Feynmann diagrams in a finite dimensional setting

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Abstract

This article aims to explain and justify the use of Feynmann diagrams as a computational tool in physics. The integrals discussed may be seen as a toybox version of the real physical case. It starts out with the basic one-dimensional Gaussian integral and then proceeds with examples of multidimensional cases. Correlators and their solutions through generating functions and Wick's theorem are shown, as well as some examples of how to relate the computations to diagrams and the corresponding rules for these diagrams.

1 Introduction

In April 1948 Richard Feynmann wrote an article that introduced a different way to describe non-relativistic quantum mechanics. In contrast to the established operator-matrix formulation of quantum mechanics, this new way was described in terms of integrals. In classical mechanics, one tries to find the trajectory which minimizes the action S between two points in time, and that trajectory is the unique trajectory the physical system will follow.

$$S(\Phi) = \int_X L(\Phi)\mathrm{d}x \tag{1}$$

In the quantum case, one instead considers all possible trajectories between those two points in time. Each trajectory has a given probability of happening with the trajectories close to the classical trajectory being the most likely ones. All of these trajectories contribute to the total probability amplitudes for events happening in the system.[1] Similarly to the classical case, one instead gets the partition function:

$$Z = \int_F e^{-kS(\Phi)}\mathcal{D}\Phi \tag{2}$$

This integral is performed over the space of fields, with a real k and a formal measure $\mathcal{D}\Phi$. One usually wants to solve these integrals in the limit $k \to \infty$. However, these infinite-dimensional measures are often ill-defined, and this is the point where a mathematician would usually stop. Physicists take it one step further and go on by developing rules for evaluating integrals like this in the finite-dimensional case, and then applies those rules to the infinite-dimensional case. Although not mathematically rigorous, it is a commonly used technique and it makes sense intuitively. If the space of fields is the real space $\mathbb{R}^d$, a well-known approximation for large values of k is that the largest contribution to Z comes from the neighbourhood around which $\frac{\partial S}{\partial x} = 0$.

[1] For a more detailed and justified treatment of this statement, refer to [RF].

It is sufficient to study Z at such a point. Taylor-expanding $S(x)$ around such a point one arrives at the following integral after a change of coordinates:

$$\int_{\mathbb{R}^d} e^{\frac{1}{2}\langle x, Ax\rangle + \frac{1}{k}U(x)} P(x)\,\mathrm{d}x \tag{3}$$

where A is a bilinear form, $U(x)$ is higher order terms and $P(x)$ are monomials in the coordinates x^i.[2] Evalutating integrals like these is messy and Feynmann diagrams provides a useful tool as a book-keeping device for all the important factors that appear in the calculations. The purpose of this article is to derive the necessary integrals in the finite-dimensional case and show how Feynmann diagrams may be used as a computational aid in these calculations. Note that detailed proofs are omitted, though some general statements about the proof may be mentioned.

2 Gaussian integrals

We are interested in calculating several integrals in finite dimensional real space. In this section they will be solved through normal analytic computation, but in section 3 the focus will be on solving them through combinatorics and algebra alone. As far as gaussian integrals are concerned the best starting point is the one-dimensional gaussian integral and its solution:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\alpha x^2}\,\mathrm{d}x = \sqrt{\frac{2\pi}{\alpha}} \tag{4}$$

This one-dimensional integral is a special case of a more general multidimensional equivalent. It may be written in both matrix-form and index-form and the solution is:

$$Z[0] \equiv \int_{\mathbb{R}^n} e^{-\frac{1}{2}\alpha x^T Q x}\,\mathrm{d}x = \int_{\mathbb{R}^n} e^{\frac{1}{2}\alpha Q_{\mu\nu}x^\mu x^\nu}\,\mathrm{d}^n x = \sqrt{\frac{(2\pi)^n}{\alpha}}\frac{1}{\sqrt{det(Q)}} \tag{5}$$

The matrix Q in the equation is assumed to be a real, symmetric and positive-definite matrix while x is now a vector of n components. One may omit the condition of positive-definiteness and just treat the integrals formally without being concerned about convergence; however, lets keep it for the sake of comparison to the one-dimensional case. The integration is still done over the domain $\mathbb{R}^n$ just like in the one dimensional case. The term $x^T Q x$ is a quadratic form, and thus gives us a sum of terms which are quadratic in the coordinates and we see that we get a multivariate equivalent of the one-dimensional gaussian integral. One may solve this integral, perhaps with some effort, by making a substitution to the eigenbasis of Q. In this basis the integral reduces to a product of several integrals of the form (1) and the result follows easily. From here on and onwards only the index-notation (along with Einsteins summation convention) will be used due to ease of notation and use.

