Degree project

GEOMETRIC TRANSFORMATION OF THE EUCLIDEAN PLANE

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Abstract

Klein defined a geometry as a study of the properties of a set S that remain invariant when its elements are subjected to the transformations of some transformation groups. The focus of this thesis is on the definitions, theorems and properties of transformations with a lot of different examples and figures. Also, this thesis aims to teach geometric transformation to future mathematics teachers at the Faculties of Education. In last sections, we will present frieze patterns which are mostly used in our daily life.

1 Introduction

Geometric transformations are one-to-one mappings of point sets onto point sets. Transformations give considerable insight into Euclidean concepts such as congruence, similarity and symmetry. Transformations are the basis for many modern applications in art, architecture, engineering, motion pictures and television. [1]

In this paper, we use the transformation approach to study Euclidean, similarity and affine geometries. We shall find matrix representations for the appropriate transformations for each geometry and use the techniques of matrix algebra to determine the effects of these transformations.

The Erlanger Program plays an important role in the development of the mathematics in the 19th century. It is the title of Klein’s famous lecture Vergleichende Betrachtungen über neuere geometrische Forschungen [A comparative Review of Recent Researches in Geometry] which was presented on the occasion of his admission as a professor at the Philosophical Faculty of the University of Erlanger in October 1872. Klein’s basic idea is that each geometry can be characterized by a group of transformations which preserve elementary properties of the given geometry.[2]

In this paper, we will present an analytic model of the Euclidean plane in first section, linear transformations of the Euclidean plane in second section, isometries in third section, direct isometries(translation and rotation) in forth section, indirect isometries(reflection and glide reflection) in fifth section, symmetry groups in sixth section, similarity transformations in seventh section and affine transformations in last section. Also, section six will explain frieze groups. You can see pictures of them in Fig.1.1.
Definition 1.1. A **geometry** is the study of those properties of a set $S$ that remain invariant (unchanged) when the elements of $S$ are subjected to the transformations of some transformation group$^1$.

Using this definition of geometry, Klein was able to give a classification of geometries in terms of groups of linear transformations. The Euclidean transformations are the motions required to carry out the mapping of figures.

In this thesis, I mainly used a book whose name is *A Course in Modern Geometries*, Judith N. Cedeberg, as a reference.

---

$^1$Here a **group** means a set, $G$, together with an operation $\cdot$ (called the group law of $G$) that combines any two elements $a$ and $b$ to form another element, denoted $a \cdot b$ or $ab$. To qualify as a group, the set and operation, $(G, \cdot)$, must satisfy four requirements known as the group axioms:

- **a) Closure**
  For all $a, b$ in $G$, the result of the operation, $a \cdot b$, is also in $G$.

- **b) Associativity**
  For all $a, b$ and $c$ in $G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

- **c) Identity element**
  There exists an element $e$ in $G$, such that for every element $a$ in $G$, the equation $e \cdot a = a \cdot e = a$ holds. The identity element of a group $G$ is often written as $1$ or $1_G$, a notation inherited from the multiplicative identity.

- **d) Inverse element**
  For each $a$ in $G$, there exists an element $b$ in $G$ such that $a \cdot b = b \cdot a = 1_G$. [4]
2 An Analytic Model of the Euclidean Plane

The analytic study of Euclidean geometry is based on the premise that each point in the plane can be assigned an ordered pair of real numbers. The usual manner in which this is done is via a Cartesian coordinate system where two perpendicular lines are used as axes. The point of intersection of these axes is assigned the ordered pair (0,0) and other points are assigned ordered pairs as shown in Fig 2.1. Rather than denote points by ordered pairs \((x,y)\) as is customary in calculus, we will use ordered pairs \((x_1,x_2)\). [5]

![Figure 2.1](image)

With this presentation of points, lines of the Euclidean plane can be represented by linear equations of the form \(a_1x_1 + a_2x_2 + a_3 = 0\) where the \(a_i\) are constant real number coefficients. Thus each ordered triple \([a_1,a_2,a_3]\), where \(a_1\) and \(a_2\) are not both zero determines the equation of a line. Notice that square brackets are used for coordinates of lines so as to distinguish them from coordinates for points. Unlike points, the coordinates of a line do not uniquely represent a line, since the equations \(a_1x_1 + a_2x_2 + a_3 = 0\) and \(ka_1x_1 + ka_2x_2 + ka_3 = 0\) represent the same line for every nonzero real number \(k\). There is, however, a one-to-one correspondence between the set of lines and the set of equivalence classes of ordered triples of real numbers defined by the following relation:

\[b_1, b_2, b_3 \sim [a_1, a_2, a_3] \iff b_i = ka_i, \ i = 1, 2, 3\] where \(k\) is a nonzero real number.

Here, this relation, denoted "\(\sim\)", is an equivalence relation if its satisfies each of the following:

(a) \(a \sim a\).
(b) If \(a \sim b\) then \(b \sim a\).
(c) If \(a \sim b\) and \(b \sim c\), then \(a \sim c\). [5]

Definition 2.1. A set of elements, all of which are pairwise related by
an equivalence relation is called an equivalence class. Any element of an equivalence class is called a representative of an equivalence class.

Since there is a one-to-one correspondence between the lines of the Euclidean plane and these equivalence classes, we can interpret lines in terms of these equivalence classes. The ordered triples \([u_1, u_2, u_3]\) belonging to a particular equivalence class will be called homogeneous coordinates of the line. If we consider one of these ordered triples to be a row matrix \([u_1, u_2, u_3]\), then the equation of the corresponding line is \(uX = 0\) where \(X = (x_1, x_2, 1)\) is a column matrix with 1 in its third entry. In particular, if \(u = [3, 2, -4]\), then \(uX = 0\) is the equation \(3x_1 + 2x_2 - 4 = 0\). This observation, together with the desire to use similar interpretations for points and lines, suggest that we interpret points in terms of equivalence classes of ordered triples of real numbers \((x_1, x_2, x_3)\), where \(x_3 \neq 0\) under the same relation. Again we will refer to elements of these equivalence classes as homogeneous coordinates of the point. In the case of points, however, since \(x_3\) is always nonzero, every ordered triple \((x_1, x_2, x_3) \sim (x_1/x_3, x_2/x_3, 1)\) so each equivalence class will have unique representative of the form \((x_1, x_2, 1)\). Sometimes I will ignore the last component and write \((x_1, x_2)\) instead of \((x_1, x_2, 1)\). And now, the terms are summarized in Table 1.

<table>
<thead>
<tr>
<th>Undefined Term</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points</td>
<td>Equivalence classes of ordered triples ((x_1, x_2, x_3)) where (x_3 \neq 0)</td>
</tr>
<tr>
<td>Lines</td>
<td>Equivalence classes of ordered triples ([u_1, u_2, u_3]) where (u_1) and (u_2) are not both 0</td>
</tr>
<tr>
<td>Incidence</td>
<td>A point (X(x_1, x_2, x_3)) is incident with a line ([u_1, u_2, u_3]) if ([u_1, u_2, u_3]\cdot(x_1, x_2, x_3) = 0) or in matrix notation (uX = 0).</td>
</tr>
</tbody>
</table>

Within the context of this analytic model, the operations of matrix algebra take on geometric significance as indicated by the following theorems. In each case the coordinates chosen to represent points will be those in which \(x_3 = 1\). The first of these theorems gives a convenient way to determine when three points are collinear (they lie on the same line).
Theorem 2.1. Three distinct points \(X(x_1, x_2, 1), Y(y_1, y_2, 1)\) and \(Z(z_1, z_2, 1)\) are collinear \(\iff\) the determinant

\[
\begin{vmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  1 & 1 & 1
\end{vmatrix} = 0
\]

Proof. \(X, Y, Z\) are collinear if and only there is a line \(u[u_1, u_2, u_3]\) such that

\[
\begin{align*}
u_1x_1 + u_2x_2 + u_3 &= 0 \\
u_1y_1 + u_2y_2 + u_3 &= 0 \\
u_1z_1 + u_2z_2 + u_3 &= 0
\end{align*}
\]

or

\[
[u_1, u_2, u_3] \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{bmatrix} = [0, 0, 0]
\]

But from linear algebra this equation has a nontrivial solution \([u_1, u_2, u_3]\) if and only if

\[
\begin{vmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  1 & 1 & 1
\end{vmatrix} = 0
\]

Since this nontrivial solution cannot have both \(u_1 = 0\) and \(u_2 = 0\), \([u_1, u_2, u_3]\) is a line containing all three points. □

Corollary: If \(A\) and \(B\) are distinct points, then the equation of line \(AB\) where \(A(a_1, a_2, 1)\) and \(B(b_1, b_2, 1)\) can be written

\[
\begin{vmatrix}
  x_1 & a_1 & b_1 \\
  x_2 & a_2 & b_2 \\
  1 & 1 & 1
\end{vmatrix} = 0
\]

Similar to Theorem 2.1 we can prove the following theorem.

Theorem 2.2. Three distinct lines \(u, v, w\) are all concurrent\(^2\) or all parallel \(\iff\) the determinant

\[
\begin{vmatrix}
  u_1 & u_2 & u_3 \\
  v_1 & v_2 & v_3 \\
  w_1 & w_2 & w_3
\end{vmatrix} = 0
\]

\(^2\)Two or more lines are said to be concurrent if they intersect in a single point.
In these theorems, it is important to note that the coordinates of points appear in columns whereas the coordinates of lines appear in rows.

