then

$$\mathcal{T}_{gh}\mathcal{T}_{hg}(x) = \lambda_{gh^{-1}}\lambda_{hg^{-1}}(x) = gh^{-1}hg^{-1} \cdot x = x,$$

and for each morphism $k \in \mathrm{St}^*(g \cdot Y)$,

$$\mathcal{T}_{gh}\mathcal{T}_{hg}(k) = \kappa_{gh^{-1}}\kappa_{hg^{-1}}(k) = gh^{-1}hg^{-1}kgh^{-1}hg^{-1} = k.$$

This completes the proof. □

Thus for each $g \in G$, the full subcategory of ${}_G X$ with object set $g\cdot Y$ is isomorphic to ${}_{\mathrm{St}^*(Y)}Y$. In view of Remark 4.12, this can be expressed by saying that groupoids arising from isomorphic group actions are isomorphic.

5. Applications to $\mathcal{A}_8$

We now apply the above to the setting of Section 3. In the light of Section 4, Proposition 3.2 may then be restated, in terms of groups and group actions, as follows.

Proposition 5.1. *For each $x \in \{l, r, c\}$, let $Y^x = \{\mathcal{G}^x(f) | f \in O_8^1\}$. With respect to the triality action,*

$$\mathrm{St}^*(Y^x) = \mathrm{St}(Y^x) = G_2$$

and $SO_8 = G_2 \sqcup \mathrm{Dest}(Y^x)$.

The definition of the functor $\mathcal{G}^x : {}_{G_2}O_8^1 \to {}_{SO_8}\mathcal{O}_8$, $x \in \{l, r, c\}$, was given in Proposition 3.2.

Proof. Let $x \in \{l, r, c\}$. Then $G_2 \subseteq \mathrm{St}(Y^x)$ by definition of $\mathcal{G}^x$. If $\phi \in SO_8$ stabilizes Y^x, then for all $f \in O_8^1$ there exists $f' \in O_8^1$ such that $\phi(\mathcal{G}^x(f)) = \mathcal{G}^x(f')$. Now fullness of $\mathcal{G}^x$ implies that $\phi \in G_2$. This implies *a fortiori* that $\mathrm{St}(Y^x) \subseteq G_2$. Altogether $G_2 = \mathrm{St}(Y^x)$, whence $G_2 = \mathrm{St}^*(Y^x)$, being a group. The dichotomy of SO_8 then follows from Theorem 4.7 as $\mathcal{G}^x$ is full, and the proof is complete. □

Our aim is now to use the methods of Section 4 in order to extend the understanding of $\mathcal{A}_8$ beyond $\mathcal{A}_8^l$, $\mathcal{A}_8^r$ and $\mathcal{A}_8^c$. We choose $\mathcal{A}_8^l$ as a point of depart for this extension. ($\mathcal{A}_8^r$ would have been an equally suitable choice, while $\mathcal{A}_8^c$ would have caused some difficulties, at least computationally, as the reader may see in the upcoming sections.)

We have seen that the category $\mathcal{A}_8^l$ is equivalent to the full subcategory of ${}_{SO_8}\mathcal{O}_8$ whose object set is

$$Y^l = \{[h, \mathrm{Id}] | h \in O_8^1\}. \tag{5.1}$$

Before continuing, we determine the set $SO_8 \cdot Y^l$ of all $[f, g] \in \mathcal{O}_8$ that are isomorphic to some element of Y^l, i.e. such that $\mathbb{O}_{f,g} \in \mathcal{A}_8^l$. Theorem 4.7 then provides additional structural information by describing how this set is partitioned into certain classes.

Proposition 5.2. *Let $f, g \in O_8$, and denote by $\cdot$ the action of SO_8 on $\mathcal{O}_8$ by triality.*

(i) $[f, g] \in SO_8 \cdot Y^l$ if and only if $g = L_a$ for some $a \in \mathbb{S}(\mathbb{O})$.

(ii) The set

$$\{[f, L_a] \in \mathcal{O}_8 | f \in O_8, a \in \mathbb{S}(\mathbb{O})\}$$

is partitioned into classes $\phi \cdot Y^l, \phi \in SO_8$.

(iii) For any $f, f' \in O_8$ and $a, a' \in \mathbb{S}(\mathbb{O})$, $[f, L_a]$ is in the same class as $[f', L_{a'}]$ if and only if $(a, f^{-1}(a^{-1})) = (\pm a', f'^{-1}(a'^{-1}))$.

The full subcategories generated by the classes of this partition are, by Corollary 4.14, all isomorphic to ${}_{G_2}Y^l$.

Proof. If $[f, g] \in SO_8 \cdot Y^l$, then there exist $\phi \in SO_8$ and $h \in O_8^1$ such that

$$[f, g] = [\phi_1 h \phi^{-1}, \phi_2 \,\mathrm{Id}\, \phi^{-1}] = [R_{\phi_2(1)^{-1}} \phi h \phi^{-1}, L_{\phi_1(1)^{-1}} \phi \,\mathrm{Id}\, \phi^{-1}]$$

whence $g = L_a$ for some $a \in \{\pm \phi_1(1)^{-1}\}$.

Conversely, assume that $g = L_a$ for some $a \in \mathbb{S}(\mathbb{O})$. Setting $u = f^{-1}(a^{-1})$ and denoting multiplication in $\mathbb{O}_{f,g}$ by $*$, we have, for each $x \in \mathbb{O}_{f,g}$, that

$$u * x = f(u) L_a(x) = f(f^{-1}(a^{-1})) L_a(x) = a^{-1}(ax) = x,$$

where juxtaposition denotes multiplication in $\mathbb{O}$, and the last equality holds since $\mathbb{O} \setminus \{0\}$ is a Moufang loop.[7] Thus u is a left unity, and $\mathbb{O}_{f,g} \in \mathcal{A}_8^l$, implying that $[f, g] \in SO_8 \cdot Y^l$ by Theorem 2.3. This proves (i), from which (ii) follows by Theorem 4.7 and Proposition 5.1.

To prove (iii), take $f \in O_8$ and $a \in \mathbb{S}(\mathbb{O})$. Then there exist $\phi \in SO_8$ and $h \in O_8^1$ such that $[f, L_a] = \phi \cdot [h, \mathrm{Id}]$. Fix a triality pair (ϕ_1, ϕ_2) of ϕ. Then

$$f = \epsilon \phi_1 h \phi^{-1} \quad \text{and} \quad L_a = \epsilon \phi_2 \phi^{-1} = \epsilon L_{\phi_1(1)^{-1}}$$

[7]Indeed, for each Moufang loop M we have $y^{-1}(yz) = z = (zy)y^{-1}$ for all $y, z \in M$.

for some $\epsilon \in C_2$, which is equivalent to

$$\phi = \epsilon f^{-1}\phi_1 h \quad \text{and} \quad \phi_1(1) = \epsilon a^{-1}. \tag{5.2}$$

Take now $f' \in O_8$ and $a' \in \mathbb{S}(\mathbb{O})$.

If $[f', L_{a'}]$ is in the same class as $[f, L_a]$, then as in (5.2) we have

$$\phi = \epsilon' f'^{-1}\phi_1 h' \quad \text{and} \quad \phi_1(1) = \epsilon' a'^{-1}$$

for some $h' \in O_8^1$ and $\epsilon' \in C_2$. We thus have $a' = \pm a$ and $\phi_1 h'(1) = \epsilon' f'\phi(1)$, and using the expression of ϕ in (5.2), we get

$$\phi_1 h'(1) = \epsilon\epsilon' f' f^{-1}\phi_1 h(1) = \epsilon\epsilon' f' f^{-1}\phi_1(1) = \epsilon' f' f^{-1}(a^{-1}).$$

As $h'(1) = 1$, the left hand side is $\epsilon' a'^{-1}$. Applying f'^{-1} to both sides we get

$$f^{-1}(a^{-1}) = f'^{-1}(a'^{-1}).$$

Conversely, assume that $(a, f^{-1}(a^{-1})) = (\epsilon'' a', f'^{-1}(a'^{-1}))$ for some $\epsilon'' \in C_2$. Then (5.2) implies that $\epsilon\epsilon'' a'^{-1} = \phi_1(1)$. Furthermore, $h' := \epsilon\epsilon''\phi_1^{-1} f'\phi \in O_8^1$, since, by (5.2) and the fact that $h \in O_8^1$,

$$h'(1) = \epsilon\epsilon''\phi_1^{-1} f'\phi(1) = \epsilon''\phi_1^{-1} f' f^{-1}\phi_1 h(1) = \epsilon''\phi_1^{-1} f' f^{-1}(\epsilon a^{-1})$$

and by assumption this is equal to

$$\epsilon''\phi_1^{-1} f' f'^{-1}(\epsilon a'^{-1}) = \phi_1^{-1}(\epsilon\epsilon'' a'^{-1}) = 1.$$

Thus

$$[f', L_{a'}] = [\epsilon\epsilon''\phi_1 h'\phi^{-1}, \epsilon\epsilon'' L_{\phi_1(1)^{-1}}] = [\phi_1 f'\phi^{-1}, L_{\phi_1(1)^{-1}}] \in \phi \cdot Y^l,$$

i.e. $[f', L_{a'}]$ is in the same class as $[f, L_a]$, and the proof is complete. □

6. Left Reflection Algebras

6.1. **Preliminaries.** We now introduce a new class of algebras in $\mathcal{A}_8$, and apply the above framework to it. First is a notational definition.