Next we define the generating function $Z[J]$ which has an added linear term in the exponential:

$$Z[J] \equiv \int_{\mathbb{R}^n} e^{\frac{1}{2}\alpha Q_{\mu\nu}x^\mu x^\nu + J_\mu x^\mu}\,\mathrm{d}^n x = Z[0]e^{\frac{1}{2\alpha}Q^{\mu\nu}J_\mu J_\nu} \tag{6}$$

Note that by convention $Q^{\mu\nu}$ represents the elements of the inverse-matrix $Q_{\mu\nu}$. Solving the integral may be done by completing the square of the exponent, and then apply (5) to it. In

[2] See [PD] for a slightly more detailed treatment.

practice one is usually not interested in calculating these specific integrals. What one would like to consider in calculations are the correlators of coordinates defined as:

$$\langle x^\sigma x^\lambda \rangle = \frac{1}{Z[0]} \int\limits_{\mathbb{R}^n} x^\sigma x^\lambda e^{\frac{1}{2}\alpha Q_{\mu\nu} x^\mu x^\nu} \, \mathrm{d}^n x \tag{7}$$

This may look like a rather nasty integral to solve at first glance, however this is where the generating function $Z[J]$ comes into play. Acute readers may have noticed that if one takes derivatives with respect to the vector J inside the integral sign, treating x and Q as constants, one ends up with an integral of the form (7). Using the fact that we have already solved the integral (6) to our advantage, one easily obtains the solution of (7) by applying the derivatives on the solution of the generating function.

$$\langle x^\sigma x^\lambda \rangle = \frac{1}{Z[0]} \frac{\partial^2}{\partial J_\sigma \partial J_\lambda} Z[J] \bigg|_{J=0} = \frac{\partial^2}{\partial J_\sigma \partial J_\lambda} e^{\frac{1}{2\alpha} Q^{\mu\nu} J_\mu J_\nu} \bigg|_{J=0} = \frac{1}{\alpha} Q^{\sigma\lambda} \tag{8}$$

(Note: One may use this to solve correlators of power series. The specifics aren't discussed here, but for the interested reader see Proposition 2.6. and the discussion before it in [PD].)

3 Wick's theorem and Feynmann diagrams

Next, consider the more general case of having a correlator of an even number of coordinates. One may easily convince themselves that the correlator is 0 for an odd number of coordinates, either by carrying out the derivatives by hand or by symmetry arguments.

$$\langle x^{\mu_1} \dots x^{\mu_{2n}} \rangle = \frac{1}{Z[0]} \frac{\partial^{2n}}{\partial J_{\mu_1} \dots \partial J_{\mu_{2n}}} Z[J] \bigg|_{J=0} \tag{9}$$

This is solved using the generating function, similarly to how equation (3) was solved though now by applying 2n derivatives. One could calculate these correlators by hand for a given n, but will probably notice that as you get to larger and larger correlators everything quickly becomes quite complex making computation by hand inefficient. Instead of wasting ones time on this, it is possible to prove the general case resulting in what is commonly known in physics as Wick's theorem (which is closely related and similar to Isserlis' theorem for the statisticians out there). Wick's theorem is stated as:

$$\langle x^{\mu_1} \dots x^{\mu_{2n}} \rangle = \frac{1}{2^n n! \alpha^n} \sum_P Q^{\mu_{P_1} \mu_{P_2}} \dots Q^{\mu_{P_{2n-1}} \mu_{P_{2n}}} \tag{10}$$

The sum runs over all possible permutations of pairs of indices. Pairing the indices with each other is also called contracting the indices. Rather than trying to explain this process in text, consider the following example:

Ex1. Consider the correlator $\langle x^{\mu_1} x^{\mu_2} x^{\mu_3} x^{\mu_4} \rangle$.

