Statistical analysis of empirical pairwise copulas for the S&P 500 stocks

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July 2012
Abstract

It is of great importance to find an analytical copula that will represent the empirical lower tail dependence. In this study, the pairwise empirical copula are estimated using data of the S&P 500 stocks during the period 2007-2010. Different optimization methods and measures of dependence have been used to fit Gaussian, t and Clayton copula to the empirical copulas, in order to represent the empirical lower tail dependence. These different measures of dependence and optimization methods with their restrictions, point at different analytical copulas being optimal. In this study the t copula with 5 degrees of freedom is giving the most fulfilling result, when it comes to representing lower tail dependence. The t copula with 5 degrees of freedom gives the best representation of empirical lower tail dependence, whether one uses the 'Empirical maximum likelihood estimator', or 'Equal $\tau$' as an approach.

Keywords: Tail dependence, Tail concentration function, Measure of similarity, Copula, Archimedean, Kendall’s tau, Spearman’s rho, Gaussian, t copula, Clayton.
Acknowledgements

I want to thank my supervisors Associate professor Tatjana Pavlenko at KTH (Royal Institute of Technology) and Dr. Rudi Schäfer at University of Duisburg-Essen, for great and valuable discussions and guidance during this Master thesis.

Stockholm, July 2012

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Chapter 1

Introduction

Copula are widely used in the risk management industry, see [1] and [3]. For example, Gaussian copula has been given part of the blame for the subprime mortgage crisis 2008, see [2] and [14].

It will be investigated which analytical copula, Gaussian, t or Clayton copula, gives the best representation of the empirical lower tail dependence. As there is no true or false answer to this, a model choice has to be based on an understanding of what is of interest. The goal of this study is to find an analytical copula that, in the long term, describes the average empirical lower tail dependence.

This thesis continues a study by M.C. Münnix and R.Schäfer, see [6], in which they have analyzed the statistical dependency structure of the S&P 500 index, in the 4-year period from 2007 to 2010, using intraday data from the New York Stock Exchanges TAQ database. As this study is a continuation of [6], it is built on the same pairwise empirical copulas.
Starting with a short introduction to Copula. The idea behind copulas is to divide the cumulative distribution function of a multivariate random variable $\mathbf{X} = [X_1, \ldots, X_n]$ into two separate components. The two components are the marginal distributions of each random variable $X_i$, where $i=1,\ldots,n$, and the pure dependence between the $X_i$’s. The relation is described in eq. (2.1).

$$\text{joint pdf} = \text{marginal pdfs } + \text{ copula } \quad (2.1)$$

By the use of the cumulative distribution function $F_i$ a new random variable can be defined, called the rank of $X_i$:

$$U_i = F_i(X_i) \quad (2.2)$$

The rank of $X_i$, which is $U_i$, is a deterministic transformation of the random variable $X_i$. $U_i$ takes values in the interval $[0,1]$.

For a multivariate random variable $\mathbf{X}$, the joint distribution of the ranks $\mathbf{U}$, is the copula.

There are three properties an n-dimensional copula $C(\mathbf{u}) = C(u_1, \ldots, u_n)$ must fulfill, in order to be a distribution function on $[0,1]^n$ with standard uniform marginal distributions. So a function $C : [0,1]^n \to [0,1]$ is a n-copula if these properties are fulfilled,
1. $C(u_1, \ldots, u_n) = 0$ if at least one $u_i = 0$, for $u_i \in [0, 1]$.

2. $C(u_1, \ldots, u_n)$ is grounded and $n$-increasing. That means that every box whose vertices lie in the $[0, 1]^n$, for the $C$-volume is non-negative.

3. The margins of $C$ satisfies $C(1, \ldots, 1, u_i, 1, \ldots, 1)=u_i$ for all $i \in \{1, \ldots, n\}$, where $u_i \in [0, 1]$.

Abe Sklar proved that a copula $C$ is unique for a given distribution $F$, if its margins are continuous, see Sklar’s theorem,

**Theorem 1. (Sklar’s theorem)** Let $F$ be an $n$-dimensional distribution function with univariate margins $F_1, \ldots, F_n$. Then there exist an $n$-copula $C$ such that, for all $x = [x_1, \ldots, x_n]$ in $\mathbb{R} = [-\infty, \infty]$,

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)). \quad (2.3)$$

If $F_1, \ldots, F_n$ are all continuous margins, then $C$ is unique; otherwise $C$ is uniquely determined on the rank $\text{Ran } F_1 \times \cdots \times \text{Ran } F_n$, where $\text{Ran } F_i = F_i(\mathbb{R})$ denotes the rank of $F_i$, for $i = 1, \ldots, n$. Conversely, if $C$ is a copula, $F$ is a joint distribution function with margins $F_1, \ldots, F_n$, where $F_1, \ldots, F_n$ are univariate distribution functions [2]

**Proof.**

$$C(u_1, \ldots, u_n) = \mathbb{P}\{U_1 \leq u_1, \ldots, U_n \leq u_n\}
= \mathbb{P}\{F_1(X_1) \leq u_1, \ldots, F_n(X_n) \leq u_n\}
= \mathbb{P}\{X_1 \leq F_1^{-1}(u_1), \ldots, X_n \leq F_n^{-1}(u_n)\}
= F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n))$$

$\Box$

As this is a study of pairwise empirical copula of S&P 500 index, it is entitled to focus on bivariate copula, that is $n = 2$. By differentiating the expression above for the bivariate case, the copula density $c(u_1, u_2)$ is obtained,
\[ c(u_1, u_2) = \partial^2_{u_1 u_2} C(u_1, u_2) = \partial^2_{u_1 u_2} F(F_1^{-1}(u_1), F_2^{-1}(u_2)) \]
\[ = \partial^2_{x_1 x_2} F(F_1^{-1}(u_1), F_2^{-1}(u_2)) du_1 F_1^{-1}(u_1) du_2 F_2^{-1}(u_2) \]
\[ = \frac{\partial^2_{x_1 x_2} F(F_1^{-1}(u_1), F_2^{-1}(u_2))}{d_1 F_1(F_1^{-1}(u_1)) d_2 F_2(F_2^{-1}(u_2))} \]
\[ = \frac{f(F_1^{-1}(u_1), F_2^{-1}(u_2))}{f_1(F_1^{-1}(u_1)) f_2(F_2^{-1}(u_2))} \]

### 2.1 Fréchet Bounds

Fréchet bounds can be viewed as the boundary of the copula, the minimum and maximum value a copula can have. The two boundaries are named Fréchet upper bound \( M(u_1, \ldots, u_n) \) and Fréchet lower bound \( W(u_1, \ldots, u_n) \). They are also called comonotonicity and countermonotonicity copula.

The bivariate comonotonicity copula is perfect dependence, see formula.

\[ M(u_1, u_2) = \min(u_1, u_2). \tag{2.4} \]

And bivariate countermonotonicity copula is perfect negative dependence, see formula,

\[ W(u_1, u_2) = \max(u_1 + u_2 - 1, 0). \tag{2.5} \]

The limits for a copula \( C \) is such that,

\[ W(u_1, u_2) \leq C(u_1, u_2) \leq M(u_1, u_2). \tag{2.6} \]

See the bivariate comonotonicity and countermonotonicity copula in fig.2.1 [10].
2.2 Elliptical copula

Elliptical copulas, such as Gaussian and t copula will be studied. Both are symmetrical and flexible for a wide range of behaviors. To see the theory of multivariate elliptical distributions, see [1].

2.2.1 Gaussian copula

If the correlation matrices \( P = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \) and \( \Sigma = \begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix} \) are such that \( P = \varphi(\Sigma) \)\(^1\) for the two random variables \( X \) and \( Y \), where \( X \) is the Gaussian random variable \( X \sim N_2(\mu_1, \mu_2, \Sigma) \) and \( Y \) is the random variable \( Y \sim N_2([0, 0], P) \), then \( X \) and \( Y \) will have the same ranks \( U \). Using eq.(2.2) to obtain the ranks,

\[
\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} F_1(X_1) \\ F_2(X_2) \end{pmatrix} = \begin{pmatrix} \Phi(Y_1) \\ \Phi(Y_2) \end{pmatrix},
\]

where \( F_i \) is the distribution function for \( X_i \), and \( \Phi \) is the distribution function for \( Y_i \), were \( i = 1, 2 \). \( \Phi \) is the standard univariate normal distribution function.

