Multi-photon emission in QED with strong background fields

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Abstract

In recent and upcoming years new lasers are being constructed with ever higher intensity. These lasers open up the possibility of probing the high intensity regime of particle physics, which will lead to either confirming our current models in this regime or the discovery of beyond standard model physics. However most previous theoretical results in this area are based old assumptions about the intensity and shape of the laser pulse that are no longer valid. In this thesis we calculate the tree-level probabilities for multi photon emission from an electron propagating in an arbitrary plane wave electromagnetic field. We show that the classical limit of our result agrees with the purely classical description of the same event. We calculate the soft emission correction to non-linear Compton scattering. We conclude that our results are infrared divergent and argue that this will be solved by including loop contributions to the process. Our results provide an important component for the theoretical predictions for the outcome of scattering experiments in high intensity background field. This thesis will add to the understanding of high intensity QED.
# Contents

1 Introduction ............................................. 1

2 Background information ................................ 2
   2.1 Conventions ........................................... 2
   2.2 Lightfront coordinates ............................... 2
   2.3 Electromagnetic plane wave field .................. 3

3 Classical radiation ...................................... 3
   3.1 Electron path ......................................... 4
   3.2 Radiation ............................................. 4
      3.2.1 Retarded solution ............................... 5
      3.2.2 At the end of time .............................. 6
      3.2.3 Regularisation ................................. 6
      3.2.4 Radiated energy ................................. 7
   3.3 Number of photons ................................... 8

4 QED ...................................................... 9
   4.1 S-matrix elements .................................... 9
      4.1.1 Electron propagator ............................. 11
      4.1.2 One vertex ....................................... 12
      4.1.3 No internal positrons ......................... 13
      4.1.4 Final expression for the general tree level S-matrix elements 15
   4.2 Classical limit ...................................... 15
      4.2.1 No radiation reaction ........................... 16
      4.2.2 S-matrix elements in the classical limit ....... 17
      4.2.3 Regularisation ................................... 18
      4.2.4 Probability of emitting $n$ photons ............ 18
      4.2.5 Too much probability ........................... 19
      4.2.6 Renormalisation of the probabilities and average number of photons ............................ 20
   4.3 Next order in $\bar{\hbar}$ ............................. 20
      4.3.1 Regularisation ................................... 21
      4.3.2 Probability ....................................... 21
      4.3.3 Remarks .......................................... 22
      4.3.4 One photon ...................................... 23
      4.3.5 Two photons ..................................... 24
   4.4 One hard photon ..................................... 24
      4.4.1 S-matrix elements for the one hard photon limit ............... 25
      4.4.2 Regularisation ................................... 27
      4.4.3 Probability ....................................... 29
      4.4.4 Infrared divergence .............................. 30
      4.4.5 Classical limit .................................. 31

5 Conclusions ............................................. 32
1 Introduction

In recent years experimentalists have been able to produce laser pulses of ever higher intensity and in the coming years these intensities will be pushed even further. High intensity lasers that are already in operation are for example the Draco laser in Dresden, Germany, the Vulcan laser in Oxford, UK and NIF in California [1]. Within a few years when the Extreme Light Infrastructure project (ELI) is built, it will be the world’s most intense light source [2].

These research facilities open up the possibility of probing a new regime of fundamental particle physics. The research that can be done at high intensity laser facilities is complementary to that of the Large Hadron Collider (LHC). While the LHC uses high energy to search for new heavy particles, high intensity lasers can be used to probe the quantum vacuum [3]. This research interest is shared with groups around the world, for example in Connecticut, Graz, Heidelberg, Munchen, Nizhnii Novgorod, Plymouth and Tsukuba. High intensity lasers can also be used to search for Axions and weakly interacting sub-eV particles (WISPS) through the phenomena of “Light shining through walls” [4, 5].

Lasers facilities of higher energy and lower intensity, such as the European XFEL which is currently under construction [6], bridge the gap between the two regimes of high energy and high intensity.

However, before anyone can start searching for evidence of beyond standard model physics in these high intensity regimes, we must first know what the standard model predicts, and to do this a better understanding of the behavior of quantum electrodynamics (QED) in this regime is needed. Most results of high intensity laser physics are derived from assumptions that are no longer conceded to be valid [7]. The breakthrough of ‘chirped pulse amplification’ has opened up the possibility of higher laser intensities then was previously thought possible [8]. The duration of the laser pulse is no longer long compared to the timescales of the physics involved, which means that the duration and shape of the pulse must be taken into account. In the recent years much attention has been paid to so-called ‘finite size’ physics originating in the geometry of the laser pulse [9, 10].

In this thesis we will calculate the emission probability of multiple photons from a single electron in a background electromagnetic field. The background field is an arbitrary plane wave and is described classically and non-perturbatively. We will start with introducing lightfront coordinates and the background field. In section 3 we will do a completely classical analysis of the process. In section 4.1 we calculate the S-matrix element for the emission with the electron and the emitted photons fully quantised, and then analyse the corresponding probabilities in three different limits. These limits are firstly to make a perturbation expansion around $\hbar$, first to zeroth order in $\hbar$ to give the classical limit and then to first order in $\hbar$ go give the first order quantum corrections. After that we explore the limit where one photon is hard and the rest are soft, which will give us the soft photon emission correction to the process of emitting one photon in the background field.

The results of this thesis will add to the understanding of QED in the high
intensity regime. Our results are also an important step in the direction of making testable predictions about what will be observed in high intensity laser experiments. It is only after carefully calculating of all the standard model predictions in these experiments that we can hope to identify any beyond standard model physics that may or may not be detectable in high intensity laser experiments.

2 Background information

This section contains conventions and definitions used in this thesis and summaries of some results we will use later on.

2.1 Conventions

The conventions used in this thesis are:

- We use natural units where $c = \hbar = 1$.
- $\epsilon$ is the electron charge $\epsilon = -|\epsilon|$.
- The signature of the metric is $(+--)$.
- $\mu, \mu_1, \mu_2, \ldots, \nu, \nu_1, \nu_2, \ldots$ are Lorenz indexes and a sum over equal Lorenz indexes is always implied.
- The index $\perp$ enumerate the two perpendicular vector components in lightfront coordinates $\perp = 1, 2$. Whenever there is a product of $\perp$-components, summation is implied $A_\perp B_\perp := A_1 B_1 + A_2 B_2$.
- A lower dot “.” denotes a 4-vector product $A.B := A^\mu B_\mu$.
- $\sigma$ and $\sigma'$ are spinor indexes and $\sigma = 1, 2$ denotes the spin states of the electron.
- $\epsilon^1, \ldots, \epsilon^n$ are the polarisation vectors of the outgoing photons and $\sum_{\epsilon_j}$ sums over the two possible polarisations of each photon.
- $\theta$ is the heavyside stepfunction $\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$.
- Unless otherwise stated “time” refers to the lightfront time.

2.2 Lightfront coordinates

Given a lightlike 4-vector $k$, we can always choose coordinates such that $k_\mu = k_+(1, 0, 0, 1)$ for some scalar $k_+$. After that we replace $x^0$ and $x^3$ with the new coordinates

$$x^\pm := x^0 \pm x^3. \quad (2.1)$$

In these coordinates $x^+$ or $k.x = k_+ x^+$ plays the role of time.
For every 4-vector $A$ we define

$$A^\pm := A^0 \pm A^3,$$

$$A_\pm := \frac{1}{2} (A_0 \pm A_3).$$

(2.2)

(2.3)

These definitions give the relations

$$A^\pm = 2 A_\mp,$$

$$A.B = A^+ B_+ + A^- B_- + A^\perp B_\perp$$

$$= 2 A_+ B_- + 2 A_- B_+ - A_\perp B_\perp.$$ 

(2.4)

(2.5)

Using the above definitions one can show that

$$d^4 x := dx^0 dx^1 dx^2 dx^3 = \frac{1}{2} dx^+ dx^- d^2 x^\perp$$

(2.6)

and

$$d^4 p := dp_0 dp_1 dp_2 dp_3 = 2 dp_+ dp_- d^2 p_\perp.$$ 

(2.7)

### 2.3 Electromagnetic plane wave field

Plane wave means that the field only depends on $k.x$ for some vector $k$. Since electromagnetic fields propagate with the speed of light $k$ must be lightlike for such a field. We can therefore use $k$ to define lightfront coordinates for the system.

The field tensor $F_{\mu\nu}$ only depends on $k.x$. This together with Maxwell’s equations in vacuum gives

$$F_{\mu\nu} = f_1(k.x)(k_{\mu}a_{1\nu} - a_{1\mu}k_{\nu}) + f_2(k.x)(k_{\mu}a_{2\nu} - a_{2\mu}k_{\nu})$$

(2.8)

where $a_1 = (0, 1, 0, 0)$ and $a_2 = (0, 0, 1, 0)$. This is exactly the field that is given by the 4-vector potential

$$A_\mu(k.x) := a_{1\mu} \int_{-\infty}^{k.x} d\phi \ f_1(\phi) + a_{2\mu} \int_{-\infty}^{k.x} d\phi \ f_2(\phi).$$

(2.9)

We assume that the electromagnetic field is a finite pulse which means that before the pulse $A = 0$ and after the pulse $A = \text{constant}$. Since $f_{1,2}$ are arbitrary there integrals are arbitrary too. This means that, apart from the constraints just mentioned, $A$ is an arbitrary function of $k.x$ with $A_+ = A_- = 0$.

### 3 Classical radiation

The aim of this section is to familiarize ourselves with the problem which we will address in the next section in QED. Many of the equations in this section will reappear in section 4.2 and it might help to have a classical understanding
of the meaning of these equations. Also we will want to compare our result in 4.2 with the result found in this section.

This section begins with the path taken by an electron under the influence of a plane wave electromagnetic field. We will calculate the electromagnetic field originating from this electron. We will calculate the total energy radiated from the electron, per frequency and direction, and we will use this result to get a fictive photon count.

In this section we will ignore any radiation reaction. We will use \( k.y = k_+y^+ \) as the time parameter, where \( y^\mu \) is the position on the electron. We will follow the presentation of Peskin & Schroder [11, Chapter 6.1].

### 3.1 Electron path

The 4-momentum of a classical electron in a plane wave electromagnetic background field such as described in section 2.3 is [12]

\[
\tilde{p}^\mu(k,y) = p^\mu - eA^\mu(k.y) + \frac{2eA(k.y).p - e^2(A(k.y))^2}{2k.p} \tag{3.1}
\]

where \( p \) is the 4-momentum when \( A = 0 \) and \( A \) is the 4-vector potential described in (2.9). Since we chose \( A \) to be zero before the laser pulse, \( p \) is the incoming electron momentum. Note that \( \tilde{p}_- \) is a constant

\[
k.\tilde{p} = k.p. \tag{3.2}
\]

The position of the particle is

\[
y^\mu(k,y) = \frac{1}{k.p} \int_{k.y}^{k.y} d\phi \, \tilde{p}^\mu(\phi) \tag{3.3}
\]

up to some constant which is given by the starting position.