With Wick's theorem we may skip calculating the integral and jump straight to the answer. If we consider each index as a point, we may join these points together in three different ways. Each edge going between two points represent a Q factor which has contracted the two indices. Thus by figure1 we may write out the sum explicitly as:

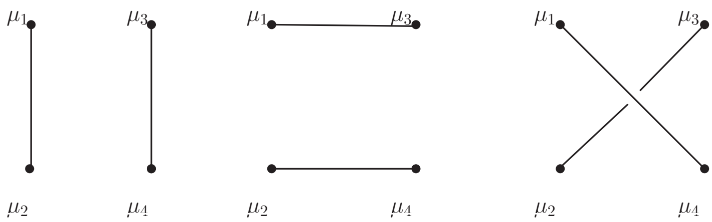

Figure 1: This shows the pairing of indices of $\langle x^{\mu_1} x^{\mu_2} x^{\mu_3} x^{\mu_4} \rangle$.

$$\langle x^{\mu_1} x^{\mu_2} x^{\mu_3} x^{\mu_4} \rangle = \frac{1}{2^2 2! \alpha^2} \left(Q^{\mu_1 \mu_2} Q^{\mu_3 \mu_4} + Q^{\mu_1 \mu_3} Q^{\mu_2 \mu_4} + Q^{\mu_1 \mu_4} Q^{\mu_2 \mu_3} + \dots \right) \tag{11}$$

One has to cover all permutations of the indices which adds up to a total of 24 terms.

End of example.

Next we would like to introduce monomials of the coordinates x:

$$V_n(x) = \frac{1}{n!} V_{\mu_1 \dots \mu_n} x^{\mu_1} \dots x^{\mu_n} \tag{12}$$

The correlator of monomials is a more general form of the correlator of coordinates. If we assume that all monomials are different for the time being, it may be expressed using Wick's theorem as:

$$\langle V_{n_1}(x) \dots V_{n_k}(x) \rangle = \frac{1}{n_1! \dots n_k!} V_{\mu_1 \dots \mu_{n_1}} V_{\nu_1 \dots \nu_{n_k}} \langle x^{\mu_1} \dots x^{\mu_{n_k}} \rangle \tag{13}$$

Notice that the indices of the V's are the same as the indices of the coordinates. Thus the indices of the V's will also be the same as the indices that each Q contracts. They are connected in this way and that connection will be used to construct our feynmann diagrams. The way of doing this revolves around the fact that we may rewrite (13) as a sum over each distinct feynmann diagram (also known as graphs) that may be produced. The weight of each such graph is related to its amount of automorphisms:

$$\langle V_{n_1}(x) \dots V_{n_k}(x) \rangle = \frac{1}{\alpha^{\frac{1}{2}(n_1 + \dots + n_k)}} \sum_{\Gamma} \frac{1}{|\mathrm{Aut}\, \Gamma|} W(\Gamma) \tag{14}$$

To construct a graph Γ, we look at Wick's theorem. First, let each index be represented by a vertex. Notice that each Q factor has contracted two indices. Let each Q be represented by an edge (an arc) running between the two vertices of the indices it has contracted. Finally, group all vertices corresponding to the indices of the V factors together into a single multivalent vertex. Refer to figure2 for a better understanding of this.

The factors $W(\Gamma)$ and $|\mathrm{Aut}\, \Gamma|$ may need some clarification. Starting with the former, $W(\Gamma)$ is simply given by the product of each V factor with each contracted Q factor, or as one may say, the product of each multivalent vertex and each edge in a given graph. At the end of this section there will be examples where this is shown for given graphs.

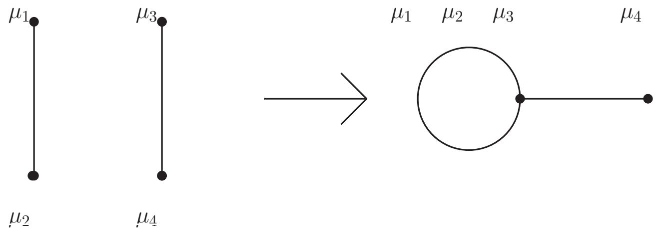

Figure 2: Grouping the indices of $\langle V_3 V_1 \rangle$ together results in a connected graph.