Definition 6.1. Let V be a Euclidean space and $U \subseteq V$ a subspace. The linear operator $\sigma_U : V \to V$ is defined as reflection in the subspace $U^\perp$, i.e. by

$$\sigma_U(v) = \begin{cases} v & \text{if } v \perp U, \\ -v & \text{if } v \in U. \end{cases}$$

Note that $\sigma_U = \sigma_U^{-1}$, being a symmetric orthogonal operator. In this section we only consider cases where $U = \mathbb{R}u$ for some $u \in V$, in which case we write σ_u instead of $\sigma_{\mathbb{R}u}$ to denote the reflection in the hyperplane $u^\perp$. We note the following basic property.

Lemma 6.2. *For each $u \in \mathbb{S}(\mathbb{O})$ and each $\phi \in SO_8$, $\phi\sigma_u\phi^{-1} = \sigma_{\phi(u)}$.*

Proof. Take any $v \in \mathbb{O}$. If $v = \mu\phi(u)$ for some $\mu \in \mathbb{R}$, then

$$\phi\sigma_u\phi^{-1}(v) = \mu\phi\sigma_u(u) = \mu\phi(-u) = -v,$$

and if $v \perp \phi(u)$, then $\phi^{-1}(v) \perp u$, whence $\sigma_u\phi^{-1}(v) = \phi^{-1}(v)$, and $\phi\sigma_u\phi^{-1}(v) = v$, proving the claim. □

Hyperplane reflections define a class of algebras as follows.

Definition 6.3. Let $u \in \mathbb{S}(\Im\mathbb{O})$. An algebra $\mathbb{O}_{f,g} \in \mathcal{A}_8$ is a *left u-reflection algebra* if $f \in O_8^1$ and $g = \sigma_u$. The full subcategory of $\mathcal{A}_8$ whose objects are all left u-reflection algebras is denoted by $\mathcal{A}_8^u$.

An algebra is called a *left reflection algebra* if it is a left u-reflection algebra for some $u \in \mathbb{S}(\Im\mathbb{O})$. We denote the set of all left reflection algebras by $\mathcal{A}_8^{\mathbb{S}(\Im\mathbb{O})}$. Right reflection algebras may be defined analogously, but will not be used here.

Remark 6.4. The terminology is in analogy with that for algebras with a left unity. Indeed, for each $u \in \mathbb{S}(\Im\mathbb{O})$ and each $\mathbb{O}_{f,g} \in \mathcal{A}_8^u$, the operator σ_u is left multiplication by the element 1.

As regards the class of all absolute valued algebras isomorphic to a left reflection algebra, Dieterich (personal communication, November 2012) has made the following observation.

Proposition 6.5. *The set $\mathcal{A}_8^{\mathbb{S}(\Im\mathbb{O})}$ of all left reflection algebras is dense in the full subcategory*

$$\mathcal{A}_8^R = \{A \in \mathcal{A}_8 | L_e \text{ is a reflection for some idempotent } e \in A\}$$

of $\mathcal{A}_8$. Moreover, $\mathcal{A}_8^R$ is closed under isomorphisms in $\mathcal{A}_8$.

Thus the objects in $\mathcal{A}_8^R$ are precisely those absolute valued algebras which are isomorphic to a left u-reflection algebra for some $u \in \mathbb{S}(\Im\mathbb{O})$.

Proof. If $A \in \mathcal{A}_8^{\mathbb{S}(\Im\mathbb{O})}$, then $A \in \mathcal{A}_8^u$ for some $u \in \mathbb{S}(\Im\mathbb{O})$, and $L_1 = \sigma_u$ by Remark 6.4. Moreover, 1 is idempotent in A since it is fixed by f and σ_u. Therefore $A \in \mathcal{A}_8^R$.

Conversely, assume that $A \in \mathcal{A}_8^R$ and let $e \in A$ be an idempotent satisfying $L_e = \sigma_v$ with $v \in \mathbb{S}(\mathbb{O})$. Consider the isotope $A_{R_e^{-1},L_e^{-1}}$ of A. This is a unital algebra with unit element e, and thus there is an isomorphism

$$\phi : A_{R_e^{-1},L_e^{-1}} \to \mathbb{O}$$

with $\phi(e) = 1$. By Proposition 4.1 in [4], the map

$$\phi : A \to \mathbb{O}_{\phi R_e \phi^{-1}, \phi L_e \phi^{-1}}$$

is then an isomorphism. Furthermore,

$$\mathbb{O}_{\phi R_e \phi^{-1}, \phi L_e \phi^{-1}} = \mathbb{O}_{\phi R_e \phi^{-1}, \phi \sigma_v \phi^{-1}} = \mathbb{O}_{f, \sigma_{\phi(v)}}$$

by Lemma 6.2 and the fact that $\phi \in SO_8$, and with $f = \phi R_e \phi^{-1}$. Now

$$f(1) = \phi R_e \phi^{-1}(1) = \phi R_e(e) = \phi(e^2) = 1$$

and $\phi(v) \in \mathbb{S}(\Im\mathbb{O})$ as $\sigma_v(e) = L_e(e) = e$ implies that $e \perp v$, whence $1 = \phi(e) \perp \phi(v)$. Thus $A \simeq \mathbb{O}_{f,\sigma_{\phi(v)}} \in \mathcal{A}_8^{\mathbb{S}(\Im\mathbb{O})}$.

Finally, if $B \in \mathcal{A}_8$ and $\psi : A \to B$ is an isomorphism, then $\psi(e)$ is idempotent in B, and $L_{\psi(e)} = \sigma_{\psi(v)}$. Thus $\mathcal{A}_8^R$ is closed under isomorphisms in $\mathcal{A}_8$. □

Left reflection algebras have no left unity, as made precise by the following result.

Proposition 6.6. *For any $u \in \mathbb{S}(\Im\mathbb{O})$, there exist no $A \in \mathcal{A}_8^u$ and $B \in \mathcal{A}_8^l$ such that $A \simeq B$.*

Proof. By Remark 6.4, left multiplication by 1 in A has determinant -1, while in B left multiplication by the left unit has determinant 1. By Proposition 1.2, A and B are therefore non-isomorphic. □

We set, for each $u \in \mathbb{S}(\Im\mathbb{O})$,

$$Y^u = \{[f,g] \in \mathcal{O}_8 | \mathbb{O}_{f,g} \in \mathcal{A}_8^u\} = \{[f,g] \in \mathcal{O}_8 | f \in O_8^1, g = \sigma_u\}.$$

This set has the following properties.

Proposition 6.7. *The triality action satisfies the following for each $u \in \mathbb{S}(\Im\mathbb{O})$:*

(i) $\mathrm{St}(Y^u) = G_2^u := \{\phi \in G_2 | \phi([u]) = [u]\} = \{\phi \in G_2 | \phi(u) = \pm u\}$,
(ii) $\mathrm{St}^*(Y^u) = \mathrm{St}(Y^u)$,
(iii) $SO_8 = \mathrm{St}(Y^u) \sqcup \mathrm{Dest}(Y^u)$, *and*
(iv) ${}_{G_2^u}Y^u$ *is a full subcategory of* ${}_{SO_8}\mathcal{O}_8$.

Proof. The set G_2^u is a subgroup of G_2. To prove (i), note that $\phi \in \mathrm{St}(Y^u)$ if and only if there exists a triality pair (ϕ_1, ϕ_2) satisfying the two conditions

$$\phi_1 f \phi^{-1} \in O_8^1, \quad \phi_2 \sigma_u \phi^{-1} = \sigma_u \tag{6.1}$$

for any $f \in O_8^1$. If $\phi \in G_2^u \leq G_2 \leq O_8^1$, then (ϕ, ϕ) is a triality pair satisfying the first condition and, as a simple computation shows, the second as well, whence $\phi \in \mathrm{St}(Y^u)$.