### 3.2 Radiation

The moving electron creates the current density \( j \)

\[
j_{\mu}(x) = e \int d\tau \, \frac{p_{\mu}}{m} \delta^{(4)}(x - y) = \frac{e}{k.p} \int d(k.y) \, \tilde{p}_{\mu} \, \delta^{(4)}(x - y) \tag{3.4}
\]

where \( \tau \) is the proper time of the particle. Fourier transformed this becomes

\[
\tilde{j}_{\mu}(k') := \int d^4x \, e^{ik'.x} j_{\mu}(x) = \frac{e}{k.p} \int d(k.y) \, \tilde{p}_{\mu} \, e^{ik'.y}. \tag{3.5}
\]

The Fourier transformed 4-vector potential of the field created by \( j_{\mu} \) the current is

\[
\tilde{A}_{\mu}'(k') = -\frac{1}{(k')^2} \tilde{j}_{\mu}(k'). \tag{3.6}
\]
This is then transformed back to give the 4-vector potential in a given point of space time

\[ A'_\mu(x) = \int \frac{d^4k'}{(2\pi)^4} e^{-ik.x} \hat{A}'_\mu(k') = \]

\[ = -\frac{e}{k.p} \int d(k.y) \, \hat{p}_\mu \int \frac{d^4k'}{(2\pi)^4} \frac{e^{ik.(y-x)}}{(k')^2}. \]  

(3.7)

### 3.2.1 Retarded solution

In light front coordinates the measure \( d^4k' \) is

\[ d^4k' = 2dk'_+ \, dk'_- \, d^2k'_\perp. \]  

(3.8)

We want to integrate over \( k'_+ \) but to do this we have to deal with the pole \( (k')^2 = 0 \). We do this by inserting the small parameter \( i\epsilon \)

\[ (k')^2 = 4k'_- \left( k'_+ - \frac{(k'_+)^2}{4k'_-} \right) \rightarrow 4k'_- \left( k'_+ - \frac{(k'_+)^2}{4k'_-} + i\epsilon \right). \]  

(3.9)

This pushes the pole just below the real axis in the \( k'_+ \)-plane and will give us the retarded solution for the field. Since we want to describe a field created by the electron and not absorbed by it, this is the solution we want.

The \( k_+ \) part of (3.7) becomes

\[ \int \frac{2dk'_+ \, e^{ik'_+(y^+-x^+)} \theta(x^+ - y^+)}{(k')^2} = \theta(x^+ - y^+ \frac{-ie^{ik'_+(y^+-x^+)}}{2k'_-} \bigg|_{k'_+ = \frac{(k'_+)^2}{4k'_-}}. \]  

(3.10)

where \( \theta \) is the heaviside step function.

So far our result includes both positive and negative frequencies. To get only positive frequencies we reinterpret the direction of the negative frequencies such that \( k' \rightarrow -k' \)

\[ \int \frac{d^2k'_+ \, dk'_- \, (-i) e^{ik.(y-x)}}{(2\pi)^2 2k'_-} = \int \frac{d^2k'_+ \, dk'_- \, (-i) e^{ik.(y-x)}}{(2\pi)^2 2k'_-} \left( \theta(k'_+) + \theta(-k'_-) \right) = \]

\[ = \int \frac{d^2k'_+ \, dk'_- \, (-i) e^{ik.(y-x)}}{(2\pi)^2} \int_0^\infty \frac{dk'_-}{2\pi} \, \frac{dk'_-}{2k'_-} \left( -ie^{ik.(y-x)} + c.c. \right). \]  

(3.11)

In the first equality we used \( \theta(k'_+) + \theta(-k'_-) = 1 \), and in the second equality \( k' \rightarrow -k' \) for the second term.

Using the above calculations (3.7) becomes

\[ A'_\mu(x) = \int \frac{d^4k'_+}{(2\pi)^2} \int_0^\infty \frac{dk'_-}{2\pi 2k'_-} \left( \frac{ie}{k.p} \int_{-\infty}^{k.x} d(k.y) \, \hat{p}_\mu e^{ik.(y-x)} + c.c. \right) \]  

(3.12)

where we have absorbed \( \theta(x^+ - y^+) \) as a upper bound for the \( k.y \)-integration.
3.2.2 At the end of time

We want to calculate the total energy radiated from the electron. To include all the energy ever radiated we choose \( x^0 \to \infty \).

\[
A_\mu'(x) = \int \frac{d^2 k'}{(2\pi)^2} \int_0^\infty \frac{dk'}{2\pi} \left[ \frac{ie}{k.y} \int d(k.y) \tilde{p}_\mu e^{ik'.(y-x)} + c.c. \right] = \int \frac{d^2 k'}{(2\pi)^2} \int_0^\infty \frac{dk'}{2\pi} \left( A(k') e^{-ik'.x} + c.c. \right)
\]  

(3.13)

where

\[
A_\mu(k') := \frac{ie}{k.p} \int d(k.y) \tilde{p}_\mu e^{ik'.y}.
\]

(3.14)

Note that \( A_\mu' \) contains a term \( ik_\mu'x^0 \) in the exponent. This term is now infinite and one might expect that this will cause problems. Fortunately this term will exactly cancel in the radiated energy. It would be more rigorous to take the limit \( x^+ \to 0 \) after calculating the radiated energy (3.28), but we do it here for convenience.

3.2.3 Regularisation

\( A \) is not a well-defined integral and we will therefore redefine it by the use of the small parameter \( \epsilon \) which we then let approach zero. \( A \) is redefined as

\[
A_\mu(k') := \frac{ie}{k.p} \int d(k.y) \tilde{p}_\mu e^{ik'.y-|k,y|\epsilon} = \frac{ie}{k.p} \int d(k.y) \frac{\tilde{p}_\mu}{i k.p \pm \epsilon} \frac{d}{d(k.y)} \left( e^{ik'.y-|k,y|\epsilon} \right) = - \frac{ie}{k.p} \int d(k.y) e^{ik'.y-|k,y|\epsilon} \frac{d}{d(k.y)} \left( \frac{\tilde{p}_\mu}{i k.p \pm \epsilon} \right) = - e \int d(k.y) e^{ik'.y} \frac{d}{d(k.y)} \left( \frac{\tilde{p}_\mu}{k'.\tilde{p}} \right).
\]

(3.15)

In the second equality we used \( d/d(k.y) \ y^\mu = p^\mu/(k.p) \), in the third equality we used partial integration and the fact that the term \( -|k,y|\epsilon \) in the exponent makes the boundary terms disappear, and in the final equality we let \( \epsilon \to 0 \).

The integral (3.15) is well defined since we assume a finite laser pulse. \( \tilde{p}_\mu/(k'.\tilde{p}) \) will be a constant before and after the laser and therefore the derivative becomes zero everywhere but in a finite interval.

Note that \( A \) is exactly the same expression as \( j \) up to a factor \( i \). So what we do in the regularisation is let the current density of the electron fade away in the very early and very late time (long before and after the pulse where we know nothing happens). This means that at the time \( x^+ \to \infty \) when we calculate
the field there is no charge and therefore no Coulomb field. Since we are only interested in the radiation field, this is exactly what we want.

Note that \( k'.A(k') = 0 \).

### 3.2.4 Radiated energy

To find the total energy we integrate the energy density of the fields over all of space

\[
\text{Energy} = \frac{1}{2} \int d^3x \ (|E(x)|^2 + |B(x)|^2). \tag{3.16}
\]

The electric and magnetic fields coming from the electron are

\[
E(x) = \int \frac{d^2k'_+}{(2\pi)^2} \int_0^\infty \frac{dk'_-}{2\pi 2k'_-} \left( E(k') e^{-ik'.x} + \text{c.c.} \right), \tag{3.17}
\]

\[
B(x) = \int \frac{d^2k'_+}{(2\pi)^2} \int_0^\infty \frac{dk'_-}{2\pi 2k'_-} \left( B(k') e^{-ik'.x} + \text{c.c.} \right) \tag{3.18}
\]

where

\[
E(k') = -i k' A_0(k') + ik'_0 A(k'), \tag{3.19}
\]

\[
B(k') = ik' \times A(k') \tag{3.20}
\]

and \( k' \) and \( A \) are the 3-vector parts of \( k'_+ \) and \( A \).

Inserting this in (3.16) we get

\[
\text{Energy} = \frac{1}{2} \int d^3x \int \frac{d^2k'_+}{(2\pi)^2} \int_0^\infty \frac{dk'_-}{2\pi 2k'_-} \int \frac{d^2k''_+}{(2\pi)^2} \int_0^\infty \frac{dk''_-}{2\pi 2k''_-} \times
\]

\[
\times \left[ \left( E(k') e^{-ik'.x} + E^*(k') e^{ik'.x} \right) \cdot \left( E(k'') e^{-ik''.x} + E^*(k'') e^{ik''.x} \right) + \right.
\]

\[
\left. + \left( B(k') e^{-ik'.x} + B^*(k') e^{ik'.x} \right) \cdot \left( B(k'') e^{-ik''.x} + B^*(k'') e^{ik''.x} \right) \right].
\]

\[
(3.21)
\]

The \( x \)-integral creates delta functions of \( k' - k'' \) and \( k' + k'' \). The \((k'_+, k''_+)-\) integral can be transformed from lightfront coordinates into an integral over \( k''_0 \)

\[
\int \frac{d^2k''_+}{(2\pi)^2} \int_0^\infty \frac{dk''_-}{2\pi 2k''_-} \delta \left( \frac{(k''_+)^2}{(2\pi)^2} \right) \theta(k''_-) = \int \frac{d^4k''}{(2\pi)^4} \delta \left( \frac{(k''_+)^2}{(2\pi)^2} \right) \theta(k''_-) = \int \frac{d^3k''}{(2\pi)^3} \theta(k''_0) = \int \frac{d^3k''}{(2\pi)^3} \theta \left( k''_0 = \sqrt{(k''_+)^2} \right). \tag{3.22}
\]
Integrating over $x$ and $k''$ we find

$$\text{Energy} = \frac{1}{4} \int \frac{d^2k'_+}{(2\pi)^2} \int_0^\infty \frac{dk'_-}{2\pi 2k'_-} \frac{1}{k'_0} \times$$

$$\times \left[ (\mathbf{E}(k') \cdot \mathbf{E}(-k') + \mathbf{B}(k') \cdot \mathbf{B}(-k')) e^{-2i x_0 k'_0} +
2 (\mathbf{E}(k') \cdot \mathbf{E}^*(k') + \mathbf{B}(k') \cdot \mathbf{B}^*(k')) +
+ (\mathbf{E}^*(k') \cdot \mathbf{E}^*(-k') + \mathbf{B}^*(k') \cdot \mathbf{B}^*(-k')) e^{2i x_0 k'_0} \right]$$

(3.23)

where

$$\mathbf{E}(k') := \mathbf{E}(k')_{k'_0 = |k'|},$$

(3.24)

$$\mathbf{B}(k') := \mathbf{B}(k')_{k'_0 = |k'|}.$$  

(3.25)

Using that $k'.\mathbf{A} = 0$ we find that

$$\mathbf{E}(k') \cdot \mathbf{E}^*(-k') = -\mathbf{B}(k') \cdot \mathbf{B}^*(-k')$$

(3.26)

and

$$\mathbf{E}(k') \cdot \mathbf{E}^*(k') = \mathbf{B}(k') \cdot \mathbf{B}^*(k') = -(k'_0)^2 A(k') A^*(k').$$

(3.27)

Finlay we find the expression for the radiation energy.

$$\text{Energy} = -\int \frac{d^2k'_+}{(2\pi)^2} \int_0^\infty \frac{dk'_-}{2\pi 2k'_-} k'_0 A(k') A^*(k').$$

(3.28)

Note that $k'.\mathbf{A} = 0$ implies that $A(k') A^*(k') \leq 0$, hence the energy is positive.