$|\mathrm{Aut}\,\Gamma|$ is the symmetry factor of Γ which is related to the amount of automorphisms it has. An automorphism of Γ is a map of its edges and vertices onto itself such that the graph is preserved. In other words, the amount of automorphisms is the same as the amount of graphs that has the same appearance as Γ but may have different indices contracted with each other. The symmetry factor of a given graph Γ is given by a product of a few factors:

$$|\mathrm{Aut}\,\Gamma| = (\#L)(\#P)(\#V) \tag{15}$$

To understand these factors it is useful to keep in mind that the automorphisms are gotten by two main operations; permutation of vertices that do not change the graph and permutation of edges that do not change the graph. So each factor is thus explained as follows:

- $(\#L) = 2^L$: This factor signifies the amount of loops in the graph (edges which start and end on the same vertex). Each loop contributes a factor 2.
- $(\#P) = p!$: This factor signifies the permutations of edges between two vertices. If there are p edges running between two vertices, you get a factor p! (note that this is also true for loops in which case the two vertices are the same vertex).
- $(\#V)$: This factor signifies the symmetry factor of vertices. This is defined as the cardinality of the subgroup of s_k that preserves the graph, disregarding orientation of edges. For the purpose of this article, we may say that if one can exchange v vertices without changing the graph, we get a factor v!.

To see that these rules work, we would first like to know how many graphs (including all automorphisms) we can have for a given correlator. This is easy and straight-forward to calculate. Note that all we have to do to get any complete graph is to pick any two indices and contract them, then take any two of the remaining indices, contract them, and repeat this process until there are no more indices left to choose from. This may be more formally written as a multinomial coefficient or as a double-factorial. For $2n$ indices the amount of graphs will be:

$$\frac{1}{n!}\binom{2n}{2_1, \ldots, 2_n} = (2n-1)!! \tag{16}$$

Now we have all the tools necessary to compute our correlators in a purely algebraic and combinatorial way. To demonstrate how Feynmann diagrams act as a great book-keeping device for the constants appearing in these calculations, check the two following examples.[3]

[3]Note, one may apply this just as well to one-dimensional correlators of the kind (9). For an example of this, see page 30 of [MZ].

Ex2. Consider the following correlator:

$$\langle V_3(x)V_5(x)\rangle = \frac{1}{3!}\frac{1}{5!}\int\limits_{\mathbb{R}^n} V_{\mu_1\mu_2\mu_3}V_{\mu_4\mu_5\mu_6\mu_7\mu_8}x^{\mu_1}x^{\mu_2}x^{\mu_3}x^{\mu_4}x^{\mu_5}x^{\mu_6}x^{\mu_7}x^{\mu_8}e^{\frac{1}{2}\alpha Q_{\mu\nu}x^\mu x^\nu}\mathrm{d}x \tag{17}$$

Grouping all indices to their respective vertices, there are only two possible (visually distinct) connected graphs one may create. For the case of two vertices, there is an easy method to generate them. Start with connecting as many edges as possible between the two vertices, and then for the vertex that has the highest degree, connect its remaining edges to itself. Each subsequent graph may then be generated by converting two of the edges the vertices share into a single loop on each vertex. This process may be repeated until one can't remove more edges between the vertices. In our example this leads to the following two graphs:

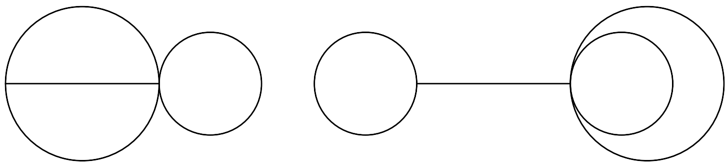

Figure 3: The graphs of $\langle V_3V_5\rangle$ demonstrating how one vertex has three legs and the other has five legs.