\(^1\)Explanation in 3.1.1 Random Vectors and their Distribution [2]
Since $X$ and $Y$ have the same correlation coefficient $\rho$, such that $\rho = \rho(X_1, X_2) = \rho(Y_1, Y_2)$, and they have the same ranks $U$, their copulas will also be the same,

$$F(F^{-1}_1(X_1), F^{-1}_2(X_2)) = \Phi_\rho(\Phi^{-1}(U_1), \Phi^{-1}(U_2)), \quad (2.7)$$

where $F$ is the joint distribution function for the multivariate random variable $X$, and $\Phi$ is the joint distribution function of $Y$. Then one can write the bivariate Gaussian copula, such that,

$$C^\text{Ga}_\rho(u_1, u_2) = \Phi_\rho(\Phi^{-1}(u_1), \Phi^{-1}(u_2)). \quad (2.8)$$

### 2.2.2 t copula

If a multivariate random variable $X$ has continuous marginal distribution functions, one can then extract an implicit copula. This is the same theory, which has been applied in the previous section Gaussian copula.

For the bivariate t copula, the formula is,

$$C^\text{t}_{\nu, \rho}(u_1, u_2) = t^{\nu, \rho}(t^{-1}_\nu(u_1), t^{-1}_\nu(u_2)), \quad (2.9)$$

where $\nu$ is degrees of freedom, $\rho$ is the correlation coefficient and $P$ is the correlation matrix, $t_\nu$ is the standard univariate t distribution for each entry $X_i$ in the multivariate random variable $X \sim t_2(\nu, [0, 0], P)$. $t^{\nu, \rho}$ is the joint distribution function of the multivariate random variable $X$.

### 2.3 Archimedean copula

A n-dimensional copula $C$ is an Archimedean copula, if it fulfills the representation,

$$C(u_1, ..., u_n) = \Psi(\Psi^{-1}(u_1) + ... + \Psi^{-1}(u_n)), \quad (2.10)$$

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where $\Psi$ is the Archimedean copula generator function. The generator function has to be a non-increasing and continuous function $\Psi(x) : [0, \infty)$ for $x \in [0, 1]$, which satisfies the conditions $\Psi(0) = 1$ and $\lim_{x \to \infty} \Psi(x) = 0$ and is strictly decreasing on $[0, \Psi^{-1}(0)]$. The inverse $\Psi^{-1}$ is given by $\Psi^{-1}(x) = \inf\{u : \Psi^{-1}(u) \leq x\}.$

2.3.1 Clayton copula

Clayton copula is a so called Archimedean copula. Clayton copula is a heavy left tail copula, with the strength parameter $\theta$, for which $\theta \in [0, \infty)$, and $u \in [0, 1]$. It’s generator function $\Psi$ is a decreasing function from $[0, 1]$ onto $[0, \infty]$, satisfying the boundary conditions $\Psi(0) = \infty$ and $\Psi(1) = 0$,

$$\Psi(u) = (1 + u)^{-1/\theta}. \quad (2.11)$$

And the inverse of the generator function is as follows,

$$\Psi^{-1}(u) = u^{-\theta} - 1. \quad (2.12)$$

The expression for bivariate Clayton copula $C^C_\theta$ is as follows,

$$C^C_\theta(u_1, u_2) = [u_1^{-\theta} + u_2^{-\theta} - 1]^{-1/\theta}. \quad (2.13)$$

To obtain the copula density function $c^C_\theta$, eq.(2.13) is differentiated,

$$c^C_\theta(u_1, u_2) = (1 + \theta)[u_1^{-\theta} + u_2^{-\theta}]^{-1-\theta}[u_1^{-\theta} + u_2^{-\theta} - 1]^{-1/\theta - 2}. \quad (2.14)$$
Chapter 3
Optimization

In this study, two optimization methods have been used to fit the analytical copula density $c^A(u_1, u_2)$ to the empirical copula density $c^E(u_1, u_2)$. Those methods are Least Square and Empirical Maximum Likelihood Estimator.

3.1 Least Square

The objective is to adjust the parameter $p$ for an analytical copula density $c^A(u_1, u_2; p)$, so that the analytical copula best fits the $t$:th dimension of the empirical copula density $c^E_t(u_1, u_2)$, where $t$ is time and $t = 1, \ldots, T$. Indexing $A$ in $c^A(u_1, u_2; p)$ stands for Analytical, $A$ will be changed to Ga for Gaussian or C for Clayton, depending on which analytical copula being optimized. Similarly $p$ will be changed to the correlation coefficient $\rho$ or the strength parameter $\theta$.

In this thesis both $c^A(u_1, u_2; p)$ and $c^E_t(u_1, u_2)$ are copula densities that are discrete. The vectors $[u_1, u_2]$ consist of $k$ elements each. Together $u_1$ and $u_2$ forms a grid, with grid points $(u_1(i), u_2(j))$, where $i, j = 1, \ldots, k$. The difference between each point in $c^E_t$ and $c^A$ at time $t$ is $\delta^A_t(u_1, u_2, p)$, such that,

$$\delta^A_t(u_1, u_2, p) = c^E_t(u_1, u_2) - c^A(u_1, u_2; p).$$  \hspace{1cm} (3.1)

The sum of absolute difference of $\delta^A_t(u_1, u_2, p)$ at time $t$ for parameter $p$, is such that,

$$E^A_t(p) = \sum_{i=1}^{k} \sum_{j=1}^{k} |\delta^A_t(u_1(i), u_2(j), p)|.$$  \hspace{1cm} (3.2)
To obtain the optimal estimator of parameter $p$, that is $\hat{p}_t$, $E_t^A(p)$ will be minimized with respect to $p$, such that,

$$\hat{p}_t = \arg \min_p E_t^A(p), \quad (3.3)$$

where eq.(3.3) is minimized for the interval of values the parameter $p$ is defined for. Using $\hat{p}_t$ to define the optimal analytical copula $\hat{C}_t^A(u_1, u_2)$ for each time $t$, such that,

$$\hat{C}_t^A(u_1, u_2) = C^A(u_1, u_2; \hat{p}_t). \quad (3.4)$$

The $t$ copula depends on two parameters, correlation coefficient $\rho$ and degree of freedom $\nu$, here the parameters will be defined as $p^a$ and $p^b$. So from here on in this section, the analytical copula density will be defined as $c^A(u_1, u_2; p^a, p^b)$. A will also be changed to $t$ for $t$ copula.

This is the same method, as the method previously used in this section,

$$\delta_t^A(u_1, u_2, p^a, p^b) = c_t^E(u_1, u_2) - c^A(u_1, u_2; p^a, p^b). \quad (3.5)$$

The sum of absolute difference of $\delta_t^A(u_1, u_2, p^a, p^b)$ at time $t$ for parameter $p^a$ and $p^b$, is such that,

$$E_t^A(p^a, p^b) = \sum_{i=1}^{n} \sum_{j=1}^{n} |\delta_t^A(u_1(i), u_2(j), p^a, p^b)|. \quad (3.6)$$

Instead of one parameter, the sum of absolute difference $E_t^A$ now depends on two parameters, which means that $E_t^A$ has to be minimized two times, with respect to each parameter, for each time $t$. Firstly $E_t^A$ will be minimized with respect to $p^a$ for each $p^b$ being fixed, such that,

$$\hat{p}^a_t(p^b) = \arg \min_{p^a} E_t^A(p^a, p^b). \quad (3.7)$$
And secondly it will be minimized with respect to $p^b$, such that,

$$
\hat{p}_t^b = \arg \min_{p^b} E_t^A(\hat{p}_t, p^b). \tag{3.8}
$$

The obtained optimal parameters, $\hat{p}_t^a(\hat{p}_t^b)$ and $\hat{p}_t^b$, is then used to define the optimal analytical copula $\hat{C}_t^A$ for each time $t$,

$$
\hat{C}_t^A(u_1, u_2) = C^A(u_1, u_2; \hat{p}_t^a(\hat{p}_t^b), \hat{p}_t^b). \tag{3.9}
$$