### 3.3 Number of photons

Classically there are no photons. However, since we have the radiated energy as an integral over wave vectors, we can easily divide the radiated energy with the photon energy to get the number of quanta for each wave vector. By doing this we find that the photon number of the radiated field is

$$N_\gamma = -\int \frac{d^2k'_+}{(2\pi)^2} \int_0^\infty \frac{dk'_-}{2\pi 2k'_-} A(k') A^*(k').$$

(3.29)

This expression is infrared divergent for $\tilde{p}(\infty) \neq \tilde{p}(-\infty)$ which is equivalent to $A(\infty) \neq 0$ [13] where $A$ is defined by (2.9). Physically this condition means that the net acceleration is non-zero. Classically the infrared divergence is of cause no problem since there are no photons. But we will see this expression reappear in section 4.2 in the $\hbar \to 0$ limit the quantum description, where the infrared divergence will be discussed further.

The energy in (3.28) is finite, therefore there this is no problem classically.
4 QED

The purpose of this section is to calculate the probability of the tree-level process of \( e^- (p) \rightarrow e^- (p') + \gamma (k^1) \cdots + \gamma (k^n) \) in a plane wave electromagnetic background field, starting with a quantum description of the electron and the emitted photons and a classical description of the background field. One of the terms in this process is illustrated in Figure 1.

![Figure 1: One of the diagrams involved in the tree-level process of \( e^- (p) \rightarrow e^- (p') + \gamma (k^1) \cdots + \gamma (k^n) \) in a background field. The whole tree-level process is described by a sum of such diagrams over every permutation of \( k^j \).](image)

4.1 S-matrix elements

We start with the S-matrix operator to \( n \)th order in the coupling constant \( e \)

\[
S^n := \frac{(ie)^n}{n!} \int d^4x_n \cdots d^4x_1 T \left\{ (\bar{\psi} A' \psi)_{x_n} \cdots (\bar{\psi} A' \psi)_{x_1} \right\}
\]

(4.1)

where \( A' \) is the photon field and \( \psi \) is the electron field in a classical electromagnetic background. Then we calculate the lowest order S-matrix elements of going from a state with no photons to a state with \( n \) photons

\[
S^n_{fi} := \langle f^n | S^n | i \rangle.
\]

(4.2)

The initial state contains one electron with momentum \( p \) and spin \( \sigma \)

\[
|i \rangle := |p, \sigma \rangle.
\]

(4.3)

The final state contains one electron with momentum \( p' \) and spin \( \sigma' \), and \( n \) photons with momenta \( \{k^j\}_{j=1}^n \) and polarisations \( \{\epsilon^j\}_{j=1}^n \)

\[
\langle f^n | := \langle p', \sigma'; k^n, \epsilon^n; \cdots; k^1, \epsilon^1 | \rangle.
\]

(4.4)
In lightfront normalisation the S-matrix elements become [13]

\[
S^n_{fi} = \frac{(ie)^n}{\sqrt{2p'_- 2p_-} \left( \prod_{j=1}^n 2k_j^2 \right)} \left( \prod_{j=1}^n d^4x_j \right) \sum \pi \times \\
\times \bar{\psi}_p(\bar{\psi}) A_{k^{\pi(n)}}(x_n) S_F(x_n, x_{n-1}) A_{k^{\pi(n-1)}}(x_{n-1}) \ldots \\
\ldots iS_F(x_1, x_1) A_{k^{\pi(1)}}(x_1) \psi_p(x_1)
\]

where \( \sum \pi \) is the sum over all possible permutations of the one-one discrete function \( \pi(j) = m \) where \( j, m \in \{1, 2, \ldots, n\} \), as illustrated in Figure 2.

Figure 2: Summation over all the permutations of \( k^j \) in the tree-level process of \( e^- (p) \to e^- (p') + \gamma (k^1) + \gamma (k^2) + \gamma (k^3) \) in a background field.

\[ \psi^\sigma_p(x) \] is the electron Volkov wave function which is the positive energy solution to \((i\partial - eA - m)\psi = 0 \) [15]

\[
\psi^\sigma_p(x) := \varphi_p(x) u^\sigma_p, \quad (4.6)
\]

\( u^\sigma_p \) is the solution to \((\bar{\psi} - m)u = 0 \) with spin \( \sigma \) and normalisation condition

\[
\sum_{\sigma} u^{\sigma}_p u^{\sigma}_p = \bar{\psi} + m \quad (4.7)
\]

and \( \varphi \) is

\[
\varphi_p(x) := \left( 1 + \frac{e k A(k, x)}{2k.p} \right) \times \\
\times \exp \left\{ -ip.x - \frac{i}{2k.p} \int^{k,x} d\phi \left( 2eA(\phi), p - e^2 A^2(\phi) \right) \right\}. \quad (4.8)
\]
Here \( A \) is the classical background field (2.9), not to be confused with \( A^\epsilon_{k/\mu} \) in (4.5) which are the emitted photons wave functions

\[
A^\epsilon_{k/\mu}(x) := \epsilon^\epsilon_{\mu} e^{ik\cdot x}. \tag{4.9}
\]

### 4.1.1 Electron propagator

The electron propagator corresponding to a plane wave electromagnetic background field is [16]

\[
S_F(x, y) := i \int \frac{d^4p}{(2\pi)^4} \varphi_p(x) \frac{\not{p} + m}{(p)^2 - m^2 + i\epsilon} \varphi_p(y) \tag{4.10}
\]

where \( \varphi_p := \gamma^0 \varphi^0_p \).

In lightfront coordinates \( d^4p = 2dp_+ dp_- d^2p_\perp \). The \( p_+ \)-part of this is

\[
\int 2dp_+ \frac{\not{p} + m}{(p)^2 - m^2 + i\epsilon} e^{-ip_+(x-y)} =
\]

\[
= \frac{1}{2p_-} \int dp_+ \frac{e^{-ip_+(x-y)}}{p_+ - \frac{(p_+)^2 + m^2}{4p_-} + \frac{i\epsilon}{4p_-} } (\not{p} + m) =
\]

\[
= \frac{(\not{p} + m)}{2p_-} \left. i2\pi \left( \theta(-p_-)\theta(-x_+ + y_+) - \theta(p_-)\theta(x_+ - y_+) \right) \right|_{p_+ = \frac{(p_\perp)^2 + m^2}{4p_-}} \times e^{-ip_+(x-y)}
\]

Suppressing the on-shell condition \( p_+ = ((p_\perp)^2 + m^2)/(4p_-) \) from now on \( iS_F \) becomes

\[
S_F(x, y) = \int \frac{d^2p_+ dp_-}{(2\pi)^2 2p_-} \varphi_p(x) (\not{p} + m) \varphi_p(y) \times \left( \theta(p_-)\theta(x_+ - y_+) - \theta(-p_-)\theta(-x_+ + y_+) \right). \tag{4.12}
\]
4.1.2 One vertex

The n-th S-matrix element is now

\[ S_{n}^{n} = \frac{(ie)^{n}}{\sqrt{2p_{-}^i 2p_{-}^- \prod_{j=1}^{n} 2k_{-}^j}} \sum_{n} \left( \prod_{j=1}^{n} \int d^{4}x_{j} \right) \left( \prod_{l=1}^{n-1} \int \frac{d^{2}p_{l}^{j} d^{2}p_{l}^{-}}{(2\pi)^{2}2p_{l}^{j}} \right) \times \]

\[ \times \bar{u}_{p}^{n} \left( \varphi_{p}^{j} A_{k_{+}^{(n)}}^{(n)} \varphi_{p_{n-1}}^{j} \right)_{x_{n}} \left( p_{n-1}^{j} + m \right) \ldots \]

\[ \cdots \left( \varphi_{p_{2}^{j}} A_{k_{+}^{(2)}}^{(2)} \varphi_{p_{1}^{j}} \right)_{x_{2}} \left( p_{1}^{j} + m \right) \left( \varphi_{p_{1}^{j}} A_{k_{+}^{(1)}}^{(1)} \varphi_{p_{1}^{j}} \right)_{x_{1}} \left( \bar{u}_{p}^{1} \right) \times \]

\[ \times \prod_{j=1}^{n-1} \theta \left( p_{j}^{+} \right) \theta \left( x_{j+1}^{+} - x_{j}^{+} \right) - \theta \left( -p_{j}^{-} \right) \theta \left( -x_{j+1}^{-} + x_{j}^{+} \right) \].

(4.13)

The dependence \( x_{j}^{+} \) and \( x_{j}^{-} \) is simple and we can therefore calculate these integrals separately for each vertex \( j \). In light front coordinates the measure \( d^{4}x \) is

\[ d^{4}x_{j} = \frac{1}{2} dx_{j}^{+} dx_{j}^{-} d^{2}x_{j}^{\perp} \].

(4.14)

For each vertex \( j \) we have a factor

\[ \int d^{2}x_{j}^{+} d^{2}x_{j}^{-} \left( \varphi_{p_{j}^{+}} A_{k_{+}^{(j)}}^{(j)} \varphi_{p_{j-1}^{+}} \right)_{x_{j}} = \]

\[ = \left( 1 + \frac{eA_{k_{+}^{(j)}}}{2k_{+}^{j}} \right) \epsilon^{\pi^{(j)}} \left( 1 + \frac{e\gamma_{\mu}}{2k_{+}^{j-1}} \right) \times \]

\[ \times \exp \left\{ i \left( p_{j}^{+} + k_{+}^{(j)} - p_{j+1}^{+} \right) x_{j}^{+} + \right. \]

\[ \left. + i \int_{k_{+}^{j}}^{k_{-}^{j}} \left( \frac{2eA_{k_{+}^{(j)}} - e_{2}A_{k_{+}^{(j)}}}{2k_{+}^{j}} - \frac{2eA_{k_{+}^{(j-1)}} - e_{2}A_{k_{+}^{(j-1)}}}{2k_{+}^{j-1}} \right) \right\} \times \]

\[ \times \int d^{2}x_{j}^{\perp} e^{i \left( p_{j}^{\perp} + k_{+}^{(j)} - p_{j+1}^{\perp} \right) x_{j}^{\perp}} \int dx_{-} = \epsilon^{\pi^{(j)}} \left( \gamma_{\mu} + \frac{e}{2k_{+}^{j}} \left( A_{k_{+}^{(j)}} \gamma_{\mu} - \gamma_{\mu} A_{k_{+}^{(j)}} \right) + \frac{e_{2}A_{k_{+}^{(j)}}}{2k_{+}^{j-1}} \right) \].

(4.15)

We evaluate the factors of this separately. The fist is

\[ \left( 1 + \frac{eA_{k_{+}^{(j)}}}{2k_{+}^{j}} \right) \epsilon^{\pi^{(j)}} \left( 1 + \frac{e\gamma_{\mu}}{2k_{+}^{j-1}} \right) = \]

\[ = \epsilon^{\pi^{(j)}} \left( \gamma_{\mu} + \frac{e}{2} \left( A_{k_{+}^{(j)}} \gamma_{\mu} - \gamma_{\mu} A_{k_{+}^{(j)}} \right) + \frac{e_{2}A_{k_{+}^{(j)}}}{2k_{+}^{j-1}} \right) \].