Conversely, assume that (ϕ_1, ϕ_2) is a triality pair satisfying the two conditions in (6.1) for *some* $f \in O_8^1$. (We will only need the existence of one $f \in O_8^1$ such that the conditions are satisfied.) The second condition implies that

$$L_{\phi_1(1)^{-1}} \phi \sigma_u \phi^{-1} = \sigma_u,$$

i.e. $\phi \sigma_u \phi^{-1} \sigma_u = L_{\phi_1(1)}$, and by Lemma 6.2 we then have $\sigma_{\phi(u)} \sigma_u = L_{\phi_1(1)}$. Thus $L_{\phi_1(1)}$ fixes any $x \in [u, \phi(u)]^\perp$, whence $\phi_1(1) = 1$ must hold since $\mathbb{O}$ is a unital division algebra. Hence $\sigma_{\phi(u)} = \sigma_u$ and $\phi(u) = \pm u$. Moreover, $\phi_1(1) = 1$ together with the first condition yields $\phi(1) = 1$. Thus $\phi \in G_2$, and then $\phi \in G_2^u$.

Statement (ii) follows from the fact that G_2^u is a group. As for (iii), assume that $\phi \notin \mathrm{Dest}(Y^u)$. Then there exist $f, f' \in O_8^1$ such that $\phi \cdot [f, \sigma_u] = [f', \sigma_u]$, which implies that the conditions in (6.1) hold for some triality pair of ϕ. By the previous paragraph we get $\phi \in G_2^u$. Finally, (iv) is equivalent to (iii) by Theorem 4.7. □

As Y^u satisfies the equivalent conditions of Theorem 4.7, we may apply the results of Section 4 to it. To begin with, we obtain the following.

Corollary 6.8. *Let $u \in \mathbb{S}(\Im\mathbb{O})$. The functors*

$$\mathcal{G}^u : {}_{G_2^u}O_8^1 \to {}_{G_2^u}Y^u, \quad \mathcal{F}|_{Y^u} : {}_{G_2^u}Y^u \to \mathcal{A}_8^u,$$

where $\mathcal{G}^u$ is defined on objects by $\mathcal{G}^u(f) = [f, \sigma_u]$ and on morphisms by $\mathcal{G}^u(\phi) = \phi$, are equivalences of categories.

Proof. Both functors are well-defined and clearly faithful. Moreover, $\mathcal{G}^u$ is dense by the definition of Y^u and full by construction, while $\mathcal{F}|_{Y^u}$ is dense by the definition of $\mathcal{A}_8^u$ and full by fullness of $\mathcal{F}$ and of ${}_{G_2^u}Y^u$ in ${}_{SO_8}\mathcal{O}_8$. □

For any $u \in \mathbb{S}(\Im\mathbb{O})$, Y^u determines the set $SO_8 \cdot Y^u$ of all $[f,g] \in \mathcal{O}_8$ such that $\mathbb{O}_{f,g}$ is isomorphic to a left u-reflection algebra, i.e. such that $\mathbb{O}_{f,g} \in \mathcal{A}_8^R$. By Theorem 4.7 and Corollary 4.14 we know that $SO_8 \cdot Y^u$ is partitioned into sets generating full subcategories each isomorphic to ${}_{G_2^u}Y^u$. While morphisms in ${}_{G_2^u}Y^u$ have trivial triality components, computing $SO_8 \cdot Y^u$ explicitly does involve triality. Instead, we consider the set $G_2 \cdot Y^u$, in view of the chain of subgroups $G_2^u \leq G_2 \leq SO_8$ and the fact that each $\phi \in G_2$ has trivial triality components.

The following proposition contains an explicit description of $G_2 \cdot Y^u$ and its partition into sets generating pairwise isomorphic subcategories. A motivation to study this set is given in the subsequent remark.

Proposition 6.9. *Let* $u \in \mathbb{S}(\Im\mathbb{O})$.

(i) Under the left action of G_2 *on* O_8^1 *by conjugation,* $\mathrm{St}(\sigma_u) = G_2^u$. *Moreover,* $G_2 = G_2^u \sqcup \mathrm{Dest}(\sigma_u)$ *and* $G_2 \cdot \sigma_u = \{\sigma_{u'} | u' \in \mathbb{S}(\Im\mathbb{O})\}$, *with* $\phi \cdot \sigma_u = \psi \cdot \sigma_u$ *if and only if* $[\phi(u)] = [\psi(u)]$.

(ii) Under the triality action,

$$G_2 \cdot Y^u = \bigcup_{u' \in \mathbb{S}(\Im\mathbb{O})} Y^{u'},$$

and $\phi \cdot Y^u = \psi \cdot Y^u \iff [\phi(u)] = [\psi(u)]$.

Thus $G_2 \cdot Y^u = \{[f,g] \in \mathcal{O}_8 | \mathbb{O}_{f,g} \in \mathcal{A}_8^{\mathbb{S}(\Im\mathbb{O})}\}$.

Proof. For (i), the stabilizer is obtained from Lemma 6.2, as $\sigma_{\phi(u)} = \sigma_u$ if and only if $\phi(u) = \pm u$. The decomposition of G_2 holds as $\{\sigma_u\}$ contains precisely one element, and the inclusion $G_2 \cdot \sigma_u \subseteq \{\sigma_{u'} | u' \in \mathbb{S}(\Im\mathbb{O})\}$ holds by Lemma 6.2. Moreover, for any $u' \in \mathbb{S}(\Im\mathbb{O})$ there are $v, v', z, z' \in \mathbb{S}(\Im\mathbb{O})$ such that (u,v,z) and (u',v',z') are Cayley triples. Thus there exists for each $u' \in \mathbb{S}(\Im\mathbb{O})$ a map $\phi \in G_2$ such that $\phi(u) = u'$, and by Lemma 6.2, $\phi \cdot \sigma_u = \sigma_{u'}$. Since u' was arbitrary, this proves the inverse inclusion. Moreover, $\sigma_{\phi(u)} = \sigma_{\psi(u)}$ holds if and only if $\phi(u) = \pm\psi(u)$, which proves the equivalence.

For (ii), the triality action of $G_2 \leq SO_8$ on Y^u is simultaneous conjugation in the sense that for all $\phi \in G_2$,

$$\phi \cdot [f, \sigma_u] = [\phi f \phi^{-1}, \phi \sigma_u \phi^{-1}] = [\phi f \phi^{-1}, \sigma_{\phi(u)}].$$

Thus $G_2 \cdot Y^u \subseteq \bigcup Y^{u'}$. Conversely, for every $u' \in \mathbb{S}(\Im\mathbb{O})$ and $[f, \sigma_{u'}] \in Y^{u'}$ there exists, by (i), $\phi \in G_2$ such that $\phi \cdot \sigma_u = \sigma_{u'}$, and then $f = \phi \cdot (\phi^{-1} f \phi)$. Thus the two sets are equal. The equivalence follows from that in (i). □

Remark 6.10. In view of the comment concluding Section 2, to classify $\mathcal{A}_8$ it suffices to consider such $[f,g] \in \mathcal{O}_8$ where $f, g \in O_8^1$. By the Cartan–Dieudonné Theorem, each element in O_8^1 can be written as a product of at most 7 reflections. Thus Y^l contains all such $[f,g]$ where g is the empty product of reflections, and for any $u \in \mathbb{S}(\Im\mathbb{O})$, Y^u exhausts, by the above proposition, all such $[f,g]$ where g is the product of precisely one reflection.

This is one motivation for attempting to classify all left reflection algebras, and Proposition 6.9 reduces this to the classification problem for the set of left u-reflection algebras for a fixed $u \in \mathbb{S}(\Im\mathbb{O})$.