The line coordinates can also be used to determine the angle between two lines using a definition given in terms of the tangent of the angle. This meaning will be explained in the following definition.

**Definition 2.2.** If \( u[u_1, u_2, u_3] \) and \( v[v_1, v_2, v_3] \) are two lines, then the measure of the angle between \( u \) and \( v \), denoted \( m(\angle(u, v)) \), is defined to be the unique angle such that

\[
\tan(\angle(u, v)) = \frac{u_1v_2 - u_2v_1}{u_1v_1 + u_2v_2}
\]

and

\[-90^\circ < m(\angle(u, v)) < 90^\circ \text{ if } u_1v_1 + u_2v_2 \neq 0
\]

\[m(\angle(u, v)) = 90^\circ \text{ if } u_1v_1 + u_2v_2 = 0\]

Here, scalar and vector products are used to write tan equation: A vector product of two vectors \( u \) and \( v \) is defined as: \( \vec{u} \times \vec{v} = |\vec{u}||\vec{v}| \sin \theta \) and a scalar product of two vectors \( u \) and \( v \) is defined as: \( u \cdot v = |\vec{u}||\vec{v}| \cos \theta \). Hence, if we divide these two equations:

\[
\frac{\vec{u} \times \vec{v}}{\vec{u} \cdot \vec{v}} = \tan \theta = \frac{u_1v_2 - u_2v_1}{u_1v_1 + u_2v_2}
\]

(see example 5)

Let’s finish this section with some examples:

1.) Let \( u \) be the line with homogeneous coordinates \([-2, 5, 7] \). (a) Find an equation for line \( u \).
(b) Find coordinates for two distinct points on \( u \).

**Solution:** (a) \([-2, 5, 7]\) \( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [0] \) and its equation \(-2x_1 + 5x_2 + 7 = 0\)
(b) By assuming \( x_1 = 1 \), we find \( x_2 = -1 \) according to equation in (a). Hence, coordinates of this point are \((1, -1, 1)\). Similarly, we can find another coordinate: for \( x_1 = 2 \), \( x_2 = -3/5 \) and coordinate are \((2, -3/5, 1)\).

2.) Let \( P \) be the point with ordered pair coordinates \((4, -7)\).
(a) Find three sets of homogeneous coordinates for \( P \). (b) Find an equation
for the lines through point \( P \). (c) Find coordinates for two lines through \( P \).

**Solution:** (a) Let’s assume this point in this form \((4k, -7k, k)\) and for \( k = 1, 2, 3 \), we obtain coordinates respectively; \((4, -7, 1)\), \((8, -14, 2)\) and \((12, -21, 3)\).

(b) \([u_1, u_2, u_3] = \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} = [0]\) so its equation is \(4u_1 - 7u_2 + u_3 = 0\).

(c) By assuming \( u_1 = u_2 = 1 \), we find \( u_3 = 3 \) hence, the first coordinates are \([1, 1, 3]\). Similarly, we can find another line: for \( u_1 = 1 \) and \( u_2 = -3 \), we find \( u_3 = -25 \) hence, the coordinates are \([1, -3, -25]\).

3.) Find the general form of the coordinates for lines through the point \((0, 0, 1)\).

**Solution:** The general linear equation is \(u_1x_1 + u_2x_2 + u_3 = 0\) and all given by lines through \((0, 0, 1)\) is \(u_10 + u_20 + u_3 = 0\) hence, we find \( u_3 = 0 \). As a conclusion the general form of the coordinates for lines through \((0,0,1)\) is \([u_1, u_2, 0]\).

4.) Find the line containing the points \((10, 2)\) and \((-7, 3)\).

**Solution:** We can solve this question by using theorem 2.2;

\[
\begin{vmatrix}
  x_1 & 10 & -7 \\
  x_2 & 2 & 3 \\
  1 & 1 & 1 \\
\end{vmatrix} = 0
\]

if we calculate this determinant \(x_1(2 - 3) - 10(x_2 - 3) - 7(x_2 - 2) = 0\)
\(x_1 + 17x_2 - 44 = 0\) hence, the line is \([1, 17, -44]\).

5.) Find the angles between between the following lines: (a) the lines \([-2, 1, 7]\) and \([3, 4, 17]\); (b) the \( x_1 \) and \( x_2 \) axes; (c) the line \( x_1 = x_2 \) and \( x_1 \) axis.

**Solution:** (a) We can calculate the angle between \( u = [-2, 1, 7] \) and \( v = [3, 4, 17] \) by using Definition 2.2;

\[
\tan(\angle(u, v)) = \frac{u_1v_2 - u_2v_1}{u_1v_1 + u_2v_2} = \frac{-2 \cdot 4 - 1 \cdot 3}{-2 \cdot 3 + 1 \cdot 4} = 11/2
\]

And so, \( \angle(u, v) = 79.7^\circ \).
(b) If $x_1$ axis, $x_2 = 0$ and its line coordinates are $u = [0, 1, 0]$, if $x_2$ axis, $x_1 = 0$ and its line coordinates are $[1, 0, 0]$ hence we find $u_1 v_1 + u_2 v_2 = 0 \cdot 1 + 1 \cdot 0 = 0$. Definition 2.2 gives $m(\angle(u, v)) = 90^\circ$. Also, as is well known, the angle between perpendicular lines is $90^\circ$.

(c) If $x_1 = x_2, x_1 - x_2 = 0$ so the line is $u = [1, -1, 0]$ and if $x_1$ axis and its line coordinate is $v = [0, 1, 0]$ hence, $\tan(\angle(u, v)) = -1$ and $\angle(u, v) = 45^\circ$. 
3 Linear Transformations of the Euclidean Plane

The analytic model of the Euclidean plane interprets points and lines in terms of equivalence classes of the vector space \( \mathbb{R}^3 \). (Refered to in section 2)

**Definition 3.1.** Let \( V \) be a vector space over \( \mathbb{R}^3 \). If \( T : V \rightarrow V \) is a function, then \( T \) is called a **linear transformation** of \( V \) if it satisfies both the following conditions:

(a) \( T(u + v) = T(u) + T(v) \) for all vectors \( u \) and \( v \) in \( V \); and

(b) \( T(ku) = kT(u) \) for all vectors \( u \) in \( V \) and scalars \( k \) in \( \mathbb{R} \).

**Definition 3.2.** A linear transformation \( T \) is **one-to-one** if whenever \( u \neq v \), \( T(u) \neq T(v) \).

**Theorem 3.1.** \( T \) is a one-to-one linear transformation of \( \mathbb{R}^3 = \{ X(x_1, x_2, x_3) : x_i \in \mathbb{R} \} \iff T(X) = AX \) where the matrix \( A = [a_{ij}]_{3 \times 3} \), \( |A| \neq 0 \) and \( a_{ij} \in \mathbb{R} \).

The matrix is on the form: \( A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    0 & 0 & 1
\end{bmatrix} \)

**Definition 3.3.** A nonempty set \( G \) of transformations of a vector space \( V \) is said to form a group under the operation of composition if it satisfies both the following conditions: (a) If \( T \in G \) then \( T^{-1} \in G \); (b) If \( T_1 \in G \) and \( T_2 \in G \), then \( T_1T_2 \in G \).

---

\( ^{3} \)A vector space over \( \mathbb{R} \) (the real numbers) is a set \( V \) that satisfies: If \( x, y, z \) are in \( V \), and \( a, b \) are scalars in \( \mathbb{R} \)

**Closure Properties**
1. \( x + y \) is in \( V \) (the sum of elements of \( V \) stays in \( V \); its "closed")
2. \( ax \) is in \( V \) (multiplication by scalars with elements in \( V \) stays in \( V \))

**Addition Properties**
1. \( x + y = y + x \) (commutativity)
2. \( x + (y + z) = (x + y) + z \) (associativity)
3. \( V \) contains an element 0, where \( x + 0 = x \) (called the zero vector)
4. For each \( x \) in \( V \) there is \( a - x \) also in \( V \) where \( x + (-x) = 0 \) (additive inverse)

**Scalar Multiplication Properties**
1. \( a(bx) = (ab)x \)
2. \( a(x + y) = ax + ay \)
3. \( (a + b)x = ax + bx \)
4. \( 1x = x \). [6]
**Theorem 3.2.** If the image of a point under a one-to-one linear transformation of $V^* = X(x_1, x_2, 1) : x_1, x_2 \in \mathbb{R}$ is given by the matrix equation $X' = AX$ then the image of a line under this same transformation is given by the matrix equation $ku' = uA^{-1}$ for some nonzero scalar $k$.

**Proof:** Consider the line $u[u_1, u_2, u_3]$ with equation $u_1x_1 + u_2x_2 + u_3x_3 = 0$; that is, $uX = 0$. Under the linear transformation, $u$ maps to $u'$, $X$ maps to $X'$, and $uX = 0 \iff u'X' = 0$. But $X' = AX$. So substituting, $u'AX = 0 = uX$. Since this must hold for all points $X$, $u = ku'A$ for a nonzero scalar $k$ or $ku' = uA^{-1}$. ■

Now, we will give several examples.

