# Free Boundary Problems of Obstacle Type, a Numerical and Theoretical Study 

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#### Abstract

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This thesis consists of five papers and it mainly addresses the theory and numerical schemes to approximate the quadrature domains, QDs.

The first paper deals with the uniqueness and some qualitative properties of the two phase QDs.

In the second paper, we present two numerical schemes to approach the one phase QDs. The first scheme is based on the properties of the given free boundary and the level set method. We use shape optimization analysis to construct the second method. We illustrate the efficiency of the schemes on a variety of numerical simulations.

We design two numerical schemes based on the finite difference discretization to approximate the multi phase QDs, in the third paper. We prove that the second method enjoys monotonicity, consistency and stability and consequently it is a convergent scheme by BarlesSouganidis theorem. We also show the efficiency of the schemes through numerical experiments.

In the fourth paper, we introduce a special class of QDs in a sub-domain of $\mathbf{R}^{\mathrm{n}}$ and study the existence and the uniqueness along with an application of the problem. We also construct a numerical scheme based on the level set method to approximate the solution.

In the last article, we study the behavior of the free boundary near the fixed boundary for a semi-linear problem. We show that it is only one phase points on the fixed boundary. From this we can conclude that the free boundary is a $C^{1}$-graph up to the fixed boundary.

Keywords: Partial differential equations, Numerical analysis, Free boundary problems

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## Sammanfattning

Denna avhandling består av fem artiklar och behandlar främst teori och numeriska metoder för att approximera kvadratur områden (quadrature domians, QDs).

Den första artikeln behandlar entydighet och allmänna egenskaper hos tvåfas QDs.
Vi presenterar två numeriska metoder för att approximera enfas QDs i andra artikeln. Den första metoden är baserad på egenskaperna hos den fria randen och så-kallade "level set" metoden. Vi använder "shape optimization" för att konstruera den andra metoden. Båda metoderna är testade i olika numeriska simuleringar.

I det tredje artikeln approximera vi flerafas QDs med hjälp av finita differens metoden. Vi visar att den andra metoden är monoton, konsistent och stabil. Från Barles-Souganidis sats följer det då att metoden är konvergent. Vi presenterar också olika numeriska simuleringar.

Vi introducerar QDs i en delmängd av $\mathbf{R}^{n}$ och studerar existens och entydighet jämte en numerisk metod baserad på nivå mängdmetoden i fjärde artikeln.

I den sista artikeln studerar vi hur den fria randen beter sig nära den fixa randen för ett semilinjärt problem. Vi visar att det enbart är enfaspunkter på den fixa randen. Från detta kan vi också dra slutsatsen att den fria randen är en $C^{1}$-graf upp till fixa randen.

The secrets eternal neither you know nor I And answers to the riddle neither you know nor I Behind the veil there is much talk about us, why When the veil falls, neither you remain nor I.
-Omar Khayyam (1048-1131)

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"Hafez! sit not a moment without wine, and the beloved Tis the season of the rose, and of the Jasmine, and of the celebration of fasting! "
-Hafez Shirazi (1325-1390 C.E.)

TO MY BELOVED FAMILY
SADNA \& VIANA

## List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Some properties of two phase quadrature domains. Joint work with C. Babaoglu. Nonlinear Anal. 74 (2011), no. 10, 3386-3396.

II Numerical approximation of one phase quadrature domains. Joint work with F. Bozorgnia. Submitted

III Numerical schemes for multi phase quadrature domains. Joint work with F. Bozorgnia. Submitted

IV Quadrature domains in a sub-domain of $\mathbf{R}^{n}$, theory and a numerical approach. Preprint

V Tangential touch between the free and the fixed boundary in a semilinear free boundary problem in two dimensions. Joint work with E. Lindgren. Accepted for publication in Arkiv för matematik.

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## Contents

Part I: INTRODUCTION ..... 17
1 Free Boundary Problems ..... 19
1.1 The One Phase Obstacle Problem ..... 19
1.2 The Two Phase Obstacle Problem ..... 22
1.3 Analysis of the Free Boundary ..... 24
1.3.1 Interior Regularity of the Function and the Free Boundary ..... 25
1.3.2 Monotonicity Formulas ..... 26
1.3.3 Optimal Regularity up to the Fixed Boundary ..... 27
2 Quadrature domains ..... 28
2.1 One Phase Case ..... 29
2.1.1 Subharmonic QDs ..... 31
2.2 Two Phase Case ..... 33
2.3 Application to Hele-Shaw flow ..... 35
2.4 Multi Phase Case ..... 37
3 Viscosity Solutions and Numerical approximations ..... 39
3.1 Viscosity Solutions ..... 39
3.2 Numerical Methods ..... 41
3.2.1 A Degenerate Elliptic Scheme ..... 41
Part II: OVERVIEW OF THE PAPERS ..... 45
4 Overview of Paper I ..... 47
4.1 Main result ..... 48
5 Overview of Paper II ..... 50
5.1 First Numerical Method ..... 50
5.2 Second Numerical Method ..... 52
6 Overview of Paper III ..... 53
6.1 Numerical Methods ..... 54
6.1.1 Degenerate elliptic scheme ..... 54
6.1.2 An iterative method for the general case ..... 55
7 Overview of Paper IV ..... 56
7.1 Main Results ..... 56
8 Overview of Paper V ..... 59
8.1 Main Result ..... 61
References ..... 62

## List of Figures

Figure 1.1:The obstacle $\psi$ touches the membrane $u$ and we are looking for the free boundary $\Gamma$ and the solution $u$. ..... 20
Figure 1.2:An example of obstacle problem. ..... 21
Figure 1.3:The inner circle is the corresponding free boundary for Figure 1.2 . ..... 22
Figure 1.4:An example of two phase obstacle problem. ..... 23
Figure 1.5:This figure illustrates the different types of the free boundary points. The point $x_{2}$ is a negative one-phase free boundary point, $x_{1}$ is a positive one-phase point, $x_{0}$ is a negative one-phase point touching the fixed boundary, $x_{3}$ is a branching point and $x_{4}$ is a two-phase point which might not be a branching point. ..... 24
Figure 2.1:An example of a one phase quadrature domain with respect to the measure $\mu=\left(2 x^{2}+y^{4}\right) \chi_{A}$ where $A$ is the annulus in the figure (a).
Figure (a) shows the first iteration and Figure (b) illustrates the numerical approximation of $\Omega$. ..... 32
Figure 2.2:Figures (a) and (b) depict $|\nabla u|$ on the outer boundary of Figure 2.1 (a) and (b) respectively. ..... 32
Figure 2.3:The surface of the function in Figure 2.1, (b). ..... 33
Figure 2.4:A two phase quadrature domain by considering two Dirac measures concentrated on two points. ..... 34
Figure 2.5:A Hele-Shaw cell. ..... 36
Figure 2.6:A five phase QD. ..... 37

## Part I: <br> INTRODUCTION

"In the middle of difficulty lies opportunity." -Albert Einstein.

## 1. Free Boundary Problems

By a "free boundary problem" we mean a boundary value problem in which we deal with solving a partial differential equation in a domain such that a part of the boundary is unknown in advance. That part of the boundary is called the free boundary. In order to solve a free boundary problem we have to provide a boundary condition which is imposed at the free boundary. One can then determine both the free boundary and the solution of the differential equation.

The theory of the free boundary problems, FBPs, have been stimulated by the increased number of mathematical model to understand a vast of actual problems in physics (plasma physics, solidification), chemistry (chemical vapor deposition), finance (American option), biology (tissue growth), industrial processes such as electro photography and other areas. The corresponding governing partial differential equations, PDEs, of FBPs exhibit a priori unknown sets, free boundaries, such as interfaces, moving boundaries, shocks, etc. The mathematical and relevant literature of this field is enormous. At this point we refer to [27] to review a number of applications of FBPs in science and industry.

In this part, a brief introduction of FBPs is given. To begin with, the reader will be given an overview of some classical examples of the topic and this is followed by the connection of FBPs and potential theory. More precisely, we will describe a class of free boundary problems which is called quadrature domains and shall study the corresponding theory in the one and the two phase cases.

We shall occasionally use the Sobolev space $W^{m, p}(\Omega)$ of functions $u$ in $\Omega$ such that $\partial^{\alpha} u \in L^{p}(\Omega)$ for all multi-indices $\alpha$ with $|\alpha| \leq m$ and its subspace $W_{0}^{m, p}(\Omega)$ which is the closure of $C_{0}^{\infty}(\Omega)$ in $W^{m, p}(\Omega)$. For $p=2$, we use $H^{m}(\Omega)$ and $H_{0}^{m}(\Omega)$ instead of $W^{m, 2}(\Omega)$ and $W_{0}^{m, 2}(\Omega)$ respectively.

### 1.1 The One Phase Obstacle Problem

The obstacle problem is the most classical example of FBPs. Suppose that a membrane is attached to two points. We also ignore gravity. Clearly if we do not push it up or down it forms like a string (in one dimension). But let us push it up with a non-flat object which is called an obstacle. Thus at some point the obstacle touches the membrane and at other points the membrane stays above the obstacle and still is a straight line there. Here the free boundary $\Gamma$, is the set of points where the membrane leaves the obstacle. In the following we can


Figure 1.1. The obstacle $\psi$ touches the membrane $u$ and we are looking for the free boundary $\Gamma$ and the solution $u$.
make a mathematical formulation of the obstacle problem. Suppose that we are given a function $\psi \in C^{2}$ as the obstacle and we have fixed the membrane on the boundary of a domain $D$ with $g$ by considering $g \geq \psi$. Moreover, we may think of the graph of $u$ as the membrane which is now forced to stay above the obstacle. The set

$$
\Lambda:=\{x \in D: u(x)=\psi(x)\}
$$

is called the coincidence set. If we set $\Omega=D \backslash \Lambda$ then the set

$$
\Gamma:=\partial \Lambda \cap D=\partial \Omega \cap D
$$

is the corresponding free boundary which is a priori unknown. Figure (1.1) illustrates the one phase obstacle problem in dimension one. In the equilibrium situation, the function $u$ is harmonic outside the contact set, i.e,

$$
\begin{equation*}
\Delta u=0 \quad \text { in } \Lambda^{c} \tag{1.1}
\end{equation*}
$$

otherwise $u=\psi$. Moreover, it is clear that $u=\psi$ on $\Gamma$ and we can see that the equilibrium is maintained when the force $\nabla u \cdot \mathbf{n}$ is the same when we approach the free boundary from any side. Hence our free boundary conditions are

$$
\begin{equation*}
u=\psi \quad \text { and } \quad \mathbf{n} \cdot \nabla u=\mathbf{n} \cdot \nabla \psi, \quad \text { on } \Gamma . \tag{1.2}
\end{equation*}
$$



Figure 1.2. An example of obstacle problem.

For simplicity let $g=0$ and set

$$
\mathcal{K}:=\left\{v \in H_{0}^{1}(D): v \geq \psi\right\}
$$

then by variational methods one can prove the following theorem, see for instance [41].

Theorem 1.1. Let $D$ be a bounded set in $\mathbf{R}^{n}$ and $\psi \in H_{0}^{1}(D)$ is given. Then the following assertions are equivalent,

1. Minimization: $u \in \mathcal{K}$ and it minimizes the energy $\|v\|^{2}=\int_{D}|\nabla v|^{2} d x$ among all $v \in \mathcal{K}$.
2. Variational inequality: $u \in \mathcal{K}$ and

$$
(v-u, u):=\int_{D} \nabla(v-u) \cdot \nabla u d x \geq 0 \quad \forall v \in \mathcal{K}
$$

3. Linear complementary problem:

$$
\begin{cases}-\Delta u \geq 0, & \text { in } D \\ u \geq \psi, & \text { in } D \\ (-\Delta u)(u-\psi)=0, & \text { in } D\end{cases}
$$



Figure 1.3. The inner circle is the corresponding free boundary for Figure 1.2.

Clearly (1.1) and (1.2) imply the complementary statement. We note that for existence and uniqueness results, the minimization and variational inequality could be good starting points and for numerical approach the complementary problem is easy to implement.