To summarize the pattern we have followed, given the triality action, we first determined a set of algebras in $\mathcal{A}_8$ corresponding to a full subset Y^u of $\mathcal{O}_8$, then formed the groupoid arising from the corestriction of the triality action to Y^u, and finally constructed a larger set of algebras whose classification reduces to that of the smaller set via the equivalence of categories in Theorem 4.7(vii). We thus have the following commutative diagram of groupoids and full functors, where the $\sim$-labeled arrows are moreover equivalences of categories, and the vertical arrows are inclusions.

$$\begin{array}{ccccc}
 & & {}_{SO_8}\mathcal{O}_8 & \xrightarrow{\sim} & \mathcal{A}_8 \\
 & & \uparrow & & \uparrow \\
 & & {}_{SO_8}(SO_8 \cdot Y^u) & \xrightarrow{\sim} & \mathcal{A}_8^R \\
 & & \wr\uparrow & & \wr\uparrow \\
 & & {}_{G_2}(G_2 \cdot Y^u) & \xrightarrow{\sim} & \mathcal{A}_8^{\mathbb{S}(\Im\mathbb{O})} \\
 & & \wr\uparrow & & \wr\uparrow \\
{}_{G_2^u}O_8^1 & \xrightarrow{\sim} & {}_{G_2^u}Y^u & \xrightarrow{\sim} & \mathcal{A}_8^u
\end{array}$$

6.2. **Reduction of the Classification Problem.** In order to classify all left reflection algebras, it remains, by Corollary 6.8, to solve the classification problem for ${}_{G_2^u}O_8^1$. First we define some group actions to be used below.[8]

Definition 6.11. For any $u \in \mathbb{S}(\Im\mathbb{O})$ and $e \in O_8^1$, the group actions β, γ, γ_u and δ_e are defined by

$$\begin{array}{rll}
\beta: & G_2 \times G_2^u \to G_2, & (\phi, \psi) \mapsto \phi\psi, \\
\gamma: & O_8^1 \times G_2 \to O_8^1, & (f, \phi) \mapsto \kappa_\phi^{-1}(f) = \phi^{-1} f \phi, \\
\gamma_u: & O_8^1 \times G_2^u \to O_8^1, & (f, \psi) \mapsto \kappa_\psi^{-1}(f), \\
\delta_e: & G_2 \times (\mathrm{St}_\gamma(e) \times G_2^u) \to G_2, & (\phi, (\chi, \psi)) \mapsto \chi^{-1}\phi\psi.
\end{array}$$

We will now deal with the classification problem. By Corollary 6.8 , this amounts to solving the normal form problem for the group action γ_u. The normal form problem for γ was solved in [7]. Since $G_2^u \leq G_2$, each γ_u-orbit is contained in a γ-orbit. In this sense, the problem at hand (properly) contains the problem solved in [7], and therefore one may ask if it is possible to use this solution in order to simplify the present problem. This is indeed the case, and the details are given in the following theorem, proven by Dieterich for the actions of Definition 6.11 (personal communication, April 2012). We present here a straight-forward generalization.

Theorem 6.12. *Let F, G and H be groups such that $H \leq G \leq F$, and let $e \in F$. Define the group actions*

$$\begin{array}{rll}
\hat\beta: & G \times H \to G, & (g, h) \mapsto gh, \\
\hat\gamma: & F \times G \to F, & (f, g) \mapsto \kappa_g^{-1}(f) = g^{-1} f g, \\
\hat\gamma_H: & F \times H \to F, & (f, h) \mapsto \kappa_h^{-1}(f), \\
\hat\delta_e: & G \times (\mathrm{St}_{\hat\gamma}(e) \times H) \to G, & (g, (s, h)) \mapsto s^{-1} g h,
\end{array}$$

and let $\hat B \subseteq G$ and $\hat C \subseteq F$ be cross-sections for $\hat\beta$ and $\hat\gamma$, respectively. Then

- *(i) the set $\{\kappa_g^{-1}(f) | g \in \hat B, f \in \hat C,\}$ exhausts the orbits of $\hat\gamma_H$,*
- *(ii) for any $f, f' \in \hat C$ and $g, g' \in \hat B$, $\kappa_g^{-1}(f) \equiv \kappa_{g'}^{-1}(f')$ with respect to $\hat\gamma_H$ if and only if $f' = f$ and $g \equiv g'$ with respect to $\hat\delta_f$, and*
- *(iii) a cross-section for $\hat\gamma_H$ is given by*

$$\bigsqcup_{f \in \hat C} \{\kappa_g^{-1}(f) | g \in \hat B \cap D_f\}$$

[8]The reason that the groups act from the right is that the orbits of β are left cosets, which, as we will see, is required for Theorem 4.7 to be applied directly.

where for each $f \in \hat{C}$, D_f is a cross-section of $\hat{\delta}_f$.

Proof. (i) Let $f' \in F$. Since $\hat{C}$ is a cross-section for $\hat{\gamma}$, there exist $f \in \hat{C}$ and $g' \in G$ such that $f' = \kappa_{g'}^{-1}(f)$. Since $\hat{B}$ is a cross-section for $\hat{\beta}$, there exist $h \in H$ and $g \in \hat{B}$ such that $g' = gh$. This proves (i), since then

$$f' = \kappa_{gh}^{-1}(f) = \kappa_h^{-1}(\kappa_g^{-1}(f)).$$

(ii) By definition, $\kappa_g^{-1}(f) \equiv_{\hat{\gamma}_H} \kappa_{g'}^{-1}(f')$ if and only if $\kappa_{g'}^{-1}(f') = \kappa_h^{-1}(\kappa_g^{-1}(f))$ for some $h \in H$, which is equivalent to

$$f' = \kappa_{ghg'^{-1}}^{-1}(f) \tag{6.2}$$

for some $h \in H$, whence $f \equiv_{\hat{\gamma}} f'$. But f and f' belong to the same cross-section of $\hat{\gamma}$, and hence coincide.

Thus (6.2) implies that $\kappa_g^{-1}(f) \equiv_{\hat{\gamma}_H} \kappa_{g'}^{-1}(f')$ if and only if $f' = f$ and there exists $h \in H$ such that $ghg'^{-1} \in \mathrm{St}_{\hat{\gamma}}(f)$. The latter statement is equivalent to $g' \in \mathrm{St}_{\hat{\gamma}}(f)gh$ for some $h \in H$. Hence

$$\kappa_g^{-1}(f) \equiv_{\hat{\gamma}_H} \kappa_{g'}^{-1}(f') \iff f' = f \wedge g' \in \mathrm{St}_{\hat{\gamma}}(f)gH,$$

which by definition of $\hat{\delta}_f$ proves (ii).

(iii) By (ii), for each $f \in \hat{C}$, the set $D_f \cap \hat{B}$ is a cross-section of $\{\kappa_g^{-1}(f) | g \in \hat{B}\}$. The claim follows since (ii) moreover implies that $\kappa_g^{-1}(f) \equiv_{\hat{\gamma}_H} \kappa_{g'}^{-1}(f')$ for some $g' \in \hat{B}$ and $f' \in \hat{C}$ only if $f' = f$.

□

We now return to the setting of Definition 6.11, where we apply the above theorem. For the remainder of this section, we fix $u \in \mathbb{S}(\Im\mathbb{O})$ and set

$$(F, G, H) = (O_8^1, G_2, G_2^u) \quad \text{and} \quad \forall e \in O_8^1, (\hat{\beta}, \hat{\gamma}, \hat{\gamma}_H, \hat{\delta}_e) = (\beta, \gamma, \gamma_u, \delta_e).$$

As a cross-section $C \subset O_8^1$ for γ, we will henceforth use the one obtained in [7], which we will partly recall in Section 7. What thus remains towards classifying left reflection algebras is to solve the normal form problem for β and that for δ_f for each $f \in C$. The solution to the first of these problems is obtained by an application of Theorem 4.7.

Proposition 6.13. *There is a bijection between the set G_2/G_2^u of orbits of β and $\mathbb{P}(\Im\mathbb{O})$, given by $\phi G_2^u \mapsto [\phi(u)]$ for all $\phi \in G_2$.*

Proof. We apply Theorem 4.7 to the left action of G_2 on O_8^1 by conjugation. By Proposition 6.9(i), there is thus a bijection ρ_1 from the set G_2/G_2^u of left cosets to the partition of $\{\sigma_{u'} | u' \in \mathbb{S}(\Im\mathbb{O})\}$ into its singleton subsets. Identifying this partition with the set itself, Theorem 4.7 asserts that the bijection is given by $\phi G_2^u \mapsto \phi\sigma_u\phi^{-1}$, and by Lemma 6.2, $\phi\sigma_u\phi^{-1} = \sigma_{\phi(u)}$. Furthermore, the map

$$\rho_2 : \{\sigma_{u'} | u' \in \mathbb{S}(\Im\mathbb{O})\} \to \mathbb{P}(\Im\mathbb{O}), \sigma_{u'} \mapsto [u']$$

is bijective, as each line through the origin in $\Im\mathbb{O}$ determines a unique reflection. This gives a bijection

$$\rho_2 \circ \rho_1 : G_2/G_2^u \to \mathbb{P}(\Im\mathbb{O}), \phi G_2^u \mapsto [\phi(u)],$$

whereby the proof is complete. □

Remark 6.14. Thus, finding a cross-section for β amounts to constructing, for each $\ell \in \mathbb{P}(\Im\mathbb{O})$, a unique representative ϕ_ℓ of the set

$$\{\phi \in G_2 | [\phi(u)] = \ell\}.$$

For the sake of definiteness, we perform this construction explicitly in Appendix B.