1.) Let $T$ be the transformation with matrix

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Find the images of points on the line $l[1, -2, 3]$. (b) Does $T$ keep any points on $l$ invariant? (c) Use the coordinates of the images of two points on $l$ to find the coordinates of $l' = T(l)$.

**Solution:**

(a) Firstly, we should find the points on the line $l$.

$$[1, -2, 3] \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = [0],$$

$x_1 - 2x_2 + 3 = 0$ and we find $X = (2x_2 - 3, x_2, 1)$. And now, we can find the images of $X$.

$$X' = T(X) = AX = \begin{bmatrix} 7x_2 - 3 \\ x_2 \\ 1 \end{bmatrix}.$$

(b) The points on $x_1$ axis are invariant since

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 1 \end{bmatrix}.$$
(the image of $x_1$ axis is on itself)

(c) We can choose two points on $X'$; let’s take $(4,1,1)$ and $(-3,0,1)$. Then the equation for the line becomes

$$\begin{vmatrix} y_1 & 4 & -3 \\ y_2 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

$y_1 - 7y_2 + 3 = 0$ and as a conclusion $l' = [1, -7, 3]$.

2.) Let $A = \begin{bmatrix} 1 & 3 & -7 \\ 2 & 5 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ be the matrix of a transformation $T$. (a) Find $P' = T(P)$ and $Q' = T(Q)$ for the points $P(1,2,1)$ and $Q(6,4,1)$. (b) Find coordinates of the lines $PQ$ and $P'Q'$.

**Solution:** (a)

$$P' = T(P) = AP = \begin{pmatrix} 1 & 3 & -7 \\ 2 & 5 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ 1 \end{pmatrix},$$

and

$$Q' = T(Q) = AQ = \begin{pmatrix} 1 & 3 & -7 \\ 2 & 5 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 36 \\ 1 \end{pmatrix}.$$

(b) For coordinates of the lines $PQ$;

$$\begin{vmatrix} x_1 & 1 & 6 \\ x_2 & 2 & 4 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

$-2x_1 + 5x_2 - 8 = 0$ and $l = [-2, 5, -8]$. For coordinates of the line $P'Q'$;

$$\begin{vmatrix} x'_1 & 0 & 11 \\ x'_2 & 16 & 36 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

$-20x'_1 + 11x'_2 - 176 = 0$ and $l' = [-20, 11, -176]$.

3.) Find invariant points under the transformation given by matrix $\begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}$.
**Solution:** The points $X(x_1, x_2)$ are an invariant points of the transformation if $AX = X$;

\[
\begin{bmatrix}
4 & 1 \\
6 & 3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
=
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

If we solve this equation, we get $4x_1 + x_2 = x_1$ and $6x_1 + 3x_2 = x_2$ which simplify to $3x_1 = -x_2$. The invariant points would lie on the line $x_2 = -3x_1$ and be of the form $(\tau, -3\tau)$. 
4 Isometries

Firstly, to understand isometries will be easier if we give an example from our everyday life; whenever we move an object without changing size and shape, when we compare ”before and after” snapshots, the distance between any two point of the object before the motion is the same as the distances between the points in their new positions. [7]

The (Euclidean) distance between two points $X(x_1, x_2, 1)$ and $Y(y_1, y_2, 1)$ is obtained by using this formula:

$$d(X, Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$  

Definition 4.1. A one-to-one linear transformation $T$ of $V^*$ onto itself is an isometry, if it preserves distance (i.e., if $d(X, Y) = d(T(X), T(Y))$ for all pairs of points $X, Y$).

There are two types of isometries: Direct and Indirect Isometry.\footnote{We recommend http://www.schooltube.com/video/24751c85d10c4583afe9/ for an illustration of isometries and dilation which is explained with in the next sections.}

4.1 Direct Isometry

We will investigate and further classify the direct isometries based on the number of points that remain invariant under the isometries. Consider a triangle $ABC$ in the plane such that the vertices $A, B, C$ occur counterclockwise around the boundary of the triangle. If you apply an isometry to the triangle, then the result will be a triangle where the vertices $A, B, C$ can occur clockwise or anticlockwise. If the orientation stays the same, then we say that the isometry is direct, but if the orientation changes, then we say that the isometry is indirect. Also, all distances are the same both for direct and indirect isometry (see Figure 4.1).
A direct isometry has the following matrix representation:

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
-a_{12} & a_{11} & a_{23} \\
0 & 0 & 1
\end{bmatrix}
\]

where \((a_{11})^2 + (a_{12})^2 = 1\)

Also, for direct isometries, if \(u'\) and \(v'\) be the images of lines \(u\) and \(v\) under an isometry; the angle \(m(\angle(u', v')) = m(\angle(u, v))\).

**Theorem 4.1.1.** A direct isometry other than the identity \(^5\), with matrix \(A = [a_{ij}]\) has exactly one invariant points iff \(a_{11} \neq 1\).

**Proof.** The point \(X(x_1, x_2, 1)\) is an invariant point of the isometry if \(AX = X\);

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
-a_{12} & a_{11} & a_{23} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
1
\end{bmatrix}
= 
\begin{bmatrix}
x_1 \\
x_2 \\
1
\end{bmatrix}
\]

or

\[\begin{align*}
(a_{11} - 1)x_1 + a_{12}x_2 + a_{13} &= 0 \\
-a_{12}x_1 + (a_{11} - 1)x_2 + a_{23} &= 0
\end{align*}\]

\(^5\)The matrix \(I\) is called an **identity matrix**, because \(IA = AI = A\) for all matrices \(A\). This is similar to the number 1, which is called the multiplicative identity, because \(1a = a1 = a\) for all real numbers \(a\). There is no matrix that works as an identity for matrices of all dimensions. For \(N \times N\) square matrices there is a matrix \(IN \times N\) that works as an identity. \([9]\)
• Case 1. $a_{11} \neq 1$. In this case, Equation (1) yields

$$x_1 = \frac{-a_{12} x_2 - a_{13}}{a_{11} - 1}$$

and Equation (2) yields

$$-a_{12} \left[ \frac{-a_{12} x_2 - a_{13}}{a_{11} - 1} \right] + (a_{11} - 1) x_2 + a_{23} = 0$$

or solving for $x_2$

$$x_2 = \frac{-a_{12} a_{13} - a_{23}(a_{11} - 1)}{a_{12}^2 + (a_{11} - 1)^2}$$

giving a unique solution.

• Case 2. $a_{11} = 1$. Then $a_{12} = 0$, since $a_{11}^2 + a_{12}^2 = 1$. So

$$AX = \begin{bmatrix} 1 & 0 & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + a_{13} \\ x_2 + a_{23} \\ 1 \end{bmatrix}$$

Thus there are no invariant points unless $a_{13} = a_{23} = 0$, in which case $A = I$. ■

4.1.1 Translation

The direct isometries with no invariant points together with the identity isometry are called translations. In a sense, a translation is a motion of a plane that moves every point of the plane a specified distance in a specified direction along a straight line. In Figure 4.1.1, $P, Q, R$ is translated to $P', Q', R'$, respectively.

![Figure 4.1.1. Translation](10)

**Theorem 4.1.1.1.** A translation $T$ has matrix representation

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix},$$

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$T^{-1}$ is also a translation and has matrix representation

$$
\begin{bmatrix}
1 & 0 & -a \\
0 & 1 & -b \\
0 & 0 & 1
\end{bmatrix}.
$$

**Theorem 4.1.1.2.** The set of translations form a group.

We talked about the properties of group in the first section, now we will show that the set of translations form a group.

a) Closure: Let $A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}$ are two translations, then the product of these translations is a translation, too.

$$
\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a+c \\ 0 & 1 & b+d \\ 0 & 0 & 1 \end{bmatrix}
$$

b) Associativity: Let $A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}$ are three translations, then they satisfy the associativity which means $(AB)C = A(BC)$.

$$
(AB)C = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a+c+e \\ 0 & 1 & b+d+f \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
A(BC) = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a+c+e \\ 0 & 1 & b+d+f \\ 0 & 0 & 1 \end{bmatrix}
$$

b) Identity Element: Since $A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $AI = IA = A$.

$$
\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}
$$
d) Inverse Element: \( A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \) and there exists matrix \( A^{-1} \) such that \( AA^{-1} = A^{-1}A = I \).

\[
\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

**Theorem 4.1.1.3.** Given a point \( X \) and a point \( Y \), there is a unique translation mapping \( X \) to \( Y \).

Let \( X = (x, y) \) and \( Y = (z, t) \) are two points. And the unique translation mapping \( X \) to \( Y \) is:

\[
\begin{bmatrix} 1 & 0 & z - x \\ 0 & 1 & t - y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} z \\ t \\ 1 \end{bmatrix}.
\]

Using the matrix representation, several characteristic properties of translations can be identified. These properties should confirm the frequently used description of a translation as sliding points along fixed lines.

**Theorem 4.1.1.4.** If a translation maps a line \( u \) to a line \( v \), then \( u \) and \( v \) are either identical or parallel (see Fig.4.1.2).