Remark 1.2. It is not hard to prove that the coincidence set $\Lambda$ is an open set, see [41]. Now, set $v:=u-\psi$, then the obstacle problem turns to

$$
\begin{cases}-\Delta v=-(\Delta \psi) \chi_{\{v>0\}} & \text { in } D,  \tag{1.3}\\ v \geq 0 & \text { in } D, \\ |\nabla v|=0 & \text { on } \Gamma,(\text { The } F B P \text { condition }) \\ v=g & \text { on } \partial D .\end{cases}
$$

### 1.2 The Two Phase Obstacle Problem

Consider a thin film or a membrane which is fixed on the boundary of a domain $D$ where a part of the membrane is under a thick liquid which is supposed to be heavier than the membrane. Now the membrane produces a pressure downward on the part of the membrane which is above the liquid, say $\lambda^{+}$. On the other hand, the part of the membrane in the liquid is also pushed up by another force, $\lambda^{-}$, due to the liquid's weight. The mathematical interpretation of the equilibrium state is the two phase obstacle problem. Mathematically, when we reach the equilibrium the solution $u$ satisfies

$$
\begin{equation*}
\Delta u=\lambda^{+} \chi_{\{u>0\}}-\lambda^{-} \chi_{\{u<0\}}, \tag{1.4}
\end{equation*}
$$

which is called the two phase obstacle problem or the two phase membrane problem.


Figure 1.4. An example of two phase obstacle problem.
Moreover, suppose that $D$ is a bounded domain with smooth boundary and $\lambda^{ \pm}$are two bounded functions on $D$. Then the Euler-Lagrange equation of the following minimization problem

$$
\text { Minimize } \quad E(v)=\int_{D}\left(\frac{1}{2}|\nabla v|^{2}+\lambda^{+} v^{+}+\lambda^{-} v^{-}\right) d x
$$

over

$$
\mathcal{K}:=\left\{v \in H^{1}(D): v-g \in H_{0}^{1}(D)\right\}
$$

is

$$
\begin{cases}\Delta u=\lambda^{+} \chi_{\{u>0\}}-\lambda^{-} \chi_{\{u<0\}}, & \text { in } D  \tag{1.5}\\ u=g, & \text { on } \partial D\end{cases}
$$

Here $u^{ \pm}=\max ( \pm u, 0)$ and the free boundary consists of the two parts

$$
\Gamma_{1}:=\partial\{u>0\} \quad \text { and } \quad \Gamma_{2}:=\partial\{u<0\} .
$$

Suppose that $u$ is a minimizer of the energy functional $E$ and $x_{0} \in \Gamma:=$ $\Gamma_{1} \cup \Gamma_{2}$. Then we divide the free boundary points into the following parts (see Figure 1.5):

1. We say that $x_{0}$ is a positive (negative) one-phase free boundary point if there exists a neighborhood of $x_{0}$ where $u$ is non-negative (non-positive) in it. In other words, $x_{0} \in \Gamma^{+} \backslash \Gamma^{-}\left(x_{0} \in \Gamma^{-} \backslash \Gamma^{+}\right)$.
2. We say that $x_{0}$ is a two-phase free boundary point if $x_{0} \in \Gamma^{+} \cap \Gamma^{-}$. Moreover, if $\left|\nabla u\left(x_{0}\right)\right|=0$ then $x_{0}$ is said to be a branching point.


Figure 1.5. This figure illustrates the different types of the free boundary points. The point $x_{2}$ is a negative one-phase free boundary point, $x_{1}$ is a positive one-phase point, $x_{0}$ is a negative one-phase point touching the fixed boundary, $x_{3}$ is a branching point and $x_{4}$ is a two-phase point which might not be a branching point.

One of the challenging issues in this case is that the interface could be considered as two parts. One part is when the gradient of $u$ vanishes and one where the gradient is non zero, and because of these two decompositions of two different types of growth it is not easy to obtain a growth estimate at points on the interface. See for instance [73] and [62] and references therein.

### 1.3 Analysis of the Free Boundary

One of the most challenging problems in FB's theory is the regularity of the function and the regularity of the free boundary. In this part, we try to briefly address several relevant techniques which are required for the regularity of the function and the free boundary that are mainly based on blow-up techniques.

To have a general discussion, we consider a general form of the problems (1.3) and (1.5). A smooth domain $D$ and a function $g$ in some appropriate class is given. We are looking for the function $u$ and a domain $\Omega=\Omega(u)$ such that

$$
\begin{cases}\Delta u=\chi_{\Omega}, & \text { in } \Omega \cap D  \tag{1.6}\\ |\nabla u|=0, & \text { on } \partial \Omega \cap D \\ u=g, & \text { on } \partial \Omega\end{cases}
$$

Since we are interested in to study the free boundary locally, we restrict ourselves to the solutions defined on the balls with centers on the free boundary.

Definition 1.3. (Local Solution) Let $x_{0} \in \Gamma$ and $M, R$ be two positive constants. For a $C^{1,1}$ solution $u$ of the free boundary problem (1.6), we say $u \in P_{R}\left(x_{0}, M\right)$ if

$$
\left\|D^{2} u(x)\right\|_{L^{\infty}\left(B_{R}\left(x_{0}\right)\right)} \leq M
$$

We will use the abbreviated notation $P_{R}(M)$ for the class $P_{R}(0, M)$. If one chooses $R=\infty$ then $u$ is defined in $\mathbf{R}^{n}$ and is called global solution. More precisely, $u \in C^{1,1}$ is a global solution if

$$
\left\|D^{2} u(x)\right\|_{L^{\infty}\left(\mathbf{R}^{n}\right)} \leq M
$$

The class $P_{\infty}(0, M)$ will also be denoted by $P_{\infty}(M)$. If $u \in P_{R}\left(x_{0}, M\right)$ then the rescaling of $u$ at $x_{0}$

$$
u_{r}(x)=u_{r, x_{0}}(x):=\frac{u\left(x_{0}+r x\right)-u\left(x_{0}\right)}{r^{2}}, \quad x \in B_{R / r}
$$

is in $P_{R / r}(M)$, see [48] section 3.3. Therefore $\left\|D^{2} u_{r}(x)\right\| \leq M$ in $B_{R / r}$ and consequently

$$
\|\nabla u(x)\| \leq M\|x\| \quad \text { and } \quad\|u(x)\| \leq \frac{1}{2} M\|x\|^{2}
$$

for all $x \in B_{R / r}$ and when $g=0$. Thus we can find a sequence $u_{i}:=u_{r_{i}}$ of solutions to problem (1.6) and $r_{i} \rightarrow 0$ such that

$$
u_{i} \rightarrow u_{0} \quad \text { in } C_{\mathrm{loc}}^{1, \alpha}\left(\mathbf{R}^{n}\right) \text { and for any } 0<\alpha<1
$$

We say that $u_{0}$ is a blow-up of $u$ at $x_{0}$. The following lemma states the properties of blow-ups. For the proof see [48], section 3.4.

Lemma 1.4. Suppose that for some $0<\alpha<1$ we have $u_{i} \rightarrow u_{0}$ in $C_{\text {loc }}^{1, \alpha}\left(\mathbf{R}^{n}\right)$ then

- $u_{0} \in P_{\infty}(M)$.
- If $u_{0}\left(x_{0}\right)$ is positive (negative) then there exists $i_{0}, \delta>0$ such that $u_{i}\left(x_{0}\right)$ is also positive (negative) for all $i>i_{0}$ in $B_{\delta}\left(x_{0}\right)$.
- $u_{0}$ solves the same problem (1.6), as $u_{i}$.
- $u_{i} \rightarrow u_{0}$ strongly in $W_{\text {loc }}^{2, p}(D)$ for any $1<p<\infty$.


### 1.3.1 Interior Regularity of the Function and the Free Boundary

For the sake of clarity let $x_{0}=0$ and $u(0)=0$ and consider a general rescaling at the origin i.e,

$$
u_{r}(x)=\frac{u(r x)}{r^{\alpha}}, \quad x \in B_{1}
$$

for some $\alpha>0$. It solves the same problem in $B_{1 / r}$ for some $\alpha$. For most of the problems the optimal regularity for $u$ is $C^{\lfloor\alpha\rfloor, \alpha-\lfloor\alpha\rfloor}$ when the scaling is $\alpha$. To prove the optimal regularity one first usually (depending on the problem) prove for example $C^{\beta}$ (or $C^{1, \beta}$ ) regularity for some $0<\beta<1$. For instance, in the one and the two phase obstacle problem, by standard elliptic theory we get $u \in C^{1, \beta}(K)$ for any $K \subset \subset D$ and $0<\beta<1$, see for instance [31]. To prove the regularity, we need an optimal growth for the solution, i.e,

$$
\sup _{B_{r}(z)}|u| \leq C r^{\alpha}
$$

when $z$ is a free boundary point and $r$ is small enough. For more information of this method we refer to [42], [62], [41] and [71].
We also need to be sure that the blow-up does not vanish identically, otherwise we lose all information in the limiting process. For most cases one is able to prove an inequality is so called non-degeneracy of the form

$$
\sup _{B_{r}(z)}|u| \geq C r^{\alpha}
$$

for some constant $C$.
The second interesting problem is the interior regularity of the free boundary. In most cases, one expects that the free boundary is the graph of a $C^{1}$ function. The optimal regularity and non-degeneracy lead us to prove that the free boundary has finite $(n-1)$-dimensional Hausdorff measure. The main idea to prove that a curve is a graph of $C^{1}$-function is to zoom in enough and find that it is close to a plane. We can interpret $u_{r}$ as the $\frac{1}{r}$-times zoomed in.
The optimal $C_{l o c}^{1,1}$ regularity for the solution to (1.5) has been proved by Uraltseva in [71] and Shahgholian in [62]. The regularity for the free boundary has been studied by Shahgholian, Uraltseva and Weiss in [63] and [65].

### 1.3.2 Monotonicity Formulas

The Weiss monotonicity formula, [72] and Alt, Caffarelli and Friedman, (ACF monotonicity formula) [3], are two significantly effective tools for analyzing the free boundary problems. Using the Weiss monotonicity formula we can often obtain that the blow-up is a homogenous function.

Theorem 1.5. (The Weiss Monotonicity Formula) Suppose that $F\left(D^{u}\right)=$ $\Delta u=f(u)$ where $f$ is a smooth function and define $G(t)=\int_{0}^{t} f(s) d s$. Then the function

$$
W(r, u, x):=\int_{B_{1}}\left(\frac{\left|\nabla u_{r}\right|^{2}}{2}+G(u)\right) d x-\alpha \int_{\partial B_{1}} u_{r}^{2} d \sigma
$$

is, under suitable assumption on $f$, a monotonically increasing function in $r$. Moreover, $W$ is constant if and only if $u$ is a homogenous function of degree $\alpha$.

Theorem 1.6. (ACF Monotonicity Formula) Let $v$ and $w$ be two non-negative subharmonic functions with disjoint support such that $w(0)=v(0)=0$. Then the function

$$
\Phi(r):=\int_{B_{r}} \frac{|\nabla v|^{2}}{|x|^{n-2}} d x \int_{B_{r}} \frac{|\nabla w|^{2}}{|x|^{n-2}} d x
$$

is a monotonically increasing function in $r$.
Because of monotonically increasing of $W$ one is able to prove that if $u_{0}$ is a blow-up then

$$
\begin{aligned}
W\left(s, u_{0}, x\right)=\lim _{i} W\left(s, u_{r_{i}}, x\right) & =\lim _{i} W\left(s r_{i}, u, x\right) \\
& =\lim _{i} W\left(r_{i}, u, x\right)=W\left(1, u_{0}, x\right)
\end{aligned}
$$

Hence $W\left(u_{0}\right)$ is constant and $u_{0}$ must be homogenous. For more information see [6], [14], [15], [41] and [65].

### 1.3.3 Optimal Regularity up to the Fixed Boundary

Another great interesting problem is the study of the behavior of the free boundary close to the fixed boundary. Roughly speaking, we deal with this problem by employing the similar tools and techniques for interior regularity such as blow-ups, optimal growth and non-degeneracy. By similar scaling techniques one is able to get the optimal regularity up to the fixed boundary by considering appropriate boundary conditions. In other words, we study the behavior of blow-ups at points near the fixed boundary instead of interior points. To be more clear, let the fixed boundary be a plane. We desire to investigate how the free boundary approaches the fixed boundary. In most cases the answer is that the free boundary touches the fixed one in a tangential manner. Then after the blow-up, we have a half space solution. We refer the reader to [4], [40], [45] and paper V in this thesis.

## 2. Quadrature domains

The English word quadrature comes from the Latin word quadratura. It means making square shaped and in general it meant to divide a land into squares. In mathematics quadrature is referred to constructive or numerical methods for determining areas. In this thesis the term quadrature has a related meaning. For example, a quadrature identity will typically be an exact formula for the integral of harmonic or analytic functions. The domain of integration is then a quadrature domain. We say a few words of the starting point of quadrature domains theory.
H. S. Shapiro and his group began to extend and generalize the concept of quadrature domains, QDs, more than thirty years ago. Some main references of QDs are [35], [66], [70] and [38].