Having done so, we have proven the following.

Corollary 6.15. *The set*

$$B = \{\phi_\ell | \ell \in \mathbb{P}(\Im\mathbb{O})\}$$

is a cross-section for β.

It remains now, by Theorem 6.12(iii), to determine for each $f \in C$ when two elements of B are in the same orbit of δ_f, and to find a cross-section of B with respect to δ_f.

Proposition 6.16. *Let $\ell, \ell' \in \mathbb{P}(\Im\mathbb{O})$ and $v, v' \in \mathbb{S}(\Im\mathbb{O})$ be such that $[v] = \ell$, and $[v'] = \ell'$ and let $f \in C$. Then $\phi_\ell \equiv \phi_{\ell'}$ with respect to δ_f if and only if $v \approx_f v'$, where*

$$v \approx_f v' \iff \exists \chi \in \mathrm{St}_\gamma(f) : \chi(v) = \pm v'. \tag{6.3}$$

Recall that we have fixed $u \in \mathbb{S}(\Im\mathbb{O})$, and that $[\phi_\ell(u)] = \ell$ and $[\phi_{\ell'}(u)] = \ell'$.

Proof. The statement that $\phi_\ell \equiv \phi_{\ell'}$ with respect to δ_f is equivalent to the existence of $\chi \in \mathrm{St}_\gamma(f)$ and $\psi \in G_2^u$ such that $\phi_\ell = \chi^{-1}\phi_{\ell'}\psi$, or, equivalently, $\chi\phi_\ell = \phi_{\ell'}\psi$.

If this holds, then

$$\chi\phi_\ell(u) = \phi_{\ell'}\psi(u) \in \{\phi_{\ell'}(\pm u)\} \subset \ell'.$$

But $\phi_\ell(u)$ is either v or $-v$, and thus $\chi(v) \in \ell'$, i.e. $\chi(v) = \pm v'$ since $\|\chi(v)\| = \|v\|$.

Conversely, if $\chi(v) = \pm v'$ for some $\chi \in \mathrm{St}_\gamma(f)$, then $\chi\phi_\ell(u) = \pm\phi_{\ell'}(u)$, and $\phi_{\ell'}^{-1}\chi\phi_\ell(u) = \pm u$. Thus $\phi_{\ell'}^{-1}\chi\phi_\ell \in G_2^u$, whence there exists $\psi \in G_2^u$ such that $\chi\phi_\ell = \phi_{\ell'}\psi$. This completes the proof. □

Remark 6.17. The condition $v \approx_f v'$ is equivalent to ℓ and ℓ' being in the same orbit of the left action of $\mathrm{St}_\gamma(f)$ on $\mathbb{P}(\Im\mathbb{O})$ by evaluation, i.e. the action defined by $\chi \cdot \ell = \{\chi(w) | w \in \ell\}$ for all $\chi \in \mathrm{St}_\gamma(f)$ and all $\ell \in \mathbb{P}(\Im\mathbb{O})$ This expresses the remainder of the classification problem as a normal form problem; nevertheless, (6.3) is a more suitable expression for computations.

7. Classifying Left Reflection Algebras

Let $u \in \mathbb{S}(\Im\mathbb{O})$ be fixed. We are now ready to compute a cross-section for the action γ_u, which would classify left reflection algebras up to isomorphism. Hence, let C be the cross-section obtained in [7] for the group action γ, and let B be the cross-section obtained in Corollary 6.15 for the action β. [9] In view of Section 6.2, what remains to be done is to compute, for each $f \in C$, a cross-section of B for δ_f. More precisely, this consists of computing $\mathrm{St}_\gamma(f)$ and thence a cross-section D'_f for the relation $\approx_f$ on $\mathbb{S}(\Im\mathbb{O})$ defined in Proposition 6.16. Indeed, by that proposition, the map $D'_f \to B, v \mapsto \phi_{[v]}$, is injective, and its image is a cross-section D_f of δ_f. Our set of representatives will thus correspond bijectively to the desired cross-section.

[9] The group actions used in this section were defined in Definition 6.11.

7.1. **Preliminaries.** We begin by summarizing a few facts from [7]. From now on we identify O_8^1 with $O_7 = O(\Im\mathbb{O})$. It is well known that for each $d \in \mathbb{N}$ and each $f \in O_d$ there exists a basis of $\mathbb{R}^d$ in which the matrix of f is block diagonal, where each block is either 1, -1, or

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some $\theta \in]0, \pi[$.

Given $f \in O_d$, Let n_f^+ and n_f^- be the dimensions of the eigenspaces of f corresponding to the eigenvalues 1 and -1, respectively. Let $R(f)$ be the (finite) set of all $\theta \in]0, \pi[$ such that R_θ is a block in the aforementioned block diagonal matrix, and for each $\theta \in R(f)$, let n_f^θ be the dimension of the *generalized eigenspace* corresponding to θ, i.e. twice the number of blocks R_θ. The set $R(f)$ and the numbers n_f^+, n_f^- and $n_f^\theta, \theta \in R(f)$, are, as indicated in [7], well-defined and basis-independent, hence invariant under conjugation by SO_d. For $d = 7$, they are hence invariant under the action γ of $G_2 \leq SO_7$. In [7], the normal form problem for γ was therefore solved separately for each possible *type*, where the type of $f \in O_7$ is the pair of sets $(\{n_\theta | \theta \in R(f)\}, \{n_f^+, n_f^-\})$. For notational consistency with [7], we write each set in the pair as a list of its elements in decreasing order, and separate the lists by a vertical line. The possible types are thus

$$\begin{array}{llll} (\emptyset|7,0), & (\emptyset|6,1), & (\emptyset|5,2), & (\emptyset|4,3), \\ (2|5,0), & (2|4,1), & (2|3,2), & \\ (4|3,0), & (4|2,1), & (2,2|3,0), & (2,2|2,1), \\ (6|1,0), & (4,2|1,0), & (2,2,2|1,0). & \end{array}$$

For simplicity we will denote, for each type T, the set $\{f \in O_7 | f \text{ is of type } T\}$ simply by T.

In this section, we will construct a cross-section for $\approx_f$ for each $f \in C$ of type

$$(\emptyset|7,0), \quad (\emptyset|6,1), \quad (\emptyset|5,2), \quad (\emptyset|4,3), \quad \text{or} \quad (2|5,0).$$

The remaining cases are more computationally demanding. We will content ourselves with computing the above to give an example; among these, the treatment of type $(\emptyset|4,3)$ relies on explicit computations of elements of G_2, while the other mainly use properties of Cayley triples and some linear algebra.

Remark 7.1. From now on we fix $v, z \in \mathbb{S}(\Im\mathbb{O})$ such that (u, v, z) is a Cayley triple, henceforth referred to as the *standard Cayley triple.* (A brief account of Cayley triples can be found in Section 1.2.) We identify $\Im\mathbb{O}$ with $\mathbb{R}^7$ and write the vectors in the basis induced by the standard Cayley triple. Likewise, we identify the set of all linear operators on $\Im\mathbb{O}$ with $\mathbb{R}^{7\times7}$, with matrices given in this basis, and deal analogously with operators on subspaces. Moreover we identify, for each $d \in \mathbb{Z}_+$, column matrices in $\mathbb{R}^{d\times1}$ with their representations as d-tuples in $\mathbb{R}^d$.

We are now ready to perform the computations.

7.2. **Type** $(\emptyset|7,0)$**.** A cross-section for type $(\emptyset|7,0)$ is given in [7] as follows.

Lemma 7.2. $C_{\emptyset|7,0} := \{\pm\mathbb{I}_7\}$ *is a cross-section of* $(\emptyset|7,0)$ *with respect to* γ.

Clearly, every $\phi \in G_2$ stabilizes the identity, and for every $u' \in \mathbb{S}(\Im\mathbb{O})$ there exists $\phi \in G_2$ such that $\phi(u) = u'$. We thus immediately arrive at the following cross-section for $\approx_f, f \in C$.

Proposition 7.3. *For any* $f \in C_{\emptyset|7,0}$,

(i) $\mathrm{St}_\gamma(f) = G_2$, *and*
(ii) $D'_f = \{u\}$ *is a cross-section of* $\mathbb{S}(\Im\mathbb{O})$ *with respect to* $\approx_f$.