**Theorem 4.1.1.5.** If a translation maps \( P \) to \( P' \) (\( P \neq P' \)), then the line \( PP' \) as well as all lines parallel to \( PP' \) are invariant (see Fig.4.1.3). No other lines are invariant.
Example: Translate the triangle which has points $A(2, 4), B(3, 1)$ and $C(5, 5)$, 4 units left and 5 units down.

Solution: We should substract 4 units from $x_1$ coordinate and 5 units from the $x_2$ coordinate like in Fig.4.1.4.

\[
A(2, 4) \Rightarrow A'(2 - 4, 4 - 5) = A'(-2, -1),
\]
\[
B(3, 1) \Rightarrow B'(3 - 4, 1 - 5) = B'(-1, -4),
\]
\[
C(5, 5) \Rightarrow C'(5 - 4, 5 - 5) = C'(1, 0).
\]

Or we can solve this problem by using Theorem 4.1.1.3.

Since Theorem 4.1.1.1, our translation matrix is $T = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$.

\[
A' = TA = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}
\]
\[
B' = TB = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}
\]
\[ C' = TC = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \]

**Figure 4.1.4**

### 4.1.2 Rotation

The direct isometries with exactly one invariant point together with the identity isometry are called **rotations**. The invariant point is called the **center of the rotation**. In a sense, a rotation is a transformation of the plane determined by rotating the plane about a fixed point, the center, by a certain amount in a certain direction. Fig.4.1.5. shows the triangle \( A \) rotated through 90°.

**Theorem 4.1.2.1.** A rotation \( R \) with center \( C(c_1, c_2, 1) \) has matrix rep-
presentation:

\[
\begin{bmatrix}
\cos \theta & -\sin \theta & c_1(1 - \cos \theta) + c_2 \sin \theta \\
\sin \theta & \cos \theta & -c_1 \sin \theta + c_2(1 - \cos \theta) \\
0 & 0 & 1
\end{bmatrix}
\]

and \(R^{-1}\) has matrix representation:

\[
\begin{bmatrix}
\cos \theta & \sin \theta & c_1(1 - \cos \theta) - c_2 \sin \theta \\
-\sin \theta & \cos \theta & c_1 \sin \theta + c_2(1 - \cos \theta) \\
0 & 0 & 1
\end{bmatrix}
\]

and is also a rotation with center \(C\). [5]

But, in this theorem, positive direction for \(\theta\) is in the anticlockwise sense.

**Theorem 4.1.2.2.** The set of all rotations with a given center \(C\) form a group.

The \(\theta\) in the matrix for \(R\) is called the measure of the angle of rotation of \(R\), or the angle of rotation of \(R\). From the preceding theorems, it follows that a rotation is uniquely determined by its angle of rotation and its center so we can denote the rotation with center \(C\) and angle \(\theta\) by \(R_{C,\theta}\). Note also that \((R_{C,\theta})^{-1} = R_{C,-\theta}\); that is, the inverse of a rotation with angle \(\theta\) is a rotation about the same center with angle \(-\theta\).

Even if the theorem 4.1.2.1 gives the matrix form for a rotation with any center \(C\), it is sufficient to remember only the simpler form for rotation with center at the point \(O(0,0,1)\). Then a translation \(T\) that maps \(O\) to \(C\) and a rotation with center \(O\) can be used as follows to find the rotation with center \(C\):

\[R_{C,\theta} = TR_{O,\theta}T^{-1}\]

**Example:** Triangle \(ABC\) has points at \(A(-10,4)\), \(B(-2,8)\) and \(C(-4,6)\). Rotate the triangle 90° clockwise about the origin.

**Solution:** We use this matrix:

\[
\begin{bmatrix}
\cos(-90^\circ) & -\sin(-90^\circ) \\
\sin(-90^\circ) & \cos(-90^\circ)
\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

\[
A' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -10 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}
\]

\[
B' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 8 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}
\]
\[
C' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.
\]

**Theorem 4.1.2.3.** Under a rotation with center \( C \) and angle \( \theta \), any point \( P \neq C \) is mapped to a point \( P' \) such that \( d(C, P) = d(C, P') \) and \( m(\angle(PCM)) = \theta \).

**Theorem 4.1.2.4.** If \( \triangle PQR \) and \( \triangle P'Q'R' \) are two triangles with \( m(\overline{PQ}) = m(\overline{P'Q'}) \), \( m(\overline{QR}) = m(\overline{Q'R'}) \), \( m(\overline{RP}) = m(\overline{R'P'}) \) and \( m(\angle PQR) = m(\angle P'Q'R') \), then there is a direct isometry mapping \( \triangle PQR \) to \( \triangle P'Q'R' \) so \( \triangle PQR \cong \triangle P'Q'R' \).

**Proof.** To show the congruence of the two triangles, it is sufficient to show that there is a direct isometry mapping \( \triangle PQR \) to \( \triangle P'Q'R' \).

Let \( T \) be a translation mapping \( P \) to \( P' \). \( T \) will also map points \( Q \) and \( R \) to points \( Q_1 \) and \( R_1 \). Let \( \theta = m(\angle Q_1P'Q') \). Then the rotation with center \( P' \) and angle \( \theta \) will map the points \( P' \) and \( Q_1 \) to \( P' \) and \( Q' \), respectively, since \( d(P', Q_1) = d(P, Q) = d(P', Q') \). Furthermore since \( m(\angle Q_1P'R_1) = m(\angle Q_1P'R') \) and \( d(P', R_1) = d(P, R) = d(P', R') \), this rotation will also map point \( R_1 \) to \( R' \). Therefore the isometry consisting of the composite \( R_{P', \theta}T \) will map \( \triangle PQR \) to \( \triangle P'Q'R' \).

**Examples:**

1.) Let \( T \) be the translation mapping \( X(1, -2, 1) \) to \( X'(3, 4, 1) \). (a) Find the matrix of \( T \) and image of line \( u[2, 3, -1] \) under \( T \). (b) Verify that lines \( u \) and \( T(u) \) are parallel.

**Solution:** (a) \( T(X) = AX = X' \), \[
\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}
\]

\[\Rightarrow a = 2, b = 6 \text{ and } T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}.\]

And now, to find image of line \( u \), we will use that \( kv = uT^{-1} \) (we will use \( v \) instead of \( T(u) \)) and we know \( T^{-1} \) from Theorem 4.1.1.1.,

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\begin{align*}
kv &= [2, 3, -1] \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} = [2, 3, -23].
\end{align*}

(b) Linear equation for \( u \): \( 2x_1 + 3x_2 - 1 = 0 \) and if we solve for \( x_2 \):
\[ x_2 = \frac{-2x_1 + 1}{3}. \]
Hence slope of line \( u \) is \(-2/3\).

Linear equation for \( v \): \( 2x_1/k + 3x_2/k - 23/k = 0 \) and if we solve for \( x_2 \):
\[ x_2 = \frac{-2x_1 + 23}{3}. \]
Hence slope of \( u = -2/3 \).

2.) (a) Verify that the following matrix is a matrix of a rotation:
\[ T = \begin{bmatrix} 3/5 & 4/5 & 2 \\ -4/5 & 3/5 & 1 \\ 0 & 0 & 1 \end{bmatrix} \]

(b) What is the center of this rotation?

**Solution:**
(a) If \( T \) is rotation, \( T \) should has these properties:

- \( a_{11}^2 + a_{12}^2 \) is should be equal to 1. \( \rightarrow (3/5)^2 + (4/5)^2 = 1 \)

- If \( a_{11} \neq 1 \), \( T \) should has one invariant point: \( TX = X \),
\[ \begin{bmatrix} 3/5 & 4/5 & 2 \\ -4/5 & 3/5 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}. \]

Hence the invariant point \( X(2, -3/2, 1) \).

(b) By using Theorem 4.1.2.1, we can find \( C \):
\[ \begin{align*}
c_1(1 - \cos \theta) + c_2 \sin \theta = c_1(1 - 3/5) + c_2(-4/5) &= 2 \\
-c_1 \sin \theta + c_2(1 - \cos \theta) = c_14/5 + c_22/5 &= 1
\end{align*} \]

If we calculate equation (3) and (4), we find \( c_1 = 2 \) and \( c_2 = -3/2 \).
Hence, \( C(2, -3/2, 1) \).

We have already found this point at (a), because for rotation, the invariant point is the center of the rotation. You can find \( C \) with two different ways as we found above.
4.2 Indirect Isometries

There are two types of indirect isometries: those that have invariant points and those that do not. There are indirect isometries that keep not just one point but every point on a particular line invariant. Such a line is said to be pointwise invariant. The adjective "pointwise" is important, since a line can be invariant without any points on it being invariant. You can see pictures of both types of indirect isometries in Fig.4.2.1.

![Fig. 4.2.1. (a) Reflection (b) Glide Reflection.][12]

An indirect isometry has the following matrix representation:

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{12} & -a_{11} & a_{23} \\
    0 & 0 & 1
\end{bmatrix}
\]

where \((a_{11})^2 + (a_{12})^2 = -1\),

Also, in indirect isometries, if \(u'\) and \(v'\) be the images of lines \(u\) and \(v\) under an isometry; the angle is \(m(\angle(u', v')) = -m(\angle(u, v))\).