The connection between Laplacian growth, especially Hele-Shaw flow, and quadrature domains has been investigated by Richardson in [68]. Before that these two theories were developing in parallel. For instance, around 1980, the construction of quadrature domains by potential theory techniques ([57] and [58]) and the theory of weak solutions for Hele-Shaw problem ([36] and [24]) were studied simultaneously and independently. For further reading we refer the reader to [38] and references therein.

We shall use the following notations in this part.
$\mathbf{R}^{n}$ The Euclidean space of dimension $N$, $\mu$ an arbitrary measure,
$\Omega$ an open subset of $\mathbf{R}^{n}$ (generally connected),
$|\Omega|$ the volume of $\Omega$,
$L^{p}(\Omega)$ the usual Lebesgue space with respect to the Lebesgue measure, $H L^{p}(\Omega)$ the subspace of $L^{p}(\Omega)$ that consists of harmonic functions in $\Omega$,
$S L^{p}(\Omega)$ the subspace of $L^{p}(\Omega)$ that consists of subharmonic functions in $\Omega$,
$\chi_{\Omega}$ the characteristic function of $\Omega$,
$C^{k}(\Omega)$ the class of $k$ times continuously differentiable functions in $\Omega$,
We always denote the fundamental solution of the Laplace equation by $G$ in $\mathbf{R}^{n}$. In other words for $x \in \mathbf{R}^{n} \backslash\{0\}$,

$$
G(x)= \begin{cases}\frac{1}{N(N-2) \omega_{N}}|x|^{2-N}, & \text { for } N \geq 3 \\ -\frac{1}{2 \pi} \ln |x|, & \text { for } N=2\end{cases}
$$

where $\omega_{N}$ is the volume of unit sphere in $\mathbf{R}^{n}$. It is known that if $\Omega$ is open and bounded then for $G(x-y)$ considered as a function of $x \in \Omega$, the following
holds

$$
\begin{array}{r}
G(x-y) \in H L^{1}(\Omega), \quad \forall y \in \Omega^{c} \\
-G(x-y) \in S L^{1}(\Omega), \quad \forall y \in \Omega \\
\pm G_{j}= \pm \frac{\partial G}{\partial x_{j}} \in S L^{1}(\Omega), \quad \forall y \in \Omega^{c}, 1 \leq j \leq N
\end{array}
$$

see [34]. Moreover, the linear combinations with positive coefficients of the functions
$\pm G_{j}(x-y), G(x-y), x \in \Omega, \forall y \in \Omega^{c} \quad$ and $\quad-G(x-y), \forall y \in \mathbf{R}^{N}$,
are dense in $S L^{1}$, and the linear combination with real coefficients of the functions $G_{j}(x-y)$ and $G(x-y)$ for $y \in \Omega^{c}$ are dense in $H L^{1}$, see [34].

### 2.1 One Phase Case

In this section we give a formal definition of a quadrature domain. First we introduce the Newtonian potential and some of its important properties. The basic sources for theses results are [5], [21] and [43].

Let $\mu$ be a measure. By $U^{\mu}$ we mean the Newtonian potential of the measure $\mu$ defined by

$$
U^{\mu}(x):=(G * \mu)(x)=\int_{\mathbf{R}^{n}} G(x-y) d \mu(y), \quad x \in \mathbf{R}^{n}
$$

Thus, $U^{\chi_{\Omega}}$ (from now on $U^{\Omega}$ for simplicity) is the Newtonian potential of $\Omega$ considered as a body with density one.

Theorem 2.1. If $\mu$ is a Radon measure with compact support then $U^{\mu}$ and $\nabla U^{\mu}$ are defined a.e. and are in $L_{l o c}^{1}$. Moreover, if $\mu$ is positive then $U^{\mu}$ is defined everywhere.

Remark 2.2. A measure $\mu$ on a measurable space $(\mathcal{X}, \Sigma)$ is inner regular if for every $A \in \Sigma$ we have $\mu(A)=\sup \{K: K$ is a compact set in $A\}$ and it is called Radon measure if it is inner regular and locally finite.

Theorem 2.3. Suppose that $\mu$ is a Radon measure with compact support then one has

$$
-\Delta U^{\mu}=\mu
$$

in the sense of distributions.
Corollary 2.4. If $\mu$ is a Radon measure with compact support then $U^{\mu}$ is harmonic in the complement of $\operatorname{supp}(\mu)$.

Theorem 2.5. If $\mu$ is a Radon measure with compact support then

$$
\left|U^{\mu}(x)\right|=O\left(|x|^{2-N}\right) \rightarrow 0 \text { as }|x| \rightarrow \infty \text { if } N \geq 3
$$

and

$$
U^{\mu}(x)=-\frac{1}{2 \pi} \ln |x| \int d \mu+O\left(|x|^{-1}\right) \text { as }|x| \rightarrow \infty \text { if } N=2
$$

Generally, if $-\Delta u=\mu$ then we can not derive that $u=U^{\mu}$, since one can add any harmonic function to $u$. But if $u$ behaves like a potential at infinity we are able to conclude $u=U^{\mu}$.

Theorem 2.6. Suppose that $\mu$ is a Radon measure with compact support and $-\Delta u=\mu$. Iffor $N \geq 3$ the function $u$ satisfies

$$
u(x) \rightarrow 0 \text { as }|x| \rightarrow \infty
$$

and

$$
u(x)=-\frac{1}{2 \pi} \ln |x| \int d \mu+O\left(|x|^{-1}\right) \text { as }|x| \rightarrow \infty \text { if } N=2
$$

then $u=U^{\mu}$.

Now we define the quadrature domain.
Definition 2.7. Suppose that $\mu$ is a measure with compact support. By a quadrature domain with respect to $\mu$ we mean an open connected set $\Omega \subset \mathbf{R}^{n}$ such that $\operatorname{supp}(\mu) \subset \Omega$ and

$$
\begin{equation*}
\int_{\Omega} h d x=\int h d \mu \tag{2.1}
\end{equation*}
$$

holds for all $h \in H L^{1}(\Omega)$. We will say $\Omega$ is a quadrature domain (QD) and write $\Omega \in Q\left(\mu, H L^{1}\right)$.

In the simplest case, it is known that the disc $D(a ; r)$ is a quadrature domain w.r.t Dirac measure (see [25]) and the quadrature identity then reduces to the ordinary mean value property for harmonic functions:

$$
h(a)|D(a ; r)|=\int_{D(a ; r)} h d x
$$

Generally, if $\Omega$ is a bounded domain in $\mathbf{R}^{n}$ and

$$
\begin{equation*}
\int_{\Omega} h d x=|\Omega| h\left(x_{0}\right) \tag{2.2}
\end{equation*}
$$

holds for all $h \in H L^{1}(\Omega)$, where $x_{0}$ is an arbitrary point, then $\Omega$ is a ball centered at $x_{0}$, see [25]. Thus a quadrature identity can be thought of as a generalized mean value property.
The quadrature identity (2.1) is equivalent to the following identities (see [34]),

$$
\begin{cases}U^{\Omega}=U^{\mu}, & \text { in } \mathbf{R}^{n} \backslash \Omega,  \tag{2.3}\\ \nabla U^{\Omega}=\nabla U^{\mu}, & \text { in } \mathbf{R}^{n} \backslash \Omega\end{cases}
$$

It has been explained in [37] and [56] that $\Omega \in Q\left(\mu, H L^{1}\right)$ is equivalent to finding a pair $(u, \Omega)$ of solution of the following one-phase free boundary problem:

$$
\begin{cases}\Delta u=\chi_{\Omega}-\mu & \text { in } \quad \mathbf{R}^{n}  \tag{2.4}\\ u=|\nabla u|=0, & \text { in } \quad \mathbf{R}^{n} \backslash \Omega\end{cases}
$$

where $u=U^{\mu}-U^{\Omega}$ is the so-called modified Schwarz potential (MSP) of the pair $(\mu, \Omega)$. We refer to [22], [28] and [44] and references therein for further reading.

### 2.1.1 Subharmonic QDs

Let $\mu$ be a measure with compact support. By a subharmonic quadrature domain we mean an open connected set $\Omega \subset \mathbf{R}^{n}$ such that $\operatorname{supp}(\mu) \subset \Omega$ and

$$
\begin{equation*}
\int_{\Omega} h d x \geq \int h d \mu \tag{2.5}
\end{equation*}
$$

holds for all $h \in S L^{1}(\Omega)$. We write $\Omega \in Q\left(\mu, S L^{1}\right)$ if (2.5) holds. M. Sakai in [58] and [59] has studied the subharmonic QDs in details.
Suppose that $\mu=\alpha \delta_{0}$ where $\delta_{0}$ is the Dirac mass at origin and $\alpha>0$. Then

$$
Q\left(\mu, H L^{1}\right)=Q\left(\mu, S L^{1}\right)=\{B(0 ; r)\},
$$

where $r \geq 0$ is determined by $|B(0 ; r)|=\alpha$, see [34].
A similar discussion shows that $\Omega \in Q\left(\mu, S L^{1}\right)$ if and only if $U^{\mu} \geq U^{\Omega}$ in $\mathbf{R}^{n}$ and $U^{\mu}=U^{\Omega}$ in $\mathbf{R}^{n} \backslash \Omega$. From a PDE point of view, $\Omega \in Q\left(\mu, S L^{1}\right)$ if and only if $u$ and $\Omega:=\{x: u(x)>0\}$ solve the following free boundary problem, (see [37])

$$
\begin{cases}\Delta u=\chi_{\Omega}-\mu & \text { in } \quad \mathbf{R}^{n}  \tag{2.6}\\ u \geq 0, & \text { in } \quad \mathbf{R}^{n} \\ u=0, & \text { in } \quad \mathbf{R}^{n} \backslash \Omega\end{cases}
$$

Example 2.8. Set $\mu=\left(2 x^{2}+y^{4}\right) \chi_{A}$ where $A$ is the annulus making by $B_{1}=B_{1}(0)$ and $B_{2}=B_{1.4}(0)$. Figure 2.1 and 2.2 illustrate the numerical approximation for the solution of (2.6) and $|\nabla u|$ on the free boundary respectively.

(a)

(b)

Figure 2.1. An example of a one phase quadrature domain with respect to the measure $\mu=\left(2 x^{2}+y^{4}\right) \chi_{A}$ where $A$ is the annulus in the figure (a). Figure (a) shows the first iteration and Figure (b) illustrates the numerical approximation of $\Omega$.


Figure 2.2. Figures (a) and (b) depict $|\nabla u|$ on the outer boundary of Figure 2.1 (a) and (b) respectively.

It is easy to give examples of (harmonic) QDs that are not subharmonic QDs.
Example 2.9. Let $\mu=\mu_{\alpha}=\alpha \rho$ where $\alpha>0$ and $\rho$ is the mass uniformly distributed on $S=\partial B(0,1)$. Define

$$
\Omega_{\beta}=\left\{x \in \mathbf{R}^{2}: \beta<\pi|x|^{2}<\beta+\alpha\right\}
$$

where $\beta \geq 0, \Omega=\Omega_{0} \cup\{0\}$ and clearly $\left|\Omega_{\beta}\right|=\alpha$. Sakai in [58] has proved that for each $0<\alpha \leq e \pi$ there exists a unique $\beta=\beta_{\alpha}$ with $\pi-\alpha<\beta_{\alpha}<\pi$ such that

$$
\int_{\Omega_{\beta_{\alpha}}} G d x=\int G d \mu_{\alpha}
$$



Figure 2.3. The surface of the function in Figure 2.1, (b).

For $0<\alpha \leq \pi$ one can prove (see [34]),

$$
Q\left(\mu_{\alpha}, H L^{1}\right)=Q\left(\mu_{\alpha}, S L^{1}\right)=\left\{\Omega_{\beta_{\alpha}}\right\}
$$

and for all $\alpha>\pi$

$$
Q\left(\mu_{\alpha}, H L^{1}\right)=\left\{\Omega, \Omega_{\beta_{\alpha}}\right\} \quad \text { and } \quad Q\left(\mu_{\alpha}, S L^{1}\right)=\left\{\Omega_{\beta_{\alpha}}\right\}
$$

For more details, see [66].

### 2.2 Two Phase Case

Two-phase quadrature domains has been introduced recently by Emamizadeh, Prajapat and Shahgholian in [23]. They have studied the existence and geometrical properties of the two phase case. In this section we briefly address some properties of this class of domains.