We expect to get a total of 105 graphs according to equation (16):

$$(8-1)!! = 7\cdot 5\cdot 3 = 105 \tag{18}$$

Let us first try to compute the correlator without the use of the symmetry factor in equation (14). We'll refer to the first graph as Γ_1 and the second as Γ_2. Now we have to find out how many ways we can contract indices according to the graph Γ_1. Starting with the 3-valent vertex, there are $5\cdot 4\cdot 3$ ways to contract its indices with those of the 5-valent vertex, and only one way to contract the remaining two indices. Thus we can contract indices in 60 ways according to Γ_1, and similarly in 45 ways according to Γ_2, giving a total of 105 graphs. Now we may use equation (13) and Wick's theorem to rewrite the correlator as:

$$\langle V_3(x)V_5(x)\rangle = \frac{1}{3!}\frac{1}{5!}\frac{1}{\alpha^4}\left(60W(\Gamma_1)+45W(\Gamma_2)\right) = \frac{1}{\alpha^4}\left(\frac{1}{12}W(\Gamma_1)+\frac{1}{16}W(\Gamma_2)\right) \tag{19}$$

The $W(\Gamma)$ are given by:

$$W(\Gamma_1) = V_{\mu_1\mu_2\mu_3}V_{\mu_4\mu_5\mu_6\mu_7\mu_8}Q^{\mu_1\mu_4}Q^{\mu_2\mu_5}Q^{\mu_3\mu_6}Q^{\mu_7\mu_8} \tag{20}$$

$$W(\Gamma_2) = V_{\mu_1\mu_2\mu_3}V_{\mu_4\mu_5\mu_6\mu_7\mu_8}Q^{\mu_1\mu_4}Q^{\mu_2\mu_3}Q^{\mu_5\mu_6}Q^{\mu_7\mu_8} \tag{21}$$

Now, let us instead try to calculate the correlator using equation (14) and symmetry factors. Using the rules we may easily derive the following values for the symmetry factors:

$$|\mathrm{Aut}\,\Gamma_1| = 3!2 = 12 \tag{22}$$

$$|\mathrm{Aut}\,\Gamma_2| = 2\cdot 2^2\cdot 2 = 16 \tag{23}$$

The factors $W(\Gamma)$ are the same as before, and thus using equation (14) we arrive at:

$$\langle V_3(x)V_5(x)\rangle = \frac{1}{\alpha^4}\left(\frac{1}{3!2}W(\Gamma_1) + \frac{1}{2\cdot 2^3}W(\Gamma_2)\right) = \frac{1}{\alpha^4}\left(\frac{1}{12}W(\Gamma_1) + \frac{1}{16}W(\Gamma_2)\right) \tag{24}$$

This is exactly the same result as (19) but with considerably less effort put into it. This is the power of Feynmann diagrams, their rules and why they are such a great tool.

End of example.

Ex3. Consider the following correlator:

$$\begin{aligned}\frac{1}{2}\langle V_5(x)V_5(x)\rangle = \frac{1}{2}\frac{1}{5!}\frac{1}{5!}\int\limits_{\mathbb{R}^n} V_{\mu_1\mu_2\mu_3\mu_4\mu_5}V_{\mu_6\mu_7\mu_8\mu_9\mu_{10}} \\ x^{\mu_1}x^{\mu_2}x^{\mu_3}x^{\mu_4}x^{\mu_5}x^{\mu_6}x^{\mu_7}x^{\mu_8}x^{\mu_9}x^{\mu_{10}}e^{\frac{1}{2}\alpha Q_{\mu\nu}x^\mu x^\nu}\,\mathrm{d}x\end{aligned} \tag{25}$$

The factor $\frac{1}{2}$ is a conventional factor because we have two vertices of the same degree. Using the same method as described in **Ex2**, we find that it is possible to make the following three graphs:

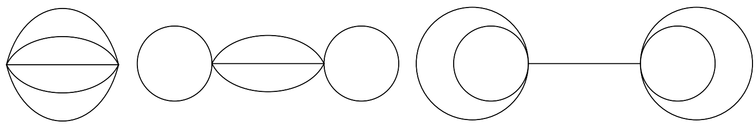

Figure 4: Three graphs of $\langle V_5V_5\rangle$ with each vertex having 5 legs.