4.2.1 Reflection

We call a reflection with axis \(m\), denoted \(R_m\), an indirect isometry that keeps line \(m\) pointwise invariant. In a sense, a reflection is an isometry in which a figure is reflected across a reflecting line, creating a mirror image. Unlike a translation or rotation, the reflection reverses the orientation of the original figure.

Also, a reflection in a line \(l\) is a transformation of a plane that pairs each point \(P\) of the plane with a point \(P'\) in such a way that \(l\) is the perpendicular bisector of \(PP'\), as long as \(P\) is not on \(l\). If \(P\) is on \(l\), then \(P = P'\).
Theorem 4.2.1.1. The matrix representation of a reflection $R_x$ with axis $x[0,1,0]$ is
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]
In general the matrix representation of a reflection $R_m$ can be found using $R_m = SR_xS^{-1}$ where $S$ is a direct isometry mapping $x$ to $m$ ($S(x) = m$).

Proof: All points on $x$ have coordinates of the form $(x_1,0,1)$. $R_x$ is then an indirect isometry that keeps each point $(x_1,0,1)$ fixed, that is, for all $x_1 \in R$
\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{12} & -a_{11} & a_{23} \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
0 \\
1 \\
\end{bmatrix}
= \begin{bmatrix}
x_1 \\
0 \\
1 \\
\end{bmatrix}
\]
So $a_{11}x_1 + a_{13} = x_1$ and $a_{12}x_1 + a_{23} = 0$. Since this must hold for all $x_1 \in R$, it follows that $a_{11} = 1$ and $a_{13} = a_{12} = a_{23} = 0$. So the matrix for $R_x$ is of the form given.

In general, there is always a direct isometry $S$, mapping $x$ to $m$. By the theorem 4.2.2.1(in the next theorems), $SR_xS^{-1}$ is an indirect isometry. Then it is sufficient to show that $SR_xS^{-1}$ keeps $m$ pointwise invariant. If $X$ is an arbitrary point on $m$, $S^{-1}(X)$ is a point on line $x$, so $R_x(S^{-1}(X)) = S^{-1}$. Thus,
\[
(SR_xS^{-1})(X) = S(S^{-1}(X)) = X
\]
and also so $X$ remains invariant. ■

Theorem 4.2.1.2. The product $R_nR_m$ of two reflections with axes $m$ and $n$ is:
(a) a translation mapping any point $P$ to a point $P'$ where $d^*(P,P') = 2d^*(m,n)$ if $n$ and $m$ are parallel ($d^*$ indicates directed distance) (like Fig.4.2.2 (a)) or
(b) a rotation with center $C$ and angle $\theta = 2|m(\angle (m,n))|$ if $n$ intersects $m$ at point $C$ (like Fig.4.2.2 (b)). Conversely, any translation or rotation can be written as the product $R_nR_m$ where lines $m$ and $n$ have the properties described in (a) and (b), respectively.
Thus, a direct isometry is the product of two reflections as illustrated in Fig.4.2.2. Here, as is seen, an indirect isometry is the product of one or three reflections (odd number of reflections). [5]

4.2.2 Glide Reflection

A glide reflection is a transformation consisting of a translation followed by a reflection in a line parallel to the slide arrow. In other words, a glide reflection with axis $l$ (see Fig.4.2.3) is the product of a reflection with axis $l$ and a nonidentity translation along $l$ (i.e., $l$ is invariant under the translation).

Reflections which keep a line pointwise invariant have been mentioned. But, glide reflections have an invariant line but no invariant points.

Examples

(1) (a) Find the matrix of $R_m$ where $m$ is the line $x_2 = (\sqrt{3}/3)x_1$. (b) Use this matrix to find $P'$, the image of the point $P(3, 7, 1)$ under this reflection.
Solution: (a) We can find this matrix by using the proof to Theorem 4.2.1.1:

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{12} & -a_{11} & a_{23} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
(\sqrt{3}/3)x_1 \\
1
\end{bmatrix}
= \begin{bmatrix}
x_1 \\
(\sqrt{3}/3)x_1 \\
1
\end{bmatrix}
\]

If we calculate this equation, we find that \(a_{11} = 1/2, a_{12} = \sqrt{3}/2\) and \(a_{13} = a_{23} = 0\). Hence, the matrix of \(R_m\) is

\[
\begin{bmatrix}
1/2 & \sqrt{3}/2 & 0 \\
\sqrt{3}/2 & -1/2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Also can be obtained by calculating \(SR_xS^{-1}\).

(b) \(P' = AP = \begin{bmatrix}
1/2 & \sqrt{3}/2 & 0 \\
\sqrt{3}/2 & -1/2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
3 \\
7 \\
1
\end{bmatrix}
= \begin{bmatrix}
(3 + 7\sqrt{3})/2 \\
(-7 + 3\sqrt{3})/2 \\
1
\end{bmatrix}
\]

(2) Find a product of a translation, a rotation, and a reflection that maps \(\triangle PQR\) to \(\triangle P'Q'R'\) where \(P(-2,5,1), Q(-2,7,1), R(-5,5,1), P'(4,3,1), Q'(6,3,1)\) and \(R'(4,0,1)\).

Solution: Firstly, we should place this triangle, \(\triangle PQR\), to the origin by a translation the triangle 2 units right and 5 units down (like in Fig.4.2.4), and then reflection with axis \(y\), and finally 90° rotation and again translate 4 units right and 3 units up.

Here, \(S\) includes both 90° rotation and the last translation.

\[
S = \begin{bmatrix}
\cos 90° & -\sin 90° & 0 \\
\sin 90° & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 4 \\
0 & 1 & 3 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 4 \\
-1 & 0 & 3 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
M = SR_yT = \begin{bmatrix}
0 & 1 & 4 \\
-1 & 0 & 3 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & -5 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & -1 \\
1 & 0 & 5 \\
0 & 0 & 1
\end{bmatrix}
\]

This matrix can be checked by using one of the points;

\[
MR = \begin{bmatrix}
0 & 1 & -1 \\
1 & 0 & 5 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-5 \\
5 \\
1
\end{bmatrix}
= \begin{bmatrix}
4 \\
0 \\
1
\end{bmatrix}
= R'
\]
(3) Reflect the point $P(3,4)$ in the $y$ axis, to get $P'$. Reflect $P'$ in the $x$ axis to get $P''$. Compare $P''$ to the result of rotating $P$ through $180^\circ$ to obtain $Q$. Do all calculations with matrices and vectors.

**Solution:** First we do the reflections

$$P' = R_yP = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

$$P'' = R_xP' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}.$$

Now for the rotation, substitute $180^\circ$ into the rotation matrix and multiply by $P$:

$$Q = R_{0(180^\circ)}P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

These calculations show that $P'' = Q$.

**Theorem 4.2.2.1.** The product of two direct or two indirect isometries is a direct isometry. The product of a direct and an indirect isometry in either order is an indirect isometry. This is coming from $detA \cdot detB = detAB$ [15].
Definition 4.2.2.1. Two sets of points $\alpha$ and $\beta$ are congruent, denoted $\alpha \simeq \beta$, if $\beta$ is the image of $\alpha$ under an isometry.

Fixed points and fixed lines of isometries are summarized in Table 3.

Table 3: Plane Isometries

<table>
<thead>
<tr>
<th>DIRECT</th>
<th>FIXED POINTS</th>
<th>FIXED LINES</th>
</tr>
</thead>
<tbody>
<tr>
<td>IDENTITY</td>
<td>✓</td>
<td>All</td>
</tr>
<tr>
<td>ROTATION</td>
<td>✓</td>
<td>One</td>
</tr>
<tr>
<td>TRANSLATION</td>
<td>✓</td>
<td>None</td>
</tr>
<tr>
<td>REFLECTION</td>
<td>X</td>
<td>all points on a line</td>
</tr>
<tr>
<td>GLIDE REFLECTION</td>
<td>X</td>
<td>None</td>
</tr>
</tbody>
</table>

Also, if you want to have a good grasp of this subject, I recommend the following worksheets.\(^6\)

Example: Find both a direct and an indirect isometry that map $X(0, 0, 1)$ and $Y(2, 0, 1)$ to $X'(1, 1, 1)$ and $Y'(3, 1, 1)$. What happens to the points $Z(1, -1, 1)$ under each of these isometries?

Solution: Firstly, we will find the direct isometry and thus we will use matrix of direct isometry:

$$A_d X = X'$$

\[ A_d = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ -a_{12} & a_{11} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}, \]

\[ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \]

\(^6\)You can find one of them in this adress http://www.superteacherworksheets.com/geometry/translation-rotation-reflection-1_TZQTQ.pdf
if we solve this matrix system, we find $a_{13} = a_{23} = 1$. And now, we will apply the same way for $Y$ and $Y'$ by writing $a_{13} = a_{23} = 1$, $A_dY = Y'$

\[
\begin{bmatrix}
a_{11} & a_{12} & 1 \\
-a_{12} & a_{11} & 1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} =
\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix},
\]

if we solve this matrix system, we find $a_{11} = 1$ and $a_{12} = 0$.

Hence, the direct isometry is $A_d =
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}$.