### 3.2 Empirical Maximum Likelihood Estimator

The objective is now to adjust the parameter $p$ for each point $(u_1(i), u_2(j))$, $i, j = 1, \ldots, k$, on the analytical copula density $c^A$, so that each $c^A(u_1(i), u_2(j))$ best fits the same point on $c_t^E(u_1(i), u_2(j))$ at time $t$. The optimal parameters at time $t$ will then be defined as $\hat{p}_t(i, j)$. The difference between $c^A$ and $c_t^E$ at each point, for the different parameter values at each time $t$ is $\delta_t^A(i, j, p)$, such that,

$$
\delta_t^A(i, j, p) = c_t^E(u_1(i), u_2(j)) - c^A(u_1(i), u_2(j); p). \tag{3.10}
$$

The value of the parameter $p$ that minimizes the absolute difference of $\delta_t^A(i, j, p)$ is the optimal parameter for the point $(u_1(i), u_2(j))$, that is,

$$
\hat{p}_t(i, j) = \arg \min_p |\delta_t^A(i, j, p)|. \tag{3.11}
$$

In this study both $F_{\hat{p}_t}(p)$ and $f_{\hat{p}_t}(p)$ are discrete functions, but the theory will be given for continuous functions. The distribution function $F_{\hat{p}_t}(p)$ of $\hat{p}_t$ is such that,

$$
F_{\hat{p}_t}(p) = P(\hat{p}_t \leq p), \tag{3.12}
$$
where $p$ in eq.(3.12) can take values in the interval defined for the parameter $p$, depending on which analytical copula being optimized. The density function $f_{\hat{p}_t}(p)$ of $\hat{p}_t$ is then,

$$f_{\hat{p}_t}(p) = \frac{d}{dp} F_{\hat{p}_t}(p).$$

(3.13)

The parameter $p$ that maximizes the density function $f_{\hat{p}_t}(p)$ is the parameter that is occurring to the largest extent, and will be defined as $\hat{p}_t^{EMLE}$, where $EMLE$ indicates that the parameter is optimized with $Empirical Maximum Likelihood approach$. The optimal parameter is such that,

$$\hat{p}_t^{EMLE} = \arg \max_p f_{\hat{p}_t}(p).$$

(3.14)

Obtaining the optimal analytical copula $\hat{C}_A^t(u_1, u_2)$ at time $t$, by the use of $\hat{p}_t^{EMLE}$, such that,

$$\hat{C}_A^t(u_1, u_2) = C_A(u_1, u_2; \hat{p}_t^{EMLE}).$$

(3.15)

The method above is appropriate, if $c^A$ depends on one parameter, but if $c^A$ is the t copula density, then it has to be optimized with respect to its second parameter as well. This parameter would be $p^b$, if $p = [p^a, p^b]$. For the t copula, $\hat{p}_t^{b,EMLE}$ will be the fixed degree of freedom for all $t$.

For each point $(u_1(i), u_2(j))$, there will be two estimators $\hat{p}_t^a(i, j; \hat{p}_t^{EMLE})$ and $\hat{p}_t^b(i, j)$. The most frequent $p^b$ at time $t$, $t = 1, \ldots, T$, will be defined as $\hat{p}_t^{b,EMLE}$. And the most frequent estimator $\hat{p}_t^{b,EMLE}$ during the period $[1,T]$ will be defined as $\hat{p}_t^{b,EMLE}$. For the t copula, $\hat{p}_t^{b,EMLE}$ will be the fixed degree of freedom for all $t$.

Obtaining the optimal analytical copula $\hat{C}_A^t(u_1, u_2)$ at time $t$, by the use of $\hat{p}_t^{EMLE}$ and $\hat{p}_t^{b,EMLE}$, such that,

$$\hat{C}_A^t(u_1, u_2) = C_A(u_1, u_2; \hat{p}_t^{EMLE}, \hat{p}_t^{b,EMLE}).$$

(3.16)
Chapter 4

Dependence

There are several ways to look at dependence and dependence structure in a copula, two of them are rank correlation and lower tail dependence. These measures will be used to get an idea of which analytical copula represents the empirical lower tail dependence.

4.1 Rank Correlation

Both Spearman’s $\rho_s$ and Kendall’s $\tau$ are rank correlation coefficients. Rank correlation is a measure of dependence that is extracted from the bivariate copula itself. The advantage is that one does not have to take marginal distributions into consideration. For Gaussian copula, a relation between Kendall’s $\tau$ and Spearman’s $\rho_s$ has been proven, see [2].

Both $\rho_s$ and $\tau$ have many similarities, such as, they take a value in the interval $[-1, 1]$. If independent, they take the value zero, but zero rank correlation does not always mean independence. For the comonotonicity copula the rank correlation takes value 1, and for the counter-monotonicity copula the rank correlation takes value -1.

4.1.1 Kendall’s $\tau$

Kendall’s $\tau$ is a measure of concordance minus discordance. Application-wise, it can be explained as, probability of the assets prices simultaneously changing in the same direction minus the probability of prices changing in opposite direction. A definition of $\tau$ can be given as follows,
\[ \tau = \mathbb{P}\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - \mathbb{P}\{(X_1 - X_2)(Y_1 - Y_2) < 0\} \]
\[ = \mathbb{P}\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - (1 - \mathbb{P}\{(X_1 - X_2)(Y_1 - Y_2) > 0\}) \]
\[ = 2\mathbb{P}\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - 1, \]

where \((X_1, Y_1)\) are random variables, and \((X_2, Y_2)\) are independent copies.

The expression for Kendall’s \(\tau\) for a bivariate copula \(C(u_1, u_2)\) is \(\tau(C)\), such that,

\[ \tau(C) = 4\int_0^1 \int_0^1 C(u_1, u_2)dC(u_1, u_2) - 1, \quad (4.1) \]

where \(u_i = F_i(x_i)\), for \(i = 1, 2\). As this thesis is a numerical study, the calculations of \(\tau(C)\), eq.(4.1) has been done with numerical integration.

Kendall’s \(\tau\) will also be used as an approach to find an analytical copula that represents the empirical lower tail dependence. This approach is simply to use the same \(\tau\) for the analytical copula as the \(\tau\) for the empirical copula, and the approach will be named ‘Equal \(\tau\).

4.1.2 Spearman’s \(\rho_s\)

Spearman’s \(\rho_s\) is a measure to quantify the degree of association between two random variables, such as \(X_1\) and \(X_2\). If these random variables \(X_1\) and \(X_2\) have a copula \(C\), such that,

\[ C(u_1, u_2) = F(x_1, x_2), \quad (4.2) \]

where \(U_i\) is the rank of \(X_i\), for \(i = 1, 2\). Then Spearman’s \(\rho_s\) for the bivariate copula \(C\), can be defined as follows,

\[ \rho_s(C) = \frac{\int \int_{[0,1]^2} C(u_1, u_2)du_1du_2 - \int \int_{[0,1]^2} \prod(u_1, u_2)du_1du_2}{\int \int_{[0,1]^2} M(u_1, u_2)du_1du_2 - \int \int_{[0,1]^2} \prod(u_1, u_2)du_1du_2}, \quad (4.3) \]
Integration of Fréchet’s upper bound, \( \int_0^1 \int_0^1 M(u_1, u_2) du_1 du_2 = 1/3 \), and integration of independence copula \( \int_0^1 \int_0^1 \prod(u_1, u_2) du_1 du_2 = 1/4 \), this gives the expression for Spearman’s \( \rho_s \) for the copula \( C \),

\[
\rho_S(C) = 12 \int_{[0,1]^2} C(u_1, u_2) du_1 du_2 - 3. \tag{4.4}
\]

### 4.1.3 Kendall’s \( \tau \) and Spearman’s \( \rho_s \) for Archimedean copula

For an Archimedean copula \( C \), Kendall’s \( \tau \) can be obtained from the Archimedean copula generator function \( \Psi(u) \), see eq.(4.5), see [3].

\[
\tau(C) = 1 + 4 \int_0^1 \frac{\Psi(u)}{\Psi'(u)} dt. \tag{4.5}
\]

It appears to be no simple expression for Spearman’s \( \rho_s \), with the use of the generator function \( \Psi(u) \), see [13].