Let $\Omega$ be an open and bounded subset of $\mathbf{R}^{n}$ and $\widetilde{H}(\Omega)$ be the set of all $U^{\eta}$ where $\eta$ is a signed Radon measure with compact support in $\Omega^{c}$. It is not difficult to show that all functions in $\widetilde{H}(\Omega)$ are harmonic in $\Omega$ and if $h \in \widetilde{H}(\Omega)$ then $h \in L_{\text {loc }}^{1}\left(\mathbf{R}^{n}\right)$. Moreover, for $x \in \Omega^{c}$ we have $G(x-\cdot)=U^{\delta_{x}} \in \widetilde{H}(\Omega)$. The proof of the next lemma could be find for instance in [5].

Lemma 2.10. Suppose that $h$ is harmonic in a bounded open set $D$ such that $\Omega \subset \subset D$. There exists a measure $\nu$ with compact support such that $\operatorname{supp}(\nu) \subset$ $D \backslash \bar{\Omega}$ and $h=U^{\nu}$ in $\Omega$.


Figure 2.4. A two phase quadrature domain by considering two Dirac measures concentrated on two points.

These useful properties of $\widetilde{H}(\Omega)$ provide us necessary tools to have the following definition for the two phase quadrature domain.

Definition 2.11. Let $\Omega^{ \pm}$be two open, disjoint and connected subsets of $\mathbf{R}^{n}$ and $\mu^{ \pm}$be two positive Radon measures with compact supports. Moreover, suppose that $\lambda^{ \pm}$are two positive constants. We say that $\Omega=\Omega^{+} \cup \Omega^{-}$is a two phase quadrature domain, with respect to $\mu^{ \pm}, \lambda^{ \pm}$and $\widetilde{H}(\Omega)$, if $\operatorname{supp}\left(\mu^{ \pm}\right) \subset \Omega^{ \pm}$, and

$$
\begin{equation*}
\int_{\Omega^{+}} \lambda^{+} h d x-\int_{\Omega^{-}} \lambda^{-} h d x=\int h\left(d \mu^{+}-d \mu^{-}\right), \quad \forall h \in \widetilde{H}(\Omega) \tag{2.7}
\end{equation*}
$$

We then write $\Omega^{ \pm} \in Q\left(\mu^{ \pm}, \widetilde{H}(\Omega)\right)$ or $\Omega \in Q(\mu, \widetilde{H}(\Omega))$ where $\mu=\mu^{+}-\mu^{-}$, see Figure 2.4.

Set $f:=\lambda^{+} \chi_{\Omega^{+}}-\lambda^{-} \chi_{\Omega^{-}}$and suppose that $y \in \Omega^{c}$. We know that $h(x):=$ $h_{y}(x)=G(x-y) \in \widetilde{H}(\Omega)$ and consequently (2.7) yields

$$
U^{f}=U^{\mu} \text { in } \mathbf{R}^{n} \backslash \Omega
$$

In addition, there is a strong connection between FB theory and two phase quadrature domains that we have studied in the first paper. By considering $\operatorname{supp}\left(\mu^{ \pm}\right) \subset \Omega^{ \pm}$we have shown that $\Omega \in \widetilde{H}(\Omega)$ if and only if $(u, \Omega)$ be a solution of the following free boundary problem

$$
\begin{cases}\Delta u=\left(\lambda^{+} \chi_{\Omega^{+}}-\mu^{+}\right)-\left(\lambda^{-} \chi_{\Omega^{-}}-\mu^{-}\right), & \text {in } \mathbf{R}^{n}  \tag{2.8}\\ u=0, & \text { in } \mathbf{R}^{n} \backslash \Omega\end{cases}
$$

Example 2.12. Let $\mu^{+}=\pi \delta_{0}$ and $\mu^{-}=3 \pi \chi_{B}$ where $B$ is a ball centered at the origin with radius $r_{0}$. One can find an appropriate $r_{0}$ such that the pair

$$
\Omega^{+}=\{x:|x|<1\} \quad \text { and } \quad \Omega^{-}=\{x: 1<|x|<2\}
$$

are the corresponding two phase quadrature domain, see [29].
Investigating the existence and the uniqueness for two phase QDs are more sophisticated compare to the one phase QD. To the best of our best knowledge, the only literature [23] and [29] deal with the existence for the two phase case. The uniqueness of the two phase case have studied in the first paper of this thesis by considering some restrictions.

It is desirable to set up accurate conditions on $\mu^{ \pm}$, even in the one phase case, to guarantee the existence however it is a challenging problem and not easy to cope with. For instance, if one consider $\mu^{-} \ll \mu^{+}$then $\Omega^{-}$shrinks and one loses the existence. It means that in general, we have to put some balance condition for $\mu^{ \pm}$to get the existence. The most accurate known condition is the Sakai's condition, see [23] and references therein.

Theorem 2.13. Let $\mu^{ \pm}$be two given measure with compact supports and $\lambda^{ \pm}$ two constants such that satisfy in the Sakai's condition, i.e.,

$$
\sup _{r>0} \frac{\mu^{ \pm}\left(B_{r}(x)\right.}{\left|B_{r}(x)\right|} \geq \lambda^{ \pm} 2^{N}, \quad x \in \operatorname{supp}\left(\mu^{ \pm}\right)
$$

Then the two phase problem (2.8) has a unique solution.

### 2.3 Application to Hele-Shaw flow

One of the most known example of Laplacian growth is viscous fluids in HeleShaw cell. By Laplacian growth we mean the dynamics of an interface between two distinct flows which is driven by a harmonic field. See for instance [17, 52]. In this subsection we study the Hele-Shaw problem.

Suppose that some incompressible fluid is confined between two parallel plates and we inject more fluid into the gap between the plates with moderate velocity. Consequently the fluid will occupy more space. We are interested in the behavior of boundary of the fluid region which is a free boundary. Richardson has formulated this problem as follows, see [52].

To model this problem mathematically let $\mu$ be a positive, finite and non zero measure with compact support and $\operatorname{supp}(\mu) \subseteq D$ where $D$ is an open subset of $\mathbf{R}^{n}$ with $C^{1}$-boundary. Let $p_{D}$ be the superharmonic function such that

$$
\begin{cases}-\Delta p_{D}=\mu & \text { in } D  \tag{2.9}\\ p_{D}=0 & \text { on } \partial D\end{cases}
$$



Figure 2.5. A Hele-Shaw cell.

We are looking for a family of regions $D_{t}$ for $t \geq 0$, such that $\partial D_{t}$ moves with the velocity $-\nabla p_{D_{t}}$ where $p_{D_{t}}$ is the unique solution of (2.9).

Let $D_{0}$ and $\mu$ be as above and $I$ be an open interval. A map $I \ni t \rightarrow$ $D_{t} \subset \mathbf{R}^{n}$ is a weak solution of the free boundary problem if the function $u_{t} \in H^{1}\left(\mathbf{R}^{n}\right)$ defined by $\chi_{D_{t}}-\chi_{D_{0}}=\Delta u_{t}+t \mu$, satisfies $u_{t} \geq 0$ and $<u_{t}, 1-\chi_{D_{t}}>=0$ where $<\cdot, \cdot>$ is the duality between $H_{0}^{1}$ and its dual space $H^{-1}$. For more details see [36].

Theorem 2.14. Suppose that $\mu$ and $D_{0}$ be as before and $T>0$. Then there exists a weak solution

$$
[0, T] \ni t \rightarrow D_{t} \subset \mathbf{R}^{n}
$$

for the problem which is unique and if $u_{t}$ is the function appearing above then $u_{t}$ is also unique and

$$
u_{t}=\int_{0}^{t} p_{D_{\tau}} d \tau
$$

Moreover, $D_{t}$ can be chosen to be

$$
D_{t}=D_{0} \cup\left\{z: u_{t}(z)>0\right\}
$$

### 2.4 Multi Phase Case

Let $\mu_{i}$ for $1 \leq i \leq m$ be measures with compact and disjoint supports. By a multi phase QD we mean $\Omega:=\bigcup_{i=1}^{m} \Omega_{i}$ where $\operatorname{supp}\left(\mu_{i}\right) \subset \Omega_{i}:=\{x:$ $\left.u_{i}(x)>0\right\}$ solve the following FB problem

$$
\begin{cases}\Delta u_{i}=\lambda_{i} \chi_{\Omega_{i}}-\mu_{i} & \text { in } \Omega_{i},  \tag{2.10}\\ u_{i}=0 & \text { on } \partial \Omega_{i}, \\ \left|\nabla u_{i}\right|=\left|\nabla u_{j}\right| & \text { on } \Gamma_{i j}:=\partial \Omega_{i} \cap \partial \Omega_{j}, \\ \left|\nabla u_{i}\right|=0 & \text { on } \partial \Omega_{i} \backslash \bigcup \Gamma_{i j},\end{cases}
$$

which is understood in the distribution sense and $\cup_{i} \partial \Omega_{i}$ is the free boundary. It turns out that $\Omega_{i} \cap \Omega_{j}=\emptyset$ for $i \neq j$. In general, the existence and the uniqueness of the solution of (2.10) are open problems, but if we consider $\mu_{i}=\delta_{i}$ where $\delta_{i}$ is Dirac measure at the point $x_{i}$, then this problem has a unique solution.


Figure 2.6. A five phase QD.
This problem could be translated into the multi phase fluid theory which is a generalization of two phase flow. For instance the problem in two phase cases is similar to the two phase Hele-Shaw flows (Muskat problem), see Figure 2.6. Here, the phases are immiscible and non chemically related. The applications of multi phase flows are in a wide variety of industries, including power, petroleum or modeling of propagating steam explosions.
To be more precise, consider a Hele-Shaw cell and inject fluids at a number of points into it. The fluids regions grow as circular discs and we initially have disjoint blobs of fluids. It is clear that before the regions meet each other the growth of any one, does not affect on the others. But they finally coalesce and
form a multi-phase quadrature domain, see Figure 2.6. Here we review some interesting questions,

- How can we construct an efficient numerical scheme for the problem?
- Do we have any hole between the phases after we get the solution?
- What kind of assumptions are sufficient to have a connected multi phase QD?
- What is the convergence rate of the numerical scheme?

For further reading about multi phase Hele-Shaw flow see [18], [19], [20], [53], [54] and [55] and references therein. Indeed, [17] is also a good reference of the application of quadrature domains in fluid dynamics.

From mathematical standpoint, the multi phase case is a complicated problem. For instance, while many explicit solutions to the single-phase Hele-Shaw problem are known, solutions to the two-phase problem (also known as the Muskat problem) are scarce. To the best of our knowledge, there is no substantial theory concerning existence, uniqueness and geometrical properties of the solution and the corresponding free boundary. However, the more general mathematical question of the global existence and well-posedness of the Muskat problem has been investigated by Siegel, Caflisch and Howison [12] and Ambrose [1].

## 3. Viscosity Solutions and Numerical approximations

Let $\mathcal{M}^{n}$ be the set of all real-valued $n \times n$ symmetric matrices. In this chapter we consider a general form of second order PDE

$$
\begin{equation*}
F\left(x, u(x), D u(x), D^{2} u(x)\right)=0, \quad \text { in } \quad \Omega . \tag{3.1}
\end{equation*}
$$

where $F: \Omega \times \mathbf{R} \times \mathbf{R}^{n} \times \mathcal{M}^{n} \rightarrow \mathbf{R}$ is continuous with respect to all variables. We also suppose that $F$ is proper which means

$$
\begin{equation*}
F\left(x, r_{1}, p, M\right) \leq F\left(x, r_{2}, p, M\right), \quad \forall r_{1} \leq r_{2}, M \in \mathcal{M}^{n} \tag{3.2}
\end{equation*}
$$

This property is very important for the uniqueness (for the comparison principle). Without this assumption the comparison principle does not hold even for the classical solutions.

### 3.1 Viscosity Solutions

It is clear that even for linear PDEs the existence of solution is not an easy problem. But one of the best method is to investigate a weaker sense of the solution by multiplying a test function and then establish the existence and regularity. However, for a non-linear PDE, this strategy does not work unless the equation is an Euler-Lagrange equation of a functional. It turns out that one need to establish a new type of weak solution. This category of solution which is so called viscosity solutions is intimately connected with numerical analysis and scientific computing and it provides efficient tools to perform convergence schemes, see next section. For further reading about viscosity solution see for instance [7] and [39] and references therein.