(4.16)
In the second factor we rewrite the integrand in the exponent as
\[ \frac{2eA.p^j - e^2(A)^2}{2k.p^j} - \frac{2eA.p^{j-1} - e^2(A)^2}{2k.p^{j-1}} = \]
\[ = -eA \left( \frac{p^{j-1}}{k.p^{j-1}} - \frac{p^j}{k.p^j} \right) + e^2A^2 \frac{k.k^{(j)}}{2k.p^j k.p^{j-1}}. \quad (4.17) \]
The last factor is
\[ \int d^2x_j^+ e^{i(p^j + k^{\pi(j)} - p^{j-1})x_j^+} \int dx_j^- e^{i(p^j + k^{\pi(j)} - p^{j-1})x_j^-} = \]
\[ = (2\pi)^3\delta_\perp \delta_- \left( p^j + k^{\pi(j)} - p^{j-1} \right). \quad (4.18) \]
Using these \( \delta \)-functions together with the on-shell conditions \( p^j_+ = \left((p^j_\perp)^2 + m^2\right)/(4p^j_-) \) and \( k^j_+ = (k^j_\perp)^2/(4k^j_-) \), we find that
\[ \left( p^j_+ + k^{\pi(j)} - p^{j-1} \right)x_j^+ = \frac{k^{\pi(j)} p^j}{k.p^{j-1}} k.x_j \quad (4.19) \]
and (4.15) finally becomes
\[ \int d^2x_j^+ dx_j^- \left( \varphi^* A^{\pi(j)}_\perp (k.x_j) \varphi p^{j-1} \right) x_j = \]
\[ = \epsilon^{\pi(j)}_{\mu} \beta^\mu (k.x_j) (2\pi)^3\delta_\perp \delta_- (p^j + k^{\pi(j)} - p^{j-1}). \quad (4.20) \]
where
\[ \beta^\mu (k.x_j) := \]
\[ := \left( \gamma^\mu + \frac{e}{2} \left( \frac{A(k.x_j)k^\mu}{k.p^j} - \frac{\gamma^\mu A(k.x_j)k}{k.p^{j-1}} \right) + e^2A^2(k.x_j)k^\mu k \right) \times \]
\[ \times \exp \left\{ i \int k.x_j d\phi \left( \frac{k^{\pi(j)} p^j}{k.p^{j-1}} + eA(\phi). \left( \frac{p^j}{k.p^j} - \frac{p^{j-1}}{k.p^{j-1}} \right) + \right. \]
\[ \left. + \frac{e^2A^2(\phi) k.k^{\pi(j)}}{2k.p^j k.p^{j-1}} \right\}. \quad (4.21) \]

4.1.3 No internal positrons

Each vertex gives \((2\pi)^3\delta_\perp \delta_- (p^j + k^{\pi(j)} - p^{j-1})\) where \( j = 1, \ldots, n \) and \( p^n := p', \) \( p^0 := p. \) These \( \delta \)-functions can be evaluated together with the \((p_\perp, p_-)\)-integrals from each propagator, leaving one overall momentum conserving \( \delta \)-function.
\[
\left( \prod_{l=1}^{n-1} \int \frac{d^2 \mathbf{p}_l \, dp_l}{(2\pi)^3 2p_l^-} \right) \left( \prod_{j=1}^{n} \delta_{\perp} \delta_{-} \left( \mathbf{p}^j + k^{\pi(j)} - p^{j-1} \right) \right) = \\
\delta_{\perp} \delta_{-} \left( \mathbf{p}' - p + \sum_{j=1}^{n} k^j \right) \prod_{l=1}^{n-1} 2p_l^-.
\]

We also get that
\[
p'_- = p_- + \sum_{m=j+1}^{n} k^{\pi(m)}.
\]

Since \( p' \) and \( k^j \) are the momenta of real particles \( p_-, k^j_+ > 0 \), which implies \( p'_- > 0 \). This means that all the internal propagators are electrons and not positrons. Any diagram containing something similar to Figure 3 is zero. More importantly, we can evaluate the step functions over \( p_+ \) in (4.13)
\[
\theta(p_+) = 1, \quad \theta(-p_-) = 0.
\]

Evaluating this with the only integrals left this we get

\[
\left( \prod_{j=1}^{n} \int dx_j^+ \right) \prod_{l=1}^{n-1} \left( \theta (p_-) \theta (x_{l+1}^- - x_l^+) - \theta (p_-) \theta (-x_{l+1}^+ + x_l^+) \right) = \\
\left( \prod_{j=1}^{n} \int dx_j^+ \right) \prod_{l=1}^{n-1} \theta (x_{l+1}^+ - x_l^+) = \\
\int_{-\infty}^{\infty} dx_n^+ \int_{-\infty}^{x_n^+} dx_{n-1}^+ \ldots \int_{-\infty}^{x_2^+} dx_1^+.
\]

Note that this forces a time ordering in lightfront time, \( x^+ \)
4.1.4 Final expression for the general tree level $S$-matrix elements

Putting everything together in (4.13) we get

$$S^\sigma_{fi} = \frac{(ie)^n (2\pi)^3 \delta_+^2 \delta_- \left(p' - p + \sum_{j=1}^n k^j\right)}{2^n \sqrt{2p'_- 2p - \prod_{j=1}^n 2k_-^j}} \left(\prod_{j=1}^n \epsilon^j_{\mu_j}\right) \times$$

$$\times \sum_{\pi} \frac{1}{\prod_{l=1}^{n-1} 2p_-^l} \int_{-\infty}^{\infty} dx_1^+ \int_{-\infty}^{x_1^+} dx_2^+ \ldots \int_{-\infty}^{x_{n-1}^+} dx_n^+ \times$$

$$\times \hat{u}_{\mu_1} \beta^{n - 1} \mu_{\pi(n)}(k, x_n) (p^{n-1} + m) \beta^{n-1} \mu_{\pi(n-1)}(k, x_{n-1}) \ldots$$

$$\ldots (p^1 + m) \beta_1 \mu_{\pi(1)}(k, x_1) u_{\mu_n}^\sigma.$$

To get a bit closer to a lorentz invariant expression we use the fact that $2p_- = k.p/k_+$ and $x^+ = k.x/k$ to re write $S^\sigma_{fi}$ as

$$S^\sigma_{fi} = \frac{(ie)^n (2\pi)^3 \delta_+^2 \delta_- \left(p' - p + \sum_{j=1}^n k^j\right)}{2^n \sqrt{k.p' k.p \prod_{j=1}^n 2k_-^j}} \left(\prod_{j=1}^n \epsilon^j_{\mu_j}\right) \times$$

$$\times \sum_{\pi} \frac{1}{\prod_{l=1}^{n-1} k.p_-^l} \int_{-\infty}^{\infty} d(k, x_n) \int_{-\infty}^{k.x_n} d(k, x_{n-1}) \ldots \int_{-\infty}^{k.x_2} d(k, x_1) \times$$

$$\times \hat{u}_{\mu_1} \beta^{n - 1} \mu_{\pi(n)}(k, x_n) (p^{n-1} + m) \beta^{n-1} \mu_{\pi(n-1)}(k, x_{n-1}) \ldots$$

$$\ldots (p^1 + m) \beta_1 \mu_{\pi(1)}(k, x_1) u_{\mu_n}^\sigma.$$

Remember that $\beta$ and $\{p^j\}_{n=1}^{n-1}$ depend on $\pi$

$$p^j = p + \sum_{m=1}^j \left(-k\pi(m) + \frac{k\pi(m) p^j}{k.p_-^{j-1}} k\right) =$$

$$= p - \sum_{m=1}^j k\pi(m) + \frac{2p. \sum_{m=1}^j k\pi(m) - \left(\sum_{m=1}^j k\pi(m)\right)^2}{2k. \left(p - \sum_{m=1}^j k\pi(m)\right)} k.$$

The fist of the above expression for $p^j$ does not look very useful since it contains $p^j$ itself, but it will turn out to be very convenient in the various limits explored later on.

The implications of this expression (4.27), in terms of probabilities will be explored in the following sections by investigating different limits.

4.2 Classical limit

We do not get much further with this expression, (4.27) without doing some simplifications. The easiest way to go is to take the complete classical limit, i.e. the limit $\hbar \to 0$. We will keep using natural units in which $\hbar = 1$ and just remembering which quantities are proportional to $\hbar$.
All the photon momenta are proportional to $\hbar$

\[ k^j = \mathcal{O}(\hbar). \quad (4.29) \]

Apart from that the only other $\hbar$ is a factor $1/\hbar$ in the exponent.

To lowest order in $\hbar$, $\beta$ becomes

\[ \beta^j \mu(k,x_j) \rightarrow \tilde{\beta}^{\pi(j)} \mu(k,x_j) := \]

\[ := \left( \gamma^\mu + \frac{eA(k,x_j)}{k.p} k^\mu - \frac{eA(\phi)}{k.p} k^\mu + \frac{e^2A^2(k,x_j)}{2(k.p)^2} k^\mu \right) \times \]

\[ \times \exp \left\{ i \int_{k,x_j} \tilde{\beta}^{\pi(j)} \frac{k^{\pi(j)} p}{k.p} + \frac{eA(\phi)p}{(k.p)^2} k^{\pi(j)} - \frac{eA(\phi)k^{\pi(j)}}{k.p} + \frac{e^2A^2(\phi)}{2(k.p)^2} k^{\pi(j)} \right\} = \]

\[ = \left( \gamma^\mu + \frac{eA(k,x_j)}{k.p} k^\mu - \frac{eA(\phi)}{k.p} k^\mu + \frac{e^2A^2(k,x_j)}{2(k.p)^2} k^\mu \right) e^{ik^{\pi(j)}\tilde{y}(k,x_j)} \quad (4.30) \]

where $\tilde{y}$ is the classical position described in (3.3). We switched the notation $y \rightarrow \tilde{y}$ for later convenience.

The exponent is expressed to first order in $k^j$. This is correct because of the factor $1/\hbar$ mentioned before.

Not that $\tilde{\beta}^j$ only depends on the one photon momentum $k^j$ corresponding to that vertex.

### 4.2.1 No radiation reaction

In the classical limit (4.28) becomes

\[ p^j = p \quad (4.31) \]

which means that there is no radiation reaction to lowest order in $\hbar$, even though this effect exists classically. [14]

Remember that we deliberately ignored the radiation reaction in our classical calculation in section 3.
4.2.2 S-matrix elements in the classical limit

In the classical limit the S-matrix elements become

\[ S^{nf}_{fi} = \frac{(ie)^n (2\pi)^3 \delta^2 \delta_-(p' - p)}{2^n (k.p)^n \prod_{j=1}^{n} c_{\mu_j}^j} \times \]

\[ \times \sum_{\pi} \int_{-\infty}^{\infty} d(k.x_n) \int_{-\infty}^{k.x_n} d(k.x_{n-1}) \ldots \int_{-\infty}^{k.x_2} d(k.x_1) \times \]

\[ \times \bar{u}_p \beta_p^{(n)} \mu_\pi(k.x_n) (\beta + m) \beta_p^{(n-1)} \mu_\pi(k.x_{n-1}) \ldots \]

\[ \ldots (\beta + m) \beta_p^{(1)} \mu_\pi(k.x) u_p^\sigma. \]

This can be simplified further with the use of

\[ (\beta + m) \beta_p^j \mu(\beta + m) = 2 \beta_p^j \mu(\beta + m) \]  

(4.33)

where

\[ \beta_p^{(j)} \mu := \beta_p^{(j)} \mu (\gamma \rightarrow p) = \bar{p}(k.x)e^{ikx} \bar{g}(k.x) \]  

(4.34)

and \( \bar{p} \) is the classical electron momentum from (3.1).