**Clayton**

From expression eq.(4.5), \( \tau(C^C) \) will be obtained, where \( C^C \) is the Clayton copula. The derivative of the generator function for Clayton copula is \( \Psi' \), such that,

\[
\Psi'(u) = -\frac{1}{\theta}(1 + u)^{-(1+\theta)}. \tag{4.6}
\]

Expression for the generator function for the Clayton copula, see eq.(2.11). Inserting eq.(4.6) & (2.11) into eq.(4.5) to obtain \( \tau(C^C) \), such that,

\[
\tau(C^C) = 1 + 4 \int_0^1 u^{\theta+1} - 1 \frac{1}{\theta} du = 1 + \frac{4}{\theta} \left( \frac{1}{\theta + 2} - \frac{1}{2} \right) = \frac{\theta}{\theta + 2}. \tag{4.7}
\]
4.2 Lower Tail Dependence

Tail dependence $\lambda$ is a measure of pairwise dependence, it only depend on the copula of a pair of random vectors $X_1$ and $X_2$, with continuous marginal distribution functions $F_1$ and $F_2$. Tail dependence is a measure of strength of the dependence, in the tail of the bivariate distribution. Lower tail dependence $\lambda_l$ is the probability that $X_2$ is equal or less then it’s $q$-quantile $F_2^{-1}(q)$, given that $X_1$ is equal or less then it’s $q$-quantile $F_1^{-1}(q)$.

Definition of tail dependence,

\[
\lambda_l := \lambda_l(X_1, X_2) = \lim_{q \to 0^+} \frac{\mathbb{P}\{X_2 \leq F_2^{-1}(q) | X_1 \leq F_1^{-1}(q)\}}{\mathbb{P}\{X_1 \leq F_1^{-1}(q)\}} = \lim_{q \to 0^+} \frac{\mathbb{P}\{X_2 \leq F_2^{-1}(q), X_1 \leq F_1^{-1}(q)\}}{\mathbb{P}\{X_1 \leq F_1^{-1}(q)\}} = \lim_{q \to 0^+} \frac{C(q, q)}{q}.
\]

This shows the relation for lower tail dependence $\lambda_l$ with the use of copula $C$, namely,

\[
\lambda_l = \lim_{q \to 0^+} \frac{C(q, q)}{q}. \quad (4.8)
\]

For numerical calculation of $\lambda_l$, the expression for $\lambda_l$ with copula $C$ in the denominator will be used. $C(1, q) = q$ is valid, because for a copula to exist, it have to fulfill the 3rd copula property in Chap 2, that gives,

\[
\lambda_l = \lim_{q \to 0^+} \frac{C(q, q)}{C(1, q)} \quad (4.9)
\]

4.3 Modeling with Measures of Dependence

In order to determine which analytical copula to use for representation of the empirical lower tail dependence, one has to use different measures of
dependence for the analytical and empirical copulas, and compare them together with their own lower tail dependence.

4.3.1 Lower Tail Dependence vs. Rank Correlation

It is important to understand the relation between lower tail dependence and correlation level for each analytical copula, and to see how they represent the empirical lower tail dependence. This can be done in several ways, two ways will be done in this study. One way is to do a scatter plot, with correlation coefficient on one axis, and lower tail dependence on the other axis. The other way is to sort the correlation coefficient for the empirical copula, and present it as a time series with lower tail dependence on one axis, and time on the other axis, where day one has the lowest correlation coefficient and the last day has the highest correlation coefficient.

In sample figure fig.4.1 lower tail dependence is plotted on the y-axis and Kendall’s $\tau$ is plotted on the x-axis for the empirical copula and the Gaussian, t and Clayton copula, optimized with Least Square.
Figure 4.1: Scatter plot for lower tail dependence vs. rank correlation. Lower tail dependence $\lambda_l$ for $q = 0.01$ on y-axis and Kendalls $\tau$ on x-axis. The black stars is empirical lower tail dependence $\lambda_l^E$ vs. it’s rank correlation coefficient $\tau(C^E)$. The analytical $\lambda_l^A$ are obtained from the analytical copulas $\hat{C}_l^A$ optimized with Least Square.

4.3.2 Lower Tail Dependence vs. Measure of Similarity

This test is a proximity test between two copulas, based on rank correlation. As it actually is Spearmans’s $\rho_s$, see eq.(4.3), for which $C_2$ is the independence copula $\prod(u_1, u_2) = u_1 \cdot u_2$. Measure of Similarity $Q_t$ is defined as follows,

$$Q(C_1, C_2) = \frac{\int \int_{[0,1]^2} C_1(u_1, u_2) du_1 du_2 - \int \int_{[0,1]^2} C_2(u_1, u_2) du_1 du_2}{\int \int_{[0,1]^2} M(u_1, u_2) du_1 du_2 - \int \int_{[0,1]^2} C_2(u_1, u_2) du_1 du_2}. \quad (4.10)$$

In eq.(4.10), $C_2$ will be the empirical copula $C_l^E$ and $C_1$ will be the analytical copula $\hat{C}_l^A$. In this study, the relation between $Q(C_1, C_2)$ and lower
tail dependence will be investigated.

Measure of Similarity $Q_t$ will also be used as an approach to find an analytical copula, that represents the empirical lower tail dependence. This approach will be named 'Minimizing $Q_t$', which is choosing the analytical copula, that minimizes the measure $Q_t$.

4.3.3 Tail Concentration Functions LR($q$) & UD($q$)

$LR(q)$ is the 'Tail Concentration Function', and it can be defined with reference to how much probability contained in the region of the rectangular $[0,0]$, $[0,q]$, $[q,0]$ and $[q,q]$, for $q \in [0,1]$. In this study though, the rectangular will be set by $[0.01,0.01]$, $[0.01,q]$, $[q,0.01]$ and $[q,q]$, for $q \in [0.01,0.99]$. $LR(q)$ is the min function of left tail concentration function $L(q)$ and right tail concentration function $R(q)$. For a copula $C(u_1, u_2)$, the two random variables, $U_1$ and $U_2$ takes values in the interval $[0,1]$. The definitions for the left and right tail concentration functions are as follows,

$$L(q) = \frac{\mathbb{P}\{U_1 < q, U_2 < q\}}{q}, \quad (4.11)$$

and,

$$R(q) = \frac{\mathbb{P}\{U_1 > q, U_2 > q\}}{(1 - q)}, \quad (4.12)$$

where $LR(q)$ is the min function of $L(q)$ and $R(q)$, such that,

$$LR(q) = min(L(q), R(q)). \quad (4.13)$$

$UD(q)$ is the Tail Concentration Function for anti-correlated events, where $U(q)$ is the Upper tail concentration function, and $D(q)$ is the Down tail concentration function, such that,

$$U(q) = 1 - \frac{\mathbb{P}\{U_1 < q, U_2 < 1 - q\}}{(1 - q)}, \quad (4.14)$$
and,

\[ D(q) = \frac{\mathbb{P}\{U_1 < 1 - q, U_2 < q\}}{q}, \quad (4.15) \]

where \( UD(q) \) is the \( \text{min} \) function of \( U(q) \) and \( D(q) \), such that,

\[ UD(q) = \text{min}(U(q), D(q)). \quad (4.16) \]

By using the tail concentration functions, it is possible to see the difference in tail behavior on probability basis between the empirical copula \( C^E_t \) and the analytical copula \( \hat{C}^A_t \). \( UD(q) \) is not of main interest in this study, but will be analyzed. These relations, \( LR(q) \) and \( UD(q) \), are from chapter 4.2, see [9].
Chapter 5

Simulation

5.1 Test of numerical methods

The empirical copula $C^E$ is an integration from empirical copula density $c^E$, and measures such as rank correlation coefficient Kendall’s $\tau$ for $C^E$, which is $\tau(C^E)$, are based on the empirical copula $C^E$. Therefore, one would like to see if this numerical solution brings an error with it. One can not use the empirical copula to find this error, if such an error exists. What one can do is to replicate the method used for the empirical data, but with the use of an analytical copula density. And then comparing the analytical solution for the analytical copula, with it’s numerical solution, the difference would then be the error.