This time we expect to get a total of 945 graphs:

$$(10-1)!! = 9\cdot 7\cdot 5\cdot 3 = 945 \tag{26}$$

As one can see, the amount of graphs increases drastically as the degree of the graph increases. Call the leftmost graph Γ_1, and the others Γ_2 and Γ_3. If we once again count all the possible ways to contract indices according to each graph, with some effort, we get:

$$\begin{aligned}\frac{1}{2}\langle V_5V_5\rangle = \frac{1}{2}\frac{1}{5!}\frac{1}{5!}\frac{1}{\alpha^5}\left(120W(\Gamma_1) + 600W(\Gamma_2) + 225W(\Gamma_3)\right) = \\ = \frac{1}{\alpha^5}\left(\frac{1}{240}W(\Gamma_1) + \frac{1}{48}W(\Gamma_2) + \frac{1}{128}W(\Gamma_3)\right)\end{aligned} \tag{27}$$

These do add up to 945 graphs as expected. For each of our graphs Γ the factors $W(\Gamma)$ (up to automorphisms) are given by:

$$W(\Gamma_1) = V_{\mu_1\mu_2\mu_3\mu_4\mu_5}V_{\mu_6\mu_7\mu_8\mu_9\mu_{10}}Q^{\mu_1\mu_6}Q^{\mu_2\mu_7}Q^{\mu_3\mu_8}Q^{\mu_4\mu_9}Q^{\mu_5\mu_{10}} \tag{28}$$

$$W(\Gamma_2) = V_{\mu_1\mu_2\mu_3\mu_4\mu_5}V_{\mu_6\mu_7\mu_8\mu_9\mu_{10}}Q^{\mu_1\mu_6}Q^{\mu_2\mu_7}Q^{\mu_3\mu_8}Q^{\mu_4\mu_5}Q^{\mu_9\mu_{10}} \tag{29}$$

$$W(\Gamma_3) = V_{\mu_1\mu_2\mu_3\mu_4\mu_5}V_{\mu_6\mu_7\mu_8\mu_9\mu_{10}}Q^{\mu_1\mu_6}Q^{\mu_2\mu_3}Q^{\mu_4\mu_5}Q^{\mu_7\mu_8}Q^{\mu_9\mu_{10}} \tag{30}$$

Let us try to compute the correlator once again using equation (14) and the rules for symmetry factors. Applying the rules to Γ_1, Γ_2 and Γ_3 again, we get:

$$|\mathrm{Aut}\,\Gamma_1| = 5!2 = 240 \tag{31}$$

$$|\mathrm{Aut}\,\Gamma_2| = 3!2 \cdot 2 \cdot 2 = 48 \tag{32}$$

$$|\mathrm{Aut}\,\Gamma_3| = 2^2 \cdot 2^2 \cdot 2 \cdot 2^2 = 128 \tag{33}$$

Using equation (14) again, we get the following expresison for the correlator:

$$\frac{1}{2}\langle V_5 V_5\rangle = \frac{1}{\alpha^5}\left(\frac{1}{240}W(\Gamma_1) + \frac{1}{48}W(\Gamma_2) + \frac{1}{128}W(\Gamma_3)\right) \tag{34}$$

Once again, the correlators (34) and (27) agree perfectly with each other. The first method works, but requires quite a lot of thinking, where as the second method just requires using the rules and is easy to handle.

End of example.

4 Conclusion

Feynmann diagrams, as they are known to most students of physics, are a pictorial tool for helping in understanding particle interactions. However, they're more than just that; they are also a great computational tool in that they allow one to algebraically solve tough and messy integrals. They, together with the derived rules, are a great book-keeping device for keeping track of all the constants that appear as one calculates the integral.

5 Sources

[RF] R. P. Feynman, "Space-Time Approach to Non-Relativistic Quantum Mechanics", Reviews of Modern Physics, Volume 20, Number 2, April 1948

[MZ] M. Zabzine, J. Qiu, "Introduction to Graded Geometry, Batalin-Vilkovisky Formalism and their Applications", [arXiv:1105.2680v2], 14 December 2011

[PD] M. Polyak, "Feynman Diagrams for Pedestrians and Mathematicians", [arXiv:math/0406251v1], 12 June 2004