The spinor part of the S-matrix then becomes

\[ \bar{u}_p \beta_p^{(n)} \mu_\pi(k.x_n) (\beta + m) \beta_p^{(n-1)} \mu_\pi(k.x_{n-1}) \ldots \]

\[ \ldots (\beta + m) \beta_p^{(1)} \mu_\pi(k.x) u_p^\sigma = \]

\[ = \frac{\bar{u}_p \beta_p^{(n)} \mu_\pi(k.x_n)}{2m} \prod_{j=1}^{n} 2 \beta_p^{(j)} \mu (k.x_j) \]  

(4.35)

and

\[ S^{nf}_{fi} = \frac{(ie)^n (2\pi)^3 \delta^2 \delta_-(p' - p)}{2^n (k.p)^n \prod_{j=1}^{n} c_{\mu_j}^j} \times \]

\[ \times \int_{-\infty}^{\infty} d(k.x_n) \int_{-\infty}^{k.x_n} d(k.x_{n-1}) \ldots \int_{-\infty}^{k.x_2} d(k.x_1) \times \]

\[ \times \sum_{\pi} \prod_{j=1}^{n} \epsilon_\pi(j) \beta_p^{(j)}(k.x_j). \]

Because of the summation over all perturbations \( \pi \), the integrand is symmetric over all \( x_j \), and therefore we can expand the integrals as

\[ \int_{-\infty}^{\infty} d(k.x_n) \int_{-\infty}^{k.x_n} d(k.x_{n-1}) \ldots \int_{-\infty}^{k.x_2} d(k.x_1) \rightarrow \frac{1}{n!} \prod_{j=1}^{n} \int_{-\infty}^{\infty} d(k.x_j). \]  

(4.37)
Then we can rename the integration variables \( x_j \rightarrow x_{\pi(j)} \) to see that every term in the \( \pi \)-sum is exactly equal, so that we can replace
\[
\sum_{\pi} \rightarrow n! .
\] (4.38)

Finally we arrive at
\[
S_{fi}^n = \frac{(ie)^n (2\pi)^3 \delta_+^2 \delta_-(p' - p)}{(k.p)^n \sqrt{\prod_{j=1}^n 2k_j^-}} \frac{\bar{u}_p u_p^n}{2m} \prod_{j=1}^n \epsilon_{\mu_j}^j \int d(k.x_j) \vec{\beta}_p^\mu (k.x_j) =
\]
\[
= \frac{(2\pi)^3 \delta_+^2 \delta_-(p' - p)}{\sqrt{\prod_{j=1}^n 2k_j^-}} \frac{\bar{u}_p u_p^n}{2m} \prod_{j=1}^n \epsilon^j . A(k^j)
\] (4.39)

where \( A \) is defined by (3.14).

### 4.2.3 Regularisation

\( A \) in (3.14) is not well defined. Again we need to regularize, and we need a regularisation that preserve gauge invariance.

\( S_{fi}^n \) is gauge invariant if \( S_{fi}^n(\epsilon^j \rightarrow k^j) = 0 \) for any \( j = 1, \ldots, n \) [11, Chapter 5]. The regularisation in (3.15) fulfills this and therefore we again redefine \( A \) as (3.15).

### 4.2.4 Probability of emitting \( n \) photons

At tree-level the probability of the electron emitting \( n \) photons is
\[
P^n = \frac{1}{2} \sum_{\sigma_1, \sigma_2} \int \frac{d^2 p_+}{(2\pi)^2} \int_0^\infty \frac{dp'_-}{2\pi} \frac{1}{n!} \left( \prod_{j=1}^n \int \frac{d^2 k_j^+}{(2\pi)^2} \int_0^\infty \frac{dk_j^-}{2\pi} \right) |S_{fi}^n|^2 \frac{1}{V} \] (4.40)

where the \((n!)^{-1}\) compensates for over counting of photons states, and the volume in light front coordinates is
\[
V := V_+ V_- = (2\pi)^3 \delta_+ \delta_- (0)
\] (4.41)
In the classical limit the probabilities become

\[
P_{\hbar \to 0}^n = \frac{1}{n!} \int \frac{d^2 p_+}{(2\pi)^2} \int_0^\infty \frac{dp_-(\sigma)}{2\pi} \frac{(2\pi)^3 \delta^3_\perp (p' - p)^2}{(2\pi)^3 \delta^3_\perp \delta_\perp (0)} \times \]

\[
\times \frac{1}{8m^2} \sum_{\sigma \sigma'} \bar{u}_\sigma^R p' u^R_{\sigma'} \times \]

\[
\times \prod_{j=1}^n \int \frac{d^2 k^j_+}{(2\pi)^2} \int_0^\infty \frac{dk^j_-}{2\pi 2k^j_-} \sum_{\epsilon, \epsilon'} \epsilon^j \cdot \mathcal{A}(k^j) \cdot \epsilon^j \cdot \mathcal{A}^*(k^j) = \]

\[
= \frac{1}{n!} \left( - \int \frac{d^2 k'_+}{(2\pi)^2} \int_0^\infty \frac{dk'_-}{2\pi 2k'_-} \mathcal{A}(k') \cdot \mathcal{A}^*(k') \right)^n = \]

\[
= \frac{1}{n!} (P_{\hbar \to 0}^1)^n. \tag{4.42} \]

In the last equality we used that \( k^j \cdot \mathcal{A}(k^j) = 0 \) which implies that \( \sum_{\epsilon} \epsilon^j \cdot \mathcal{A}(k^j) \cdot \epsilon^j \cdot \mathcal{A}^*(k^j) = -\mathcal{A}(k^j) \cdot \mathcal{A}^*(k^j) \). This also means that \( \mathcal{A} \) has to be lightlike or spacelike which means that

\[
\mathcal{A}(k') \cdot \mathcal{A}^*(k') \leq 0 \tag{4.43} \]

and thus we see that \( P_{\hbar \to 0}^n \geq 0 \).

Note that \( P_{\hbar \to 0}^1 = N_\gamma \) where \( N_\gamma \) is the classical photon count in (3.29).

### 4.2.5 Too much probability

The tree-level the probability of emitting no photons is

\[
P^0 = \frac{1}{2} \sum_{\sigma \sigma'} \int \frac{d^2 p_+}{(2\pi)^2} \int_0^\infty \frac{dp_-(\sigma)}{2\pi} \frac{|\langle p', \sigma' | p_\sigma \rangle|^2}{V} = 1. \tag{4.44} \]

Note that this result is independent of taking the classical limit \( \hbar \to 0 \).

This gives us a total probability \( \sum_{n=0}^\infty P_{\hbar \to 0}^n \geq 1 \) which is clearly unphysical. In addition we have an infrared divergence for \( P^1 \) if \( \mathcal{A}(-\infty) \neq 0 \) [13]. The reason for this result is that there are lots of processes involving loop diagrams that we have not taken into account.

In perturbative scattering, loop diagrams actually give a negative contribution to the probability that precisely cancel the infrared divergence in every observable [11, Chapter 6]. It is reasonable to believe that if one calculates the loop contribution one will find a negative probability contribution that exactly cancels out this anomaly. These loop diagrams ought to be calculated to know this for certain, but doing this is outside the scope of this thesis.
4.2.6 Renormalisation of the probabilities and average number of photons

We will now calculate the average number of photons $N_\gamma$ to compare with our result (3.29) from the classical calculations. For this purpose we will assume that the tree-level process described by $P_{\hbar \to 0}^n$, $n = 0, 1, \ldots$ are the only ones possible and renormalise the probabilities according to that. We define the new renormalized probability $\mathcal{P}^n$ as [14]

$$\mathcal{P}^n := \frac{P^n}{Z} = \frac{(P_{\hbar \to 0}^1)^n}{n! Z}$$

and demand that $\sum_{n=0}^{\infty} \mathcal{P}^n = 1$. This gives that

$$Z = \sum_{n=0}^{\infty} P_{\hbar \to 0}^n = e^{P_{\hbar \to 0}^1}$$

and

$$\mathcal{P}^n = \frac{1}{n!} (P_{\hbar \to 0}^1)^n e^{-P_{\hbar \to 0}^1}.$$  

From these renormalised probabilities we find that the average number of photons is

$$N_\gamma = \sum_{n=1}^{\infty} n \mathcal{P}^n = P_{\hbar \to 0}^1$$

which agrees with the classical result in (3.29).

4.3 Next order in $\hbar$

In this section we will calculate the first order quantum correction to the classical limit. We will do this by doing the same calculations as in section 4.2 but to the next order in $\hbar$.

The explicit calculations of this section are rather long and involve no new ideas. In this section we will therefore just state some results. The main result is presented in section 4.3.2 and discussed in section 4.3.3.

All the calculations in this section are to order $O(\hbar)$. Just as in the previous section, we continue to keep natural units and just remember that $k^j = O(\hbar)$. The only other $\hbar$ are the $1/\hbar$ in all exponents. In (4.51), (4.55) and (4.57) we have expanded part of the exponent as a sum and we therefore choose to write out the associated $1/\hbar$ explicitly.

In this approximation

$$p^j = p + \sum_{m=1}^{j} \left( -k \pi(m) + \frac{k^{\pi(m)} - k^j}{k.p.k} \right)$$

and

$$p^j.p^l = m^2$$

where $j, l = 1, \ldots, n$. 

20
4.3.1 Regularisation

We will not attempt any explicit regularisation. Instead we will just assume that there is a way, similar to that in sections 3.2.3 and 4.4.2, to make the S-matrix elements well defined in such a way that $S^n_{ij} (\epsilon^j \rightarrow k^j) = 0$ for any $k = 1, \ldots, n$.