5.1.1 Kendall’s $\tau$, $\tau(C^A)$

For an analytical copula $C^A$, the difference between the numerically calculated $\tau^n(C^A)$ and the analytical solution for $\tau^a(C^A)$ is defined as $\delta \tau(C^A)$, where $n$ indicates numerical solution in $\tau^n$, and $a$ indicates analytical solution in $\tau^a$. Normalizing this difference $\delta \tau(C^A)$ with the analytical solution $\tau^a(C^A)$ and defining this normalized difference as $\Delta \tau(C^A)$, such that,

$$
\Delta \tau(C^A) = \frac{\tau^n(C^A) - \tau^a(C^A)}{\tau^a(C^A)} = \delta \tau(C^A) / \tau^a(C^A),
$$

(5.1)

where $\tau^n(C^A)$ is numerically integrated, based on eq.(5.2),

$$
\tau^n(C^A) = 4 \int_0^1 \int_0^1 C^A(u_1, u_2) dC^A(u_1, u_2) - 1.
$$

(5.2)
The analytical expression of $\tau^a(C^A)$ for Gaussian, t and Clayton copula can be seen in tab.5.1.

<table>
<thead>
<tr>
<th>Analytical Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\pi}{2} \arcsin(\rho)$</td>
</tr>
<tr>
<td>$\frac{\pi}{2} \arcsin(\rho)$</td>
</tr>
<tr>
<td>$\frac{\pi}{\theta + 2}$</td>
</tr>
</tbody>
</table>

Table 5.1: Analytical expression for Kendall’s $\tau$ for the analytical copula, based on the correlation coefficient $\rho$ or the strength parameter $\theta$, depending on which copula.

$\Delta \tau(C^A)$ and $\delta \tau(C^A)$ are presented in fig.5.1 - 5.4, for Gaussian, t and Clayton copula. Noties that values of $\Delta \tau(C^A)$ is given on left y-axis, while values of $\delta \tau(C^A)$ is given on right y-axis.
Figure 5.1: On the left y-axis the normalized difference $\Delta \tau(C^{Ga})$ and on the right y-axis the difference $\delta \tau(C^{Ga})$. On the x-axis the correlation coefficient $\rho$. Here one can see how the difference between the numerically calculated $\tau^n(C^{Ga})$ and the analytical solution for $\tau^a(C^{Ga})$ behaves for different $\rho$. 
Figure 5.2: On the left y-axis the normalized difference $\Delta \tau(C_t)$ and on the right y-axis the difference $\delta \tau(C_t)$. On the x-axis the correlation coefficient $\rho$. Here one can see how the difference between the numerically calculated $\tau^n(C_t)$ and the analytical solution for $\tau^a(C_t)$ behaves for different $\rho$. This is for $t$ copula $C_{5,\rho}$.

In tab.5.1 the analytical expression $\tau^a(C^{Ga}) = \tau^a(C^t)$, which would mean that Kendall’s $\tau$ for Gaussian and $t$ copula are the same for the same correlation coefficient. As one an see, this is not the case, compare fig.(5.1) with fig.(5.2). But as degree of freedom $\nu$ increase for $t$ copula, the difference between $\tau^a(C^t)$ and $\tau^n(C^t)$ decrease. See fig.(5.3) in which $\nu = 100$, and compare with fig.(5.1).
Figure 5.3: On the left $y$-axis the normalized difference $\Delta \tau(C_t)$ and on the right $y$-axis the difference $\delta \tau(C_t)$. On the $x$-axis the correlation coefficient $\rho$. Here one can see how the difference between the numerically calculated $\tau^n(C_t)$ and the analytical solution for $\tau^a(C_t)$ behaves for different $\rho$. This is for $t$ copula $C^t_{100,\rho}$. 
Figure 5.4: On the left y-axis the normalized difference $\Delta \tau(C^C)$ and on the right y-axis the difference $\delta \tau(C^C)$. On the x-axis the strength parameter $\theta$. Here one can see how the difference between the numerically calculated $\tau^a(C^C)$ and the analytical solution for $\tau^a(C^C)$ behaves for different $\theta$.

Fréchet’s Bound

It is also important that Fréchet’s lower and upper bounds have the value -1 and 1, for both Spearman’s $\rho$, and Kendall’s $\tau$. $\rho_s$ is numerically integrated, based on eq.(5.3), and $\tau$ is numerically integrated, based on the equations eq.(5.8).

**Spearman’s $\rho$**

Numerical integration of eq.(5.3), such that,

$$\rho_s(C) = \frac{\int \int_{[0,1]^2} C(u_1, u_2) du_1 du_2 - \int \int_{[0,1]^2} \Pi(u_1, u_2) du_1 du_2}{\int \int_{[0,1]^2} M(u_1, u_2) du_1 du_2 - \int \int_{[0,1]^2} \Pi(u_1, u_2) du_1 du_2}, \quad (5.3)$$

where $C$ will be exchanged to comonotonicity copula $M$ and counter-monotonicity copula $W$. 
Here the difference between the integrated continuous comonotonicity copula and the numerically integrated discrete comonotonicity copula will be shown. Integration of the continuous comonotonicity copula is such that,

$$\int \int_{[0,1]^2} M(u_1, u_2) du_1 du_2 = \frac{1}{3},$$  \hspace{1cm} (5.4)

for the integration limits $[0,1]^2$. By changing the integration limits to the discrete copula limit $U \in [0.01, 0.99]$, such that,

$$\int \int_{[0.01,0.99]^2} M(u_1, u_2) du_1 du_2 = 0.3233.$$

Obviously eq.(5.4) and eq.(5.5) give different results, because of different integration limits. Numerical integration of discrete comonotonicity copula for $U \in [0.01, 0.99]$ and $i, j = 1, \ldots, k$ for $k = 50$, is such that,

$$\sum_{i=1}^{k} \sum_{i=1}^{k} \frac{1}{k^2} M(u_1(i), u_2(j)) = 0.3334.$$

What is interesting though is if the numerically integrated eq.(5.6) deviates from eq.(5.5), which it does. By increasing $k$, one is able to decrease the difference between the outcome of eq.(5.6) and eq.(5.5). This deviation would obviously result in a Spearman’s $\rho$ greater than 1 for comonotonicity copula and smaller than -1 for countermonotonicity copula, which is not valid.

By modifying eq.(5.3), the numerically integrated Spearman’s $\rho$ is such that,

$$\rho^N_S(C) = \frac{\sum_{i=1}^{k} \sum_{i=1}^{k} \frac{1}{k^2} C(u_1(i), u_2(j)) - \sum_{i=1}^{k} \sum_{i=1}^{k} \frac{1}{k^2} \Pi(u_1(i), u_2(j))}{\sum_{i=1}^{k} \sum_{i=1}^{k} \frac{1}{k^2} M(u_1(i), u_2(j)) - \sum_{i=1}^{k} \sum_{i=1}^{k} \frac{1}{k^2} \Pi(u_1(i), u_2(j))},$$

(5.7)
where $n$ in $\rho_n^s(C)$ indicates numerical. The comonotonicity copula will now take the value $\rho_n^s(M) = 1$ and the countermonotonicity copula will take the value $\rho_n^s(W) = -1$.

**Kendall’s $\tau$**

Kendall’s $\tau$ is obtained by numerical integration of eq.(5.8), such that,

$$\tau(C) = 4 \int_0^1 \int_0^1 C(u_1, u_2)dC(u_1, u_2) - 1.$$  \hspace{1cm} (5.8)

One can get hold of comonotonicity copula and countermonotonicity copula by the Gaussian copula. Comonotonicity copula is the Gaussian copula with $\rho = 1$, such that $M = C_1^{Ga}$ and $m = c_1^{Ga}$. And countermonotonicity copula is the Gaussian copula with $\rho = -1$, such that $W = C_{-1}^{Ga}$ and $w = c_{-1}^{Ga}$.