Secondly, we will find indirect isometry by using matrix of indirect isometry:

$A_iX = X'$ (i denotes indirect isometry)

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{12} & -a_{11} & a_{23} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} =
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},
\]

if we solve this matrix system, we find $a_{13} = a_{23} = 1$. And now, we will apply the same way for $Y$ and $Y'$ by writing $a_{13} = a_{23} = 1$, $A_iY = Y'$

\[
\begin{bmatrix}
a_{11} & a_{12} & 1 \\
a_{12} & -a_{11} & 1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} =
\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix},
\]

if we solve this matrix system, we find $a_{11} = 1$ and $a_{12} = 0$.

Hence, the indirect isometry is $A_i =
\begin{bmatrix}
1 & 0 & 1 \\
0 & -1 & 1 \\
0 & 0 & 1
\end{bmatrix}$.

Finally, we will find $Z'$ under $A_d$ and $A_i$ (see Fig.4.2.5);

$A_dZ = Z'$,

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} =
\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},
\]

$A_iZ = Z'$,

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & -1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} =
\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.
\]
Our goal in this section is to define and illustrate a method for characterizing the kind of symmetry a pattern has. To make things easy, we will stick to patterns like the one in Figure 5.1, which are called strip patterns. A strip pattern lies between two parallel horizontal lines and repeats itself if we move along just the right amount from left to right.

A symmetry transformation is an isometry that is identified with a specific figure. For example, the rotation $R = R_{C,90}$ maps each point of the design shown in Fig. 5.2 onto another point of the design. Furthermore, each rotation in the set $I, R, R^2, R^3$ is also a symmetry of this figure. This group of symmetries is known as the symmetry group of this figure.

**Definition 5.1.** If $\alpha$ is a set of points and $T$ is an isometry such that $T(\alpha) = \alpha$, then $T$ is a symmetry of $\alpha$.

**Theorem 5.1.** The set of all symmetries of a set of points form a group.
To find the symmetry group of a line segment $\overline{PQ}$, we must determine which isometries keep $\overline{PQ}$ invariant. Other than the identity, the only direct isometry that has this property is the rotation with center at the midpoint of $\overline{PQ}$ and angle $180^\circ$. Such rotations are known as half-turns.

**Half-turns:** A rotation with center $C$ and angle $180^\circ$ is a half-turn with center $C$, denoted $H_C$.

The only indirect symmetries of $\overline{PQ}$ are the two reflections with axes $m = PQ$ and $n$=perpendicular bisector of $\overline{PQ}$.

**Definition 5.2.** Let $\alpha$ be a set of points. If $H_P$ is a half-turn with center $P$ such that $H_P(\alpha) = \alpha$, then $P$ is a point of symmetry for $\alpha$. If $R_m$ is a reflection with axis $m$ such that $R_m(\alpha) = \alpha$, then $m$ is a line of symmetry of $\alpha$.

**Definition 5.3.** If every element of a group $G$ is a product of the elements $T_1, T_2, ..., T_n$ then $G$ is generated by $T_1, T_2, ..., T_n$, denoted by $G = \langle T_1, T_2, ..., T_n \rangle$.

These symmetry groups cannot contain a nonidentity translation or glide reflection, their only elements are rotations and reflections. It is possible to show that any of these finite groups can be generated by either one or two symmetries. For this reason, they are known as *cyclic* and *dihedral groups*. [5]

These are figures whose symmetry groups do include translations. A pattern characterized by remaining invariant under some shortest nonidentity translation is known as a *frieze pattern*. Frieze patterns are commonly found in wallpaper borders, designs on pottery, decorative designs on buildings,
needlepoint stitches, ironwork railings and in many other places.

A group of isometries that keeps a given line $c$ invariant and translations form an infinite cyclic subgroup is a **frieze group** with center $c$.

The translation that generates the cyclic subgroup of a frieze group is the "shortest" translation referred to in the description of frieze patterns. In the remainder of this section, we will denote this shortest translation by $\tau$.

If $T_{A,B}$ is a translation mapping $A$ to $B$, then $d(A, B)$ is called the **length of the translation**. If $d(A, B) < d(C, D)$, then $T_{A,B}$ is shorter than $T_{C,D}$.

**The Seven Possible Frieze Groups**

1. $F_1$ contains only a translation symmetry.

   ![F1 Pattern](image)

   $F_1$ Pattern [18]

2. $F_2$ contains translation and rotation (by a half-turn) symmetries.

   ![F2 Pattern](image)

   $F_2$ Pattern

3. $F_3$ contains translation and horizontal reflection symmetries.

   ![F3 Pattern](image)

   $F_3$ Pattern

4. $F_4$ contains translation and vertical reflection symmetries.
5. $F_5$ contains all symmetries (translation, horizontal and vertical reflection, and rotation).

6. $F_6$ contains translation, glide and vertical reflection and rotation (by a half-turn) symmetries.

7. $F_7$ contains translation and glide reflection symmetries.
The seven possible frieze patterns determined by these frieze groups along with their types of symmetry are displayed in Table 4.

<table>
<thead>
<tr>
<th>Frieze pattern</th>
<th>Point</th>
<th>Center Line</th>
<th>Perpendicular Line</th>
<th>Glide Reflection</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>✔</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F_2$</td>
<td></td>
<td>✔</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F_3$</td>
<td></td>
<td></td>
<td>✔</td>
<td></td>
</tr>
<tr>
<td>$F_4$</td>
<td></td>
<td></td>
<td></td>
<td>✔</td>
</tr>
<tr>
<td>$F_5$</td>
<td>✔</td>
<td>✔</td>
<td></td>
<td>✔</td>
</tr>
<tr>
<td>$F_6$</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>$F_7$</td>
<td>✔</td>
<td></td>
<td></td>
<td>✔</td>
</tr>
</tbody>
</table>

We will refer to half-turn reflection instead of point type of symmetry, horizontal reflection instead of center line type of symmetry and vertical reflection instead of perpendicular line type of symmetry differently from Table 4.

For another example of frieze groups see Fig.5.3.

When we look at Fig.5.3, the first figure is frieze pattern $F_1$ (because, it has just translation), the second figure is frieze pattern $F_7$ (because, it has translation and glide reflection), the third figure is frieze pattern $F_3$ (because, it has translation and horizontal reflection), the forth figure is frieze pattern
$F_4$ (because it has translation and vertical reflection), the fifth figure is frieze pattern $F_7$ (because it has translation and half-turn reflection), the sixth figure is frieze pattern $F_6$ (because, it has translation, half-turn reflection, vertical reflection and glide reflection) and the last figure is frieze pattern $F_5$ (because, it has translation, half-turn reflection, horizontal reflection, vertical reflection and glide reflection).

6 Similarity Transformations

In this section we will determine which properties of $V^*$ remain invariant under one-to-one linear transformations that preserve ratios of distance. Two figures are called similar if they have the same shape but have different sizes. A similarity transformation is a rigid motion, or isometry, together with a rescaling. In other words, a similarity transformation may alter both position and size, but preserves shape (see Fig. 6.1.).

![Figure 6.1. Matryoshka Dolls [20]](image)

A similarity with ratio $r$ is one-to-one linear transformation $T$ of $V^*$ onto itself such that for each pairs of points, $P, Q$ $d^*(T(P), T(Q)) = rd^*(P, Q)$ for some nonzero real number $r$ where $d^*$ denotes directed distance.

**Example:** In Fig.6.2, $\triangle ABC$ is similar with $\triangle AED$ and ratio:

$$r = \frac{d^*(A, E)}{d^*(A, B)} = \frac{d^*(A, D)}{d^*(A, C)} = \frac{d^*(D, E)}{d^*(C, B)} = \frac{3}{2}$$

Every isometry is a similarity with ratio $\pm 1$. Since similarities are one-to-one linear transformations of $V^*$ they have also $3 \times 3$ matrix representations with corresponding point and line equations $X' = AX$ and $ku' = uA^{-1}$. There are also direct and indirect similarities and the set of similarities form a group.
Theorem 6.1. A similarity with ratio $r$ has one of the following matrix representations;

$$
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  -a_{12} & a_{11} & a_{23} \\
  0 & 0 & 1
\end{bmatrix}
$$

or

$$
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{12} & -a_{11} & a_{23} \\
  0 & 0 & 1
\end{bmatrix}
$$

where $a_{21}^2 + a_{12}^2 = r^2$.

Figures that correspond to each other under a similarity are said to be similar. The verification that similar triangles do indeed have angles of the same measure and sides of proportional measure is nearly a replication of the proofs of comparable theorems for congruent triangles (see Section 4).

Let $u'$ and $v'$ be the images of lines $u$ and $v$ under a similarity. If the similarity is direct then $m(\angle(u', v')) = m(\angle(u, v))$. If the similarity is indirect then $m(\angle(u', v')) = -m(\angle(u, v))$.

Theorem 6.2. If $\triangle PQR \sim \triangle P'Q'R'$ then there exist an $r \neq 0$ such that $m(\overline{PQ'}) = rm(\overline{PQ})$, $m(\overline{QR'}) = rm(\overline{QR})$, $m(\overline{RP'}) = rm(\overline{RP})$, $m(\angle P'Q'R') = \mp m(\angle PQR)$, $m(\angle Q'R'P') = \mp m(\angle QRP)$ and $m(\angle R'P'Q') = \mp m(\angle RPQ)$.