7.3. **Type** $(\emptyset|6,1)$**.** A cross-section for γ is given in [7] as follows.

Lemma 7.4. $C_{\emptyset|6,1} := \left\{ \pm \begin{pmatrix} 1 & \\ & -\mathbb{I}_6 \end{pmatrix} \right\}$ *is a cross-section of* $(\emptyset|6,1)$ *with respect to* γ.

We can then compute the following cross-section for $\approx_f$.

Proposition 7.5. *For any* $f \in C_{\emptyset|6,1}$,

(i) $\mathrm{St}_\gamma(f) = \left\{ \begin{pmatrix} \pm 1 & \\ & \chi' \end{pmatrix} | \chi' \in O_6 \right\} \cap G_2$, *and*
(ii) $D'_f = \{\xi u + \eta v | (\xi, \eta) \in \mathbb{S}([0,1]^2)\}$ *is a cross-section of* $\mathbb{S}(\Im\mathbb{O})$ *with respect to* $\approx_f$.

Proof. By a result from linear algebra (see [11], p. 223), if $\chi \in \mathrm{St}_\gamma(f)$ for some $f \in C_{\emptyset|6,1}$ (i.e. if χ commutes with f), then χ is of the form

$$\left\{ \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} | \chi_1 \in \mathbb{R}, \chi_2 \in \mathbb{R}^{6\times 6} \right\},$$

and the converse holds by direct verification. The fact that $\mathrm{St}_\gamma(f) \subseteq G_2$ then implies that $\chi_1 \in O_1$ and $\chi_2 \in O_6$, proving (i).

To prove (ii), given any $w \in \mathbb{S}(\Im\mathbb{O})$ there exist $\xi, \eta \in [0,1]$ with $\xi^2 + \eta^2 = 1$, and $v' \perp u$ with $\|v'\| = 1$, such that $w = \pm(\xi u + \eta v')$. Moreover there exists $z' \in \mathbb{S}(\Im\mathbb{O})$ such that (u, v', z') is a Cayley triple, and hence there exists $\chi \in G_2$ mapping (u, v, z) to (u, v', z'). By (i), $\chi \in \mathrm{St}_\gamma(f)$, and $\chi(\xi u + \eta v) = \pm w$. Hence D'_f is exhaustive.

To show that D'_f is irredundant, assume that

$$\xi u + \eta v \approx_f \xi' u + \eta' v.$$

for some $(\xi, \eta), (\xi', \eta') \in \mathbb{S}([0,1]^2)$. Then $\chi(\xi u + \eta v) = \xi' u + \eta' v$ for some $\chi \in \mathrm{St}_\gamma(f)$, and from (i) it follows that $|\xi| = |\xi'|$, whence $|\eta| = |\eta'|$, and thus $(\xi, \eta) = (\xi', \eta')$, completing the proof. □

7.4. **Types** $(\emptyset|5,2)$ **and** $(2|5,0)$**.** We treat these types simultaneously due to computational similarities. The cross-sections given in [7] are as follows.

Lemma 7.6. *(i)* $C_{\emptyset|5,2} := \left\{ \pm \begin{pmatrix} \mathbb{I}_2 & \\ & -\mathbb{I}_5 \end{pmatrix} \right\}$ *is a cross-section of* $(\emptyset|5,2)$ *with respect to* γ.
(ii) $C_{2|5,0} := \left\{ \pm \begin{pmatrix} R_\theta & \\ & -\mathbb{I}_5 \end{pmatrix} | \theta \in]0, \pi[\right\}$ *is a cross-section of* $(2|5,0)$ *with respect to* γ.

Without further ado, we find cross-sections for these types with respect to $\approx_f$.

Proposition 7.7. *For any* $f \in C_{\emptyset|5,2} \cup C_{2|5,0}$,

(i) $\mathrm{St}_\gamma(f) = \left\{ \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix} | \chi_1 \in O_2, \chi_2 \in O_5 \right\} \cap G_2$, *and*

(ii) $D'_f = \{\xi u + \eta uv + \zeta z | (\xi, \eta, \zeta) \in \mathbb{S}([0,1]^3)\}$ *is a cross-section of* $\mathbb{S}(\Im\mathbb{O})$ *with respect to* $\approx_f$.

Proof. The proof of (i) is analogous to that for type $(\emptyset|6,1)$. For (ii), take any $w \in \mathbb{S}(\Im\mathbb{O})$. Then there exist $(\xi, \eta, \zeta) \in \mathbb{S}([-1,1]^3)$ and $u', z' \in \mathbb{S}(\Im\mathbb{O})$ satisfying $u' \in [u, v]$ and $z' \perp [u, v, uv]$, such that

$$w = \pm(\xi u' + \eta uv + \zeta z'),$$

and u' and z' can be chosen so that ξ, η and ζ are all positive. By definition of the multiplication in $\mathbb{O}$, there is $v' \in u'^{\perp} \cap [u, v]$ such that $u'v' = uv$. Then (u', v', z') is a Cayley triple, and there exists $\chi \in G_2$ mapping (u, v, z) to (u', v', z'); (i) then implies that $\chi \in \mathrm{St}_\gamma(f)$. Furthermore, $\chi(\xi u + \eta uv + \zeta z) = \pm w$, whence D'_f is exhaustive.

If $\chi(\xi u + \eta uv + \zeta z) = \pm(\xi' u + \eta' uv + \zeta' z)$ for some $\chi \in \mathrm{St}_\gamma(f)$ and $(\xi, \eta, \zeta), (\xi', \eta', \zeta') \in \mathbb{S}([0,1]^3)$, then by the block decomposition in (i) we have $\xi = \xi'$, and

$$\eta\chi(uv) + \zeta\chi(z) = \pm(\eta' uv + \zeta' z). \tag{7.1}$$

The block decomposition further implies that $\chi(u), \chi(v) \in [u, v]$. As $\chi \in G_2$ and the product of any two mutually orthogonal unit vectors in $[u, v]$ belongs to $[uv]$, we have

$$\chi(uv) = \chi(u)\chi(v) = \epsilon uv \tag{7.2}$$

for some $\epsilon \in C_2$, whence (7.1) implies that

$$(\epsilon\eta \mp \eta')uv = \pm\zeta' z - \zeta\chi(z). \tag{7.3}$$

Now by (7.2) along with χ being orthogonal, $uv \perp [z, \chi(z)]$. Thus (7.3) implies that $\epsilon\eta \mp \eta' = 0$, and then $(\xi, \eta, \zeta) = (\xi', \eta', \zeta')$. Thus D'_f is irredundant. □

7.5. **Type** $(\emptyset|4,3)$**.** To begin with, we introduce, for each $\theta \in [0, \pi/2]$, the matrix

$$\hat{R}_\theta = \begin{pmatrix} \mathbb{I}_2 & & \\ & R_\theta & \\ & & \mathbb{I}_3 \end{pmatrix}$$

and the sets

$$C_\theta = \left\{ \pm \hat{R}_\theta^{-1} \begin{pmatrix} \mathbb{I}_3 & \\ & -\mathbb{I}_4 \end{pmatrix} \hat{R}_\theta \right\}$$

and

$$T_\theta = \left\{ \hat{R}_\theta^{-1} \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix} \hat{R}_\theta | \chi_1 \in O_3, \chi_2 \in O_4 \right\} \cap G_2.$$

These are then used to express the cross-section of $(\emptyset|4,3)$ computed in [7].

Lemma 7.8. $C_{\emptyset|4,3} := \bigcup_{\theta \in [0,\pi/2]} C_\theta$ *is a cross-section of* $(\emptyset|4,3)$ *with respect to* γ.

The matrices in C_θ are not block-diagonal with respect to the standard basis when $\theta \neq 0$, and to compute a cross-section for $\approx_f, f \in C_\theta$, we need to express $\mathrm{St}_\gamma(f)$ more explicitly than in the previous cases. The details are given in the following lemma, where $\times$ denotes the (standard) vector product in $\mathbb{R}^3$.