More precisely, by considering suitable assumptions on the PDE and appropriate boundary conditions the classical solutions are in general unique but they might not exist. For example, for the Eikonal equation $\left|u^{\prime}(x)\right|=1$ in $\Omega=(-1,1)$ with boundary conditions $u(-1)=u(1)=0$ there is no clas$\operatorname{sic} C^{1}$ solution. One could consider that the Lipschitz solution instead of $C^{1}$ which implies that the derivative does not exist at every point but just almost every where. In this case we get the existence but not uniqueness. Crandall and Lions introduced a different notion of weak solution in 1982 which works very well for many first- and second-order nonlinear PDEs. This class of solution are also well-setup for the existence, uniqueness and stability of solutions in the viscosity sense.

Definition 3.1. A continuous function $u$ is called

- a viscosity sub(super)solution for the equation (3.1) if for every $\psi \in$ $C^{2}(\Omega)$ and local maximum (minimum) point $x_{0} \in \Omega$ of $u-\psi$ then

$$
F\left(x_{0}, u\left(x_{0}\right), D \psi\left(x_{0}\right), D^{2} \psi\left(x_{0}\right)\right) \leq(\geq) 0
$$

- $a$ viscosity solution of (3.1) if and only if it is both viscosity subsolution and supersolution.

Some important remarks:

1. For a PDE of order $k$ one can make similar definition by considering test functions in $C^{k}$.
2. For viscosity subsolution (supersolution) it is enough to consider $u$ being upper semi continuous (lower semi continuous).
3. We note that the equation $F=0$ and $-F=0$ are different in viscosity sense. For instance, we can show that $u(x)=-|x|+1$ is a viscosity solution for $\left|u^{\prime}\right|-1=0$ in $(-1,1)$ but not for the $-\left|u^{\prime}\right|+1=0$.
4. If $u$ solves (3.1) in e the viscosity sense then $-u$ solves

$$
-F\left(x,-u(x),-D u(x),-D^{2} u(x)\right)=0
$$

Definition 3.2. The fully non-linear second order partial differential equation (3.1) is called degenerate elliptic if for $M_{1}, M_{2} \in \mathcal{M}^{n}$ with $M_{1} \leq M_{2}$ then

$$
F\left(x, r, p, M_{2}\right) \leq F\left(x, r, p, M_{1}\right)
$$

where $M_{1} \leq M_{2}$, means $M_{2}-M_{1}$ is a nonnegative definite symmetric matrix.
Theorem 3.3. Suppose that $u \in C^{2}$ is a classical solution of (3.1) then it is a viscosity solution if one of the following statements holds:

1. The equation (3.1) is a first order PDE.
2. The equation (3.1) is degenerate elliptic.

Proof. Let $\psi \in C^{2}$ and $u-\psi$ has a local maximum at $x_{0}$ then clearly $D u\left(x_{0}\right)=$ $D \psi\left(x_{0}\right)$ and $D^{2} u\left(x_{0}\right) \leq D^{2} \psi\left(x_{0}\right)$. First suppose that the equation (3.1) does not depend on $D^{2} u$. Hence

$$
0=F\left(x_{0}, u\left(x_{0}\right), D u\left(x_{0}\right)\right)=F\left(x_{0}, u\left(x_{0}\right), D \psi\left(x_{0}\right)\right)
$$

which shows that $u$ is a viscosity subsolution. Similarly we can prove that $u$ is also a viscosity supersolution.

Now if $F$ is a degenerate elliptic then by the definition (3.2)

$$
0=F\left(x_{0}, u\left(x_{0}\right), D u\left(x_{0}\right), D^{2} u\left(x_{0}\right)\right) \geq F\left(x_{0}, u\left(x_{0}\right), D u\left(x_{0}\right), D^{2} \psi\left(x_{0}\right)\right)
$$

If we consider that $u-\psi$ has a local minimum at $x_{0}$ then $D^{2} u\left(x_{0}\right) \geq D^{2} \psi\left(x_{0}\right)$ and we obtain

$$
0=F\left(x_{0}, u\left(x_{0}\right), D u\left(x_{0}\right), D^{2} u\left(x_{0}\right)\right) \leq F\left(x_{0}, u\left(x_{0}\right), D u\left(x_{0}\right), D^{2} \psi\left(x_{0}\right)\right)
$$

Remark 3.4. This theorem asserts that the viscosity solution is consistent with that of a classical solution. Theorem 1 in chapter 10 of [26] shows that the statement is correct at any point where the viscosity solution is differentiable.

### 3.2 Numerical Methods

### 3.2.1 A Degenerate Elliptic Scheme

In this section we shall highlight parts of the theory of a convergent finite difference method for obstacle type problems without going into the technical details. We build a convergent numerical scheme for degenerate elliptic equations for which the uniqueness of viscosity solution is quite well known, see [16]. This class of equations includes free boundary problems, Hamilton Jacobi equations and fully non-linear elliptic equations. More precisely we aim to build such a scheme that preserves ellipticity. The discrete ellipticity property is called monotonicity. For convergence we have to provide monotonicity, consistency and stability to guarantee the convergence, see [49].

Let $F[u](x):=F\left(x, u(x), D u(x), D^{2} u(x)\right)$ be a degenerate elliptic operator. Consider a grid on a domain $D$ that consist of $N$ points $x_{i} \in D$. Let $\mathcal{N}_{i}$ denotes the set of all neighbors of $x_{i}$. At each grid point we present a scheme by $F^{i}[u]$ for $F$ and we suppose that $u\left(x_{i}\right)=u_{i}$ is a grid function. For simplicity and from now on, we write a scheme in the following form

$$
F^{i}[u]:=F^{i}\left(u_{i}, u_{i}-u_{j}\right) \quad \text { where } u_{j}=u\left(x_{j}\right) \text { and } x_{j} \in \mathcal{N}_{j} .
$$

We will discuss a class of nonlinear schemes which are monotone. This class is called degenerate elliptic schemes and it provides a strong form of stability and enjoys non-expansivity in the max norm. For the reader's connivence we provide the most important definition of the concepts used in this part.

By stability we mean that for every $h>0$ the scheme has a solution $u_{h}$ which is uniformly bounded independently of $h$.

Definition 3.5. Suppose that $F$ is a degenerate elliptic equation with a solution mapping $S$ that maps continuous boundary data $g$, to a continuous solution $u$.

- We say that $S$ is monotone if for all continuous functions $g_{1}, g_{2}$ defined on $\partial D$

$$
g_{1}(x) \leq g_{2}(x) \text { on } \partial D \text { then } S\left(g_{1}\right)(x) \leq S\left(g_{2}\right)(x), \quad x \in D
$$

- The scheme $F^{i}$ is consistent at $x$ iffor every $\psi \in C^{2}$ which is defined in a neighborhood of $x$

$$
F^{i}[\psi](x) \rightarrow F[\psi](x) \text { as } h \rightarrow 0
$$

where $h$ is the mesh spacing.

- The scheme $F$ is degenerate elliptic if $F^{i}$ is a non decreasing function of $u_{i}$ and the differences $u_{i}-u_{j}$.

Barles and Souganidis have shown that a consistent, stable approximation scheme converges uniformly on compact subsets to the unique viscosity solution provided it is monotone, see [8].

Remark 3.6. In the work of Barles and Souganidis, they do not indicate how to build such schemes and in general constructing a monotone scheme is the most challenging part. A natural way for building a monotone scheme is to implement the finite difference method.

As an example we look at a simple problem and we try to construct a convergent scheme. Let $u \in L^{\infty}(D)$ be the solution of the classical obstacle problem

$$
\begin{cases}\Delta u=f(x) \chi_{\Omega} & \text { in } D  \tag{3.3}\\ u>0 & \text { in } \Omega \\ u=|\nabla u|=0 & \text { in } D \backslash \Omega \\ u=g & \text { on } \partial D\end{cases}
$$

for $f \in L^{\infty}(D)$ and $g$ a nonnegative function on the boundary of $D$. It is easy to see that $u$ satisfies in

$$
\begin{align*}
F\left(x, u, D u, D^{2} u\right): & =\max \left(-\operatorname{tr}\left(D^{2} u\right)+f, u-g\right) \\
& =\max (-\Delta u+f, u-g)=0 \tag{3.4}
\end{align*}
$$

One can easily verify that (3.4) is a degenerate elliptic equation and its viscosity solution is a weak solution of (3.3) by Theorem 3.3.

For simplicity we continue in dimension two and assume that $D$ is a rectangle. Let $g=0$ and $\mathcal{N}$ be a uniform grid on $D$ with mesh spacing $\triangle x=\triangle y=$ $h$. We use the grid function $U_{i, j}$ for the approximation of $u_{i j}:=u\left(x_{i}, y_{j}\right)$ where $\left(x_{i}, y_{j}\right) \in \mathcal{N}$ for $1 \leq i \leq p, 1 \leq j \leq q$. To discretize (3.4) we use the five stencil points for the Laplacian and get the discrete form of (3.4)

$$
\max \left(U_{i, j}-\overline{U_{i, j}}+\frac{f_{i, j} h^{2}}{4}, U_{i, j}\right)=0
$$

where $\overline{U_{i, j}}=\left(U_{i-1, j}+U_{i+1, j}+U_{i, j-1}+U_{i, j+1}\right) / 4$. By a simple calculation one obtains

$$
\begin{equation*}
F^{i, j}[u]:=\min \left(\frac{f_{i, j} h^{2}}{4}-\overline{U_{i, j}}, 0\right) \tag{3.5}
\end{equation*}
$$

It is easily verified that this scheme is monotone, stable and consistent and consequently this method is convergent. Moreover, the viscosity solution of (3.3) that is also a weak solution and vice versa.

A very general form of the obstacle type problems is

$$
\max \left(F\left(x, u, D u, D^{2} u\right), u-g\right)=0
$$

or $\min \left(F\left(x, u, D u, D^{2} u\right), u-g\right)=0$ or even

$$
\max \left(\min \left(F\left(x, u, D u, D^{2} u\right), u-g_{1}\right), u-g_{2}\right)=0
$$

It is possible to obtain a degenerate elliptic scheme for these formulations, see [49].

## Part II: <br> OVERVIEW OF THE PAPERS

"Anyone who has never made a mistake has never tried anything new." -Albert Einstein.

## 4. Overview of Paper I

In this article, we study a two phase free boundary problem which is the generalization of the one phase quadrature domains, QDs. We prove the uniqueness along other qualitative properties of the solution.

Suppose that two measures $\mu^{ \pm}$with compact supports and two constants $\lambda^{ \pm}>0$ are given. Let $u$ and two domains $\Omega^{ \pm}$solve the following free boundary problem

$$
\begin{cases}\Delta u=\left(\lambda^{+} \chi_{\Omega^{+}}-\mu^{+}\right)-\left(\lambda^{-} \chi_{\Omega^{-}}-\mu^{-}\right), & \text {in } \mathbf{R}^{n}  \tag{4.1}\\ u=0, & \text { in } \mathbf{R}^{n} \backslash \Omega\end{cases}
$$

where $\operatorname{supp}\left(\mu^{ \pm}\right) \subset \Omega^{ \pm}$and $\Omega=\Omega^{+} \cup \Omega^{-}$. The main result of this article is to prove the uniqueness of the solution to (4.1). We also study the link between the theory of the quadrature domain and (4.1).

The first step is to provide an appropriate definition for the two phase QDs by considering a proper class of test functions.

Let $\Omega$ be an open and bounded subset of $\mathbf{R}^{n}$. Suppose that $\widetilde{H}(\Omega)$ denotes the set of all potentials $U^{\eta}$, where $\eta$ is a signed Radon measure with compact support in $\Omega^{c}$. One can easily verify if $h \in \widetilde{H}(\Omega)$ then $h \in L_{\text {loc }}^{1}\left(\mathbf{R}^{n}\right)$ and clearly all functions in $\widetilde{H}(\Omega)$ are harmonic in $\Omega$. Moreover, for $x \in \Omega^{c}$ we have $G(x-\cdot)=U^{\delta_{x}} \in \widetilde{H}(\Omega)$.

Definition 4.1. Let $\Omega^{ \pm}$be two open, disjoint and connected subsets of $\mathbf{R}^{n}$ and $\mu^{ \pm}$be two positive Radon measures with compact supports. Moreover, suppose that $\lambda^{ \pm}$are two positive constants. We say that $\Omega=\Omega^{+} \cup \Omega^{-}$is a two phase quadrature domain, with respect to $\mu^{ \pm}, \lambda^{ \pm}$and $\tilde{H}(\Omega)$, if $\operatorname{supp}\left(\mu^{ \pm}\right) \subset \Omega^{ \pm}$, and

$$
\begin{equation*}
\int_{\Omega^{+}} \lambda^{+} h-\int_{\Omega^{-}} \lambda^{-} h=\int h\left(d \mu^{+}-d \mu^{-}\right), \quad \forall h \in \widetilde{H}(\Omega) \tag{4.2}
\end{equation*}
$$

We then write $\Omega^{ \pm} \in Q\left(\mu^{ \pm}, \widetilde{H}(\Omega)\right)$ or $\Omega \in Q(\mu, \widetilde{H}(\Omega))$ where $\mu=\mu^{+}-\mu^{-}$.
The most trivial examples of two phase QDs arise when the positivity part and the negativity part are two disjoint one phase QDs with respect to two measures. For non-trivial examples, see [29].