Definition 6.1. Let $C$ be an arbitrary point and $r$ a nonzero real number. A dilation with center $C$ and ratio $r$, denoted $D_{C,r}$, is a direct similarity with ratio $r$ and invariant point $C$ that maps any point $P$ to a point $P'$ on line $CP$ (see Fig.6.3). Dilations are also called dilatations or central similarities.
**Theorem 6.3.** Under a dilation \( D_{C,r} \) the point \( C \) and each line incident with \( C \) are invariant.

**Theorem 6.4.** A dilation with center \( O(0,0,1) \) and ratio \( r \) has matrix representation

\[
\begin{bmatrix}
  r & 0 & 0 \\
  0 & r & 0 \\
  0 & 0 & 1 \\
\end{bmatrix},
\]

A dilation with center \( C(c_1,c_2,1) \) has matrix representation

\[
\begin{bmatrix}
  r & 0 & c_1(1-r) \\
  0 & r & c_2(1-r) \\
  0 & 0 & 1 \\
\end{bmatrix}.
\]

**Theorem 6.5.** If \( D_{C,r} \) is a dilation with \( r \neq 1 \) and \( m \) is a line not incident with \( C \), then \( D_{C,r}(m) = m' \) is a distinct line parallel to \( m \).

**Proof.** The line equation of this dilation requires the matrix \((D_{C,r})^{-1}\). Since this transformation is also a dilation with center \( C \) and ratio \( r' = 1/r \), its matrix representation is given by Theorem 6.4. Using this matrix in the line equation of the dilation, we can find \( m' \), the image of the line \( m[m_1,m_2,m_3] \) as follows:

\[
[m_1,m_2,m_3] \begin{bmatrix}
  r' & 0 & c_1(1-r') \\
  0 & r' & c_2(1-r') \\
  0 & 0 & 1 \\
\end{bmatrix} = [r'm_1,r'm_2,m'_3]
\]

where \( m'_3 = m_1c_1(1-r') + m_2c_2(1-r') + m_3 \).

Clearly, \( m' = [r'm_1,r'm_2,m'_3] \) is equal to \( m \) iff \( m'_3 = r'm_3 \), that is, iff

\[
m_1c_1(1-r') + m_2c_2(1-r') + m_3 = r'm_3
\]

or

\[
m_1c_1(1-r') + m_2c_2(1-r') + m_3(1-r') = 0
\]

or

\[
m_1c_1 + m_2c_2 + m_3 = 0 \text{ since } r' \neq 1;
\]
but this is exactly the condition that makes $m$ incident with $C$. Thus if
$m$ is not incident with $C$, $m'$ is necessarily a distinct line parallel to $m$. ■

**Theorem 6.6.** Every similarity can be expressed as the product of a
dilation and an isometry (see the next example 3).

**Theorem 6.7.** If $\triangle PQR$ and $\triangle P'Q'R'$ are two triangles with $m(\overline{PQ}) = rm(\overline{PQ})$, $m(\overline{QR}) = rm(\overline{RP})$, and also $m(\angle P'Q'R') = \mp m(\angle PQR)$, $m(\angle Q'R'R') = \mp m(\angle QRP)$ and $m(\angle R'P'Q') = \mp m(\angle RPQ)$, then there is a similarity mapping $\triangle PQR$ to $\triangle P'Q'R'$.

We will summarize isometries and dilation in Table 5.

<table>
<thead>
<tr>
<th>Line Reflection</th>
<th>Point Reflection</th>
<th>Translations</th>
<th>Rotations</th>
<th>Dilations</th>
<th>Glide Reflection</th>
</tr>
</thead>
<tbody>
<tr>
<td>Indirect Isometry</td>
<td>Direct Isometry</td>
<td>Direct Isometry</td>
<td>Direct Isometry</td>
<td>Not Isometry</td>
<td>Indirect Isometry</td>
</tr>
<tr>
<td>1. distance</td>
<td>1. distance</td>
<td>1. distance</td>
<td>1. distance</td>
<td>1. distance</td>
<td>1. distance</td>
</tr>
<tr>
<td>2. angle measure</td>
<td>2. angle measure</td>
<td>2. angle measure</td>
<td>2. angle measure</td>
<td>2. angle measure</td>
<td>2. angle measure</td>
</tr>
<tr>
<td>3. parallelism</td>
<td>3. parallelism</td>
<td>3. parallelism</td>
<td>3. parallelism</td>
<td>3. parallelism</td>
<td>3. parallelism</td>
</tr>
<tr>
<td>Reverse Orientation</td>
<td>Same Orientation</td>
<td>Same Orientation</td>
<td>Same Orientation</td>
<td>Same Orientation</td>
<td>Reverse Orientation</td>
</tr>
</tbody>
</table>

Table 5 [21]

**Examples**

(1) Let $C$, $P$ and $P'$ be points with coordinates $C(3, -2, 1)$, $P(1, 0, 1)$ and $P'(7, -6, 1)$.
(a) Show that these three points are collinear. (b) Find the matrix of a dilation with center $C$ that maps $P$ to $P'$. (c) Find the image of lines $m[1, 1, -1]$ and $n[1, 1, 1]$ under this dilation.

**Solution:** (a) If these three points are collinear, the following determi-
nant should be equal to zero (see Fig.6.4):
\[
\begin{vmatrix} 3 & 1 & 7 \\ -2 & 0 & -6 \\ 1 & 1 & 1 \end{vmatrix} = 0
\]

Figure 6.4

(b) Firstly, we should find ratio \( r \). If we calculate the distance between \( P \) and \( C \);
\[
d(C, P) = \sqrt{(c_1 - p_1)^2 + (c_2 - p_2)^2} = \sqrt{(3 - 1)^2 + (-2 - 0)^2} = 2\sqrt{2}
\]
and if we calculate the distance between \( C \) and \( P' \);
\[
d(C, P') = \sqrt{(c_1 - p'_1)^2 + (c_2 - p'_2)^2} = \sqrt{(3 - 7)^2 + (-2 - (-6))^2} = 4\sqrt{2}.
\]

The distance between \( P \) and \( C \) is two times of the distance between \( C \) and \( P' \), but in reverse direction. We mean, \( d(CP') = 2d(CP) \) and \( P \) and \( P' \) lie on opposite sides of \( C \), so \( r = -2 \). The matrix of a dilation with center \( C \) can be found by using Theorem 6.4.
\[
\begin{bmatrix} r & 0 & c_1(1-r) \\ 0 & r & c_2(1-r) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 3(1+2) \\ 0 & -2 & -2(1+2) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 9 \\ 0 & -2 & -6 \\ 0 & 0 & 1 \end{bmatrix}
\]
(c) We can find the image of lines by using Theorem 6.5. 
(here, \( r' = 1/r = -1/2 \))

\[
\begin{bmatrix}
m_1', m_2', m_3' \\
\end{bmatrix} = [1, 1, -1]
\begin{bmatrix}
-1/2 & 0 & 9/2 \\
0 & -1/2 & -3 \\
0 & 0 & 1 \\
\end{bmatrix}
= [-1/2, -1/2, 1/2]
\]

So the line is invariant under the transformation.

\[
\begin{bmatrix}
n_1', n_2', n_3' \\
\end{bmatrix} = [1, 1, 1]
\begin{bmatrix}
-1/2 & 0 & 9/2 \\
0 & -1/2 & -3 \\
0 & 0 & 1 \\
\end{bmatrix}
= [-1/2, -1/2, 5/2]
\]

As a conclusion, in accordance with Theorem 6.5. \( n' \) is parallel to \( n \).

(2) Show that a rotation with angle 180° is a dilation with center is (0, 0) and \( r = -1 \).

Solution:

\[
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\cos(-180)^\circ & -\sin(-180)^\circ & 0 \\
\sin(-180)^\circ & \cos(-180)^\circ & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
r & 0 & 0 \\
0 & r & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

(3) Find the product of a translation, rotation, and dilation that maps \( \triangle PQR \) to \( \triangle P'Q'R' \) where \( P(3, 6, 1), Q(-2, 5, 1), R(-3, -1, 1), P'(0, 0, 1), Q'(2, -10, 1), R'(14, -12, 1). \) [Hint: firstly, translate \( P \) to \( P' \)]

Solution: Firstly, we will translate the triangle 3 units left and 6 units down and so the translation matrix (see Fig.6.5):

\[
\begin{bmatrix}
1 & 0 & -3 \\
0 & 1 & -6 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Secondly, we will rotate with angle 90° and so the rotation matrix:

\[
\begin{bmatrix}
\cos(-90)^\circ & -\sin(-90)^\circ & 0 \\
\sin(-90)^\circ & \cos(-90)^\circ & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]
Thirdly, we will find the dilation with center $O(0,0,1)$ and ratio $r = 2$:

$$
\begin{bmatrix}
    r & 0 & 0 \\
    0 & r & 0 \\
    0 & 0 & 1 \\
\end{bmatrix} =
\begin{bmatrix}
    2 & 0 & 0 \\
    0 & 2 & 0 \\
    0 & 0 & 1 \\
\end{bmatrix}.
$$

And the last, we will find the product of all matrices:

$$
\begin{bmatrix}
    2 & 0 & 0 \\
    0 & 2 & 0 \\
    0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
    0 & -1 & 0 \\
    1 & 0 & 0 \\
    0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
    1 & 0 & -3 \\
    0 & 1 & -6 \\
    0 & 0 & 1 \\
\end{bmatrix} =
\begin{bmatrix}
    0 & -2 & 12 \\
    2 & 0 & -6 \\
    0 & 0 & 1 \\
\end{bmatrix}.
$$

Let us finally check if $R$ maps to $R'$.

$$
\begin{bmatrix}
    0 & -2 & 12 \\
    2 & 0 & -6 \\
    0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
    -3 \\
    -1 \\
    1 \\
\end{bmatrix} =
\begin{bmatrix}
    14 \\
    -12 \\
    1 \\
\end{bmatrix} = R'.
$$

Figure 6.5
7 Affine Transformation

An affine transformation is any transformation of $V^*$ that preserves collinearity (i.e., all points lying on a line initially still lie on a line after transformation) and ratios of distances (e.g., the midpoint of a line segment remains the midpoint after transformation). In other words, affine transformation preserves parallelness among lines. So, if $l$ and $m$ are parallel lines and $\alpha$ is an affine transformation, then lines $\alpha(l)$ and $\alpha(m)$ are parallel. An affine transformation is also called an affinity.