4.3.2 Probability

The probability of emitting $n$ photon, calculated at tree-level and to order $\mathcal{O}(\hbar)$ is

$$P_n^{\mathcal{O}(\hbar)} = P_n^{\hbar \rightarrow 0} +$$

$$+ \left( \frac{-e^2}{(k.p)^2} \right)^n \left( \prod_{j=1}^{n} \int \frac{d^2 k_x^j}{(2\pi)^2} \int_0^\infty \frac{dk_y^j}{2\pi 2k_y^j} \right) \times$$

$$\times \sum \int_{-\infty}^{\infty} d(k.x_n) \int_{-\infty}^{k.x_1} d(k.x_{n-1}) \ldots \int_{-\infty}^{k.x_2} d(k.x_1) \times$$

$$\times \int_{-\infty}^{\infty} d(k.y_n) \int_{-\infty}^{k.y_1} d(k.y_{n-1}) \ldots \int_{-\infty}^{k.y_2} d(k.y_1) \times$$

$$\times \left( \prod_{j=1}^{n} e^{ik_{j}} \left( \tilde{p}(k.x_{\pi} - 1(j)) - \tilde{p}(k.y_{j}) \right) \right) \times$$

$$\times \left( \frac{i}{\hbar} \sum_{j=1}^{n} \int^{k.x_j} d\phi \: \alpha_{2}^{i} (\phi) - \frac{i}{\hbar} \sum_{j=1}^{n} \int^{k.y_j} d\phi \: \alpha_{1}^{i} (\phi) \right) \times (4.51)$$

$$\times \prod_{j=1}^{n} \tilde{p}(k.x_{\pi} - 1(j)) \tilde{p}(k.y_{j}) +$$

$$+ \frac{1}{k.p} \left( \sum_{j=1}^{n} k.k_j + \sum_{l=1}^{n-1} (n - l) \left( k.k_{\pi} (l) + k.k_{l} \right) \right) \times$$

$$\times \prod_{j=1}^{n} \tilde{p}(k.x_{\pi} - 1(j)) \tilde{p}(k.y_{j}) +$$

$$+ \sum_{j=1}^{n} d^j (k.x_{\pi} - 1(j), k.y_{j}) \prod_{m \neq j} \tilde{p}(k.x_{\pi} - 1(m)) \tilde{p}(k.y_{m})$$
where
\[
\alpha^j,\pi(\phi) := \sum_{m=1}^{j} \left( \frac{k.k(\pi(j)) k(\pi(m))}{(k.p)^2} + \frac{k.k(\pi(j)) k(\pi(m))}{(k.p)^2} - \frac{k(\pi(j)) k(\pi(m))}{(k.p)^2} \right) + eA(\phi) \left[ \left( \frac{k.k(\pi(j))}{(k.p)^2} p - \frac{k.k(\pi(j))}{(k.p)^2} k(\pi(j)) + \sum_{m=1}^{j-1} \left( \frac{2k.k(\pi(j)) k(k(\pi(m)))}{(k.p)^3} p - \frac{k.k(\pi(j))}{(k.p)^2} k(\pi(j)) - \frac{k.k(\pi(j))}{(k.p)^2} k(\pi(m)) \right) \right] + e^2 A^2(\phi) \left( \frac{(k.k(\pi(j))^2}{2(k.p)^3} + \sum_{m=1}^{j-1} \frac{k.k(\pi(j)) k(k(\pi(m)))}{(k.p)^3} \right) \right)
\frac{\pi_1 - \pi_1}{j} \sum_{m=1}^{j} \left( k.k(\pi(m)) - \frac{k.k(\pi(m))}{k.p} - k.m \right) \right) + e^2 A^2(\phi) \left( \sum_{m=1}^{j} \frac{k.k(\pi(m))}{k.p} - \sum_{m=1}^{j} k.k(m) \right)
\end{equation}
\]

and
\[
\alpha^j := \alpha^j,\pi(\phi) (\pi(j) \rightarrow j, \pi(m) \rightarrow m)
\end{equation}

\(\alpha^j,\pi\) is purely \(O(\tilde{\hbar}^2)\), but note that the term is divided by \(\tilde{\hbar}\) which makes it \(O(\tilde{\hbar})\). \(\tilde{p}\) is a classical value defined in (3.1) and therefore purely \(O(1)\).

\[d^j(k,x,k.y) :=
\begin{align*}
&= e \left( A(k,x) - A(k,y) \right) \left( \sum_{m=1}^{j} \left( \frac{k.k(\pi(m))}{k.p} p - k(\pi(m)) \right) - \sum_{m=1}^{j} \left( \frac{k.k(m)}{k.p} p - k.m \right) \right) + e^2 \frac{k.p}{2} \left( A^2(k,x) - A^2(k,y) \right) \left( \sum_{m=1}^{j} k.k(\pi(m)) - \sum_{m=1}^{j} k.k(m) \right)
\end{align*}
\]

\(d^j\) is purely \(O(\hbar)\).

4.3.3 Remarks

Note that there is only one sum over perturbations in the probability. \(|S^j_{fi}|\) includes two such sums, one from \(S^n_{fi}\) and one from \(S^n_{fi}^\ast\). But in integrating over every \(k^j\), they become dummy variables that can be renamed after our liking. In the calculation leading up to (4.51) we chose that the first \(k^j\) in \(S^n_{fi}^\ast\) where always to be named \(k^1\), the second one \(k^2\), and so on. By “first” we mean the one emitted first in lightfront time. Or in other words we chose to rename \(k^j\) in such a way that the perturbation \(\pi\) belonging to \(S^n_{fi}^\ast\), is equal to \(\pi(j) \equiv j\) in every term.
Since there is a sum over every perturbation of $k^j$ in $S^n_{j}$, it will be the same independently of how we rename the $k^j$. Therefore by renaming the $k^j$ to always get the same perturbation $\pi(j) \equiv j$ in $S^n_{j}$, for every term, the sum over perturbation in $S^n_{j}$, could be swapped for an $n!$.

In addition, this renaming of $k^j$ means that the perturbation of the $k^j$ in $S^n_{j}$, which in the remaining one in (4.51), is the perturbation of $k^j$ in relation to their order in $S^n_{j}$. This means that the one term in the sum where $\pi(j) \equiv j$ is contains the diagonal corrections while all the other terms in the perturbation sum contains corrections belonging to inference terms.

In the classical limit the diagonal terms and the interference terms were all equal. This is clearly not the case to the next order in $\hbar$.

### 4.3.4 One photon

For $n = 1$ we find that

$$P_{\Omega(h)}^1 = P_{\hbar \rightarrow 0}^1 + \frac{-e^2}{(k.p)^2} \int \frac{d^2 k_1}{(2\pi)^2} \int_0^\infty \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} d(k.x) \int_{-\infty}^{\infty} d(k.y) \times$$

$$\times \left( \frac{k.k_1}{k.p} + \frac{i}{\hbar} \int_{k.y}^{k.x} d\phi \alpha^1(\phi) \right) \hat{p}(k.x) \hat{p}(k.y) e^{ik_1 \cdot (\tilde{y}(k.x) - \tilde{y}(k.y))} \quad (4.55)$$

where

$$\alpha^1(\phi) := \frac{2k.k_1 k_1.p}{(k.p)^2} - \frac{k_1.k_1}{(k.p)^2} + eA(\phi) \left( \frac{(k.k_1)^2}{(k.p)^3} p - \frac{k.k_1}{(k.p)^2} k_1^2 \right) +$$

$$+ \frac{e^2 A^2(\phi)}{2(k.p)^3} (k.k_1)^2. \quad (4.56)$$
4.3.5 Two photons

For \( n = 2 \) we find that

\[
P_{\text{O(h)}}^2 = \frac{\pi^2}{2k^2_p} + \frac{\pi^2}{2k^2_p} \int_0^{\infty} \frac{d\phi}{2\pi^2} \int_0^{\infty} \frac{d\phi'}{2\pi^2} \int_0^{\infty} \frac{d\phi_1}{2\pi^2} \int_0^{\infty} \frac{d\phi_2}{2\pi^2} \times
\]

\[
\times \int_{-\infty}^{\infty} d(k.x_2) \int_{-\infty}^{\infty} d(k.x_1) \int_{-\infty}^{\infty} d(k.y_2) \int_{-\infty}^{\infty} d(k.y_1) \times
\]

\[
\times \left[ \frac{k^2 + 3k^1_k}{k^2} + \frac{\pi^2}{2k^2_p} \int_{k.y_2}^{k.x_2} d\phi \ alpha^2(\phi) + \frac{\pi^2}{2k^2_p} \int_{k.y_1}^{k.x_1} d\phi \ alpha^1(\phi) \right] \times
\]

\[
\times \tilde{p}(k.x_2) \tilde{p}(k.y_2) \tilde{p}(k.x_1) \tilde{p}(k.y_1) +
\]

\[
+ d^2(k.x_2, k.y_2) \tilde{p}(k.x_1) \tilde{p}(k.y_1) + d^2(k.x_1, k.y_1) \tilde{p}(k.x_2) \tilde{p}(k.y_2) \right] \times
\]

\[
\times \exp \left\{ ik^2 \left( \tilde{y}(k.x_2) - \tilde{y}(k.y_2) \right) + ik^1 \left( \tilde{y}(k.x_1) - \tilde{y}(k.y_1) \right) \right\} +
\]

\[
+ \left[ \frac{2k^2 + 2k^1_k}{k^2} + \frac{\pi^2}{2k^2_p} \int_{k.y_2}^{k.x_2} d\phi \ alpha^1(\phi) + \frac{\pi^2}{2k^2_p} \int_{k.y_1}^{k.x_1} d\phi \ alpha^2(\phi) -
\right.
\]

\[
- \frac{\pi^2}{2k^2_p} \int_{k.y_2}^{k.x_2} d\phi \ alpha^2(\phi) - \frac{\pi^2}{2k^2_p} \int_{k.y_1}^{k.x_1} d\phi \ alpha^1(\phi) \right] \times
\]

\[
\times \tilde{p}(k.x_1) \tilde{p}(k.y_2) \tilde{p}(k.x_2) \tilde{p}(k.y_1) +
\]

\[
+ d^2(k.x_1, k.y_2) \tilde{p}(k.x_2) \tilde{p}(k.y_1) + d^2(k.x_2, k.y_1) \tilde{p}(k.x_1) \tilde{p}(k.y_2) \right] \times
\]

\[
\times \exp \left\{ ik^2 \left( \tilde{y}(k.x_1) - \tilde{y}(k.y_2) \right) + ik^1 \left( \tilde{y}(k.x_2) - \tilde{y}(k.y_1) \right) \right\} \right]\] (4.57)

where the fist term is the diagonal correction and the second term is the interferes correction. \( \pi \) in the second term is \( \pi(1) = 2, \pi(2) = 1 \).

4.4 One hard photon

Another possible way to explore the proes of photon emission in a background field is to investigate the limit where only one photon is hard and the others are soft. This means that \( k_0^1 > \Lambda \) and \( k_0^m < \Lambda \) for \( m > 1 \) where \( \Lambda \) is some low
energy cut-off, which could be the lower limit for the photon energy detectable by experimental equipment. $\Lambda$ is assumed to be small enough so that we can approximate everything to lowest order in $\Lambda$.

Since $k^m = O(\Lambda)$, $m > 1$ the calculation of this section will resemble those of the limit $h \to 0$ in section 4.2. The difference is the one hard photon and that the $1/h$ in the exponent does not matter in this approximation. But since before the regularisation the exponents of $\beta^j$ defined in (4.21) can have arbitrary large absolute value for $k.x_j = \pm \infty$, we can not disregard the exponent completely. Up till regularisation we will keep the exponent of $\beta^m$, $m > 1$ to their lowest order in $\Lambda$, even though this is one order higher than the rest of the expression.