Lemma 7.9. *Let* $\theta \in]0, \pi/2]$. *Then* $T_\theta = T'_\theta$, *where*

$$T'_\theta = \left\{ \hat{R}_\theta^{-1} \begin{pmatrix} x_1 & x_2 & \epsilon x_1 \times x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (x_1 \times x_2)^* & \epsilon x_2^* & -\epsilon x_1^* & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \epsilon \end{pmatrix} \hat{R}_\theta | \right.$$
$$\left. x_1 \perp x_2 \in \mathbb{S}^2, \epsilon \in E_\theta \right\},$$

with $x^* = (x^3, x^2, -x^1)$ *for each* $x \in \mathbb{R}^3$, *and* $E_\theta = C_2$ *if* $\theta = \pi/2$, *and* $\{1\}$ *if not.*

Proof. If $\chi \in T_\theta \subset G_2$, then χ respects octonion multiplication. Computing $\chi(u)\chi(v)$ thus determines $\chi(uv)$ in terms of the first two columns of χ. The fourth column is then obtained from the properties of G_2 upon analysing the entrywise effect of conjugation by $\hat{R}_\theta$. These determine the remaining columns, and carrying out the computations one sees that $\chi \in T'_\theta$. Conversely, if $\chi \in T'_\theta$, denote by χ_{*j} the j^{th} column of χ. Then $(\chi_{*1}, \chi_{*2}, \chi_{*4})$ is a Cayley triple, inducing, as can be directly verified, the basis $(\chi_{*1}, \ldots, \chi_{*7})$. Thus $\chi \in G_2$, and then clearly $\chi \in T_\theta$. □

A cross-section for $\approx_f$ is then given by the following result.

Proposition 7.10. *Let* $\theta \in [0, \pi/2]$ *and* $f \in C_\theta$. *Then the following holds.*

(i) $\mathrm{St}_\gamma(f) = T_\theta$.

(ii) If $\theta = 0$, *then* $D'_f = \{\xi u + \eta z | (\xi, \eta) \in \mathbb{S}([0,1]^2)\}$ *is a cross-section of* $\mathbb{S}(\Im\mathbb{O})$ *with respect to* $\approx_f$.

(iii) If $\theta \neq 0$, *then*

$$D'_f = \{\xi(u \sin\omega + v\cos\omega) + \eta uz + \zeta(uv)z | (\omega, \xi, \eta, \zeta) \in [0, \pi] \times \mathbb{S}([0,1]^3),$$
$$\xi\eta = 0 \Rightarrow \omega = 0, \theta = \pi/2 \Rightarrow \omega \leq \pi/2\}$$

is a cross-section of $\mathbb{S}(\Im\mathbb{O})$ *with respect to* $\approx_f$.

Proof. (i) follows from the fact that

$$\left\{ \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix} | \chi_1 \in O_3, \chi_2 \in O_4 \right\} \cap G_2$$

is the stabilizer of

$$\begin{pmatrix} \mathbb{I}_3 & \\ & -\mathbb{I}_4 \end{pmatrix},$$

which holds by arguments analogous to those in the proofs of the preceding propositions.

To prove (ii), given any $w \in \mathbb{S}(\Im\mathbb{O})$, we have

$$w = \xi u' + \eta z'$$

for some $(\xi, \eta) \in \mathbb{S}([0,1]^2)$ and $u', z' \in \mathbb{S}(\Im\mathbb{O})$ with $u' \in [u, v, uv]$ and $z' \perp [u, v, uv]$. Now for any $v' \in u'^{\perp} \cap [u, v, uv]$ we have $u'v' \in [u, v, uv]$.[10] Thus for any such v', (u', v', z') is a Cayley triple, and there is $\chi \in G_2$ mapping (u, v, z) to (u', v', z'). Then $\chi(\xi u + \eta z) = w$ and, as $\theta = 0$ implies that $\hat{R}_\theta = \mathbb{I}_7$, (i) gives that $\chi \in \mathrm{St}_\gamma(f)$, whence D'_f is exhaustive.

[10] Indeed, $[u, v, uv]$ is a subalgebra of $\mathbb{O}$ isomorphic to $\mathbb{H}$.

To show that D'_f is irredundant, assume that $\chi(\xi u + \eta z) = \xi' u + \eta' z$ for some $(\xi, \eta), (\xi', \eta') \in \mathbb{S}([0,1]^2)$ and $\chi \in \mathrm{St}_\gamma(f)$. Since $\hat{R}_\theta = \mathbb{I}_7$, we have

$$\chi = \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix}$$

with $\chi_1 \in O_3$ and $\chi_2 \in O_4$. Then $\chi(\xi u+\eta z) = \xi' u+\eta' z$ implies that $(\xi, \eta) = (\xi', \eta')$.

For (iii), given $w \in \mathbb{S}(\Im\mathbb{O})$, we shall construct $(\omega, \xi, \eta, \zeta)$ such that $(\xi \sin\omega, \xi\cos\omega, 0, 0, \eta, 0, \zeta) \in D'_f$, and, in view of Lemma 7.9, a matrix

$$X = \begin{pmatrix} x_1 & x_2 & \epsilon x_1 \times x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (x_1 \times x_2)^* & \epsilon x_2^* & -\epsilon x_1^* & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \epsilon \end{pmatrix}, x_1 \perp x_2 \in \mathbb{S}^2, \epsilon \in E_\theta,$$

such that $\chi = \hat{R}_\theta^{-1} X \hat{R}_\theta$ maps $(\xi\sin\omega, \xi\cos\omega, 0, 0, \eta, 0, \zeta)$ to $\pm w$. To this end, set

$$y_1 = (w^1, w^2, w^3\cos\theta - w^4\sin\theta) \quad \text{and} \quad y_2 = (-w^6, w^5, w^3\sin\theta + w^4\cos\theta).$$

Then there are four possible, mutually excluding, cases.

If $y_1 = y_2 = 0$, set $\epsilon = 1$ and

$$x_1 = (1,0,0), \quad x_2 = (0,1,0), \quad (\omega, \xi, \eta, \zeta) = (0,0,0,1).$$

If $y_1 \neq 0$ and $y_2 = 0$, take any $z \in \mathbb{S}^2$, $z \perp y_1$, and set $\epsilon = 1$, and

$$x_1 = z, \quad x_2 = \tfrac{\mathrm{sgn}(w^7)}{\|y_1\|} y_1, \quad (\omega, \xi, \eta, \zeta) = (0, \|y_1\|, 0, |w^7|).$$

If $y_2 \neq 0$ and $y_1 = \nu y_2$ for some $\nu \in \mathbb{R}$, take any $z \in \mathbb{S}^2$, $z \perp y_2$, and set $\epsilon = \mathrm{sgn}(\nu)$ if $\theta = \pi/2$, and $\epsilon = 1$ otherwise, and

$$x_1 = z, \quad x_2 = \tfrac{\mathrm{sgn}(w^7)}{\|y_2\|} y_2, \quad (\cos\omega, \xi, \eta, \zeta) = (\epsilon\,\mathrm{sgn}(\nu), \|y_1\|, \|y_2\|, |w^7|).$$

If $y_1 \times y_2 \neq 0$, let ϵ and ω be given by

$$\epsilon\cos\omega = \tfrac{\langle y_1, y_2\rangle}{\|y_1\|\|y_2\|}, \quad \langle y_1, y_2\rangle = 0 \Rightarrow \epsilon = 1,$$

where $\langle , \rangle$ denotes the Euclidean inner product, and furthermore set

$$x_1 = \tfrac{\mathrm{sgn}(w^7)}{\sin\omega}\big(\tfrac{\epsilon}{\|y_1\|} y_1 - \cos\omega \tfrac{1}{\|y_2\|} y_2\big) \quad x_2 = \tfrac{\mathrm{sgn}(w^7)}{\|y_2\|} y_2$$

and

$$(\xi, \eta, \zeta) = (\|y_1\|, \|y_2\|, |w^7|).$$

Then in all four cases, using the convention that $\mathrm{sgn}(0) = 1$, $(\omega, \xi, \eta, \zeta)$ is uniquely defined by the condition $(\xi\sin\omega, \xi\cos\omega, 0, 0, \eta, 0, \zeta) \in D'_f$. Moreover, $\chi := \hat{R}_\theta^{-1} X \hat{R}_\theta \in T_\theta$ by Lemma 7.9, and χ maps $(\xi\sin\omega, \xi\cos\omega, 0, 0, \eta, 0, \zeta)$ to $\pm w$. Thus D'_f is exhaustive.