The numerically integrated Kendall’s $\tau$ for the discrete comonotonicity copula is such that,

$$\tau^n(M) = 4 \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{1}{k^2} M(u_1(i), u_2(j))m(u_1(i), u_2(j)) - 1 = 0.9987,$$  \hspace{1cm} (5.9)

and for countermonotonicity copula, Kendall’s $\tau$ is such that,

$$\tau^n(W) = 4 \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{1}{k^2} W(u_1(i), u_2(j))w(u_1(i), u_2(j)) - 1 = -0.9987.$$  \hspace{1cm} (5.10)

The values of $\rho_n^s$ and $\tau^n$ for comonotonicity and countermonotonicity copula, where $n$ indicates numerical can be seen in tab.5.2.

### 5.1.2 Tail Concentration Function LR(q) & UD(q)

To see if the tail concentration function $LR(q)$ and $UD(q)$ gives valid results, these functions will be calculated for Fréchet bounds. If the tail concentration function gives valid results, it should give $LR(q) = 1$ and $UD(q) = 0$ for all $q \in [0, 1]$ for Fréchet upper bound. And it should give $LR(q) = 0$ and
Rank correlation for Fréchet bounds

<table>
<thead>
<tr>
<th>Comonotonicity copula</th>
<th>Countermonotonicity copula</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_n^s )</td>
<td>1</td>
</tr>
<tr>
<td>( \tau_n^s )</td>
<td>-0.9987</td>
</tr>
</tbody>
</table>

Table 5.2: Rank correlation coefficients for Fréchet bounds, calculated with numerical integration. The deviation from the analytical solution, may be because of the numerical integration limits \([0.01, 0.99]^2\), while the limit for the analytical solution goes from \([0, 1]^2\).

\( UD(q) = 1 \) for all \( q \in [0, 1] \) for Fréchet lower bound. Result for \( LR(q) \) and \( UD(q) \) for Fréchet bounds, see fig. 5.5.

Figure 5.5: Tail concentration function \( LR(q) \) and \( UD(q) \) for \( q \in [0, 1] \) for Fréchet bounds, numerically calculated. The empirical tail concentration functions are based on the average empirical copula, for the time period 2007-2010.
5.2 Lower Tail Dependence

One is interested to know if there is a relation between rank correlation coefficient Kendall’s $\tau$ for the empirical copula, that is $\tau(C^E)$ and its lower tail dependence $\lambda^E_l$. The mean value $\mu_{\lambda_l}$ and the standard deviation $\sigma_{\lambda_l}$ for each year has been calculated for the empirical tail dependence $\lambda^E_l$. Mean value and standard deviation for $\tau(C^E)$ for each year, has also been calculated. A comparison between the behavior of $\lambda^E_l$ and $\tau(C^E)$ has been done. What one can see in tab.A.3 & A.4 is that the correlation level on average is almost the same for all years, except 2007, $\mu_{\tau}$ is slightly lower. For 2007, $\mu_{\lambda_l}$ is also lower, than $\mu_{\lambda_l}$ for other years. A behavior like that is something an analytical copula would indicate too, lower correlation level gives lower tail dependence. What is interesting though is the comparison between year 2008 and 2009. Both years have almost equal $\mu_{\tau}$ and $\sigma_{\lambda_l}$. The difference though between the two years is that $\sigma_{\tau_{2008}} > \sigma_{\tau_{2009}}$, while $\mu_{\lambda_{2008}} < \mu_{\lambda_{2009}}$. That sort of difference indicates that tail dependence is random for the empirical copula, or that one has to take other parameters in consideration.

A simple comparison between the analytical copulas and the empirical copulas will be done, through visual comparison between their normalized histograms for lower tail dependence. See fig.A.1-A.4 for the different analytical copulas and approaches used, such as Least Square or Empirical Maximum Likelihood Estimator, 'Equal $\tau$' and 'Minimizing $Q_t$'. See also sample fig.5.6. In this comparison correlation coefficients are not taken in consideration, it is rather a way of seeing what the different analytical copulas with different approaches can perform, in describing empirical lower tail dependence. Normalized histogram of lower tail dependence is plotted for $q = [0.01, 0.03, 0.05]$

As one can see, lower tail dependence $\lambda^G_l$ for Gaussian copulas underestimates the lower tail dependence $\lambda^E_l$ for empirical copula, on average for every $q$ value plotted. Although $\lambda^G_l$ is giving a better representation of the empirical lower tail dependence for $q = 0.05$, compared to $q = 0.01$.

The t copula has almost the same shape of its lower tail dependence density function as the empirical lower tail dependence density function, for all approaches used, such as Least Square, Empirical Maximum Likelihood Estimator, 'Equal $\tau$' and 'Minimizing $Q_t$', see fig.A.1-A.4.

Lower tail dependence for Clayton copula optimized with Least Square, estimate the lower tail dependence for the empirical copula fairly good. For the other approaches, Clayton copula over estimates empirical lower tail dependence.
See fig. 5.6, normalized histogram for lower tail dependence.

Figure 5.6: Normalized histogram for lower tail dependence for Gaussian copula, optimized with Least Square for \( q = [0.01(\text{left}), 0.03(\text{middle}), 0.05(\text{right})] \). Black contour is the lower tail dependence for empirical copula. Histogram is normalized with numbers of \( \lambda_l \) outcomes.

By the use of fig. 5.6 and figures alike, one can determine if an analytical copula with a certain approach, systematically under- or overestimates the empirical lower tail dependence, \( \lambda_l^E \).

5.3 Measure of Similarity, between two copulas

Using eq. (4.10), and by replacing \( C_2 \) with \( C_t^E \) and \( C_1 \) with \( \hat{C}_t^A \), which has been optimized with either Least Square, Empirical Maximum Likelihood Estimator or 'Equal \( \tau \)', one obtains the measure of similarity \( Q_t \), at time \( t \). In fig. A.11, one can see the values of \( Q_t \) as a time series.

Eq. (4.10) can be written as follows,

\[
Q_t(\hat{C}_t^A, C_t^E) = \frac{\int_{[0,1]^2} \hat{C}_t^A(u_1, u_2)du_1du_2 - \int_{[0,1]^2} C_t^E(u_1, u_2)du_1du_2}{\int_{[0,1]^2} M(u_1, u_2)du_1du_2 - \int_{[0,1]^2} C_t^E(u_1, u_2)du_1du_2}.
\]

In tab. 5.3, \( Q_t \)'s mean value \( \mu_Q \) and standard deviation \( \sigma_Q \) for the period 2007-2010 are being compared for the different approaches and for the different analytical copulas. A comparison like the one in tab. 5.3, show how well the different analytical copulas and approaches imitate the empirical copula, with emphasis on rank correlation.
Table 5.3: Mean value and standard deviation for Measure of Similarity $Q_t$, for the different approaches and different analytical copulas.

Measure of Similarity $Q_t$ will be compared with lower tail dependence.
5.4 Lower Tail Dependence vs Rank Correlation

An analytical copula $C^A$ have the same lower tail dependence for a certain correlation coefficient, regardless of optimization method. The effect of the optimization method, is that it determines the correlation coefficient or the strength parameter. So optimization methods that are resulting in increasing correlation coefficients or strength parameters on average, are also resulting in analytical copulas $\hat{C}^A_t$, with increased lower tail dependence $\lambda_t^A$ on average.

For scatter plots on Lower Tail Dependence vs. Rank Correlation, see fig.A.5-A.8. Sample figure of a scatter plot of Lower Tail Dependence vs. Kendall’s $\tau$, see fig.5.7.

In these scatter plots, one is able to see how well the analytical copulas represent the empirical lower tail dependence, for different values on the correlation coefficient. And also which values of the correlation coefficient that occurs most frequently for the empirical copulas.