Remark 4.2. It is a challenging problem to find appropriate conditions on the measures to guarantee the existence of QDs even in the one phase case. For
the one phase case Sakai in [60] has established such a condition and Gardiner and Sjödin have considered a stronger version of Sakai's condition to get the existence for the two phase case, see [29].

Obviously one has to make a balance between the measures to obtain the existence. In other words, if the intensity of one of the measures is much smaller than the other then the bigger one will cover the smaller one and we lose the existence.

### 4.1 Main result

Let $f=\lambda^{+} \chi_{\Omega^{+}}-\lambda^{-} \chi_{\Omega^{-}}$and $\mu=\mu^{+}-\mu^{-}$. For $y \in \Omega^{c}$ we know that $h(x)=h_{y}(x)=G(x-y) \in \widetilde{H}(\Omega)$ and consequently (4.2) implies

$$
U^{f}=U^{\mu}, \text { in } \mathbf{R}^{n} \backslash \Omega
$$

For $u=U^{f}-U^{\mu}$, we can prove that $\Omega \in \widetilde{H}(\Omega)$ if and only if $(u, \Omega)$ is a solution of the following free boundary problem

$$
\begin{cases}\Delta u=\left(\lambda^{+} \chi_{\Omega^{+}}-\mu^{+}\right)-\left(\lambda^{-} \chi_{\Omega^{-}}-\mu^{-}\right), & \text {in } \mathbf{R}^{n},  \tag{4.3}\\ u=0, & \text { in } \mathbf{R}^{n} \backslash \Omega \\ \operatorname{supp}\left(\mu^{ \pm}\right) \subset \Omega^{ \pm} . & \end{cases}
$$

The main contribution of this paper is to answer the following question.
If $\Omega^{ \pm}$and $D^{ \pm}$satisfy (4.2) then what is the relation between these domains?
Lemma 4.3. If $\Omega^{ \pm}$and $D^{ \pm}$satisfy (4.2) then for $\Omega:=\Omega^{+} \cup \Omega^{-}$and $D:=$ $D^{+} \cup D^{-}$, there is a measure $\nu$ with compact support such that $\Omega, D \in$ $Q(\nu, \widetilde{H})$ and $\operatorname{supp}(\nu) \subset \overline{\Omega \cap D}$. Moreover, $\bar{\Omega} \cap \bar{D} \neq \emptyset$ and

$$
\begin{equation*}
\lambda^{+}\left|\Omega^{+}\right|-\lambda^{-}\left|\Omega^{-}\right|=\lambda^{+}\left|D^{+}\right|-\lambda^{-}\left|D^{-}\right| \tag{4.4}
\end{equation*}
$$

Concerning the uniqueness we take into account another important properties for the domains. We assume that the domains are solid otherwise one loses the uniqueness, see [34] and [59]. Now by considering the sign conditions

$$
\Omega^{ \pm}:=\{ \pm u>0\}, \quad D^{ \pm}:=\{ \pm v>0\}
$$

then it is easy to get $u=v$ and $\Omega^{ \pm}=D^{ \pm}$, just by applying the maximum principle. The main theorem of this paper is as follows.

Theorem 4.4. Suppose that $u$ and $v$ are two solutions of (4.3) with the corresponding domains $\Omega^{ \pm}$and $D^{ \pm}$. Let $\Omega^{-} \subset\{u<0\}$ and $D^{+} \subset\{v>0\}$ and suppose that $\overline{\Omega^{-} \cup D^{+}}{ }^{c}$ is connected. Then $\Omega^{ \pm}=D^{ \pm}$.

We briefly explain the idea of the proof. First we can see that $v$ is nonnegative in $\mathbf{R}^{n} \backslash D^{-}$and $u$ is non-positive in $\mathbf{R}^{n} \backslash \Omega^{+}$. Then let $\Omega \cup D \subset B_{R}$ and $w=u-v$. We can prove that

$$
\begin{cases}\Delta w \geq 0 & \text { in } L=B_{R} \backslash \overline{\left(D^{+} \cup \Omega^{-}\right)} \\ w \leq 0 & \text { on } \partial L\end{cases}
$$

Finally by using the strong maximum principle it is verified that $u=v$ in $L$. For the last part it is easy to show that

$$
\Omega^{+} \cup D^{-} \subset D^{+} \cup \Omega^{-}
$$

On the other hand we have

$$
\Omega^{+} \cap \Omega^{-}=D^{+} \cap D^{-}=\emptyset
$$

Hence $\Omega^{+} \subset D^{+}$and $D^{-} \subset \Omega^{-}$which contradict (4.4) and this proves the theorem.

## 5. Overview of Paper II

The main contribution of this paper is to find numerical schemes for the solution of the following FBP, arising from the one phase quadrature domain theory.

Find $u$ and $\Omega=\{x: u(x)>0\}$ such that

$$
\begin{cases}\Delta u=\chi_{\Omega}-\mu, & \text { in } \quad \mathbf{R}^{n},  \tag{P}\\ u \geq 0, & \text { in } \quad \mathbf{R}^{n}, \\ u=\|\nabla u\|=0, & \text { in } \quad \mathbf{R}^{n} \backslash \Omega \\ \operatorname{supp}(\mu) \subset \Omega . & \end{cases}
$$

Here $\mu$ is a Radon measure with compact support and $\Omega$ is its corresponding quadrature domain. The function $u$ and $\Omega$ are unknown and $\partial \Omega$ is the free boundary.

The first numerical approach is based on blow-up techniques and we present a link between the theoretical and numerical parts of the free boundary theory. The second method is constructed by employing the shape optimization techniques. See [50], [51] and [61] for further reading about the level set method and [67] for the shape optimization analysis.

### 5.1 First Numerical Method

This method is mainly based on the properties of the free boundary and the level set method. In order to explain the idea, we look at the problem (P) in one dimension. We have

$$
\begin{equation*}
u^{\prime \prime}=1, \quad \text { in } \Omega \backslash \operatorname{supp}(\mu) . \tag{5.1}
\end{equation*}
$$

Now if we consider $x_{f}$ as the free boundary point and multiply (5.1) by $u^{\prime}$ and then integrate over $\left(x, x_{f}\right)$ we will end up with

$$
\frac{1}{2}\left(u^{\prime}\right)^{2}(x)=u(x)
$$

In higher dimensions, we prove that $\frac{|\nabla u(x)|}{\sqrt{2 u(x)}}$ goes to one when $x$ approaches the free boundary.

Theorem 5.1. Let $x_{0}$ be a regular free boundary point and $x \in\{u>0\}$ then

$$
\begin{equation*}
\lim _{x \rightarrow x_{0}} \frac{|\nabla u(x)|}{\sqrt{2 u(x)}}=1 \tag{5.2}
\end{equation*}
$$

To prove this theorem, one needs some techniques such as optimal growth, non-degeneracy and blow-up. For full description of these concepts, see [48].

Let $x_{0}=0$ be a regular point and $u_{0}$ be a global homogenous solution of degree two, then by Theorem 3.22 in [48] we deduce that $u_{0}(x)=\frac{1}{2}\left[(x \cdot e)^{+}\right]^{2}$ where $e$ is a unit vector. Therefore by rescalling the problem we get

$$
\left\|\frac{u(r x)}{r^{2}}-\frac{1}{2}\left[(x \cdot e)^{+}\right]^{2}\right\| \rightarrow 0
$$

which leads us to (5.2).
Now we would like to construct a sequence $\left(\Omega_{k}, u_{k}\right)$ which converges to the solution of Problem (P). To begin with, consider the following problem

$$
\begin{cases}\Delta u_{k}=1-\mu, & \text { in } \Omega_{k}  \tag{5.3}\\ \frac{\partial u_{k}}{\partial n_{k}}=-\theta u_{k}, & \text { on } \partial \Omega_{k}\end{cases}
$$

Theorem 5.1 states that $\theta u_{k}$ should behave as $\sqrt{2 u_{k}}$, so we choose

$$
\begin{equation*}
\theta=\left(\frac{2}{\sup _{\partial \Omega_{k-1}} u_{k-1}}\right)^{1 / 2} \tag{5.4}
\end{equation*}
$$

We mention that to get an appropriate value for $\theta$ in step $k$, let $\theta$ be as in (5.4) and solve (5.3) to obtain the value of $u_{k}$ on $\partial \Omega_{k}$. Then iterate the formula (5.4) to converge and obtain the optimal value for $\theta$.

By the level set method, the displacement of the boundary $\Omega(t)$ can be obtained by considering the following equation:

$$
\begin{equation*}
\frac{\partial \phi}{\partial t}+\zeta \sqrt{2 u(t)}=0, \quad \text { on } \partial \Omega(t) \tag{5.5}
\end{equation*}
$$

In order to use the level set method, one has to extend the velocity field. Let $v$ be the velocity extension to a big domain $\mathcal{T}$ such that $\operatorname{supp}(\mu) \subset \mathcal{T}$. This leads us to extract the level set formulation

$$
\begin{equation*}
\frac{\partial \phi}{\partial t}+v(t)=0, \quad \text { in } \mathcal{T} \backslash \operatorname{supp}(\mu) \tag{5.6}
\end{equation*}
$$

The first algorithm is given as follows. Choose a tolerance such that $T O L \ll$ 1 and let $\Omega_{0}$ be an initial guess. Compute $u_{k}$ on $\Omega_{k}$ which is the solution of (5.3). Then extend the velocity and find the level set function $\phi$ from (5.6) and get $\Omega_{k+1}$. If $\sup _{\partial \Omega_{k+1}}\left|u_{k+1}\right|<\mathrm{TOL}$, then stop otherwise, set $k=k+1$ and iterate the previous steps.

### 5.2 Second Numerical Method

One of the essential tools to design and construct an industrial structure is shape optimization. The shape optimization consist of finding a geometry which minimizes a given functional with specific constraint. From a mathematical point of view, in shape sensitivity we analyze how the solution of a PDE changes when the domain is changing with a velocity field. For further reading, see for instance [67].

To begin with, let $\Sigma \subset D$ where $D$ is an approximation for the corresponding quadrature domain w.r.t the measure $\mu$. Sakai in [60] has proved the existence of such a $D$. Consider the following minimization problem

$$
\begin{equation*}
\min _{\Sigma \subset D} E(\Sigma):=\int_{\Sigma} \frac{1}{2}\left|\nabla u_{\Sigma}\right|^{2} d x-\int_{\Sigma}(1-\mu) u_{\Sigma} d x \tag{5.7}
\end{equation*}
$$

We clarify the link between Problem $(\mathrm{P})$ and the minimization problem (5.7) by classical shape calculus. It is easy to derive that if $\Sigma$ has a sufficiently smooth boundary and $\mathbf{V}$ is a smooth velocity field then

$$
\begin{equation*}
d E(\Sigma, \mathbf{V})=-3 / 2 \int_{\Sigma} \operatorname{div}\left(\left|\nabla u_{\Sigma}\right|^{2} \mathbf{V}\right) d x=-3 / 2 \int_{\partial \Sigma}\left|\nabla u_{\Sigma}\right|^{2} \mathbf{V} \cdot \mathbf{n} d s \tag{5.8}
\end{equation*}
$$

We conclude that for a solution of Problem (P), $\nabla u_{\Sigma}$ is vanishing on $\partial \Sigma$. Therefore $d E(\Sigma, \mathbf{V})=0$, which means that the solution of Problem (P) is a critical point of the shape functional $E$.
Now in order to construct a scheme we consider an evolution $\Sigma_{t}=\Sigma(t)=$ $\{\varphi(x, t)<0\}$. Let $\partial \Sigma=\{\varphi(x, t)=0\}$ and we desire that if $t$ increases then $\Sigma_{t}$ converges to the solution of Problem (P), $\Omega$. This can be done by finding an appropriate velocity field. If $\Sigma_{t} \subset \Omega$ then $\frac{\partial u_{\Sigma_{t}}}{\partial \mathbf{n}}$ is negative on $\partial \Sigma_{t}$ because $d E$ changes sign at $\partial \Omega$. Hence we choose

$$
\mathbf{V}(\mathbf{t}) \cdot \mathbf{n}=-\frac{\partial u_{\Sigma_{t}}}{\partial \mathbf{n}}
$$

and consequently

$$
\begin{equation*}
d E\left(\Sigma_{t}, \mathbf{V}(\mathbf{t})\right)=3 / 2 \int_{\partial \Sigma_{t}}\left|\nabla u_{\Sigma_{t}}\right|^{2} \underbrace{\frac{\partial u_{\Sigma_{t}}}{\partial \mathbf{n}}}_{<0} d s<0 \tag{5.9}
\end{equation*}
$$

The second algorithm is as follows.
Set $t=0$ and choose an initial domain $\Sigma_{0}$ such that $\operatorname{supp}(\mu) \subset \Sigma_{0}$. The next step is to solve $\Delta u_{t}=1$ in $\Sigma_{t} \backslash \operatorname{supp}(\mu)$ with Dirichlet boundary condition $u_{t}=0$ on $\Gamma_{t}$ and compute $\mathbf{V} \cdot \mathbf{n}=-\nabla u_{t} \cdot \mathbf{n}_{\Gamma_{t}}$ where $u_{t}=u_{\Sigma_{t}}$. We need a stopping condition so if $\left\|\nabla u_{t}\right\|_{L^{2}(\Gamma)} \ll \epsilon$ then stop, otherwise move the free boundary in the normal direction. For instance, in dimension one we can apply $x_{t+1}=x_{k}-u^{\prime}\left(x_{t}\right)$ and get the new shape $\Sigma_{t+1}$, with the free boundary $\Gamma_{t+1}$. By iterating these processes we will get the numerical approximation.