To show that D'_f is irredundant, assume that

$$w := (\xi\sin\omega, \xi\cos\omega, 0, 0, \eta, 0, \zeta) \approx_f (\xi'\sin\omega', \xi'\cos\omega', 0, 0, \eta', 0, \zeta') =: w',$$

i.e. that there exists $\chi \in \mathrm{St}_\gamma(f)$ such that $\chi(w) = \pm w'$. By (i) and Lemma 7.9, this is equivalent to the existence of $x_1 \perp x_2 \in \mathbb{S}^2$, $\epsilon \in E_\theta$ and $\delta \in C_2$ such that, componentwise,

$$\begin{array}{rcl} \delta\xi'\sin\omega' &=& \xi(x_1^1\sin\omega + x_2^1\cos\omega) \\ \delta\xi'\cos\omega' &=& \xi(x_1^2\sin\omega + x_2^2\cos\omega) \\ 0 &=& \xi(x_1^3\sin\omega + x_2^3\cos\omega)\cos\theta + \epsilon\eta x_2^3\sin\theta \\ 0 &=& -\xi(x_1^3\sin\omega + x_2^3\cos\omega)\sin\theta + \epsilon\eta x_2^3\cos\theta \\ \delta\eta' &=& \epsilon\eta x_2^2 \\ 0 &=& -\epsilon\eta x_2^1 \\ \delta\zeta' &=& \epsilon\zeta. \end{array}$$

From the bottom three lines, together with a $\sin\theta$-multiple of the third line added to a $\cos\theta$-multiple of the fourth, we deduce that $|\zeta| = |\zeta'|$ and $|\eta| = |\eta'|$, whence $(\xi,\eta,\zeta) = (\xi',\eta',\zeta')$. Thus $w = w'$ if $\xi\eta = 0$. If $\xi\eta \neq 0$, we get $x_2 = (0, \delta\epsilon, 0)$, which implies that $x_1^2 = 0$. Then the second line gives $\cos\omega' = \epsilon\cos\omega$, implying $\omega' = \omega$, which completes the proof. □

8. Conclusion and Future Perspectives

The procedure hitherto employed gives, if completed, an explicit classification of left reflection algebras. By the Cartan–Dieudonné Theorem, each $g \in O_8^1$ is the product of n reflections for some $0 \leq n \leq 7$. The cases $n = 0$ and $n = 1$ having been treated in [7] and above, respectively, one may attempt to use the above techniques to investigate the set of all *left n-reflection algebras*, i.e. algebras $\mathbb{O}_{f,g}$ where $f, g \in O_8^1$ and g is the product of n reflections, $2 \leq n \leq 7$. When doing so, two issues arise for larger n. To begin with, as the number of reflections is not invariant under isomorphism, one must exclude such left n-reflection algebras that are isomorphic to left n'-reflection algebras for some $n' < n$. Secondly, the above work was simplified by the fact that for any $u \in \mathbb{S}(\Im\mathbb{O})$, each left reflection algebra is isomorphic, by a G_2-morphism, to $\mathbb{O}_{f,\sigma_u}$. For $n \geq 3$, the situation becomes increasingly complicated, due to the restrictive properties of G_2.

These generalizations are, however, beyond the scope of this paper, and it is the author's hope to be able to treat them in a forthcoming publication.

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Appendix A. On the Category $\mathcal{C}_8^1$

In this appendix we will show that the full subcategory $\mathcal{C}_8^1 \subseteq {}_{SO_8}\mathcal{O}_8$, defined in the end of Section 2, does not arise from the triality action of any subgroup of SO_8 on $\mathcal{O}_8^1$. In other words we will show that the subcategory ${}_{\mathrm{St}^*(\mathcal{O}_8^1)}\mathcal{O}_8^1 \subseteq {}_{SO_8}\mathcal{O}_8$ is not full. By Theorem 4.7, this is equivalent to showing that $\mathcal{O}_8^1$ is not full in $\mathcal{O}_8$ with respect to the triality action, i.e that there exists $\phi \in SO_8 \setminus (\mathrm{St}(\mathcal{O}_8^1) \cup \mathrm{Dest}(\mathcal{O}_8^1))$.

To this end, fix a Cayley triple $(u, v, z) \in \mathbb{O}^3$, set $\theta = 2\pi/3$, and define $\phi \in SO_8$ as the rotation with angle 2θ in the $(1, u)$-plane, i.e. let ϕ be given by

$$\phi(1) = \cos(2\theta)1 + \sin(2\theta)u, \quad \phi(u) = -\sin(2\theta)1 + \cos(2\theta)u,$$

and $\phi(x) = x$ for each $x \in [1, u]^{\perp}$. In this case one can easily compute a triality pair (ϕ_1, ϕ_2) of ϕ. Expressed in the basis induced by (u, v, z), this is given by

$$\phi_i = \begin{pmatrix} \cos\theta & -\sin\theta & & & & & & \\ \sin\theta & \cos\theta & & & & & & \\ & & \cos\theta & \epsilon_i \sin\theta & & & & \\ & & -\epsilon_i \sin\theta & \cos\theta & & & & \\ & & & & \cos\theta & \epsilon_i sin\theta & & \\ & & & & -\epsilon_i \sin\theta & \cos\theta & & \\ & & & & & & \cos\theta & -\epsilon_i \sin\theta \\ & & & & & & \epsilon_i \sin\theta & \cos\theta \end{pmatrix}$$

for each $i \in \underline{2}$, where $\epsilon_i = (-1)^i$.

Then $[\mathrm{Id}, \mathrm{Id}] \in \mathcal{O}_8^1$ satisfies $\phi \cdot [\mathrm{Id}, \mathrm{Id}] \notin \mathcal{O}_8^1$, implying that $\phi \notin \mathrm{St}(\mathcal{O}_8^1)$, while $[\sigma_u, \sigma_u] \in \mathcal{O}_8^1$ satisfies $\phi \cdot [\sigma_u, \sigma_u] \in \mathcal{O}_8^1$, and hence $\phi \notin \mathrm{Dest}(\mathcal{O}_8^1)$. This completes the proof.

Appendix B. Constructing Certain Elements in G_2

Let $u \in \mathbb{S}(\Im\mathbb{O})$. In this appendix, we describe, for the sake of fixing a choice, a general and explicit method to construct, for each line ℓ through the origin in $\Im\mathbb{O}$, an element $\phi_\ell \in G_2$ such that $[\phi_\ell(u)] = \ell$. We thus fix a Cayley triple (u, v, z), and let $\mathfrak{B}$ be the basis induced by it (see Remark 7.1). Our aim is then to construct a Cayley triple (u', v', z') such that $[u'] = \ell$. To this end let x be the orthogonal projection onto ℓ of the first element in $\mathfrak{B}$ not orthogonal to ℓ. Let y_1 be the first $y \in \mathfrak{B}$ such that $y \notin \ell$. By performing the Gram-Schmidt procedure on (x, y_1) we obtain an orthonormal set (u', v') such that $[u'] = \ell$. Finally, let y_2 be the first $y \in \mathfrak{B}$ such that $y \notin [u', v', u'v']$. As $u'v' \perp [u', v']$ and $\|u'v'\| = 1$, the Gram-Schmidt procedure applied to $(u', v', u'v', y_2)$ yields an orthonormal quadruple $(u', v', u'v', z')$, with (u', v', z') a Cayley triple as required.

References

[1] A. A. Albert, Absolute Valued Real Algebras. Ann. of Math. 48 (1947), pp. 495–501.

[2] S. Alsaody, Morphisms in the Category of Finite-Dimensional Absolute Valued Algebras. Colloq. Math. 125 (2011), pp. 147–174.

[3] J. Baez, The Octonions. Bull. Amer. Math. Soc. 39 (2002), pp. 145–205.

[4] A. Calderón, A. Kaidi, C. Martín, A. Morales, M. Ramírez and A. Rochdi, Finite-Dimensional Absolute Valued Algebras. Israel J. Math. 184 (2011), 193–220.

[5] C. Chevalley, *The Algebraic Theory of Spinors*. Columbia University Press, New York, NY (1954).

[6] J. H. Conway & D. A. Smith, *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry.* A K Peters Ltd., Natick, MA (2003).

[7] J. A. Cuenca Mira, E. Darpö & E. Dieterich, Classification of the Finite Dimensional Absolute Valued Algebras having a Non-Zero Central Idempotent or a One-Sided Unity. Bull. Sci. Math. 134 (2010), pp. 247–277.

[8] E. Darpö & E. Dieterich, The Double Sign of a Real Division Algebra of Finite Dimension Greater than One. Math. Nachr. 285 (2012), pp. 1635–1642.

[9] E. Dieterich, A General Approach to Finite Dimensional Division Algebras. Colloq. Math. 126 (2012), pp. 73–86.

[10] L. Forsberg, Four-Dimensional Absolute Valued Algebras. Department of Mathematics, Uppsala University, Uppsala, Sweden (2009).

[11] F. R. Gantmacher, *The Theory of Matrices, Volume One*. Chelsea Publishing Company, New York, NY (1960).

[12] H. Saltzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen and M. Stroppel, *Compact Projective Planes*. Walter de Gruyter, Berlin, Germany (1995).

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