The relation between Spearman’s $\rho_s$ and Kendalls $\tau$ is not linear. For sample figure of a scatter plot of Lower Tail Dependence vs Spearman’s $\rho$, see fig.5.8.
Figure 5.7: Lower tail dependence $\lambda_q^A$ and $\lambda_q^E$, for $q = 0.01$ for $q \in [0, 1]$, vs. $\tau(C_q^A)$ and $\tau(C_q^E)$, where the analytical copulas has been optimized with Least Square.
Figure 5.8: Lower tail dependence $\lambda_{1}^{A}$ and $\lambda_{1}^{E}$, for $q = 0.01$ for $q \in [0, 1]$, vs. $\rho_{s}(C_{t}^{A})$ and $\rho_{s}(C_{t}^{E})$, where the analytical copulas has been optimized with Least Square.
5.5 Lower Tail Dependence vs. Measure of Similarity

Lower Tail dependence vs. Measure of similarity $Q_t$, see Measure of similarity in chapter 5.3. $\lambda_l$ vs. $Q_t$ is a good approach to see whether the size of $Q_t$ has any linear relationship with lower tail dependence, see scatterplots in fig.A.9-A.10. What one can notice, is that at a certain value of $Q_t$, the analytical $\lambda_l$, does not increase significantly. This $Q_t$ value is different for different models and approaches. Size of $\lambda_l$ will stagnate at certain value of $Q_t$, for a specific analytical copula and approach, even though $Q_t$ increase significantly. And since $Q_t$ is based on the calculation of rank correlation, it actually describes rank correlation with empirical copula as benchmark. For sample figures see fig.5.9 & 5.9.

![Graph showing $\lambda_l$ vs. $Q_t$ Least Square](image)

Figure 5.9: On x-axis, $Q_t$ for the different analytical copulas, optimized with Least Square. On y-axis, lower tail dependence $\lambda_l$, for the analytical and empirical copulas. This is for $q = 0.01$ for $q \in [0, 1]$.

The different approaches Least Square, Empirical Maximum Likelihood Estimator, 'Equal $\tau$' & 'Minimizing $Q_t$' give notable differences when comparing $\lambda_l$ and $Q_t$. To see the scatter plots, look at fig.A.9-A.10.
Figure 5.10: On x-axis, $Q_t$ for the different analytical copulas, optimized with Empirical Maximum Likelihood Estimator. On y-axis, lower tail dependence $\lambda_l$ for the analytical and empirical copulas. This is for $q = 0.01$ for $q \in [0, 1]$.

5.6 Lower Tail Dependence sorted by Rank Correlation

In the scatter plots fig.A.5-A.8, there is no notable connection between empirical output and analytical output at time $t$. To visualize the connection between outputs at time $t$, $\tau(C_t^E)$ will be sorted by size,

$$A = [\min(\tau(C_t^E)) < \cdots < \max(\tau(C_t^E))],$$

Each $\tau(C_t^E)$ occurs at a date. In order to keep track of the new order of dates, a time vector $T$ is established,

$$T = [t_{\min(\tau(C_t^E))}, \cdots, t_{\max(\tau(C_t^E))}].$$

Each analytical copula and empirical copula has a specific $\lambda_l$, at each
time $t$, this is vector $\lambda^A_1(T)$ for the analytical copula and $\lambda^E_1(T)$ for the empirical copula, see fig.A.12-A.15. This method clearly shows how volatile the empirical tail dependence is, and the size of the deviation does not seem to depend on the size of rank correlation, but this would need further investigation.

It is now easier to get an understanding visually for the behavior of $\lambda_l$ for the different approaches, Least Square, Empirical Maximum Likelihood Estimator, ‘Equal $\tau$’ & ‘Minimizing $Q_t$’. One can see which value each model have compared to the empirical, for the same date.

To see the average lower tail dependence $\mu_{\lambda_l}$ for the different approaches and analytical copulas compared to the empirical copula, see tab.A.1.

In the industry $q = 0.01$ and $q = 0.05$ are of interest, for $q \in [0,1]$. Now, for $q = 0.01$, no analytical copula is giving a satisfying result, when it comes to $\lambda_l$, closest are Clayton copula optimized with Least Square. Over all, $t_{\nu=5}$ copula seems to be the most appropriate analytical copula, both with Empirical Maximum Likelihood Estimator and ‘Equal $\tau$’, their representations of $\lambda_l$ are fulfilling on average, see tab.A.1.

## 5.7 Tail Concentration Functions $LR(q)$ & $UD(q)$

$LR(q)$ and $UD(q)$ are tail concentration functions, where $q \in [0,1]$. They enable visualization of probability through out the model, and makes it comparable with the empirical copula, for a better understanding see chap 4.3.3. The functions give good visualization of the probability structure. The $UD(q)$ also tells us important things about the anti-correlated events, which are of great interest in hedging.

The visual aspect gives an idea of how good an analytical model is with a certain approach, see sample fig.5.11 or fig.A.16-A.19. In the sample figure it is obvious that $LR(q)$ for Gaussian copula, optimized with Least Square, is not giving a good representation for $q \leq 0.05$, about 50 % under empirical $LR(q)$.

To get a better idea of which analytical copula that represents the empirical copula best, with respect to $LR(q)$ and $UD(q)$, see tab.A.2, for eq.(5.12) & (5.13). $LR^A(q)$ is for the analytical copula and $LR^E(q)$ is for the empirical copula, it is equivalent for the $UD(q)$ function.

$$\Delta LR(q) = LR^A(q) - LR^E(q) \quad (5.12)$$
\[ \Delta UD(q) = UD^A(q) - UD^E(q) \] (5.13)

From tab.A.2 it is clear that for the \( LR(q) \), the \( t \) copula \( C_{5,\rho}^t \), optimized with Least Square, is the most suitable choice. And for the \( UD(q) \), one would choose \( t \) copula \( C_{5,\rho}^t \) optimized with 'Minimizing \( Q_t \)'.

Figure 5.11: Upper plot: On x-axis, $q$ for $q \in [0.01, 0.99]$. On y-axis $LR(q)$. Lower plot: On x-axis, $q$ for $q \in [0.01, 0.99]$. On y-axis $UD(q)$. Both plots are based on the average empirical copula and the average Gaussian copula, optimized with Least Square.
Chapter 6

Discussion

The difference between analytical and numerical results shown in chapter 5.1 has not been used to correct or compensate the numerical results, neither for the analytical copulas nor the empirical copulas. The difference in results may come from the fact that the ranks for the analytical solution of the analytical copula is $U_a^i \in [0, 1]$, while the ranks for the numerical solution of the analytical copula is $U_n^i \in [0.01, 0.99]$ where $i = 1, 2$.

The tail concentration functions $LR(q)$ and $UD(q)$, are based on the average empirical copula. This is because we are interested in an analytical copula which will give a good approximation of tail dependence in the long term.

When deciding which analytical copula best represents the empirical copula, with respect to lower tail dependence, the easiest would be if all measures used in this thesis, suggested the choice of the same analytical copula. The majority of measures indicates that the most suitable analytical copula would be $t$ copula $C_{5, \rho}$ optimized with ‘Equal $t$’. The one exception from this is the tail concentration function $LR(q)$, which indicates $t$ copula $C_{3, \rho}$ optimized with Least Square.

In tab.A.3 - A.4, some statistical measures were considered, to see if the empirical lower tail dependence is random, or if it might be possible to find an analytical relationship for the empirical lower tail dependence. The results in tab.A.3 - A.4 are interpreted such that the empirical lower tail dependence is random. Although it is of importance to point out that the sample size of data points for the year 2010 in tab.A.3 & A.4, is half the sample size compared to the other years.
Chapter 7

Conclusion

The empirical copulas exhibit different lower tail dependence, even when the correlation coefficient is the same. An analytical copula gives a specific lower tail dependence for a specific correlation coefficient, that is why it is difficult to decide an analytical copula to represent the empirical lower tail dependence.

As mentioned the empirical copulas behave differently, the most probable reason for this, when it comes to financial data, is non-stationarity. That is, besides the estimation error, we are really dealing with a different copula each time.

One could use several analytical copulas, combine them as a function, and weigh them differently in the function. This could be an idea to simulate lower tail dependence. But the beauty of using copulas, is the simplicity and the small amount of computational time and effort needed.

The decision basis in this study for deciding if an analytical copula is useful, is as follows: It has to give a good representation of the lower tail dependence, on average, for the time period 2007-2010.