## 6. Overview of Paper III

In this article, we investigate the following problem.
Let $\mu_{i}, i=1, \cdots, m$ be finite measures with compact supports and $\lambda_{i}(x)$ be non-negative Lipschitz continuous functions. Find functions $u_{i}$ and domains $\Omega_{i}:=\left\{x \in \mathbf{R}^{n} \mid u_{i}(x)>0\right\}$ for $i=1, \cdots, m$ such that $\operatorname{supp}\left(\mu_{i}\right) \subset \Omega_{i}$ and

$$
\begin{cases}\Delta u_{i}=\lambda_{i} \chi_{\Omega_{i}}-\mu_{i} & \text { in } \Omega_{i}  \tag{6.1}\\ u_{i}=0 & \text { on } \partial \Omega_{i} \\ \left|\nabla u_{i}\right|=\left|\nabla u_{j}\right| & \text { on } \Gamma_{i j}:=\partial \Omega_{i} \cap \partial \Omega_{j} \\ \left|\nabla u_{i}\right|=0 & \text { on } \partial \Omega_{i} \backslash \bigcup \Gamma_{i j}\end{cases}
$$

There is a vast literature on the problem for $k=1$ but to the best of our knowledge there is not much work for the case $k=2$.These cases arise from the theory of the quadrature domain which is quite well studied for the one phase case. In this article, we design two numerical methods and prove that the numerical approximation converges to the viscosity solution of the given problems (6.1) in the cases $k=1,2$. First we reformulate the problem and extract degenerate elliptic equations for the one and the two phase cases.

Suppose that $\operatorname{supp}(\mu) \subset \Omega$. We can prove that the viscosity solution of

$$
L\left(x, u, D u, D^{2} u\right):=\min (-\Delta u+\lambda-\mu, u)=0, \quad \text { (Min-Formula) }
$$

is a solution of

$$
\begin{cases}\Delta u=\lambda-\mu & \text { in } \Omega=\{u>0\}  \tag{6.2}\\ u \geq 0 & \text { in } \mathbf{R}^{n}\end{cases}
$$

and vice versa. Now we try to state a similar statement for the two phase case. Suppose that $u$ is the solution of

$$
\left\{\begin{array}{l}
\Delta u=\lambda^{+} \chi_{\Omega^{+}}-\mu^{+}-\left(\lambda^{-} \chi_{\Omega^{-}}-\mu^{-}\right), \quad \text { in } \mathbf{R}^{n}  \tag{6.3}\\
\Omega^{ \pm}=\{ \pm u \geq 0\}
\end{array}\right.
$$

We prove that $u$ satisfies the following non-linear equation in $\Omega=\Omega^{+} \cup \Omega^{-}$, which is called Min-Max formula,

$$
\begin{equation*}
L u:=\min \left(-\Delta u+\lambda^{+}-\mu^{+}, \max \left(-\Delta u-\lambda^{-}+\mu^{-}, u\right)\right)=0 \tag{6.4}
\end{equation*}
$$

We establish the relation between the two phase case of the problem and the Min-Max formula as follows,

- Equation (6.4) is a degenerate elliptic equation and has a unique viscosity solution.
- Suppose that $\mu^{ \pm}$are two Dirac measures. The weak solution of (6.3) is a viscosity solution of (6.4) and vice versa.


### 6.1 Numerical Methods

### 6.1.1 Degenerate elliptic scheme

Generally, at each grid point, $x_{i}$, we write a finite difference scheme, $L$ as follows

$$
L_{h}^{i}[u]=L_{h}\left[u_{i}, u_{i}-\left.u_{j}\right|_{j=N(i)}\right], \quad i=1, \ldots, N
$$

where $u_{j}=u\left(x_{j}\right)$ is shorthand for the list of neighbors $\left.u_{j}\right|_{j=N(i)}$. Also by $\overline{u_{i}}$ we mean the average of $\left.u_{j}\right|_{j=N(i)}$. For the sake of simplicity, we drop $h$ and write

$$
L^{i}[u]=L\left[u_{i}, u_{i}-u_{j}\right],
$$

The scheme $L$ is degenerate elliptic if each component $L^{i}[u]=L\left[u_{i}, u_{i}-u_{j}\right]$ is non-decreasing in each variable.

For example, we discretize Min formula in two dimensions and obtain the following degenerate elliptic scheme

$$
\begin{aligned}
0=L^{i}[u]: & =\min \left(-\Delta_{h} u_{i}+\lambda_{h}-\mu_{h}, u_{i}\right) \\
& =\min \left(4\left(u_{i}-\overline{u_{i}}\right)+\left(\lambda_{h}-\mu_{h}\right) h^{2}, u_{i}\right) \\
& =u_{i}+\min \left(-\overline{u_{i}}+\left(\lambda_{h}-\mu_{h}\right) h^{2} / 4,0\right),
\end{aligned}
$$

where $\mu_{h}$ is an appropriate discretization of $\mu$. Similarly it is easy to find the following discretization for Min-Max problem (6.4) where $\mu_{h}^{ \pm}$are appropriate discretizations of $\mu^{ \pm}$,

$$
\begin{align*}
L^{i}[u] & =L\left[u_{i}, u_{i}-u_{j}\right] \\
& =\min \left(\sum_{j=N(i)}\left(u_{i}-u_{j}\right)+\left(\lambda_{h}^{+}-\mu_{h}^{+}\right) h^{2},\right.  \tag{6.5}\\
& \left.\max \left(\sum_{j=N(i)}\left(u_{i}-u_{j}\right)-\left(\lambda_{h}^{-}-\mu_{h}^{-}\right) h^{2}, u_{i}\right)\right)=0 .
\end{align*}
$$

Lemma 6.1. The scheme (6.5) is degenerate elliptic, monotone, stable and consistent.

Corollary 6.2. Suppose that $\mu^{ \pm}$are Dirac measures. As a corollary of the previous lemma and Barles-Souganidis Theorem we obtain that the scheme (6.5) converges to the unique viscosity solution of (6.4).

We construct the first numerical algorithm for the two phase case based on Min-Max formulation. Suppose that a tolerance, TOL, and a big enough set $D$ are given. First of all, we find a discretization formula for the function $\lambda^{ \pm}$and the measures $\mu^{ \pm}$and apply the finite difference scheme (6.5)

$$
u_{i}^{k+1}=\max \left(\overline{u_{i}^{k}}+\frac{\mu_{h}^{+}-\lambda_{h}^{+}}{4} h^{2}, \min \left(\overline{u_{i}^{k}}+\frac{\lambda_{h}^{-}-\mu_{h}^{-}}{4} h^{2}, 0\right)\right)
$$

If $\inf _{x_{i} \in \Omega_{k}}\left|u_{i}^{k+1}-u_{i}^{k}\right| \leq$ TOL then stop, otherwise iterate the previous step. We note that to have a better acceleration for convergence one can apply the multi-grid method, see [32].

### 6.1.2 An iterative method for the general case

In general case of the problem we can construct another iterative method. Suppose that $u_{i}$ for $i=1, \cdots, m$ are the solutions for the problem (6.1) with the corresponding positivity sets $\Omega_{i}$. Similar to the first method we discretize $\Delta u_{i}=\lambda_{i} \chi_{\Omega_{i}}-\mu_{i}$ and iterate by the following process. If $X_{i} \in \operatorname{supp}\left(\mu_{i}\right)$ then

$$
\begin{equation*}
u_{i}^{(k+1)}\left(X_{i}\right)=\max \left(\bar{u}_{i}^{(k)}\left(X_{i}\right)-\sum_{j \neq i} \bar{u}_{j}^{(k)}\left(X_{i}\right)+\frac{\left(\mu_{j}-\lambda_{j}\right) h^{2}}{4}, 0\right) \tag{6.6}
\end{equation*}
$$

otherwise

$$
\begin{equation*}
u_{i}^{(k+1)}\left(X_{i}\right)=\max \left(\bar{u}_{i}^{(k)}\left(X_{i}\right)-\sum_{j \neq i} \bar{u}_{j}^{(k)}\left(X_{i}\right)-\frac{\lambda_{j} h^{2}}{4}, 0\right) \tag{6.7}
\end{equation*}
$$

This process leads us to a numerical approximation of the problem (6.1). The main idea of this method arises from the discretization of the two phase case. More precisely, if we discretize $\Delta\left(u^{+}-u^{-}\right)=\lambda^{+} \chi_{\Omega^{+}}-\mu^{+}-\left(\lambda^{-} \chi_{\Omega^{-}}-\mu^{-}\right)$ where $u^{ \pm}=\max ( \pm u, 0), \Omega=\{ \pm u>0\}$ and obtain $u^{ \pm}$by imposing the condition $u^{+}\left(X_{i}\right) u^{-}\left(X_{i}\right)=0$, then we end up with the discretization formulas (6.6) and (6.7), see [11].

## 7. Overview of Paper IV

In this paper, we study the quadrature domains in a subdomain of $\mathbf{R}^{n}$. Suppose that a measure $\mu$ with compact support and a fixed domain $K \subset \mathbf{R}^{n}$ are given. We investigate the existence and the uniqueness of the following problem

$$
\begin{cases}\Delta u=c \lambda L_{\Omega}-\mu, & \text { in } K  \tag{7.1}\\ u=0, & \text { in } \bar{K} \backslash \Omega \\ u \geq 0, & \text { in } K \\ \operatorname{supp}(\mu) \subset K . & \end{cases}
$$

Here $\Omega:=\{u>0\}, c>0$ and $\lambda$ is the Lebesgue measure. We always consider $\Gamma_{0}:=\partial \Omega \cap \partial K \neq \emptyset$, otherwise the problem has been studied by Shahgholian and Gustafsson, see [37]. The problem is also similar to the obstacle problem when $|\nabla u|=0$ on $\Gamma_{0}$. We use some potential theory techniques such as balayage to obtain the existence and the uniqueness of $\Omega$ along with some of its properties. We study an application of this problem and a numerical scheme to approximate the solution is also given.

For any $\Omega \subset K$ and a signed Radon measure $\nu$ with compact support, we define

$$
H_{K}(\Omega, \nu)=\left\{U_{K}^{\nu}: \operatorname{supp}(\nu) \subset K \backslash \Omega\right\}
$$

and $S_{K}(\Omega, \nu)=\left\{U_{K}^{\nu}, \operatorname{supp}(\nu) \subset K\right.$ and $\nu \leq 0$ in $\left.\Omega,\right\}$ where $U_{K}^{\mu}$ denotes the Green potential of $\mu$ on $K$.

Suppose that $K \subset \mathbf{R}^{n}$ and $\mu$ is a measure with compact support in $K$. We say $\Omega \subseteq K$ is a $K-(S) Q D$ if

$$
\begin{equation*}
\int_{\Omega} h d x=(\geq) \int h d \mu, \quad \forall h \in H_{K}(\Omega),\left(h \in S_{K}(\Omega)\right) \tag{7.2}
\end{equation*}
$$

and $\operatorname{supp}(\mu) \subset \Omega$. We call (7.2) a $K-(s u b)$ quadrature identity and $\Omega$ a $K$ (sub) quadrature domain and write $\Omega \in Q(\mu, K)(\Omega \in S Q(\mu, K))$. Obviously, $S Q(\mu, K) \subseteq Q(\mu, K)$.