In Fig.7.1, an enjoyable illustration of affinity is shown.

![Figure 7.1](image)

Dilation, reflection, rotation, shear, similarity transformations, and translation are all affine transformations, as are their combinations. In general, an affine transformation is a composition of rotations, translations, dilations, and shears. While an affine transformation preserves proportions on lines, it does not necessarily preserve angles or lengths. [22]

The affinities maps points according to the matrix equation $X' = AX$ in $V^*$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad |A| \neq 0. \]

**Theorem 7.1.** If $T$ is an affinity and $P, Q$ and $R$ are three distinct collinear points such that

$$ \frac{d(Q,P)}{d(Q,R)} = k \quad \text{then} \quad \frac{d(T(Q),T(P))}{d(T(Q),T(R))} = k. $$

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The images are also collinear (See Fig.7.2 for an example).

![Figure 7.2](image)

Affinities also preserve segments and their midpoints.

There are two specific types of affinities: Shears and Strains.

### 7.1 Shear

A shear with axis \( m \), denoted \( S_m \), is an affinity that keeps \( m \) pointwise invariant and maps every other point \( P \) to a point \( P' \) so that the line \( PP' \) is parallel to \( m \) (See Fig.7.3).

![Figure 7.3](image)

**Theorem 7.1.1** The matrix representation of a shear with axis \( x[0,1,0] \) is

\[
\begin{bmatrix}
1 & j & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

In general the matrix representation of a shear \( S_m \) can be found using \( S_m = SS_xS^{-1} \) where \( S \) is a direct isometry mapping \( x \) to \( m \) \((S(x) = m). [5]

### 7.2 Strain

A strain with axis \( m \), denoted \( T_m \), keeps \( m \) pointwise invariant and maps every other point \( P \) to a point \( P' \) so that the line \( PP' \) is perpendicular to \( m \).
Theorem 7.2.1 The matrix representation of a strain with axis $x[0, 1, 0]$ is
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & k & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]
In general, the matrix representation of strain $T_m$ can be found using $T_m = ST_xS^{-1}$ where $S$ is a direct isometry mapping $x$ to $m$ ($S(x) = m$).

Theorem 7.2.2 Any affinity can be written as the product of a shear, a strain and a direct similarity.

Proof: We can verify this theorem by merely demonstrating that the following product does indeed yield the matrix of a general affinity as indicated:
\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
a_{11} & -a_{21} & a_{13} \\
a_{21} & a_{11} & a_{23} \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & k & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & j & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
where $j = \frac{a_{11}a_{22} + a_{21}a_{22}}{a_{11}a_{21} + a_{21}^2}$ and $k = \frac{a_{11}a_{22} - a_{12}a_{22}}{a_{11}a_{21} + a_{21}^2}$. ■

Note that a strain is a reflection if $k < 0$.

Examples

1.) Show that a dilation with center $O$ is the product of strains with axes $x[0, 1, 0]$ and $y[1, 0, 0]$.

Solution: The strain with axis $x$: \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & k & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
and the strain with axis $y$: \[
\begin{bmatrix}
k & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
and the product of them is a dilation;
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & k & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & k & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
k & 0 & 0 \\
0 & k & 0 \\
0 & 0 & 1 \\
\end{bmatrix}.
\]

2.) Find the matrix of an affinity mapping \(P(1, -1, 1), Q(2, 1, 1)\) and \(R(3, 0, 1)\) to \(P'(0, 1, 1), Q'(1, 2, 1)\) and \(R'(0, 3, 1)\), respectively.

**Solution:** We can denote with \(A\) the matrix of an affinity mapping and \(A\) provide the following: \(AP = P', AQ = Q', AR = R'\). Now, we can find this matrix;
\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
-1 \\
1 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
1 \\
1 \\
\end{bmatrix},
\]
\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
2 \\
1 \\
1 \\
\end{bmatrix}
= \begin{bmatrix}
1 \\
2 \\
1 \\
\end{bmatrix},
\]
and
\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
3 \\
0 \\
1 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
3 \\
1 \\
\end{bmatrix}.
\]

After tedious calculations, we find the matrix \(A\):
\[
A = \begin{bmatrix}
-1/3 & 2/3 & 1 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}.
\]

Also, we can solve this problem in another way: (see Fig.7.5)

- First step: Translation (-1,1) and the matrix is \(M_1 = \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}\).

We can find \(P_1, Q_1, R_1\) by using \(M_1P = P_1, M_1Q = Q_1, M_1R = R_1\) (Theorem 3.1). Hence, \(P_1 = (0, 0), Q_1 = (1, 2), R_1 = (2, 1)\).
• Second step: Rotation approximately 63.4° and the matrix is

\[ M_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

We can find \( P_2, Q_2, R_2 \) by using \( M_2P_1 = P_2, M_2Q_1 = Q_2, M_2R_1 = R_2 \). Hence, \( P_2 = (0, 0), Q_2 = (-3/\sqrt{5}, 4/\sqrt{5}), R_2 = (0, \sqrt{5}) \).

• Third step: Reflection \( R_y \) with axis \( y \) and the matrix is \( M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

We can find \( P_3, Q_3, R_3 \) by using \( M_3P_2 = P_3, M_3Q_2 = Q_3, M_3R_2 = R_3 \). Hence, \( P_3 = (0, 0), Q_3 = (3/\sqrt{5}, 4/\sqrt{5}), R_3 = (0, \sqrt{5}) \).

• Forth step: Shear with axis \( y \) and the matrix is

\[ \begin{bmatrix} 1 & 0 & 0 \\ j & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

We can find \( j \) by using \( Q_3; \)

\[ \begin{bmatrix} 1 & 0 & 0 \\ j & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3/\sqrt{5} \\ 4/\sqrt{5} \\ 1 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{5} \\ 4/\sqrt{5} + j3/\sqrt{5} \\ 1 \end{bmatrix} \]

\( 4/\sqrt{5} + j3/\sqrt{5} = \sqrt{5}/2 \), and so \( j = -0.5 \).

Hence, \( M_4 = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \).

We can find \( P_4, Q_4, R_4 \) by using \( M_4P_3 = P_4, M_4Q_3 = Q_4, M_4R_3 = R_4 \). Hence, \( P_4 = (0, 0), Q_4 = (3/\sqrt{5}, 5/\sqrt{5}/2), R_4 = (0, \sqrt{5}) \).

• Fifth step: Strain with axes \( x \) and \( y \) and the matrix is \( M_5 = \begin{bmatrix} \sqrt{5}/3 & 0 & 0 \\ 0 & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

We can find \( P_5, Q_5, R_5 \) by using \( M_5P_4 = P_5, M_5Q_4 = Q_5, M_5R_4 = R_5 \). Hence, \( P_5 = (0, 0), Q_5 = (1, 1), R_5 = (0, 2) \).

• Sixth step: Translation \((0,1)\) and the matrix is \( M_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \)

We can find \( P_6, Q_6, R_6 \) by using \( M_6P_5 = P_6, M_6Q_5 = Q_6, M_6R_5 = R_6 \). Hence, \( P_6 = (0, 1) = P', Q_6 = (1, 2) = Q', R_6 = (0, 3) = R' \).
Hence, the product of these 6 matrices is

\[
\begin{bmatrix}
-1/3 & 2/3 & 1 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} = A = M_6 M_5 M_4 M_3 M_2 M_1.
\]
8 Conclusion

This paper has attempted to provide general overview of geometric transformations. In addition to theoretical knowledge, we focused on visuality. It is enjoyable for everybody to learn this subject. The paper also described some applications in daily life where transformations could be commonly used. Because, after this research we realized that everything around us was related with geometric transformations. For example; chinaware, carpets, wallpapers, iron gates, etc... Even if, while looking to the sea or lake, we can see transformations, especially reflections. This paper was intended to convey further understanding of geometrical transformations.

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References


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[15] https://www2.bc.edu/~reederma/llinalg3.pdf