4.4.1 S-matrix elements for the one hard photon limit

We start with the general S-matrix elements from (4.27)

$$S_{f_i} = \frac{(ie)^n (2\pi)^3 \delta^2 \delta_-(p' - p + \sum_{j=1}^n k^j)}{2^n \sqrt{k.p'} k.p \prod_{j=1}^n 2k^j} \times \left( \prod_{j=1}^n e^j \right) \times \sum_{\pi} \prod_{j=1}^n k.p' \int_{-\infty}^{\infty} d(k.x_n) \int_{-\infty}^{k.x_n} d(k.x_{n-1}) \ldots \int_{-\infty}^{k.x_2} d(k.x_1) \times (4.58)$$

$$\times \bar{u}^\sigma_{p'} \beta^\mu_{\pi(s)}(k.x_n)(p^{n-1} + m) \beta^\mu_{\pi(s-1)}(k.x_{n-1}) \ldots \beta^1 \beta^1_{\pi(1)}(k.x_1) u^\sigma_p$$

where $p^j$ now is

$$p^j = \begin{cases} p & j < \pi^{-1}(1) \\ p' & j \geq \pi^{-1}(1) \end{cases}. \quad (4.59)$$

In this approximation $\beta$ becomes

$$\beta^j \mu = \begin{cases} \tilde{\beta}^\pi(j) \mu & j < \pi^{-1}(1) \\ \tilde{\beta}^\mu & j = \pi^{-1}(1) \\ \tilde{\beta}^{\pi(j)} \mu & j < \pi^{-1}(1) \end{cases} \quad (4.60)$$

where $\tilde{\beta}$ is given by (4.30), $\tilde{\beta}': \tilde{\beta} (p \to p')$ and $\tilde{\beta} : \beta^j (p^j \to p', p^{j-1} \to p, k^{\pi(j)} \to k^{1})$, $\beta^j$ is defined in (4.21).

It might seem inconsistent to use $\tilde{\beta}^j$ in this limit since the spinor factor of this expression is taken to $O(1)$ and the exponent is taken to $O(\Lambda)$. Since we are taking the soft limit this time and not the classical limit the factor $1/h$ in the exponent does not matter. However, before renormalisation the absolute value of exponent can become arbitrary large and we must therefore keep it to lowest non-zero order which happens to be $O(\Lambda)$.
Using (4.33) the spinor part of the S-matrix element becomes

\[ \bar{u}_p' \beta^{\mu_{\pi(n)}}(k.x_n)(p^{\pi-1} + m)\beta^{\mu_{\pi(n-1)}}(k.x_{n-1}) \ldots \]

\[ \ldots (p^1 + m)\beta^1 \mu_{\pi(1)}(k.x_1)u^\sigma_p = \]

\[ \left( \prod_{m=\pi-1(1)+1}^n \frac{2\beta^{\pi(m)}_{\mu_{\pi(m)}}(k.x_m)}{k.p'} \right) \left( \prod_{m=1}^{\pi-1(1)-1} \frac{2\beta^{\mu_{\pi(m)}}_p(k.x_m)}{k.p} \right) \times \]

\[ \bar{u}_p' \beta^{\mu_{\pi(1)}(k.x_{\pi-1(1)})}u^\sigma_p \]

(4.61)

where \( \beta^{\mu}_{p'} := \beta'(\gamma \rightarrow p') \) and \( \beta_p := \beta(\gamma \rightarrow p) \). The full S-matrix element becomes

\[ S^n_{fi} = \left( i e \right)^n \left( 2\pi \right)^3 \delta_\perp \delta_\perp (p' - p + k^1) \left( \prod_{j=1}^n e^\mu_{\mu_j} \right) \times \]

\[ \times \sum \int_{-\infty}^{\infty} d(k.x_n) \int_{-\infty}^{k.x_n} d(k.x_{n-1}) \ldots \int_{-\infty}^{k.x_2} d(k.x_1) \times \]

\[ \times \left( \prod_{m=\pi-1(1)+1}^n \frac{1}{k.p'} \beta^{\pi(m)}_{\mu_{\pi(m)}}(k.x_m) \right) \times \]

\[ \times \left( \prod_{m=1}^{\pi-1(1)-1} \frac{1}{k.p} \beta^{\mu_{\pi(m)}}_p(k.x_m) \right) \times \]

\[ \times \frac{1}{2} \bar{u}_p' \beta^{\mu_{\pi(1)}(k.x_{\pi-1(1)})}u^\sigma_p = \]

(4.62)

where in the last equality we renamed \( x_{\pi-1(m)} \rightarrow x_m \).

26
where \( \hat{A} \) is the classical value \( A \) described in section 3.2, but for an electron making a sudden change of momentum \( \tilde{p} \rightarrow \tilde{p}' := \tilde{p} (p \rightarrow p') \) when \( k.y = k.x \)

\[
\hat{A}_{k.x_1} (k^j) := ie \int d(k.y) \frac{\hat{p}_{k.x_1}(k.y)}{k^j} \hat{p}_{k.x_1}(k.y) e^{ik^j (\hat{y}_{k.x_1}(k.y))} \tag{4.64}
\]

where \( \hat{p} \) is the classical momentum

\[
\hat{p}_{k.x_1}(k.y) := \theta(k.x_1-k.y) \left( p^\mu - eA^\mu(k.y) + \frac{2eA(k.y).p - e^2 (A(k.y))^2}{2k.p} k^\mu \right) + \theta(k.y-k.x_1) \left( p'^\mu - eA^\mu(k.y) + \frac{2eA(k.y).p' - e^2 (A(k.y))^2}{2k.p'} k^\mu \right) \tag{4.65}
\]

and \( \hat{y} \) the classical path

\[
\hat{y}_{k.x_1}^\mu (k.y) = \int^{k.y} d\phi \frac{\hat{p}_{k.x_1}(\phi)}{k.\hat{p}(\phi)} . \tag{4.66}
\]

The above equations should be compared with (3.14), (3.1) and (3.3).

Remember that \( \hat{y} \) in this expression does not originate from a classical position although the equation happens to agree. \( \hat{y} \) comes from the phase in the Volkov wave function. Demanding \( \hat{y} \) to be continuous is the same as demanding that the phase of the Volkov wave function (4.6) is continuous over \( \psi_\sigma^\mu (x_1) \rightarrow \psi_{\sigma'}^\mu (x_1) \).

What we have here is exactly the classical limit \( \hbar \rightarrow 0 \) of this process taken only for the soft photons. This interim result is an artefact of the similarity between the soft limit and the classical limit, together with the fact that we could not take the full limit in the exponent due to arbitrary large \( k.y \).

In the next section we will see that choosing \( \hat{y} \) to be continuous is required for gauge independence in the classical limit of \( \beta \) and \( \beta' \). In the full soft limit however this phase will disappear completely.

### 4.4.2 Regularisation

We want to redefine \( \hat{A} \) so that it becomes well defined and so that \( k'.\hat{A}(k') = 0 \) which is the condition for \( S_{j_0}^n \) being gauge invariant. To do this we follow the
procedure of section 3.2.3 and (3.15). We redefine \( \hat{A} \) as

\[
\hat{A}_{k,x_1}(k^j) := ie \int d(k,y) \frac{\hat{p}_{k,x_1}}{k} \hat{y}_{k,x_1} e^{ik^j\hat{y}} e^{-|k,y|} =
\]

\[
= e \left( \frac{\hat{p}}{k^j,\hat{p}} - \frac{\hat{p}'}{k^j,\hat{p}'} \right) e^{ik^j\hat{y}} \bigg|_{k,x_1} - e \int_{-\infty}^{k,x_1} d(k,y) e^{ik^j\hat{y}} \frac{d}{dk} \left( \frac{\hat{p}}{k^j,\hat{p}} \right) - e \int_{k,x_1}^{\infty} d(k,y) e^{ik^j\hat{y}} \frac{d}{dk} \left( \frac{\hat{p}'}{k^j,\hat{p}'} \right).
\]

(4.67)

It is trivial to see that this fulfills \( k'^j \hat{A}(k') = 0 \), but this is only true because we made the choice to let \( \hat{y} \) be continuous at \( k,x_1 \).

Since the background field is assumed to be a finite pulse, \( A \) and therefore \( \hat{p} \) and \( \hat{p}' \) will be constant before and after the pulse. The integrand is therefore only non-zero at a finite interval and the absolute of the phase can no longer become arbitrary large. This means that we can expand \( e^{ik^j\hat{y}} \) to lowest order in \( \Lambda \) meaning that \( e^{ik^j\hat{y}} = 1 \) for \( j > 1 \). This gives

\[
\hat{A}_{k,x_1}(k^j) = e \left( \frac{p}{k^j,\hat{p}} - \frac{\hat{p}'}{k^j,\hat{p}'} \right) \bigg|_{k,x_1} - e \int_{-\infty}^{k,x_1} d(k,y) \frac{d}{dk} \left( \frac{\hat{p}}{k^j,\hat{p}} \right) -
\]

\[
e \int_{k,x_1}^{\infty} d(k,y) \frac{d}{dk} \left( \frac{\hat{p}'}{k^j,\hat{p}'} \right) =
\]

(4.68)

which does no longer depend on \( k,x_1 \) and can therefor be moved out of the \( k,x_1 \)-integral so that (4.63) becomes

\[
S^f_{ji} = \frac{i e^n (2\pi)^3 \delta^2_+ \delta_+ (p' - p + k^1)}{\sqrt{k.p' k.p} \prod_{j=1}^{n} 2k^j} \left( \prod_{m=2}^{n} e^m \left( \frac{p}{k^j,\hat{p}} - \frac{\hat{p}'}{k^j,\hat{p}'} \right) \right) \times
\]

\[
\times e^1 \int_{-\infty}^{\infty} d(k,x_1) \frac{1}{2} \bar{\alpha}_{p'}^j \hat{\beta}^j (k,x_1) u_p^m =
\]

(4.69)

\[
= S^f_{ji} \prod_{m=2}^{n} \frac{e}{2k^m} e^m \left( \frac{p}{k^j,\hat{p}} - \frac{\hat{p}'}{k^j,\hat{p}'} \right)
\]

We also need to regularise the \( k,x_1 \)-integral in the S-matrix element. We
will do this in the exact same way as for the soft photon

\[ i \int d(k.x_1) \bar{u}_p^{\sigma'} \hat{\beta}^{\mu}(k.x_1) u_p^{\sigma} := i \int d(k.x_1) \bar{u}_p^{\sigma'} \hat{\beta}^{\mu}(k.x_1) u_p^{\sigma} e^{-|k.x_1|^2} = \]

\[ = \int d(k.x_1) \exp \left\{ \int^{k.x_1} d\phi \left( \frac{k^1.p'}{k.p} + e A(\phi) \left( \frac{p'}{k.p} - \frac{p}{k.p} \right) + \frac{e^2 A^2(\phi)}{2} k.k \right) \right\} \times \]

\[ \times \frac{d}{d(k.x_1)} \left( \frac{\gamma^{\mu} - \frac{e}{2} \left( \frac{\hat{A}(k.x_1)k^{\mu}}{k.p} - \frac{\hat{A}(k.x_1)k^{\mu}}{k.p} \right) + \frac{e^2 A^2(k.x_1)k.k}{2 k.p k.p} }{k^1.p'} + e A(\phi) \left( \frac{p'}{k.p} - \frac{p}{k.p} \right) + \frac{e^2 A^2(\phi) k.k}{2 k.p k.p} \right) u_p^{\sigma}. \] 

(4.70)

It is not difficult to check that the scalar product between \( k^1 \) and (4.70) is zero.