The behavior for t copula $C_{5,\rho}$, optimized with Empirical Maximum Likelihood Estimator or with the use of 'Equal $\tau$' as an approach, is quite similar. The difference in average lower tail dependence is between 2.8-4.5% for the two methods. The small difference in outcome between the methods, suggest the choice t copula $C_{5,\rho}'$ with the approach 'Equal $\tau$'.

Even though the separate years behave differently for the empirical data, t copula $C_{5,\rho}'$ still seems to be the optimal choice, when it comes to lower tail dependence for these analytical copulas.
There are probably other factors that need to be counted in, to get a better understanding of the lower tail dependence for the empirical copula, such as the index liquidity, interest rates, commodity and energy prices and other prices that could effect the companies contained in the index.

The reason why $t$ copula $C_{5,\rho}^t$ gives the best fit of these analytical copulas, is because of the adaptational ability degrees of freedom, $\nu$, provides. The empirical lower tail dependence behaves sufficiently similar for all years 2007-2010, which makes it possible to have a fixed degree of freedom.
Bibliography


Appendix A

Appendix A
### Table A.1: Mean lower tail dependence for empirical and analytical copulas.

Empirical mean lower tail dependence is highlighted in black. Analytical mean lower tail dependence closest to empirical mean lower tail dependence is highlighted in italic and the specific color code, used throughout this thesis.

<table>
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**Least Square**

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**Empirical Maximum Likelihood Estimator**

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**Equal \( \tau \)**

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**Minimizing \( Q_t \)**

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Table A.2: The average difference between the average analytical and average empirical tail concentration function $LR(q)$ and $UD(q)$, for $q \in [0, 1]$. 
Table A.3: Statistical numbers for empirical lower tail dependence, for the different years.

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Table A.4: Statistical numbers for empirical lower tail dependence, and Kendall’s $\tau$ for the empirical copula, for the different years.

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Figure A.1: Normalized histogram for lower tail dependence, for Gaussian copula (1st from above), t copula (2nd), Clayton copula (3rd), optimized with Least Square, for $q = [0.01(Left), 0.03(Middle), 0.05(Right)], q \in [0, 1]$. 
Figure A.2: Normalized histogram for lower tail dependence, for Gaussian copula (1st from above), $t$ copula (2nd), Clayton copula (3rd), optimized with Empirical Maximum Likelihood Estimator, for $q = [0.01\,(Left), 0.03\,(Middle), 0.05\,(Right)], q \in [0, 1]$. 
Figure A.3: Normalized histogram for lower tail dependence, for Gaussian copula (1st from above), t copula (2nd), Clayton copula (3rd), optimized with Equal $\tau$, for $q = [0.01\text{(Left)}, 0.03\text{(Middle)}, 0.05\text{(Right)}]$, $q \in [0, 1]$. 
Figure A.4: Normalized histogram for lower tail dependence, for Gaussian copula (1st from above), t copula (2nd), Clayton copula (3rd), optimized with Minimizing $Q_t$, for $q = [0.01 (Left), 0.03 (Middle), 0.05 (Right)], q \in [0, 1]$. 
Figure A.5: Left: Lower tail dependence vs. Kendall’s τ. Optimized with Least Square. Right: Lower tail dependence vs. Spearman’s ρ. Optimized with Least Square. q = [0.01(Upper), 0.03(Middle), 0.05(Lower)], q ∈ [0, 1].
Figure A.6: Left: Lower tail dependence vs. Kendall’s $\tau$. Optimized with Empirical Maximum Likelihood Estimator. Right: Lower tail dependence vs. Spearman’s $\rho_s$. Optimized with Empirical Maximum Likelihood Estimator. $q = [0.01(Upper), 0.03(Middle), 0.05(Lower)], q \in [0, 1]$. 
Figure A.7: Left: Lower tail dependence vs. Kendall’s $\tau$. Optimized with Equal $\tau$. Right: Lower tail dependence vs. Spearman’s $\rho_s$. Optimized with Equal $\tau$. $q = [0.01(Upper), 0.03(Middle), 0.05(Lower)]$, $q \in [0, 1]$. 
Figure A.8: Left: Lower tail dependence vs. Kendall's $\tau$. Optimized with Minimizing $Q_t$. Right: Lower tail dependence vs. Spearman's $\rho$. Optimized with Minimizing $Q_t$. $q = [0.01(\text{Upper}), 0.03(\text{Middle}), 0.05(\text{Lower})]$, $q \in [0, 1]$. 
Figure A.9: Left: Lower tail dependence vs. Measure of Similarity, optimized with Least Square. Right: Lower tail dependence vs. Measure of Similarity, optimized with Empirical Maximum Likelihood Estimator, for $q = [0.01(Upper), 0.03(Middle), 0.05(Lower)]$, $q \in [0, 1]$. 
Figure A.10: Left: Lower tail dependence vs. Measure of Similarity, optimized with Equal $\tau$. Right: Lower tail dependence vs. Measure of Similarity, optimized with Minimizing $Q_t$, for $q = [0.01(Upper), 0.03(Middle), 0.05(Lower)]$, $q \in [0, 1]$. 

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Figure A.11: On x-axis is time $t$. On y-axis is Measure of similarity $Q(C_t^A, C_t^E)$. Gaussian (1st from top), t copula (2nd) and Clayton (3rd).
Figure A.12: Empirical lower tail dependence sorted by size of correlation coefficient for empirical copula. Lower tail dependence for analytical copula, optimized with Least Square, plotted for the same dates as the empirical copula, for $q = [0.01(Upper), 0.03(Middle), 0.05(Lower)]$, $q \in [0, 1]$. 

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Figure A.13: Empirical lower tail dependence sorted by size of correlation coefficient for empirical copula. Lower tail dependence for analytical copula, optimized with Empirical Maximum Likelihood Estimator, plotted for the same dates as the empirical copula, for $q = [0.01(Upper), 0.03(Middle), 0.05(Lower)], \, q \in [0, 1]$. 
Figure A.14: Empirical lower tail dependence sorted by size of correlation coefficient for empirical copula. Lower tail dependence for analytical copula, with Equal $\tau$, plotted for the same $\tau$ as the empirical copula, for $q = [0.01(Upper), 0.03(Middle), 0.05(Lower)], q \in [0, 1]$. 
Figure A.15: Empirical lower tail dependence sorted by size of correlation coefficient for empirical copula. Lower tail dependence for analytical copula, optimized with Minimizing $Q_1$, plotted for the same dates as the empirical copula, for $q = [0.01(Upper), 0.03(Middle), 0.05(Lower)]$, $q \in [0, 1]$. 
Figure A.16: Right: Tail concentration function $UD(z)$ for average [Gaussian copula (Upper), $t$ copula (Middle), Clayton copula (Lower)], optimized with Least Square. Left: Tail concentration function $LR(z)$ for average [Gaussian copula (Upper), $t$ copula (Middle), Clayton copula (Lower)], optimized with Least Square.
Figure A.17: Right: Tail concentration function $UD(z)$ for average [Gaussian copula (Upper), t copula (Middle), Clayton copula (Lower)], optimized with Empirical Maximum Likelihood Estimator. Left: Tail concentration function $LR(q)$ for average [Gaussian copula (Upper), t copula (Middle), Clayton copula (Lower)], optimized with Empirical Maximum Likelihood Estimator.
Figure A.18: Right: Tail concentration function $UD(z)$ for average \{Gaussian copula (Upper), $t$ copula (Middle), Clayton copula (Lower)\}, with Equal $\tau$. Left: Tail concentration function $LR(q)$ for average \{Gaussian copula (Upper), $t$ copula (Middle), Clayton copula (Lower)\}, with Equal $\tau$. 
Figure A.19: Right: Tail concentration function $UD(z)$ for average [Gaussian copula (Upper), $t$ copula (Middle), Clayton copula (Lower)], optimized with Minimizing $Q_t$. Left: Tail concentration function $LR(z)$ for average [Gaussian copula (Upper), $t$ copula (Middle), Clayton copula (Lower)], optimized with Minimizing $Q_t$. 