### 7.1 Main Results

We can prove that if the measure is concentrated enough then $S Q(\mu, K) \neq \emptyset$.

Theorem 7.1. Suppose that $K \subset \mathbf{R}^{n}$ and $\mu$ is a positive measure with compact support in $K$ which satisfies

$$
\limsup _{r \rightarrow 0} \frac{\mu\left(B_{r}(x)\right)}{\lambda\left(B_{r}(x)\right)} \geq 2^{n}, \quad \text { for all } x \in \operatorname{supp}(\mu)
$$

Then $\operatorname{supp}(\mu) \subset \omega_{K}^{\mu}$ and $S Q(\mu, K) \neq \emptyset$.
The idea is to prove that $\omega_{K}^{\mu}:=\left\{V_{K}^{\mu}<U_{K}^{\mu}\right\}$ is a $K$ - SQD where $-\Delta V_{K}^{\mu}$ is called the partial balayage of $\mu$ on $K$. As a result, we can show that if $\Omega_{1}, \Omega_{2} \in S Q(\mu, K)$ then $\Omega_{1} \equiv \Omega_{2}$ up to a Lebesgue null set. The next lemma clarifies the relation between the PDE formulation (7.1) and the $K$ - SQDs.

Lemma 7.2. If $u:=U_{K}^{\mu}-V_{K}^{\mu}$, and $\Omega=\{u>0\}$ satisfy the problem (7.1) then $\Omega \in S Q(\mu, K)$.

To be more precise we can prove that

$$
\Omega=\{u>0\} \in S Q(\mu, K) \equiv \begin{cases}\Delta u=\chi_{\Omega}-\mu & \text { in } K \\ u=0 & \text { on } \partial \Omega \cup \partial K \\ |\nabla u|=0 & \text { on } \partial \Omega \cap K \\ u>0 & \text { in } \Omega\end{cases}
$$

The proof is a direct consequence of the definition of the balayage and the properties of the function $u=U_{K}^{\mu}-V_{K}^{\mu}$.

An application of this class of QDs is in the rock mechanics. Suppose that $K$ is a porous medium, like a rock, and $\operatorname{supp}(\mu)$ is a source of a fluid located in the rock. It is clear that the saturated part is unknown in advance and it depends on the time. One could apply Darcy's law and Baiocchi transform to derive a PDE formulation for the problem. Gustafsson in [33] has studied the behavior of the boundary of the saturated set when the source is located on $\partial K$. For $K=\mathbf{R}_{+}^{n}:=\left\{x \in \mathbf{R}^{n}: x_{1}>0\right\}$, the problem becomes the Hele-Shaw problem in the half space, see [9] and [10].

Theorem 7.3. Suppose that $K$ is a bounded region, as a rock, and let $\mu$ be a positive finite measure such that $\operatorname{supp}(\mu) \subset K$. If $\Omega$ is the saturated part of the rock at the time $t_{0}$, then $\Omega$ is a $K-S Q D$.

We also construct a numerical scheme based on the level set method. The main idea is to construct a sequence of domains $\left\{\Omega_{i}\right\}$, which satisfies a sequence of elliptic PDEs, see (7.3), and update $\partial \Omega_{i}$ with an appropriate velocity field. More precisely, if $\Omega_{i}$ is an approximation of $\Omega$ such that $\partial \Omega_{i} \cap \partial K \neq \emptyset$
then we obtain the corresponding function $u_{i}$ by solving

$$
\begin{cases}\Delta u_{i}=\chi_{\Omega_{i}}-\mu & \text { in } K  \tag{7.3}\\ u_{i}=0 & \text { on } \partial K \\ \left|\nabla u_{i}\right|=0 & \text { on } \partial \Omega_{i} \cap K\end{cases}
$$

To find an updated domain $\Omega_{i+1}$, we move $\partial \Omega_{i}$ with the velocity field $u_{i}$ in the normal direction and obtain $\partial \Omega_{i+1}$.

## 8. Overview of Paper V

Suppose that $B_{1}$ is the unit ball, $\Pi=\left\{x \in \mathbf{R}^{n}: x_{1}=0\right\}$ and $B_{1}^{+}=B_{1} \cap$ $\left\{x: x_{1}>0\right\}$. Let $\lambda^{ \pm}$be two positive constants, $0<p<1$ and $u^{ \pm}:=$ $\max \{ \pm u, 0\}$. In this paper we study minimizers of the functional

$$
\begin{equation*}
E(u):=\int_{B_{1}^{+}}\left(|\nabla u|^{2}+2\left(\lambda^{+}\left(u^{+}\right)^{p}+\lambda^{-}\left(u^{-}\right)^{p}\right)\right) d x \tag{8.1}
\end{equation*}
$$

over

$$
\mathcal{K}=\left\{u \in W^{1,2}\left(B_{1}^{+}\right): u=0 \text { on } B_{1} \cap \Pi \text { and } u=f \text { on } \partial B_{1}^{+} \backslash \Pi\right\}
$$

where $f \in W^{1,2}\left(B_{1}\right) \cap L^{\infty}\left(B_{1}\right)$. Let $M$ and $R$ be two positive constants and consider $0 \in \Gamma \cap \Pi$. Define

$$
P_{R}(M):=\left\{u: u \text { is a minimizer of } E \text { in } B_{R}^{+} \text {and }\|u\|_{L^{\infty}\left(B_{R}^{+}\right)} \leq M .\right\}
$$

The main theorem of this paper is as follows.
Theorem. Let $u \in P_{1}(M)$ in dimension two. Then, in a neighborhood of the origin, $u$ does not change the sign. Moreover, the free boundary is a $C^{1}$ graph with a modulus of continuity depending only on $M, \lambda^{ \pm}$and $p$.

In other words, the free boundary approaches the fixed one in a tangential manner. It is not hard to prove that the corresponding Euler Lagrange formulation of (8.1) is

$$
\begin{cases}\Delta u=p\left(\lambda^{+}\left(u^{+}\right)^{p-1} \chi_{\{u>0\}}-\lambda^{-}\left(u^{-}\right)^{p-1} \chi_{\{u<0\}}\right) & \text { in } B_{1}^{+} \cap\{u \neq 0\}  \tag{8.2}\\ u=f & \text { on } \partial B_{1}^{+} \backslash \Pi \\ u=0 & \text { on } B_{1} \cap \Pi\end{cases}
$$

Equation (8.2) reduces to the two-phase obstacle problem when $p=1$.
In [30] it is proved that the minimizer is in $C_{l o c}^{1, \beta-1}$ when $\beta=2 /(2-p)$ and in [46] it has been proved the $C^{1}$ regularity of the free boundary in dimension two. We use the notation $\Omega^{+}=\{u>0\}, \Omega^{-}=\{u<0\}, \Gamma^{ \pm}=\partial \Omega^{ \pm}$, $\Gamma=\Gamma^{+} \cup \Gamma^{-}$and refer to $\Gamma$ as the free boundary.

First we prove that any minimizer is Hölder continuous up to the fixed boundary by applying the Morrey's embedding theorem, see [29]. We use the following lemma to get the $C^{1, \alpha}$ regularity up to the fixed boundary, see [47]. The idea of the proof could be find in [13].

Lemma 8.1. Let $u \in H^{1}\left(B_{1}^{+}\right)$. Assume there exist two constants $C$ and $\alpha$ such that for each $x_{0} \in B_{\frac{1}{2}}^{+}$there is a vector $A\left(x_{0}\right)$ with the property

$$
\begin{equation*}
\int_{B_{r}\left(x_{0}\right) \cap B_{1}^{+}}\left|\nabla u(x)-A\left(x_{0}\right)\right|^{2} d x \leq C r^{n+2 \alpha}, \quad \text { for every } r<\frac{1}{2} \tag{8.3}
\end{equation*}
$$

Then $u \in C^{1, \alpha}\left(\overline{B_{\frac{1}{2}}^{+}}\right)$and we have the estimate

$$
\|u\|_{C^{1, \alpha}\left(\overline{B_{\frac{1}{2}}}\right)} \leq C_{0}(C) .
$$

Let $x_{0} \in \Gamma$ and consider the following rescaled function for a given minimizer $u$ of (8.1)

$$
u_{x_{0}, r}(x)=\frac{u\left(x_{0}+r x\right)}{r^{\beta}}, \quad \beta=\frac{2}{2-p}, \quad r>0
$$

In the case $x_{0}=0$, we use the notation $u_{r}=u_{0, r}$. Consequently if we can find a sequence $u_{x_{0}, r_{j}}, r_{j} \rightarrow 0$ such that

$$
u_{x_{0}, r_{j}} \rightarrow u_{0} \quad \text { in } C_{\mathrm{loc}}^{1}\left(\mathbf{R}^{n} \cap\left\{x_{1}>0\right\}\right)\left(\text { or } C_{\mathrm{loc}}^{1}\left(\mathbf{R}^{n}\right)\right)
$$

we obtain $u_{0}$ as the blow-up of $u$ at $x_{0}$. It is easy to see that $u_{0}$ is a global minimizer of (8.1), i.e., a minimizer in $\mathbf{R}^{n} \cap\left\{x_{1}>0\right\}$ or in $\mathbf{R}^{n}$.

In this case the Weiss's monotone function is considered as
$W\left(r, x_{0}, u\right)=r^{-2 \beta} \int_{B_{r}^{+}\left(x_{0}\right)}\left(|\nabla u|^{2}+2 G(u)\right) d x-\frac{\beta}{r^{1+2 \beta}} \int_{\partial B_{r}^{+}\left(x_{0}\right)} u^{2}(x) d s$,
for $G(u)=2 \lambda^{+}\left(u^{+}\right)^{p}+2 \lambda^{-}\left(u^{-}\right)^{p}$ and $r>0$. Then as we mentioned before it implies that any blow-up is homogenous of degree $\beta$. We also can prove a nondegeneracy property. Suppose that $u$ is a minimizer of (8.1) and $x_{0} \in \Gamma^{+} \cap \Pi$. Then for some constant $c^{+}=c^{+}\left(\lambda^{+}\right)$

$$
\begin{equation*}
\sup _{\partial B_{r}^{+}\left(x_{0}\right) \cap \Omega^{+}} u \geq c^{+} r^{\beta}, \quad 0<r<\frac{1}{2} \tag{8.4}
\end{equation*}
$$

Consequently we obtain that the limit of free boundary points are always free boundary points. Then by using these two important properties in dimension two, we deduce that for $u \in P_{\infty}(M)$ one of the following holds:

1. $u(x)=c^{+}\left(x_{1}^{+}\right)^{\beta}$, for the one phase non-negative points.
2. $u(x)=-c^{-}\left(x_{1}^{-}\right)^{\beta}$, for the one phase non-positive points.

By the above categorization we can prove that the origin is a one phase point for $u \in P_{1}(M)$.

### 8.1 Main Result

We note that any free boundary point, say the origin, that touches the flat part is a one phase point. It means that the function $u$ has a sign close to the origin. We can prove that the free boundary has a normal close to $e_{1}$ when it approaches П.

Proposition 8.2. Let $u \in P_{1}(M)$. For any $\delta>0$ there are $\varepsilon=\varepsilon\left(\lambda^{ \pm}, p, M, \delta\right)$ and $\rho=\rho\left(\lambda^{ \pm}, p, M, \delta\right)$ so that $x \in \Gamma$ and $x_{1}<\varepsilon$ imply

$$
\Gamma \cap B_{\rho}^{+}(x) \subset K_{\delta}(x) \cap B_{\rho}^{+}(x)
$$

Here $K_{\delta}(z)=\left\{\left|x_{1}-z_{1}\right|<\delta \sqrt{\left(x_{2}-z_{2}\right)^{2}+\cdots+\left(x_{n}-z_{n}\right)^{2}}\right\}$. To prove the main theorem we have to show that the normal of $\Gamma$ at a point $x$, i.e., $\nu_{x}$ is uniformly continuous. Alt and Philips in [2] proved that the free boundary is a $C^{1}$-graph far from the flat part and on the other hand by the previous proposition we know that the normal of the free boundary approaches $e_{1}$ as we approach $\Pi$. These all statements, assure that the free boundary is a uniform $C^{1}$-graph up to $\Pi$.

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