Rewriting \( k^1 \) in terms of \( p \) and \( p' \) with the help of the delta function we find that

\[ k^1_{\mu} \bar{u}_p^{\sigma'} \left( \gamma^{\mu} + \frac{e}{2} \left( \frac{\hat{A}(k.x_1)k^{\mu}}{k.p} - \frac{\hat{A}(k.x_1)k^{\mu}}{k.p} \right) + \frac{e^2 A^2(k.x_1)k.k}{2 k.p k.p} \right) u_p^{\sigma} = \]

\[ = \left( \frac{k^1.p'}{k.p} + e A(\phi) \left( \frac{p'}{k.p} - \frac{p}{k.p} \right) + \frac{e^2 A^2(\phi) k.k}{2 k.p k.p} \right) u_p^{\sigma} \hat{k} u_p^{\sigma} \] 

(4.71)

which exactly cancels the denominator inside the derivative in (4.70) so that the whole derivative becomes zero.

4.4.3 Probability

To get the probability we will also have to integrate over all possible probabilities. Remembering the cutoff this becomes

\[ P_{\text{one hard}}^{n} = \frac{1}{2} \sum_{\sigma' \sigma \in \epsilon} \int d^2 p_{\perp} \int_{0}^{\infty} \frac{d p'}{2\pi} \int_{k^1_{\perp} > \Lambda} \frac{d^2 k_{\perp} k^1_{\perp}}{(2\pi)^3} \times \]

\[ \times \frac{1}{(n-1)!} \left( \prod_{j=2}^{n} \int_{0 < l^1_{\perp} < \Lambda} \frac{d^2 l_{\perp} l^1_{\perp}}{(2\pi)^3} \right) \left| S_{fi} \right|^{2} \frac{1}{V}. \] 

(4.72)

Note that the symmetry factor in (4.72) is \( 1/(n-1)! \) compared to \( 1/n \) in (4.40).

This is because it is no longer possible to get multiple counting of terms by interchanging the \( k^1 \)-photon with the others because they exist in different regions of momentum space. We now only have \( (n-1) \) identical particles.

Using the S-matrix element from (4.69) with regularised \( \beta \) we find that

\[ P_{\text{one hard}}^{n} = \int_{k^1_{\perp} > \Lambda} \frac{d^2 k_{\perp} k^1_{\perp}}{(2\pi)^3} P_{\text{one hard}}^{1}(k^1) \times \]

\[ \times \frac{1}{(n-1)!} \left( -e^2 \int_{0 < l^1_{\perp} < \Lambda} \frac{d^2 l_{\perp} l^1_{\perp}}{(2\pi)^3} \left( \frac{p}{k^1.p} - \frac{\hat{p}'(\infty)}{k^1.p} \right)^2 \right)^{n-1} \] 

(4.73)
where $P^1_{\text{one hard}}$ is the probability density of emitting one hard photon of momentum $k^1$ and no soft photons

$$P^1_{\text{one hard}}(k^1) = -e^2 \frac{1}{k.p' \cdot k.p} \int d(k.x) \int d(k.y) \times \frac{1}{8} Tr \left \{ (p' + m) \beta^\mu(k.x)(p + m) \beta^\dagger_\mu(k.y) \right \} .$$

(4.74)

Remember that $\tilde{p}'$ (through $p'$) and $\hat{\beta}$ depend on $k_1$.

The probability of emitting one hard photon and any number of soft photons is

$$\sum_{n=1}^{\infty} P^1_{\text{one hard}} = \int_{k_1^1 > \Lambda} \frac{d^2k_1^1 \cdot dk_1^1}{(2\pi)^2 k_1^1} P^1_{\text{one hard}}(k^1) \times \exp \left \{ -e^2 \int_{0 < k_0^1 < \Lambda} \frac{d^3k_0^1 \cdot dk_0^1}{(2\pi)^3 k_0^1} \left ( \frac{p}{k'.p} - \frac{\tilde{p}'(\infty)}{k'.\tilde{p}'(\infty)} \right )^2 \right \} .$$

(4.75)

In a real measurement situation, there will always be a limit to how soft photons can be measured. Let $\Lambda$ be that limit, then $\sum_{n=1}^{\infty} P^1_{\text{one hard}}$ together with any loop corrections describes the probability of measuring one single photon emission.

![Figure 4: The tree-level process of $e^- (p) \rightarrow e^- (p') + \gamma (k^1) + \text{(any number of soft photons)}$ in a background field.](image)

4.4.4 **Infrared divergence**

$P^n_{\text{one hard}}, n > 1$ has an infrared divergence that comes from

$$\int_{0 < k_0^1 < \Lambda} \frac{d^2k_0^1 \cdot dk_0^1}{(2\pi)^2 k_0^1} \left ( \frac{p}{k'.p} - \frac{\tilde{p}'(\infty)}{k'.\tilde{p}'(\infty)} \right )^2 .$$

(4.76)

This is easiest seen by using the method of (3.22) to transform the $k'$-integral

$$\int_{0 < k_0^1 < \Lambda} \frac{d^2k^1 \cdot dk'_0}{(2\pi)^2 k'_0} = \int_{0 < k_0^1 < \Lambda} \frac{d^3k'}{(2\pi)^3 k'_0} = \frac{1}{2(2\pi)^2} \int d\Omega k'_0 \int_0^\Lambda dk'_0 k'_0 .$$

(4.77)
where \( \int d\Omega_{k'} \) is the integral over all possible directions of \( k' \).

By defining the vector \( n \) such that \( k' \mu = \hat{k}'_0 n^\mu \) we can rewrite the integrand of (4.76) as

\[
\left( \frac{p}{k'.p} - \frac{\hat{p}'(\infty)}{k'.\hat{p}'(\infty)} \right)^2 = \frac{1}{(k'_0)^2} \left( \frac{p}{n'.p} - \frac{\hat{p}'(\infty)}{n'.\hat{p}'(\infty)} \right)^2. \tag{4.78}
\]

Then

\[
\int_{0<k'_0<\Lambda} d^2k'_i \frac{d^2k'_j}{(2\pi)^32k'_-} \left( \frac{p}{k'.p} - \frac{\hat{p}'(\infty)}{k'.\hat{p}'(\infty)} \right)^2 = \frac{1}{2(2\pi)^2} \int d\Omega_{k'} \left( \frac{p}{n'.p} - \frac{\hat{p}'(\infty)}{n'.\hat{p}'(\infty)} \right)^2 \int_0^\Lambda dk'_0 \frac{1}{k'_0} \tag{4.79}
\]

which is zero if \( \hat{p}'(\infty) = p \) and infinite otherwise.

If the net acceleration on the electron from both the external electromagnetic field and the emission of the hard photon is exactly zero, then there are no infrared diverges, but otherwise there are. Just as in section 4.2.5 it is a reasonable to believe that this divergences will disappear in every observable when including loop corrections in the calculation.

### 4.4.5 Classical limit

As discussed earlier there is a difference between the soft and the classical limits. The difference lies mostly in the interpretation of the result, but also in the exponent of the equations. We will now present what happens if we take the classical limit of all photons but one.

In the soft limit we let the exponent of \( \hat{A} \to 0 \) after regularisation, in the classical limit we do not do that. The probability of emitting one hard and any number of classical photons is

\[
\sum_{n=1}^\infty P_{\text{one hard}}^n = -e^2 \int_{k'_0>\Lambda} \frac{d^2k'_i}{(2\pi)^32k'_-} \frac{1}{k'.p} \int d(k.x) \int d(k.y) \times \]

\[
\times \frac{1}{8} Tr \left\{ \{\hat{p}' + m, \hat{A}'(k.x)\}(\hat{p}' + m, \hat{A}'(k.y)) \right\} \times \exp \left\{- \int_{0<k'_0<\Lambda} \frac{d^2k'_i}{(2\pi)^32k'_-} \hat{A}_{k,x}(k') \cdot \hat{A}_{k,y}(k') \right\} \tag{4.80}
\]

where \( \hat{A} \) is defined by (4.68). Since we do not get rid of the \( k.x (k.y) \) dependence of \( \hat{A}_{k,x} (\hat{A}_{k,y}) \) in this limit, we can not separate the \( \hat{A}_{k,x} (\hat{A}_{k,y}) \) from the \( k.x \)-integral (\( k.y \)-integral).

This extra \( k.x (k.y) \) dependence complicates the regularisation of the hard photon integral of the S-matrix element since (4.70) is no longer applicable. If \( A(\infty) = 0 \) there is only one term in the integral that diverges and we can regularise this by demanding \( S^p_{\phi_i} (\epsilon^1 \to k^1) = 0 \) which is the same as
\[ k_\mu^1 \int d(k.x_1) \frac{1}{2} \bar{u}^\mu_P \beta^\mu_P (k.x_1) u^\mu_P \prod_{m=2}^n \epsilon^m \hat{A}_{k,x_1} (k^m) = 0. \] (4.81)

This implies that the divergent part coming from the integral over \( \gamma \)-term of \( \hat{\beta} \) is redefined as

\[
\int d(k.x) \exp\{\ldots\} \prod_{m=2}^n \epsilon^m \hat{A}_{k,x_1} (k^m) :=
\]

\[
:= -\frac{k.p}{k^1.p'} \int d(k.x_1) \left( eA(k.x_1) \left( \frac{p'}{k.p'} - \frac{p}{k.p} \right) + \frac{e^2 A^2 (k.x_1) k.k^1}{2 k.p' k.p} \right) \times
\]

\[
\times \exp\{\ldots\} \prod_{m=2}^n \epsilon^m \hat{A}_{k,x_1} (k^m)
\]

(4.82)

where \( \exp\{\ldots\} \) is the exponent of \( \hat{\beta} \)

\[
\exp\{\ldots\} :=
\]

\[
:= \exp \left\{ \int^{k.x_1} d\phi \left( \frac{k^1.p'}{k.p} + eA(\phi) \left( \frac{p'}{k.p'} - \frac{p}{k.p} \right) + \frac{e^2 A^2 (\phi) k.k^1}{2 k.p' k.p} \right) \right\}.
\]

(4.83)

(4.82) is well defined since the integrand is only non-zero in a finite interval.

Remember that (4.82) is only true if \( A = 0 \) when \( k.x \to \infty \) and there is no guarantee for that. Remember how \( A \) were first defined in (2.9). But even if (4.82) is not true, there will exist a way to regularise such that everything is well defined and \( k.x \to \) gives \( S_f i = 0 \). However, calculating the explicit form of such general regularisation is much more complicated and outside the scope of this thesis.

5 Conclusions

We have calculated the radiation from a single electron in a plane wave electromagnetic background field, both classically and in quantum field theory. Our calculations have been non-perturbative with respect to the background field. We have shown that under certain assumptions of renormalisation of the probabilities the classical limit of the quantum calculation can be shown to give the correct classical result. We have calculated the first order quantum corrections to this classical limit.

We have calculated the probability that a single electron in a plane wave electromagnetic background field will emit one hard and any number of soft photons. In doing so we provided a result for the correction to the emission
probability of one hard photon due to soft photon emission. This result needs to be complemented with the loop corrections for the same process to give the correct observable.

We have concluded that our results are generally infrared divergent. However, we argued that it is reasonable to believe that this will resolve itself when loop corrections are added to the results.

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